Dynamic topological logics over spaces with continuous functions

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abstract. Dynamic topological logics are combinations of topological and temporal modal logics that are used for reasoning about dynamical systems consisting of a topological space and a continuous function on it. Here we partially solve a major open problem in the field by showing (by reduction of the ω-reachability problem for lossy channel systems) that the dynamic topological logic over arbitrary topological spaces as well as those over $\mathbb{R}^n$, for each $n \geq 1$, are undecidable. Actually, we prove this result for the natural and expressive fragment of the full dynamic topological language where the topological operators cannot be applied to formulas containing the temporal eventuality. Using Kruskal's tree theorem we also show that the formulas of this fragment that are valid in arbitrary topological spaces with continuous functions are recursively enumerable, which is not the case for spaces with homeomorphisms.

Keywords: dynamic topological logic, spatial logic, temporal logic.

1 Introduction

Dynamic topological logics were introduced in [11, 12, 14, 2] with the aim of logical modelling of and reasoning about the asymptotic properties of continuous functions on topological spaces, that is, orbits $\{f(x), f^2(x), \dots\}$ of points x in a topological space under a continuous function f. The full dynamic topological language $\mathcal{DTL}$ of these logics can be regarded as a natural combination of the standard modal language of $\mathcal{S}4$ interpreted over topological spaces and the propositional temporal language of $\mathcal{LTL}$ interpreted over $\langle \mathbb{N}, < \rangle$. For instance, the $\mathcal{DTL}$-formulas

$$\bigcirc \mathbf{I} \varphi \qquad \text{and} \qquad \mathbf{I} \Box_F \varphi \tag{1}$$

are interpreted in a dynamical system $\langle \langle T, \mathbb{I} \rangle, f \rangle$ (where $\mathbb{I}$ is the interior operator on the space T and f is a continuous function on $\langle T, \mathbb{I} \rangle$) under a valuation $\mathfrak{V}$ as, respectively, the sets

$$f^{-1}(\mathbb{I}\mathfrak{V}(\varphi)) \qquad \text{and} \qquad \mathbb{I} \bigcap_{n=1}^{\infty} f^{-n}(\mathfrak{V}(\varphi)).$$

The available body of knowledge in this area can be roughly classified as follows.

The fragment $\mathcal{DTL}^{\bigcirc}$ of $\mathcal{DTL}$ with sole temporal operator $\bigcirc$:

- The sets of $\mathcal{DTL}^{\bigcirc}$-formulas that are valid in dynamic topological systems (DTSs, for short) with *homeomorphisms* based on (i) arbitrary topological spaces, (ii) Aleksandrov spaces, (iii) $\mathbb{R}^n$, for each $n \geq 1$, coincide, enjoy the finite model property (fmp), are finitely axiomatisable and decidable [2, 14, 13, 9].

- The sets of $\mathcal{DTL}^{\bigcirc}$-formulas that are valid in DTSs (with *continuous functions*, not only homeomorphisms) based on (i) topological spaces and (ii) Aleksandrov spaces coincide, enjoy the fmp, are finitely axiomatisable, and so decidable [2, 13, 5]. An interesting and challenging open problem here is to investigate the set of $\mathcal{DTL}^{\bigcirc}$-formulas that are valid in $\mathbb{R}$.

The fragment $\mathcal{DTL}_0$ of $\mathcal{DTL}$ where no temporal operator can occur in the scope of a topological operator: The logic in this language over arbitrary topological spaces and continuous functions was axiomatised in [13]. It is easy to see, in fact, that this logic is decidable and coincides with the corresponding logics over Aleksandrov spaces and $\mathbb{R}^n$, $n \geq 1$, no matter whether the functions are continuous or homeomorphisms only.

Full $\mathcal{DTL}$:

- The sets of $\mathcal{DTL}$-formulas that are valid in DTSs with *homeomorphisms* based on (i) topological spaces, (ii) Aleksandrov spaces, (iii) $\mathbb{R}^n$, for each $n \geq 1$, are all pairwise distinct and are not recursively enumerable [9]. It is of interest to note that the logics over (i)–(iii) with *finitely many iterations* of homeomorphisms coincide, but are still not recursively enumerable [9].

- Only two results have been known for the logics of DTSs with *continuous functions*. First, the logics over arbitrary and Aleksandrov spaces with *finitely many iterations* coincide and are decidable, but not in primitive recursive time [8]. And second, the logic over Aleksandrov spaces turns out to be undecidable [10]. The major open problem has been to investigate the computational properties (decidability, axiomatisability, etc.) of the logic over arbitrary topological spaces and the Euclidean spaces $\mathbb{R}^n$.

In this paper we present a partial solution to this open problem by showing that the logics in question are undecidable.

One key observation that has led to this result was the following fact discovered in [7] (and probably elsewhere) in a somewhat different context: the fragment $\mathcal{DTL}_1$ of $\mathcal{DTL}$, where we are not allowed to apply the topological operators to formulas containing $\Box_F$ or $\Diamond_F$ (as in the latter formula in (1)) but can have $\bigcirc$ in the scope of $\mathbf{I}$, is still expressive enough to encode various undecidable problems. But, on the other hand, it is not sufficiently strong to distinguish between arbitrary and Aleksandrov topological spaces. Clearly,

$\mathcal{DTL}_1$ extends both $\mathcal{DTL}^{\bigcirc}$ and $\mathcal{DTL}_0$. It follows from [9] that the sets of $\mathcal{DTL}_1$-formulas valid in (i) topological spaces, (ii) Aleksandrov spaces, (iii) $\mathbb{R}^n$, for each $n \geq 1$, with *homeomorphisms* are still not recursively enumerable (even for systems with finitely many iterations).

The first main result of this paper is that the undecidable ω-reachability problem for lossy channel systems can be reduced, via $\mathcal{DTL}_1$-formulas, to satisfiability in Aleksandrov (and so arbitrary topological) spaces. On the other hand, using Kruskal's tree theorem, we show that the set of valid $\mathcal{DTL}_1$-formulas is recursively enumerable (we actually conjecture that it is finitely axiomatisable). However, it is still an open problem whether the set of all valid $\mathcal{DTL}$-formulas is axiomatisable.

The second key idea is that the technique of embedding Aleksandrov spaces into $\mathbb{R}$ from [3] can be used to show that, given an arbitrary $\mathcal{DTL}_1$-formula φ, one can construct a formula $\varphi^{\mathbb{R}}$, the *relativisation* of φ, such that φ is satisfiable if and only if $\varphi^{\mathbb{R}}$ is satisfiable in the DTS $\langle\langle\mathbb{R}, \mathbb{I}_{\mathbb{R}}\rangle, x \mapsto x+1\rangle$. Therefore, the ω-reachability problem is in fact reduced to the satisfiability of relativised $\mathcal{DTL}_1$-formulas in Euclidean spaces. For the reader's convenience we summarise the results discussed above in the table below where merged cells mean that the corresponding logics coincide.

<table>
<tr><th rowspan="2">language</th><th rowspan="2">functions</th><th colspan="3">space</th></tr>
<tr><th>arbitrary</th><th>Aleksandrov</th><th>$\mathbb{R}^n$</th></tr>
<tr><td rowspan="2">$\mathcal{DTL}^{\bigcirc}$</td><td>continuous</td><td colspan="2">fmp, finite axiom. [2, 13, 5]</td><td>?</td></tr>
<tr><td>homeomorphisms</td><td colspan="3">fmp, finite axiom. [2, 14, 13, 9]</td></tr>
<tr><td rowspan="2">$\mathcal{DTL}_0$</td><td>continuous</td><td colspan="3" rowspan="2">decidable, finite axiom. (partly in [13])</td></tr>
<tr><td>homeomorphisms</td></tr>
<tr><td rowspan="2">$\mathcal{DTL}_1$</td><td>continuous</td><td colspan="2">undecidable, but r.e.</td><td>undecidable</td></tr>
<tr><td>homeomorphisms</td><td colspan="2">non-r.e. [9]</td><td>non-r.e. [9]</td></tr>
<tr><td rowspan="2">$\mathcal{DTL}_1$ / $\mathcal{DTL}$</td><td>continuous/
finite iterations</td><td colspan="2">decidable in
non-prim. recursive time [8]</td><td>?</td></tr>
<tr><td>homeomorphisms/
finite iterations</td><td colspan="3">non-r.e. [9]</td></tr>
<tr><td rowspan="2">$\mathcal{DTL}$</td><td>continuous</td><td>undecidable</td><td>undecidable</td><td>undecidable</td></tr>
<tr><td>homeomorphisms</td><td>non-r.e. [9]</td><td>non-r.e. [9]</td><td>non-r.e. [9]</td></tr>
</table>

2 $\mathcal{DTL}$

The full language $\mathcal{DTL}$ of *dynamic topological logic* is constructed in the usual way from a countably infinite set $\{p_1, p_2, \dots\}$ of *spatial variables*, the Booleans $\neg$, $\wedge$ (and their standard derivatives), the modal (or rather topological) operators $\mathbf{I}$ and $\mathbf{C}$, and the temporal operators $\bigcirc$ (next-time), $\Box_F$ (always in the future) and $\Diamond_F$ (eventually). More precisely, $\mathcal{DTL}$-*formulas* are given by the following definition:

$$\varphi \quad ::= \quad p_i \quad | \quad \neg\varphi \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \mathbf{I}\varphi \quad | \quad \bigcirc\varphi \quad | \quad \Box_F\varphi.$$

We use $\mathbf{C}\varphi$ and $\Diamond_F\varphi$ as abbreviations for $\neg\mathbf{I}\neg\varphi$ and $\neg\Box_F\neg\varphi$, respectively.

A *dynamic topological structure* (DTS, for short) is a pair of the form $\mathfrak{F} = \langle \mathfrak{T}, f \rangle$, where $\mathfrak{T} = \langle T, \mathbb{I} \rangle$ is a topological space with the interior operator $\mathbb{I}$, and $f \colon T \to T$ is a continuous function on $\mathfrak{T}$, that is, $f^{-1}(A)$ is open whenever $A \subseteq T$ is open (alternatively, $f^{-1}(\mathbb{I}A) \subseteq \mathbb{I}f^{-1}(A)$, for all $A \subseteq T$). A *dynamic topological model* (DTM, for short) is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, where $\mathfrak{F}$ is a DTS and $\mathfrak{V}$ a *valuation* assigning to each variable p_i a subset $\mathfrak{V}(p_i)$ of T. The *truth-relation* $(\mathfrak{M}, w) \models \varphi$, for $w \in T$, is defined inductively as follows:

- $(\mathfrak{M}, w) \models p_i \quad \text{iff} \quad w \in \mathfrak{V}(p_i)$,
- $(\mathfrak{M}, w) \models \neg\varphi \quad \text{iff} \quad (\mathfrak{M}, w) \not\models \varphi$,
- $(\mathfrak{M}, w) \models \varphi_1 \wedge \varphi_2 \quad \text{iff} \quad (\mathfrak{M}, w) \models \varphi_1$ and $(\mathfrak{M}, w) \models \varphi_2$,
- $(\mathfrak{M}, w) \models \mathbf{I}\varphi \quad \text{iff} \quad w \in \mathbb{I}\{v \in T \mid (\mathfrak{M}, v) \models \varphi\}$,
- $(\mathfrak{M}, w) \models \bigcirc\varphi \quad \text{iff} \quad (\mathfrak{M}, f(w)) \models \varphi$,
- $(\mathfrak{M}, w) \models \Box_F\varphi \quad \text{iff} \quad (\mathfrak{M}, f^n(w)) \models \varphi$ for all $n > 0$.

For a class $\mathcal{K}$ of DTSs, we denote by $\mathsf{Log}\,\mathcal{K}$, the *dynamic topological logic of* $\mathcal{K}$, the set of all $\mathcal{DTL}$-formulas φ such that $(\mathfrak{M}, w) \models \varphi$ for every DTM $\mathfrak{M}$ based on a DTS from $\mathcal{K}$ and every point w in $\mathfrak{M}$. If we are only interested in the restriction of $\mathsf{Log}\,\mathcal{K}$ to a certain fragment $\mathcal{DTL}_i$ of $\mathcal{DTL}$, then we write $\mathsf{Log}_i\,\mathcal{K}$ for $\mathsf{Log}\,\mathcal{K} \cap \mathcal{DTL}_i$.

In this paper, we deal with (i) the class $\mathcal{T}$ of DTSs based on arbitrary topological spaces, (ii) the classes $\mathcal{R}^n$ of DTSs based on the Euclidean $\mathbb{R}^n$, for $n \geq 1$, and (iii) the class $\mathcal{A}$ of DTSs based on Aleksandrov spaces. If we restrict these classes to DTSs with homeomorphisms then we write $\mathcal{T}_h$, $\mathcal{R}^n_h$, and $\mathcal{A}_h$, respectively.

We remind the reader that every quasi-order $\mathfrak{G} = \langle W, R \rangle$ (R is a reflexive and transitive relation on W) gives rise to the topological space $\mathfrak{T}_\mathfrak{G}$ over W consisting of all R-closed subsets of W. In other words, the interior operator $\mathbb{I}_\mathfrak{G}$ on $\mathfrak{T}_\mathfrak{G}$ can be defined by taking, for every $X \subseteq W$,

$$\mathbb{I}_\mathfrak{G}\, X = \{w \in X \mid \forall v \in W \ (wRv \to v \in X)\}.$$

Such spaces are known as *Aleksandrov spaces.* Alternatively they can be defined as topological spaces where arbitrary (not only finite) intersections of open sets are open; for details see [1, 4]. Clearly, for $\mathfrak{M} = \langle \langle \mathfrak{T}_\mathfrak{G}, f \rangle, \mathfrak{V} \rangle$,

$$(\mathfrak{M}, w) \models \mathbf{I}\varphi \quad \text{iff} \quad (\mathfrak{M}, v) \models \varphi \text{ for every } v \in W \text{ with } wRv,$$

It should be also clear that a function $f \colon W \to W$ is *continuous* on $\mathfrak{T}_\mathfrak{G}$ iff

$$wRv \qquad \text{implies} \qquad f(w)Rf(v),$$

for all $w, v \in W$. We call such a function f *monotonous.* A bijection f is a *homeomorphism* on $\mathfrak{T}_\mathfrak{G}$ iff both the implication above and its converse hold.

3 $\mathcal{DTL}_1$

Let us consider now the sublanguage $\mathcal{DTL}_1$ of $\mathcal{DTL}$ where no $\Box_F$ and $\Diamond_F$ can occur in the scope of $\mathbf{I}$ ($\bigcirc$ is allowed); more precisely, *$\mathcal{DTL}_1$-formulas* φ are defined as follows:

$$\psi \ ::= \ p_i \ | \ \neg\psi \ | \ \psi_1 \wedge \psi_2 \ | \ \mathbf{I}\psi \ | \ \bigcirc\psi, \tag{2}$$
$$\varphi \ ::= \ \psi \ | \ \neg\varphi \ | \ \varphi_1 \wedge \varphi_2 \ | \ \bigcirc\varphi \ | \ \Box_F\varphi. \tag{3}$$

This language turns out to be quite interesting and useful. Note first that the formulas used in the proof of Theorem 9 from [9] belong to $\mathcal{DTL}_1$, and so we obtain the following:

THEOREM 1. *None of the logics* $\mathsf{Log}_1\,\mathcal{T}_h$, $\mathsf{Log}_1\,\mathcal{A}_h$, $\mathsf{Log}_1\,\mathcal{R}^n_h$, *for* $n \geq 1$, *is recursively enumerable.*

On the other hand, $\mathcal{DTL}_1$ does not distinguish between arbitrary topological and Aleksandrov spaces, and, moreover, enjoys a kind of 'local finite model property.' These features of $\mathcal{DTL}_1$ are formulated and proved in Lemmas 2, 5, and 6. They will be heavily used later on in this paper.

LEMMA 2. *Every satisfiable $\mathcal{DTL}_1$-formula φ is satisfiable in a DTM based on an Aleksandrov topological space.*

Moreover, one can choose a DTM $\mathfrak{M}_{\mathfrak{G}}$ satisfying φ in such a way that it is based on the Aleksandrov space $\mathfrak{T}_{\mathfrak{G}}$ induced by a quasi-order $\mathfrak{G} = \langle W, R\rangle$ where,

(max) *for all $x \in W$ and all $\mathcal{DTL}_1$-formulas ψ with $(\mathfrak{M}_{\mathfrak{G}}, x) \models \psi$, the set $A_{x,\psi} = \{y \in W \mid xRy \text{ and } (\mathfrak{M}_{\mathfrak{G}}, y) \models \psi\}$ contains an R-maximal point (i.e., a point z such that if zRz' for some $z' \in A_{x,\psi}$ then $z'Rz$).*

To illustrate the idea of the proof suppose that φ is satisfied in a DTM $\mathfrak{M}$ based on a topological space $\mathfrak{T}$. Then we consider the DTM $\mathfrak{M}_{\mathfrak{G}}$ based on the Aleksandrov space $\mathfrak{T}_{\mathfrak{G}}$ of all ultrafilters over $\mathfrak{T}$ and show by induction on the structure of φ that it is satisfied in $\mathfrak{M}$ iff is is satisfied in $\mathfrak{M}_{\mathfrak{G}}$. There is a subtlety in the proof though: the inductive step for subformulas of the form (2) goes through for *all* ultrafilters whereas that for subformulas of the form (3) for *principal* ultrafilters only. It also follows from the proof that the claim of Lemma 2 is true for DTMs with homeomorphisms.

COROLLARY 3. $\mathsf{Log}_1\,\mathcal{T} = \mathsf{Log}_1\,\mathcal{A}$ *and* $\mathsf{Log}_1\,\mathcal{T}_h = \mathsf{Log}_1\,\mathcal{A}_h$.

It is known [17] that $\mathsf{Log}_1\,\mathcal{T} \subsetneqq \mathsf{Log}_1\,\mathcal{R}^n$, $n \geq 1$, while the question whether the equality $\mathsf{Log}_1\,\mathcal{T}_h = \mathsf{Log}_1\,\mathcal{R}^n_h$ holds remains open.

REMARK 4. Note that neither equality in the corollary above holds for full $\mathcal{DTL}$. For instance, $\varphi = \Box_F\mathbf{I}p \to \mathbf{I}\Box_F p$ is valid in all DTMs based on Aleksandrov spaces (as infinite intersections of open sets are open there). However, $\neg\varphi$ can be satisfied in a DTM $\mathfrak{M}$ based on $\mathbb{R}$: if one takes the continuous function $f(x) = 2x$ and $\mathfrak{V}(p) = (-1, 1)$ then $\bigcap_{k=1}^{\infty} f^{-1}(\mathbb{I}_{\mathbb{R}}\mathfrak{V}(p)) = \{0\}$ but $\mathbb{I}_{\mathbb{R}} \bigcap_{k=1}^{\infty} f^{-1}(\mathfrak{V}(p)) = \emptyset$, and so $(\mathfrak{M}, 0) \models \neg\varphi$.

The following lemma shows that, for $\mathcal{DTL}_1$, DTMs with continuous functions are closely connected to expanding domain product models introduced in [8]:

LEMMA 5. *Let φ be a $\mathcal{DTL}_1$-formula and $\mathfrak{M}$ a DTM based on an Aleksandrov space and satisfying* **(max)** *such that $(\mathfrak{M}, x_0) \models \varphi$ for some x_0. Then there exist a DTM $\mathfrak{M}^{ep}$ based on the Aleksandrov space induced by a quasi-order $\langle W^{ep}, R^{ep}\rangle$ and a continuous function f^{ep} on it such that $(\mathfrak{M}^{ep}, r_0) \models \varphi$ for some $r_0 \in W^{ep}$ and*

(fin) *$\langle W^{ep}, R^{ep}\rangle$ is the disjoint union of finite quasi-orders $\langle W^{ep}_n, R^{ep}_n\rangle$, for $n \geq 0$;*

(root) *for $n \geq 0$, $r_{n+1} = f^{ep}(r_n)$ and $W^{ep}_n = \{y \in W^{ep} \mid r_n R^{ep} y\}$;*

(inj) *f^{ep} is injective;*

(size) *$|W^{ep}_n| \leq ((1 + \ell(\varphi))!)^{n+1}$, for $n \geq 0$, where $\ell(\varphi)$ is the length of φ.*

The proof of this lemma is similar to that of Lemma 2.2 in [8].

Before proceeding to the next lemma, we remind the reader that, for a quasi-order R on W, a set $C \subseteq W$ is called a *cluster* in $\langle W, R\rangle$ if, for some $x \in W$, $C = \{y \in W \mid xRy \ \& \ yRx\}$; in this case we also say that C is the cluster *generated by* x and denote it by $C(x)$. A *tree of clusters* is a rooted quasi-order $\langle W, R\rangle$ such that, for all $x, y, z \in W$, if xRz and yRz, then xRy or yRx. A cluster $C(y)$ is said to be an *immediate strict successor* of a cluster $C(x)$ if xRy, $C(x) \neq C(y)$ and whenever $xRzRy$ then either $C(z) = C(x)$ or $C(z) = C(y)$.

LEMMA 6. *Let φ be a $\mathcal{DTL}_1$-formula satisfiable in a DTM based on an Aleksandrov space and meeting conditions* **(fin)**, **(root)**, **(inj)**. *Then there exist a recursive function $F(\varphi, n)$ and a DTM $\mathfrak{M}^{ex}$ based on the Aleksandrov space induced by a quasi-order $\langle W^{ex}, R^{ex}\rangle$ and a continuous function f^{ex} on it such that $(\mathfrak{M}^{ex}, r^{ex}) \models \varphi$ for some $r^{ex} \in W^{ex}$, $\mathfrak{M}^{ex}$ satisfies* **(inj)** *and*

(tree) *$\langle W^{ex}, R^{ex}\rangle$ is the disjoint union of finite trees of clusters $\langle W^{ex}_n, R^{ex}_n\rangle$, for $n \geq 0$;*

(top) *for $n \geq 0$, $y \in W^{ex}_n$, $x \in W^{ex}_{n+1}$ with $xR^{ex}f^{ex}(y)$, there is $z \in W^{ex}_n$ such that $zR^{ex}y$ and $f^{ex}(z) = x$; in particular, for all $x, y \in W^{ex}$, $f^{ex}(x)R^{ex}f^{ex}(y)$ implies $xR^{ex}y$;*

(size′) *$|W^{ex}_n| \leq F(\varphi, n)$.*

A DTM $\mathfrak{M}^{ex}$ satisfying the conditions of the lemma is constructed by a kind of 'unravelling,' and the proof can be found in Appendix A.

We use Lemma 6 and Kruskal's tree theorem to prove, in a way similar to the proof of Theorem 4 from [10], the following (cf. Theorem 1):

THEOREM 7. $\mathsf{Log}_1\,\mathcal{T}$ *is recursively enumerable.*

The details of the proof can be found in Appendix B.

4 Embedding DTMs into the real line

THEOREM 8. *A $\mathcal{DTL}_1$-formula φ is satisfiable iff $\varphi^{\mathbb{R}}$, the relativisation of φ, is satisfiable in the DTS $\langle\langle\mathbb{R}, \mathbb{I}_{\mathbb{R}}\rangle, x \mapsto x+1\rangle$, where*

$$\varphi^{\mathbb{R}} \;=\; dom \;\wedge\; \Box_F^+ \mathbf{I}(dom \to \bigcirc dom) \;\;\wedge\;\; \varphi^{dom},$$

dom is a fresh variable, and φ^{dom} is the result of replacing every occurrence of a subformula of the form $\mathbf{I}\psi$ in φ with $\mathbf{I}(dom \to \psi)$.

Proof. Suppose that $\varphi^{\mathbb{R}}$ is satisfied in $\langle\langle\mathbb{R}, \mathbb{I}_{\mathbb{R}}\rangle, x \mapsto x+1\rangle$. Then, by Lemma 2, $\varphi^{\mathbb{R}}$ is satisfied in a DTM $\mathfrak{M}$ based on an Aleksandrov space. It is readily seen that φ is satisfied in the DTM $\mathfrak{M}'$ obtained from $\mathfrak{M}$ by removing all those points where *dom* is false.

Conversely, suppose that φ is satisfiable. Let $\mathfrak{M}^{ex} = \langle\langle\mathfrak{T}^{ex}, f^{ex}\rangle, \mathfrak{V}^{ex}\rangle$ be the model provided by Lemma 6 and satisfying φ. Our plan is (1) to extend $\mathfrak{M}^{ex}$ to a DTM $\widehat{\mathfrak{M}} = \langle\langle\widehat{T}, \widehat{f}\rangle, \widehat{\mathfrak{V}}\rangle$ that satisfies $\varphi^{\mathbb{R}}$ and is based on ω-trees of clusters of finite depth, and then (2) to embed $\widehat{\mathfrak{M}}$ into a model based on $\langle\langle\mathbb{R}, \mathbb{I}_{\mathbb{R}}\rangle, x \mapsto x+1\rangle$.

We remind the reader that a tree of clusters $\langle W, R\rangle$ is *of depth n* if n is the length of the longest sequence $C(x_1), \dots, C(x_n)$ of clusters in $\langle W, R\rangle$ such that $C(x_{i+1})$ is an immediate strict successor of $C(x_i)$. A cluster C is called *final* if it has no strict successor. A tree of clusters is called an ω-tree if every non-final cluster has ω distinct strict immediate successors.

(1) Fix some $k \geq 0$. We know that by **(tree)**, $\langle W_k^{ex}, R_k^{ex}\rangle$ is a finite tree of clusters. We construct an ω-tree of clusters $\widehat{\mathfrak{G}}_k = \langle\widehat{W}_k, \widehat{R}_k\rangle$ by attaching an ω-tree of depth 2 to every cluster in $\langle W_k^{ex}, R_k^{ex}\rangle$. More precisely, let

$$\widehat{W}_k \;=\; W_k^{ex} \cup \{(C(x), n) \mid x \in W_k^{ex},\ n \in \mathbb{N}\}$$

and let $\widehat{R}_k$ be the transitive and reflexive closure of

$$R_k^{ex} \cup \{(x, (C(x), n)) \mid x \in W_k^{ex},\ n \in \mathbb{N}\}.$$

Note that the $\widehat{W}_k$ are pairwise disjoint. Let $\widehat{W} = \bigcup_{k=0}^{\infty} \widehat{W}_k$, $\widehat{R} = \bigcup_{k=0}^{\infty} \widehat{R}_k$, and let $\widehat{\mathfrak{T}}$ be the Aleksandrov space induced by $\widehat{\mathfrak{G}} = \langle\widehat{W}, \widehat{R}\rangle$.

Define a function $\widehat{f}\colon \widehat{W} \to \widehat{W}$ by taking, for all $x \in W^{ex}$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\widehat{f}(x) &\;=\; f^{ex}(x),\\
\widehat{f}((C(x), n)) &\;=\; \begin{cases} y_n, & \text{if } n < m,\\ (C(f^{ex}(x)), n-m), & \text{otherwise,}\end{cases}
\end{aligned}$$

where $\{y_0, \dots, y_{m-1}\} \subseteq W_{k+1}^{ex} \setminus \{f(z) \mid z \in W_k^{ex}\}$ and $C(y_0), \dots, C(y_{m-1})$ are all the distinct strict immediate successors of $C(f^{ex}(x))$.

Clearly, $\widehat{f}$ is a well-defined function on $\widehat{W}$. Informally, (i) each $x \in W_k^{ex}$ is mapped onto its image $f^{ex}(x)$; (ii) for each $x \in W_k^{ex}$, its first m successors of the form $(C(x), n)$ from $\widehat{W}_k \setminus W_k^{ex}$ are mapped onto the roots of the m

distinct trees that are attached to $C(f^{ex}(x))$ at step $k+1$ and the remaining successors of this sort are simply renumbered. Since every $\widehat{\mathfrak{G}}_{k+1}$ contains an isomorphic copy of $\widehat{\mathfrak{G}}_k$ and in view of **(top)**, the function $\widehat{f}$ is monotonous.

Clearly, $\varphi^{\mathbb{R}}$ is satisfied in $\widehat{\mathfrak{M}}$, where the valuation $\widehat{\mathfrak{V}}$ defined by

$$\widehat{\mathfrak{V}}(dom) = W^{ex} \qquad \text{and} \qquad \widehat{\mathfrak{V}}(p_i) = \mathfrak{V}^{ex}(p_i), \quad \text{for every variable } p_i \text{ in } \varphi.$$

(2) For the second step we require the following generalisation of a result from [3]:

LEMMA 9. *For every open interval $I \subseteq \mathbb{R}$ and every ω-tree of clusters $\mathfrak{G} = \langle W, R\rangle$ of finite depth, there is a surjective open and continuous function $f_I^{\mathfrak{G}} \colon I \to \mathfrak{T}_{\mathfrak{G}}$. Moreover, for every initial ω-subtree $\mathfrak{G}' = \langle W', R'\rangle$ of $\mathfrak{G}$ (where each cluster of $\mathfrak{G}'$ either inherits all strict immediate successors from $\mathfrak{G}$ or none of them) and every point $z \in I$,*

(cons) *if $f_I^{\mathfrak{G}'}(z)$ is not in a final cluster of $\mathfrak{G}'$, then $f_I^{\mathfrak{G}}(z) = f_I^{\mathfrak{G}'}(z)$;*

(final) *if $f_I^{\mathfrak{G}'}(z)$ is in a final cluster of $\mathfrak{G}'$, then there is an open interval I_z with $z \in I_z$ such that $f_I^{\mathfrak{G}}(I_z) = \{y \in W \mid f_I^{\mathfrak{G}'}(z) R y\}$.*

The above result can be proved by extending Theorem 16 in [3] so that in [3, Lemma 15], an enumeration of all countably many immediate strict successors is chosen in such a way that each of them occurs infinitely often in the enumeration.

Thus, Lemma 9 provides us with a surjective open and continuous function $f_{(0,1)}^{\widehat{\mathfrak{G}}_k} \colon (0,1) \to \widehat{\mathfrak{G}}_k$, for each $k \geq 0$. Define $g : \bigcup_{k=0}^{\infty}(k, k+1) \to \widehat{\mathfrak{G}}$ by taking:

$$g(k+z) = f_{(0,1)}^{\widehat{\mathfrak{G}}_k}, \qquad \text{for all } k \in \mathbb{N} \ \text{ and } \ z \in (0,1).$$

Finally, define a valuation $\mathfrak{V}_{\mathbb{R}}$ in $\mathbb{R}$ by

$$\mathfrak{V}_{\mathbb{R}}(dom) = \{z \in \textstyle\bigcup_{k=0}^{\infty}(k, k+1) \mid g(z) \in \widehat{\mathfrak{V}}(dom)\},$$
$$\mathfrak{V}_{\mathbb{R}}(p_i) = \{z \in \textstyle\bigcup_{k=0}^{\infty}(k, k+1) \mid g(z) \in \widehat{\mathfrak{V}}(p_i)\},$$

for every variable p_i of φ. First, we show that, for all $z \in \mathbb{R}$,

$$\text{if } z \in \mathfrak{V}_{\mathbb{R}}(dom) \quad \text{then} \quad (z+1) \in \mathfrak{V}_{\mathbb{R}}(dom) \text{ and } \widehat{f}(g(z)) = g(z+1). \tag{4}$$

Indeed, suppose that $z \in \mathfrak{V}_{\mathbb{R}}(dom)$. Then $g(z) \in \widehat{\mathfrak{V}}(dom)$ and therefore, $g(z) \in W_k^{ex}$, for some $k \geq 0$. Then $\widehat{f}(g(z)) \in W_{k+1}^{ex} \subseteq \widehat{\mathfrak{V}}(dom)$. Moreover, by **(cons)**, $\widehat{f}(g(z)) = g(z+1)$. Hence, $z+1 \in \mathfrak{V}_{\mathbb{R}}(dom)$. Note that it is essential that $g(z)$ is not in a final cluster of $\widehat{\mathfrak{G}}_k$; and this is precisely the reason why at step **(1)** we needed to extend $\langle W^{ex}, R^{ex}\rangle$ to $\langle \widehat{W}, \widehat{R}\rangle$ by ω-trees of depth 2 (for each final cluster of each $\widehat{\mathfrak{G}}_k$ the attached ω-tree of depth 2 provides countably many 'placeholders' for the trees that will be attached to this cluster in $W^{ex}_{k+1}, W^{ex}_{k+2}, \dots$; note also that these placeholders do not belong to the 'domain').

Now, we prove by induction on the construction of subformulas ψ of φ^{dom} that, for every $z \in \mathfrak{V}_{\mathbb{R}}(dom)$,[1]

$$g(z) \in \widehat{\mathfrak{V}}(\psi) \qquad \text{iff} \qquad z \in \mathfrak{V}_{\mathbb{R}}(\psi). \tag{5}$$

The case of variables follows immediately from the definition and the cases of the Booleans are trivial.

Case $\psi = \mathbf{I}(dom \to \psi')$. Since $z \in \mathfrak{V}_{\mathbb{R}}(dom)$, there is $k \in \mathbb{N}$ such that $z \in (k, k+1)$. Then

$$\begin{array}{ll}
z \in \mathfrak{V}_{\mathbb{R}}(\mathbf{I}(dom \to \psi')) & \text{iff [def. of open set]} \\
\text{there is an open } U \subseteq (k, k+1) \text{ such that} & \\
\quad z \in U \text{ and } U \subseteq \mathfrak{V}_{\mathbb{R}}(dom \to \psi') & \text{iff [definition]} \\
\text{there is an open } U \subseteq (k, k+1) \text{ such that} & \\
\quad z \in U \text{ and } U \cap \mathfrak{V}_{\mathbb{R}}(dom) \subseteq \mathfrak{V}_{\mathbb{R}}(\psi') & \text{iff [IH]} \\
\text{there is an open } U \subseteq (k, k+1) \text{ such that} & \\
\quad z \in U \text{ and } g(U) \cap \widehat{\mathfrak{V}}(dom) \subseteq \widehat{\mathfrak{V}}(\psi') & \text{iff } [V = g(U) \\
 & \quad g \text{ open and cont.]} \\
\text{there is an open } V \subseteq \widehat{W}_k \text{ such that} & \\
\quad g(z) \in V \text{ and } V \cap \widehat{\mathfrak{V}}(dom) \subseteq \widehat{\mathfrak{V}}(\psi') & \text{iff [definition]} \\
g(z) \in \widehat{\mathfrak{V}}(\mathbf{I}(dom \to \psi')) &
\end{array}$$

Case $\psi = \bigcirc\psi'$. Then

$$\begin{array}{lllll}
z \in \mathfrak{V}_{\mathbb{R}}(\bigcirc\psi') & \text{iff} & z+1 \in \mathfrak{V}_{\mathbb{R}}(\psi') & \text{iff} & [\text{IH}] \\
 & & g(z+1) \in \widehat{\mathfrak{V}}(\psi') & \text{iff} & [(4)] \\
 & & \widehat{f}(g(z)) \in \widehat{\mathfrak{V}}(\psi') & \text{iff} & g(z) \in \widehat{\mathfrak{V}}(\bigcirc\psi').
\end{array}$$

Case $\psi = \Box_F \psi'$ is considered in the same way.

By (4) and (5), $\varphi^{\mathbb{R}}$ is satisfied in the DTS $\langle\langle \mathbb{R}, \mathbb{I}_{\mathbb{R}}\rangle, x \mapsto x+1\rangle$ under the valuation $\mathfrak{V}_{\mathbb{R}}$. ❑

COROLLARY 10. *A $\mathcal{DTL}_1$-formula φ is satisfiable iff $\varphi^{\mathbb{R}}$ is satisfiable in the DTS $\langle\langle \mathbb{R}^n, \mathbb{I}_{\mathbb{R}^n}\rangle, (x_1, \dots, x_{n-1}, x_n) \mapsto (x_1, \dots, x_{n-1}, x_n + 1)\rangle$, $n \geq 1$.*

It should be noted that although the relativisation $\varphi^{\mathbb{R}}$ is satisfied in a DTS based on an Aleksandrov space iff φ is satisfied in some (in general, different) DTS based on an Aleksandrov space, analogous statement fails for the Euclidean spaces: φ may not be satisfiable in a DTS based on $\mathbb{R}$ even if its relativisation $\varphi^{\mathbb{R}}$ is satisfiable (see, e.g., a counterexample in [17]). The explanation lies in the variable *dom* which 'carves' out of the model based on $\mathbb{R}$ a subspace that topologically resembles an Aleksandrov space.

[1] For a DTM $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$, we denote by $\mathfrak{V}(\varphi)$ the set $\{w \in \mathfrak{F} \mid (\mathfrak{M}, w) \models \varphi\}$.

5 Undecidability

As usual, for a finite alphabet Σ, the set of all finite words over Σ (including the empty word ϵ) is denoted by Σ^*. Given a word $w = a_1 \cdot a_2 \cdot \dots \cdot a_n$, we denote its length n by $|w|$ and refer to its elements using the bracket notation: $w(i) = a_i$, for $1 \leq i \leq n$. For a pair of words u and w, we say that u is a *subword* of w and write $u \sqsubseteq_h w$ if there is a strictly monotone map $h\colon \{1, \dots, |u|\} \to \{1, \dots, |w|\}$ such that $u(i) = w(h(i))$, for every i, $1 \leq i \leq |u|$. We also write $u \sqsubseteq w$ if $u \sqsubseteq_h w$ for some h.

A *single channel system* is a triple $S = \langle Q, \Sigma, \Delta \rangle$, where $Q = \{q_1, \dots, q_n\}$ is a set of *control states*, $\Sigma = \{a_1, \dots, a_m\}$ is an alphabet of *messages* and $\Delta \subseteq Q \times \{?, !\} \times \Sigma \times Q$ is a set of *transitions*.

A *configuration* of S is a pair $\gamma = \langle q, w \rangle$, where $q \in Q$ and $w \in \Sigma^*$. Define a *lossy relation* and two types of *perfect transition relations for S* between configurations by taking:

$$\textbf{(l)} \quad \langle q, w \rangle \to_\ell \langle q, w' \rangle \quad \text{iff} \quad w' \sqsubseteq w,$$

$$\textbf{(s)} \quad \langle q, w \rangle \xrightarrow{\langle q,!,a,q' \rangle}_p \langle q', a \cdot w \rangle, \qquad \textbf{(r)} \quad \langle q, w \cdot a \rangle \xrightarrow{\langle q,?,a,q' \rangle}_p \langle q', w \rangle .$$

(Here **s** stands for 'send' and **r** for 'receive.') Finally, define *lossy transition relations for S* as the following compositions:

$$\textbf{(ls)} \quad \langle q, w \rangle \xrightarrow{\langle q,!,a,q' \rangle}_\ell \langle q', w' \rangle \quad \text{iff} \quad \langle q, w \rangle \xrightarrow{\langle q,!,a,q' \rangle}_p \langle q', a \cdot w \rangle \to_\ell \langle q', w' \rangle ,$$

$$\textbf{(lr)} \; \langle q, w \cdot a \rangle \xrightarrow{\langle q,?,a,q' \rangle}_\ell \langle q', w' \rangle \quad \text{iff} \quad \langle q, w \cdot a \rangle \xrightarrow{\langle q,?,a,q' \rangle}_p \langle q', w \rangle \to_\ell \langle q', w' \rangle .$$

For a binary relation $\to$ on configurations of S, a sequence $\gamma_0, \dots, \gamma_m$ of configurations of S is called a *$\to$-computation of S* if $\gamma_k \to \gamma_{k+1}$, for $k < m$.

The undecidable 'master problem' that will be used to prove the undecidability of the logics under consideration is the *ω-reachability problem for channel systems*: given a channel system $S = \langle Q, \Sigma, \Delta \rangle$, two states $q_0, q_{rec} \in Q$, and a relation $\to$ in the interval

$$\bigcup_{\delta_k \in \Delta} \xrightarrow{\delta_k}_p \quad \subseteq \quad \to \quad \subseteq \quad \bigcup_{\delta_k \in \Delta} \xrightarrow{\delta_k}_\ell \tag{6}$$

decide whether for every $n \in \mathbb{N}$ there exists a $\to$-computation of S starting with $\langle q_0, \epsilon \rangle$ and reaching q_{rec} at least n times. The following lemma can be proved by a reduction of the undecidable boundedness problem for lossy channel systems [15]. The reduction was suggested by Ph. Schnoebelen:

LEMMA 11. *The ω-reachability problem is undecidable.*

We are now in a position to prove the main result of the paper:

THEOREM 12. *None of $\mathsf{Log}_1\,\mathcal{T}$ and $\mathsf{Log}_1\,\mathcal{R}^n$, for $n \geq 1$, is decidable.*

Proof. By Corollary 10, it suffices to consider (the complement of) $\mathsf{Log}_1\,\mathcal{T}$. Given a single channel system $S = \langle Q, \Sigma, \Delta \rangle$ and states $q_0, q_{rec} \in Q$, we construct a $\mathcal{DTL}_1$-formula $\varphi_{S,q_0,q_{rec}}$ such that

(⇒) if $\varphi_{S,q_0,q_{rec}}$ is satisfiable then, for every $n \in \mathbb{N}$, there is a *lossy* computation starting with $\langle q_0, \epsilon\rangle$ and reaching q_{rec} at least n times;

(⇐) if, for every $n \in \mathbb{N}$, there is a *perfect* computation starting with $\langle q_0, \epsilon\rangle$ and reaching q_{rec} at least n times then $\varphi_{S,q_0,q_{rec}}$ is satisfiable in a DTM.

By Lemma 11 this will imply that $\mathsf{Log}_1\,\mathcal{T}$ is undecidable.

The formula $\varphi_{S,q_0,q_{rec}}$ is the conjunction of four formulas σ, θ_{S,q_0}, τ_S and $\rho_{S,q_{rec}}$ and will be constructed in four steps $(\boldsymbol{\sigma})$, $(\boldsymbol{\theta})$, $(\boldsymbol{\tau})$ and $(\boldsymbol{\rho})$, respectively.

(⇒) To explain the meaning of each of the four conjuncts, let us assume that $(\mathfrak{M}, x_0) \models \varphi_{S,q_0,q_{rec}}$, for some DTM $\mathfrak{M} = \langle\langle\mathfrak{T}, f\rangle, \mathfrak{V}\rangle$, where $\mathfrak{T}$ is the Aleksandrov topological space induced by a quasi-order $\langle W, R\rangle$.

For every $n \geq 0$, let $W_n = \{y \in W \mid f^n(x_0)Ry\}$ and $R_n = R \cap (W_n \times W_n)$. Clearly, the $\langle W_n, R_n\rangle$ are quasi-orders. By Lemma 5, we may also assume that the W_n are *finite* and f is injective.

$(\boldsymbol{\sigma})$ First we introduce a new operator $\mathbf{S}$ (to be interpreted as an almost 'irreflexive' diamond in $\langle W, R\rangle$). Namely, for every $\mathcal{DTL}^{\bigcirc}$-formula ψ, we put

$$\mathbf{S}\psi \;=\; (s \wedge \mathbf{C}(\neg s \wedge \mathbf{C}\psi)) \vee (\neg s \wedge \mathbf{C}(s \wedge \mathbf{C}\psi))$$

for some fresh variable s. Define new relations $\overline{R}_n$ on W_n, $n \geq 0$, by taking, for all $x, y \in W_n$,

$$x\overline{R}_n y \quad \text{iff} \quad \exists z \in W_n \; \big(xR_n zR_n y \;\; \text{and} \;\; (\mathfrak{M}, x) \models s \Leftrightarrow (\mathfrak{M}, z) \models \neg s\big).$$

It can be checked that $\overline{R}_n$ is transitive, $\overline{R}_n \subseteq R_n$ and, for every $x \in W_n$,

$$(\mathfrak{M}, x) \models \mathbf{S}\psi \qquad \text{iff} \qquad \text{there is } y \in W_n \text{ such that } x\overline{R}_n y \text{ and } (\mathfrak{M}, y) \models \psi.$$

Let

$$\sigma \;=\; \Box_F^{+}\mathbf{I}(s \leftrightarrow \bigcirc s). \tag{7}$$

By the monotonicity of f, one can also show that if $(\mathfrak{M}, x_0) \models \sigma$ then, for all $x, y \in W_n$,

$$\text{if} \quad x\overline{R}_n y \quad \text{then} \quad f(x)\overline{R}_{n+1}f(y).$$

$(\boldsymbol{\theta})$ Next, we encode infinitely many computations along the orbit of x_0. Each computation is encoded backwards (from its end to the beginning), and the variable m delimits computations in the sense that each computation starting in q_0 is encoded between two consecutive occurrences of m. For every transition $\delta_k \in \Delta$, we introduce a fresh variable $\mathsf{tr}\delta_k$ and, for every state $q_i \in Q$, a variable q_i.[2] Let θ_{S,q_0} be the conjunction of the following

[2]Note the different font we use to denote these variables: their values will be needed only along the orbit of x_0 (in contrast to s and other variables below).

formulas:

$$\bigwedge_{\delta_i \neq \delta_j} \Box_F^+ \neg(\mathsf{tr}\delta_i \wedge \mathsf{tr}\delta_j) \quad \wedge \quad \bigwedge_{q_i \neq q_j} \Box_F^+ \neg(\mathsf{q}_i \wedge \mathsf{q}_j), \tag{8}$$

$$\Box_F^+\big(\neg\mathsf{m} \to \bigvee_{\delta_i \in \Delta} \mathsf{tr}\delta_i\big), \tag{9}$$

$$\bigwedge_{\delta_k = \langle q_i, *, a, q_j \rangle \in \Delta} \Box_F^+\big(\mathsf{tr}\delta_k \to \mathsf{q}_j \wedge \bigcirc \mathsf{q}_i\big), \tag{10}$$

$$\Box_F^+\big(\mathsf{m} \to \mathsf{q}_0\big), \tag{11}$$

$$\mathsf{m} \wedge \Box_F^+ \Diamond_F \mathsf{m}, \tag{12}$$

where $*$ denotes either of $\{?, !\}$. One can show that if $(\mathfrak{M}, x_0) \models \theta_{S,q_0}$ then there is an infinite sequence of natural numbers $0 = M_0 < M_1 < \dots$ such that, for every $n \geq 0$,

$$(\mathfrak{M}, f^{M_n}(x_0)) \models \mathsf{m}.$$

Let $N_n = M_n - (M_{n-1} + 1)$, for $n > 0$. Clearly, $N_n \geq 1$, for $n > 0$. Let $N_0 = 0$. For every $n \geq 0$, there are also unique sequences

$$q_{i_0^n}, \dots, q_{i_{N_n}^n} \qquad \text{and} \qquad \delta_{k_0^n}, \dots, \delta_{k_{N_n - 1}^n}$$

of, respectively, states from Q and transitions from Δ such that

- $(\mathfrak{M}, f^{M_n - j}(x_0)) \models \mathsf{q}_{i_j^n}$, for each $0 \leq j \leq N_n$,
- $(\mathfrak{M}, f^{M_n - (j+1)}(x_0)) \models \mathsf{tr}\delta_{k_j^n}$, for each $0 \leq j < N_n$,
- $\delta_{k_j^n} = \big\langle q_{i_j^n}, *, a, q_{i_{j+1}^n} \big\rangle$ and $q_{i_0^n} = q_0$.

($\boldsymbol{\tau}$) Our next step is to encode words in the 'topological dimension.' For every message $a_i \in \Sigma$, we introduce a fresh variable a_i and let $\lambda = \bigvee_{a_i \in \Sigma} a_i$. Intuitively, we encode a word in $\overline{R}_k$-connected points of W_k.

Denote by τ_S the conjunction of the following formulas:

$$\bigwedge_{a_i \neq a_j} \Box_F^+ \mathbf{I} \neg(a_i \wedge a_j), \tag{13}$$

$$\Box_F^+ \mathbf{I}(\lambda \to \mathbf{I}\lambda), \tag{14}$$

$$\bigwedge_{a_i \in \Sigma} \Box_F^+ \mathbf{I}(a_i \to \bigcirc(\lambda \to a_i)), \tag{15}$$

$$\Box_F^+(\mathsf{m} \to \mathbf{I}\neg\lambda), \tag{16}$$

$$\bigwedge_{\delta_k = \langle q, !, a, q' \rangle \in \Delta} \Box_F^+\big(\mathsf{tr}\delta_k \ \to \ \mathbf{I}(\lambda \to \neg\mathbf{S}\bigcirc\neg\lambda) \wedge \mathbf{I}(\lambda \wedge \bigcirc\neg\lambda \to a)\big), \tag{17}$$

$$\bigwedge_{\delta_k = \langle q, ?, a, q' \rangle \in \Delta} \Box_F^+\big(\mathsf{tr}\delta_k \ \to \ \mathbf{I}(\lambda \to \bigcirc\lambda) \wedge \mathbf{I}(\neg\mathbf{S}\lambda \to \bigcirc\mathbf{S}\lambda) \wedge \bigcirc\mathbf{I}(\neg\mathbf{S}\lambda \to a)\big). \tag{18}$$

CLAIM 13. Let $y \in W_n$ be such that $(\mathfrak{M}, y) \models \lambda$. Then $y\overline{R}_n y$ does not hold.

Proof of claim. Suppose that $y\overline{R}_n y$. By (16), $(\mathfrak{M}, f^n(x_0)) \models \neg\mathsf{m}$ and thus, by (9), $(\mathfrak{M}, f^n(x_0)) \models \mathsf{tr}\delta_k$ for some $\delta_k \in \Delta$. Consider two cases:

- if $\delta_k = \langle q, ?, a, q' \rangle$ then $(\mathfrak{M}, f(y)) \models \lambda$, by the first conjunct of (18);
- if $\delta_k = \langle q, !, a, q' \rangle$ then, as $y\overline{R}_n y$, we have, by (17), $(\mathfrak{M}, f(y)) \models \lambda$.

By the monotonicity of f, we obtain $f(y) \in W_{n+1}$ and $f(y)\overline{R}_{n+1} f(y)$; therefore, one can repeat the above step for $f(y)$. However, this process cannot continue infinitely long because there is $M_j > n$ such that $(\mathfrak{M}, f^{M_j}(x_0)) \models \mathsf{m}$, and so, by (16), $(\mathfrak{M}, y) \models \neg\lambda$ for all $y \in W_{M_j}$. ❑

Say that a finite sequence $\vec{y} = (y_1, y_2, \dots, y_k)$ of elements of W_n with $y_1\overline{R}_n y_2 \overline{R}_n \dots$ *carries the* Σ*-word* w and write $val(\vec{y}) = w$ if $(\mathfrak{M}, y_i) \models w(i)$, for all $1 \le i \le |w|$. Note that, by (14), if $(\mathfrak{M}, y) \models \lambda$ for some $y \in W_n$ then, for every $z \in W_n$ with $y\overline{R}_n z$, we will also have $(\mathfrak{M}, z) \models \lambda$. Let $val(\vec{y}) = \epsilon$ if $\vec{y}$ is the empty sequence. We say that a sequence $\vec{y}\,' = (y'_1, \dots, y'_r)$ with $y'_i \overline{R}_n y'_{i+1}$ is an *extension* of $\vec{y} = (y_1, \dots, y_k)$ with $y_i \overline{R}_n y_{i+1}$ if there is a strictly monotone function $h \colon \{1, \dots, k\} \to \{1, \dots, r\}$ (i.e., $h(i) > h(j)$, for $i > j$) such that $y_i = y'_{h(i)}$, for $1 \le i \le k$. A sequence $\vec{y} = (y_1, \dots, y_k)$ with $y_i \overline{R}_n y_{i+1}$ is said to be *maximal carrying a* Σ*-word* if no extension of $\vec{y}$ carries a Σ-word.

Suppose $(\mathfrak{M}, f^n(x_0)) \models \neg\mathsf{m}$. Then, by (9), $(\mathfrak{M}, f^n(x_0)) \models \mathsf{tr}\delta_k$ for some transition $\delta_k \in \Delta$. Let $\vec{y}\,' = (y_1, \dots, y_l)$ with $y_i \overline{R}_n y_{i+1}$ be a maximal sequence carrying a Σ-word and $val(\vec{y}\,') = d_1 \dots d_l$. First, consider the case when $\vec{y}\,'$ is empty. This means that $(\mathfrak{M}, y) \models \neg\lambda$ for all $y \in W_n$. Take any maximal sequence $\vec{y}$ in W_{n+1} carrying a Σ-word. Clearly, $\vec{y}$ can be regarded as an extension of $f(\vec{y}\,') = \epsilon$ and, as ϵ is a subword of any word, we have $\langle q, val(\vec{y}) \rangle \xrightarrow{\delta_k}_{\ell} \langle q', \epsilon \rangle$, for any transition $\delta_k = \langle q, *, a, q' \rangle \in \Delta$ (this corresponds to a situation when everything is lost after the transition).

Suppose now that $\vec{y}\,'$ is not empty. Let $f(\vec{y}\,') = (f(y_1), \dots, f(y_l))$. Note that points of this sequence are all distinct since f is injective. Consider the cases of sending and receiving messages:

Case $\delta_k = \langle q, !, a, q' \rangle$. Three subcases are possible:

(0) There is no m, $1 \le m \le l$, such that $(\mathfrak{M}, y_m) \models \bigcirc\lambda$. Then, by the first conjunct of (17), $\vec{y}\,' = (y_1)$ and, by its last conjunct, $val(\vec{y}\,') = a$. Take any maximal sequence $\vec{y}$ in W_{n+1} carrying a Σ-word. As $\vec{y}$ does not contain $f(y_1)$, it can trivially be regarded as an extension of $f(\vec{y}\,')$. Thus $\langle q, val(\vec{y}) \rangle \xrightarrow{\delta_k}_{\ell} \langle q', a \rangle$ (everything but the sent message is lost).

(1) Let $(\mathfrak{M}, y_1) \models \bigcirc\lambda$. Then $(\mathfrak{M}, f(y_1)) \models \lambda$ and, by (15), $val(\vec{y}\,') = val(f(\vec{y}\,'))$. Take any maximal extension $\vec{y}$ of $f(\vec{y}\,')$ carrying a Σ-word. Then $val(\vec{y}) \sqsupseteq val(\vec{y}\,')$ and $\langle q, val(\vec{y}) \rangle \xrightarrow{\delta_k}_{\ell} \langle q', val(\vec{y}\,') \rangle$ (the sent message is lost; other messages are possibly lost but not completely).

(2) Let $(\mathfrak{M}, y_1) \models \bigcirc\neg\lambda$ but $(\mathfrak{M}, y_2) \models \bigcirc\lambda$. By the last conjunct of (17), we have $(\mathfrak{M}, y_1) \models a$ and, by (15), $\mathit{val}(\vec{y}\,') = a \cdot \mathit{val}(f(\vec{y}\,'))$. Take any maximal extension $\vec{y}$ of $f(\vec{y}\,')$ carrying a Σ-word. Then we have $\langle q, \mathit{val}(\vec{y})\rangle \xrightarrow{\delta_k}_{\ell} \langle q', \mathit{val}(\vec{y}\,')\rangle$ (the sent message is not lost; other messages are possibly lost but not completely).

By the first conjunct of (17), no other option is possible.

Case $\delta_k = \langle q, ?, a, q'\rangle$. By the maximality of $\vec{y}$ and Claim 13, $(\mathfrak{M}, y_l) \models \lambda \wedge \neg\mathbf{S}\lambda$. Then, by the first and second conjuncts of (18), $(\mathfrak{M}, f(y_l)) \models \lambda \wedge \mathbf{S}\lambda$. By Claim 13, the set $\{z \in W_{n+1} \mid f(y_l)\overline{R}_{n+1}z, \quad (\mathfrak{M}, z) \models \lambda\}$ contains no infinite ascending $\overline{R}_{n+1}$-chain, and so we can find some $y \in W_{n+1}$ such that $f(y_l)\overline{R}_{n+1}y$ and $(\mathfrak{M}, y) \models \lambda \wedge \neg\mathbf{S}\lambda$. By the last conjunct of (18), we have $(\mathfrak{M}, y) \models a$. As, by the first conjunct of (18) and (15), $\mathit{val}(f(\vec{y}\,')) = \mathit{val}(\vec{y}\,')$, we obtain $\mathit{val}(f(\vec{y}\,') \cdot y) = \mathit{val}(\vec{y}\,') \cdot a$. Finally, take any maximal extension $\vec{y}$ of $f(\vec{y}\,') \cdot y$ carrying a Σ-word. Clearly, $\langle q, \mathit{val}(\vec{y})\rangle \xrightarrow{\delta_k}_{\ell} \langle q', \mathit{val}(\vec{y}\,')\rangle$.

Now, given $n \geq 1$, we can find a lossy computation. Let $(q_{i_0^n}, \dots, q_{i_{N_n}^n})$ and $(\delta_{k_0^n}, \dots, \delta_{k_{N_n-1}^n})$ be the unique sequences of states and transitions, respectively, defined by the model $\mathfrak{M}$ as described in step $(\boldsymbol{\theta})$. We start from the tail: let $j = N_n$ and take any maximal sequence $\vec{y}_j$ in $W_{M_n - j}$ carrying a Σ-word. We know from the considerations that there is a maximal sequence $\vec{y}_{j-1}$ in $W_{M_n-(j-1)}$ carrying a Σ-word such that

$$\big\langle q_{i_{j-1}^n}, \mathit{val}(\vec{y}_{j-1})\big\rangle \xrightarrow{\delta_{k_j^n}}_{\ell} \big\langle q_{i_j^n}, \mathit{val}(\vec{y}_j)\big\rangle.$$

By repeating this procedure sufficiently many times, we arrive at $\langle q_0, \epsilon\rangle$, where the n-th computation starts (here we use (11) and (16)).

$(\boldsymbol{\rho})$ At the final step we enforce computations to visit the state q_{rec} more and more often. We require two fresh variables *light* and *on*. Let $\rho_{S,q_{rec}}$ be the conjunction of the formulas:

$$\mathit{light} \ \wedge\ \Box_F\big(\mathsf{m} \to \mathbf{I}(\mathit{light} \to \bigcirc\mathbf{S}\mathit{light})\big), \tag{19}$$

$$\Box_F^+\mathbf{I}(\mathit{light} \to \bigcirc\mathit{light}), \tag{20}$$

$$\Box_F^+\big(\mathsf{m} \to \bigcirc\mathbf{I}(\mathit{light} \to \mathit{on})\big), \tag{21}$$

$$\Box_F^+\big(\mathbf{C}(\mathit{light} \wedge \mathit{on} \wedge \bigcirc\neg\mathit{on}) \to \mathsf{q}_{rec}\big), \tag{22}$$

$$\Box_F^+\mathbf{I}\big((\mathit{light} \wedge \mathit{on} \wedge \bigcirc\neg\mathit{on}) \to \neg\mathbf{S}(\mathit{light} \wedge \mathit{on} \wedge \bigcirc\neg\mathit{on})\big), \tag{23}$$

$$\Box_F^+\big(\mathsf{m} \to \mathbf{I}(\mathit{light} \to \neg\mathit{on})\big). \tag{24}$$

The above conjunction works as follows. By (19), with every start of a computation (i.e., whenever m is true) a fresh point x is picked up and the variable *light* is true at it; we say in this case that a new light is created. By (20), *light* is also true at all images $f^i(x)$ of x. At the next iteration, by (21), all lights are switched on, i.e., *on* is true at every point with *light*. By (22), whenever a light is switched off the current state of the computation must be q_{rec}. Finally, by (24), all lights must be switched off at the next occurrence of m. It should also be noted that, by (19), the root x_0 is always a light and whenever m is true (for $n > 0$) a fresh light is created as an $\overline{R}_n$-successor for every existing light. Therefore, after k occurrences of m there will be an $\overline{R}_n$-sequence of lights of length $\geq k$. As, by (23), along every $\overline{R}_n$-sequence only one light may be switched off at a time and, by (23), this may only happen in the state q_{rec}, the variable m cannot become true sooner than the computation visits q_{rec} at least k times. This completes the encoding of the ω-reachability problem.

($\Leftarrow$) For the converse direction we refer the reader to Appendix C. ❑

As a consequence we obtain the following

THEOREM 14. *None of* $\mathsf{Log}\,\mathcal{T}$, $\mathsf{Log}\,\mathcal{A}$, $\mathsf{Log}\,\mathcal{R}^n$, *for* $n \geq 1$, *is decidable.*

Appendix

A Proof of Lemma 6

Proof. Let $(\mathfrak{M}, r) \models \varphi$, where $\mathfrak{M} = \langle\langle \mathfrak{T}, f\rangle, \mathfrak{V}\rangle$, $\mathfrak{T}$ is induced by $\langle W, R\rangle$ and $\mathfrak{M}$ satisfies **(fin)**, **(root)**, **(inj)**. The DTM $\mathfrak{M}^{ex}$ satisfying the conditions of the lemma will be constructed by a kind of 'unravelling' $\mathfrak{M}$.

A sequence $\vec{C} = (C_0, \dots, C_n)$, $n \geq 0$, is called a *cluster sequence over* $\langle W, R\rangle$ iff the C_i are clusters in $\langle W, R\rangle$ such that C_{i+1} is an immediate strict successor of C_i. Define a partial order $\leq$ on cluster sequences by taking

$$(C_0, \dots, C_n) \leq (C'_0, \dots, C'_m) \quad \text{iff} \quad n \leq m \text{ and } C_i = C'_i, \text{ for } 0 \leq i \leq n.$$

For a point $x \in W$, let $T(x)$ be the set of all pairs $\boldsymbol{y} = \langle \vec{C}, y\rangle$ of cluster sequences $\vec{C} = (C_0, \dots, C_n)$ in $\langle W, R\rangle$ such that $C_0 = C(x)$ and $C_n = C(y)$. Such pairs are ordered by the relation

$$\langle \vec{C}, y\rangle \leq \langle \vec{C}', y'\rangle \qquad \text{iff} \qquad \vec{C} \leq \vec{C}'.$$

Clearly, $\langle T(x), \leq\rangle$ is a tree of clusters, and every y with xRy is represented by at least one point $\boldsymbol{y}$ in this tree. Moreover, for every $\boldsymbol{y} = \langle \vec{C}, y\rangle$ and $z \in C(y)$, $T(x)$ contains $\boldsymbol{z} = \langle \vec{C}, z\rangle$ and both $\boldsymbol{y}$ and $\boldsymbol{z}$ are in the same cluster in $\langle T(x), \leq\rangle$. It should be noted that if $\langle W, R\rangle$ is a tree of clusters then, for every $x \in W$, $\langle T(x), \leq\rangle$ is isomorphic to the quasi-order $\langle W_x, R_x\rangle$ generated by x.

The new model $\mathfrak{M}^{ex}$ will be based on the Aleksandrov space induced by a quasi-order on sequences of pairs of the form $\langle \vec{C}, y\rangle$. We define it inductively: let

$$W_0^{ex} = \{(\boldsymbol{y}) \mid \boldsymbol{y} \in T(r)\},$$
$$(\boldsymbol{y})R_0^{ex}(\boldsymbol{y'}) \quad \text{iff} \quad \boldsymbol{y} \leq \boldsymbol{y'}.$$

Suppose now that $\langle W_k^{ex}, R_k^{ex}\rangle$, $k \geq 0$, has already been defined. Let W_{k+1}^{ex} be the union of two sets:

$$\overline{W}_k^{ex} = \{(\boldsymbol{y_0},\dots,\boldsymbol{y}_k, f(\boldsymbol{y_k})) \mid (\boldsymbol{y_0},\dots,\boldsymbol{y}_k) \in W_k^{ex} \text{ and } \boldsymbol{y_k} = \langle \vec{C}_k, y_k\rangle\},$$

$$X_{k+1}^{ex} = \bigcup_{(\boldsymbol{y_0},\dots,\boldsymbol{y_k})\in W_k^{ex}} \big\{(\boldsymbol{y_0},\dots,\boldsymbol{y_k},\boldsymbol{y}) \mid \boldsymbol{y_k} = \langle \vec{C}_k, y_k\rangle \text{ and } \boldsymbol{y} \in T(f(y_k))\big\},$$

where we write $f(\boldsymbol{y_k})$ for $\langle \vec{C}_k, f(y_k)\rangle$ whenever $\boldsymbol{y_k} = \langle \vec{C}_k, y_k\rangle$. Note that W_{k+1}^{ex} contains sequences of length $k+2$ only and, for every sequence $(\boldsymbol{y_0},\dots,\boldsymbol{y_k}) \in W_k^{ex}$ with $\boldsymbol{y_k} = \langle \vec{C}_k, y_k\rangle$, y_k indicates the element of W this sequence represents.

Let R_k^{ex} be the transitive closure of the union of the following three relations:

$$\begin{aligned}
&\big\{((\boldsymbol{y_0},\dots,\boldsymbol{y_k}, f(\boldsymbol{y_k})),(\boldsymbol{y'_0},\dots,\boldsymbol{y'_k}, f(\boldsymbol{y'_k}))) \in \overline{W}_k^{ex} \times \overline{W}_k^{ex} \;\mid \\
&\qquad\qquad\qquad\qquad (\boldsymbol{y_0},\dots,\boldsymbol{y_k}) R_k^{ex} (\boldsymbol{y'_0},\dots,\boldsymbol{y'_k})\big\},\\
&\big\{((\boldsymbol{y_0},\dots,\boldsymbol{y}_k, f(\boldsymbol{y_k})),(\boldsymbol{y_0},\dots,\boldsymbol{y}_k,\boldsymbol{y})) \in \overline{W}_k^{ex} \times X_{k+1}^{ex} \;\mid\; \boldsymbol{y} \in T(f(y_k))\big\},\\
&\big\{((\boldsymbol{y_0},\dots,\boldsymbol{y_k},\boldsymbol{y}),(\boldsymbol{y_0},\dots,\boldsymbol{y_k},\boldsymbol{y'})) \in X_{k+1}^{ex} \times X_{k+1}^{ex} \;\mid\; \boldsymbol{y} \leq \boldsymbol{y}'\big\}.
\end{aligned}$$

Informally, the first one says that $\overline{W}_k^{ex}$ is an isomorphic copy of W_k^{ex}, whereas the second and the third say together that to each point $(\boldsymbol{y_0},\dots,\boldsymbol{y}_k, f(\boldsymbol{y_k}))$ of $\overline{W}_k^{ex}$ a tree is attached and that tree is an isomorphic copy of the tree generated by $f(y_k)$ in W. It is readily checked that $\langle R_{k+1}^{ex}, W_{k+1}^{ex}\rangle$ is a tree of clusters.

The W_k^{ex} are pairwise disjoint. Let $W^{ex} = \bigcup_{k=0}^{\infty} W_k^{ex}$, $R^{ex} = \bigcup_{k=0}^{\infty} R_k^{ex}$, $\mathfrak{T}^{ex}$ be the Aleksandrov space induced by $\langle W^{ex}, R^{ex}\rangle$. Define f^{ex} by taking

$$f^{ex}((\boldsymbol{y_0},\dots,\boldsymbol{y_k})) = (\boldsymbol{y_0},\dots,\boldsymbol{y_k}, f(\boldsymbol{y_k})),$$

for all $(\boldsymbol{y_0},\dots,\boldsymbol{y_k}) \in W^{ex}$. Clearly, f^{ex} is monotonous, injective and satisfies **(top)** (in fact, it is a homeomorphism between each pair W_k^{ex} and $\overline{W}_k^{ex}$). Finally, define a valuation $\mathfrak{V}^{ex}$ by taking, for every variable p,

$$\mathfrak{V}^{ex}(p) = \{(\boldsymbol{y_0},\dots,\boldsymbol{y_k}) \in W^{ex} \mid \boldsymbol{y_k} = \langle \vec{C}_k, y_k\rangle \text{ and } y_k \in \mathfrak{V}(p)\}.$$

Let $\mathfrak{M}^{ex} = \langle\langle \mathfrak{T}^{ex}, f^{ex}\rangle, \mathfrak{V}^{ex}\rangle$ and $r^{ex} = \langle C(r), r\rangle$. It is not difficult to see that $(\mathfrak{M}^{ex}, r^{ex}) \models \varphi$. ❑

B Sketch of the proof of Theorem 7

We present here only a sketch of the proof. To apply Kruskal's tree theorem, we require a representation of the models of Lemma 6 as sequences of labelled trees. This can be achieved in a straightforward way using Hintikka-type structures of the following form. For a $\mathcal{DTL}_1$-formula φ, let $cl\varphi$ be the set of all subformulas of φ and their negations. A *full φ-type t* is Boolean-closed subset of $cl\varphi$, that is,

- $\neg\psi \in t$ iff $\psi \notin t$, for every subformula ψ of φ,
- $\psi_1 \wedge \psi_2 \in t$ iff $\psi_1, \psi_2 \in t$, for every subformula $\psi_1 \wedge \psi_2$ of φ.

A subset s of a φ-type t is a (*simple*) *φ-type* if it does not contain formulas with occurrences of $\Box_F$ but satisfies the conditions above for all subformulas of φ not containing $\Box_F$.

A *quasistate* $\mathfrak{S} = \langle W, R, l\rangle$ is a finite transitive *irreflexive* tree $\langle W, R\rangle$ with a labelling function l which assigns

- a set $l(x)$ of simple φ-types to every $x \in W$ that is not the root of $\langle W, R\rangle$ (this set is supposed to encode the cluster represented by x),
- a set $l(r) = \{l^1(r)\} \cup l^2(r)$, where $l^1(r)$ is a full φ-type and $l^2(r)$ is a set of simple φ-types, to the root r of $\langle W, R\rangle$

such that the following condition holds for all $\mathbf{I}\psi \in \varphi$, $x \in W$, and $t \in l(x)$: we have $\mathbf{I}\psi \in t$ iff $\psi \in s$ for every $s \in l(y)$ with $x = y$ or xRy.

A sequence $\mathfrak{S}_n = \langle W_n, R_n, l_n\rangle$, $n < \omega$, of quasistates is called a *quasimodel* for φ if there are injective maps $f_n : W_n \to W_{n+1}$ such that

(1) for all $x, y \in W_n$, xR_ny iff $f_n(x)R_{n+1}f_n(y)$,

(2) $\varphi \in l_0^1(r_0)$,

(3) $|\mathfrak{S}_n| \leq F(\varphi, n)$, where $F(\varphi, n)$ is the recursive function from Lemma 6,

(4) $f(r_n) = r_{n+1}$ for the roots r_n and r_{n+1} of $\langle W_n, R_n\rangle$ and $\langle W_{n+1}, R_{n+1}\rangle$, respectively,

(5) if $f_n(x) = y$, then for every $t \in l_n(x)$ there exists $t' \in l_{n+1}(y)$ such that for every $\bigcirc\psi \in cl\varphi$, $\bigcirc\psi \in l_n(x)$ iff $\psi \in l_{n+1}(y)$; if $x = r_n$ and $t = l_n^1(r_n)$, then we can choose $t' = l_{n+1}^1(r_{n+1})$,

(6) for every $\Box_F\psi \in cl\varphi$, $\neg\Box_F\psi \in l_n^1(r_n)$ iff there exists $m > n$ such that $\neg\psi \in l_m^1(r_m)$; if m is minimal with this property, then we say that m *realises the eventuality* $\neg\Box_F\psi$.

The proof of the following lemma is easy and left to the reader.

LEMMA 15. *A $\mathcal{DTL}_1$-formula φ is satisfiable in a DTM iff there exists a quasimodel for φ.*

We remind the reader of the following weak version of *Kruskal's tree theorem*. An injective function $i : \langle W_1, R_1, l_1\rangle \to \langle W_2, R_2, l_2\rangle$ between labelled trees is called an *embedding* if, for all $x, y \in W_1$, xR_1y iff $i(x)R_2i(y)$, and $l_1(x) = l_2(i_2(x))$. We assume that the labels are taken from some finite set.

THEOREM 16 (Kruskal). *For any sequence $\mathfrak{S}_n$, $n < \omega$, of labelled trees, there exists $i < j$ such that $\mathfrak{S}_i$ is embeddable into $\mathfrak{S}_j$.*

Consider now a quasimodel $\mathfrak{S}_n$, $n < \omega$, for φ. Clearly, if there are $i < j$ such that $\mathfrak{S}_i$ is embeddable into $\mathfrak{S}_j$, then the sequence

$$\mathfrak{S}_0, \dots, \mathfrak{S}_{i-1}, \mathfrak{S}_j, \mathfrak{S}_{j+1} \dots$$

is a quasimodel for φ as well. Moreover, we can 'prune' the quasistates $\mathfrak{S}_j, \mathfrak{S}_{j+1} \dots$ in such a way that the bound from (3) above still holds.

Now we can recursively enumerate the $\mathcal{DTL}_1$-formulas φ which are not satisfiable as follows: for every φ enumerate all finite sequences $\mathfrak{S}_0, \dots, \mathfrak{S}_m$ such that conditions (1)–(5) above and the following modification of condition (6) hold:

- For every $\Box_F\psi \in cl\varphi$ and every $n < m$, we have $\neg\Box_F\psi \in l_n^1(r_n)$ iff $\neg\Box_F\psi \in l_k^1(r_m)$ or there exists k, $m \geq k > n$, such that k realises the eventuality $\neg\Box_F\psi$.

- Let $0 = k_0 < k_1 < k_2 \cdots < k_n \leq m$ be the minimal numbers such that every $\neg\Box_F\psi \in l_{k_i}^1(r_{k_i})$ is realised until k_{i+1}. If no $\neg\Box_F\psi$ exists in $l_{k_i}^1(r_{k_i})$, then set $k_{i+1} = k_i + 1$. Then, if there are m_1, m_2 with $k_i < m_1 < m_2 < k_{i+1}$ or $k_n < m_1 < m_2 < m$ such that $\mathfrak{S}_{m_1}$ is embeddable into $\mathfrak{S}_{m_2}$, then there exists a $\neg\Box_F\psi \in l_{k_1}^1(r_{k_1})$ which is realised somewhere in the interval $[m_1, m_2)$.

Now suppose that the enumeration of such sequences does not terminate for φ (i.e., infinite many such sequences are generated). Then, by König's lemma, there exists an infinite sequence whose finite initial segments satisfy the conditions above. But then, by Kruskal's tree theorem, this sequence is a quasimodel for φ, and so φ is satisfiable. Conversely, if the enumeration of such finite sequences terminates (i.e., there are only finitely many sequences), then there is no quasimodel for φ. Hence φ is not satisfiable. It follows that the non-satisfiable formulas (and therefore the valid formulas) are recursively enumerable.

C Proof of the soundness part of Theorem 12

($\Leftarrow$) For the converse direction, suppose that, for every $n \geq 1$, we have a *perfect* computation of S starting with $\langle q_0, \epsilon\rangle$ and reaching q_{rec} at least n times. Let N_n be the number of transitions in the n-th computation; the 0-th computation is considered to be empty and $N_0 = 0$. For $n > 0$, let $M_n = \sum_{k=1}^{n}(N_k + 1)$ and $M_0 = 0$ (for technical reasons we assume $M_{-1} = -1$).

Fix $n \geq 0$ and consider the n-th computation:

$$\langle q_0, \epsilon\rangle \overset{\delta_{k_1^n}}{\to}_p \left\langle q_{i_1^n}, w_1^n \right\rangle \overset{\delta_{k_2^n}}{\to}_p \left\langle q_{i_2^n}, w_2^n \right\rangle \overset{\delta_{k_3^n}}{\to}_p \cdots \overset{\delta_{k_{N_n}^n}}{\to}_p \left\langle q_{i_{N_n}^n}, w_{N_n}^n \right\rangle.$$

Let H_0^n be the number of send (!) transitions in the above sequence. Clearly, the number of messages that can be held in the channel during the above computation is bounded by H_0^n because receive (?) transitions only remove messages from the channel.

We will inductively define sequences H_j^n and L_j^n (w_j^n will be written between L_j^n and H_j^n). As H_0^n is already defined, let $L_0^n = H_0^n$. Suppose that we have defined such sequences for $j-1$. Then

$$(L_j^n, H_j^n) = \begin{cases} (L_{j-1}^n, H_{j-1}^n - 1), & \text{if } \delta_{i_j^n} = \langle q_{i_{j-1}^n}, ?, a, q_{i_j^n} \rangle, \\ (L_{j-1}^n - 1, H_{j-1}^n), & \text{if } \delta_{i_j^n} = \langle q_{i_{j-1}^n}, !, a, q_{i_j^n} \rangle. \end{cases}$$

We are in a position now to define a satisfying model for $\varphi_{S,q_0,q_{rec}}$ based on the computations above. First, for each computation number $n \in \mathbb{N}$ and a step j in it, $0 \le j \le N_n$, let

$$U_j^n = \{0, 1, \dots, H_j^n\} \times \{0, 1\}$$

be the chunk of the model required to encode this step of the computation: one extra point is needed for a light that is created for this computation; moreover, points are duplicated to make the strict modality **S** work properly. Note that U_0^n is the largest among them and includes all of the U_j^n.

Next, for every $m \in \mathbb{N}$, let

$$W_m = \{m\} \times \Big[\bigcup_{n=0}^{M_n < m} \big(\{n\} \times U_0^n\big) \quad \cup \quad \big(\{n_0\} \times U_{M_{n_0} - m}^{n_0}\big) \Big],$$

where n_0 is such that $M_{n_0 - 1} < m \le M_{n_0}$ (i.e., n_0 is the number of the computation at iteration m and $j = M_{n_0} - m$ is the step number inside that computation). Note that the computations are encoded backwards—from the end to its start. Let R_m be the lexicographic order on W_m, i.e.,

$$(m, n, k, q) R_m (m, n', k', q') \quad \text{iff} \quad n < n' \text{ or} \\ \big(n = n' \text{ and } (k < k' \text{ or } (k = k' \text{ and } q \le q')\big)$$

for all quadruples $(m, n, k, q), (m, n', k', q') \in W_m$ (n and n' are for computation numbers, k and k' for elements to write words on, and q and q' range $\{0,1\}$ for duplication). Note that $r_m = (m, 0, 0, 0)$ is the root of $\langle W_m, R_m \rangle$.

Let $W = \bigcup_{m=0}^{\infty} W_m$, $R = \bigcup_{m=0}^{\infty} R_m$ and let $\mathfrak{T}$ be the Aleksandrov space induced by $\langle W, R \rangle$. Define f by taking

$$f((m, n, k, q)) = (m+1, n, k, q)$$

for all $(m, n, k, q) \in W$. Clearly, f is monotonous on $\langle W, R \rangle$. Finally, we define a valuation $\mathfrak{V}$:

$$\begin{aligned}
&\mathfrak{V}(s) = \{(m, n, k, 0) \in W\}, \\
&\mathfrak{V}(\mathsf{m}) = \{(m, 0, 0, 0) \in W \mid \exists n \in \mathbb{N} \quad C(m; n, 0)\}, \\
&\mathfrak{V}(\mathsf{tr}\delta_k) = \{(m, 0, 0, 0) \in W \mid \exists n, j \in \mathbb{N} \quad C(m; n, j+1) \text{ and } \delta_{k_j^n} = \delta_k\}, \\
&\mathfrak{V}(\mathsf{q}_i) = \{(m, 0, 0, 0) \in W \mid \exists n, j \in \mathbb{N} \quad C(m; n, j) \text{ and } q_{i_j^n} = q_i\}, \\
&\mathfrak{V}(a_i) = \{(m, n, k, q) \in W \mid \exists j \in \mathbb{N} \quad C(m; n, j) \text{ and } w_j^n(L_j^n + k) = a_i\}, \\
&\mathfrak{V}(\mathit{light}) = \{(m, n, 0, q) \in W\}, \\
&\mathfrak{V}(\mathit{on}) = \{(m, n, 0, q) \in W \mid \exists j \in \mathbb{N} \quad C(m; n, j) \text{ and} \\
&\qquad\qquad\qquad q_{rec} \text{ occurs } > n \text{ times among } q_{i_{N_n}^n}, \dots, q_{i_j^n}\},
\end{aligned}$$

where the predicate $C(m; n, j)$ is true iff $m = M_n - j$ and $0 \le j \le N_n$.

It can be readily verified that $(\langle\langle \mathfrak{T}, f \rangle, \mathfrak{V} \rangle, r_0) \models \varphi_{S,q_0,q_{rec}}$.

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D Proof of Lemma 2

Proof. Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be a DTM, where $\mathfrak{F} = \langle \mathfrak{T}, f \rangle$ and $\mathfrak{T} = \langle T, \mathbb{I} \rangle$. Denote by $Uf(T)$ the set of ultrafilters over T. For any two ultrafilters $\boldsymbol{x}, \boldsymbol{y} \in Uf(T)$, set

$$\boldsymbol{x} R \boldsymbol{y} \qquad \text{iff} \qquad \forall A \subseteq T \quad (\mathbb{I}A \in \boldsymbol{x} \to A \in \boldsymbol{y}).$$

It is well-known that R is a quasi-order on $Uf(T)$ ($\mathbb{I}A \subseteq A$ and $\mathbb{I}A \subseteq \mathbb{I}\mathbb{I}A$ for reflexivity and transitivity, respectively). The Kripke frame $\mathfrak{G} = \langle Uf(T), R \rangle$ gives rise to the Aleksandrov space $\mathfrak{T}_{\mathfrak{G}} = \langle Uf(T), \mathbb{I}_{\mathfrak{G}} \rangle$ where

$$\mathbb{I}_{\mathfrak{G}} \boldsymbol{X} = \{ \boldsymbol{x} \in Uf(T) \mid \forall \boldsymbol{y}\ (\boldsymbol{x} R \boldsymbol{y} \to \boldsymbol{y} \in \boldsymbol{X}) \},$$

for all $\boldsymbol{X} \subseteq Uf(T)$. Let $f_{\mathfrak{G}}(\boldsymbol{x}) = \{ A \subseteq T \mid f^{-1}(A) \in \boldsymbol{x} \}$.[3] We show that $f_{\mathfrak{G}}(\boldsymbol{x})$ is an ultrafilter over T:

- $f_{\mathfrak{G}}(\boldsymbol{x}) \neq \emptyset$ (for instance, $T \in f_{\mathfrak{G}}(\boldsymbol{x})$ as $f^{-1}(T) = T \in \boldsymbol{x}$);
- if $A, B \in f_{\mathfrak{G}}(\boldsymbol{x})$ the $f^{-1}(A), f^{-1}(B) \in \boldsymbol{x}$ and, as $\boldsymbol{x}$ is an ultrafilter, $f^{-1}(A) \cap f^{-1}(B) \in \boldsymbol{x}$, whence $f^{-1}(A \cap B) \in \boldsymbol{x}$ and $A \cap B \in f_{\mathfrak{G}}(\boldsymbol{x})$;
- if $A \in f_{\mathfrak{G}}(\boldsymbol{x})$ and $A \subseteq B$ then $f^{-1}(A) \in \boldsymbol{x}$ and $f^{-1}(A) \subseteq f^{-1}(B)$, from which, as $\boldsymbol{x}$ is an ultrafilter, $f^{-1}(B) \in \boldsymbol{x}$, and so $B \in f_{\mathfrak{G}}(\boldsymbol{x})$;
- if $A, T \setminus A \in f_{\mathfrak{G}}(\boldsymbol{x})$, for some $A \subseteq T$, then $f^{-1}(A), f^{-1}(T \setminus A) \in \boldsymbol{x}$ contrary to $\boldsymbol{x}$ being an ultrafilter.

The function $f_{\mathfrak{G}}$ is monotonous with respect to R. Indeed, suppose $\boldsymbol{x} R \boldsymbol{y}$. Take any $A \subseteq T$ with $\mathbb{I}A \in f_{\mathfrak{G}}(\boldsymbol{x})$, i.e., $f^{-1}(\mathbb{I}A) \in \boldsymbol{x}$. As f is continuous, $f^{-1}(\mathbb{I}A) \subseteq \mathbb{I}f^{-1}(A)$, and since $\boldsymbol{x}$ is an ultrafilter, we have $\mathbb{I}f^{-1}(A) \in \boldsymbol{x}$. From $\boldsymbol{x} R \boldsymbol{y}$ we conclude $f^{-1}(A) \in \boldsymbol{y}$. Therefore, $A \in f_{\mathfrak{G}}(\boldsymbol{y})$ and as the choice of A is arbitrary, $f_{\mathfrak{G}}(\boldsymbol{x}) R f_{\mathfrak{G}}(\boldsymbol{y})$.

It follows that $f_{\mathfrak{G}}$ is continuous in $\mathfrak{T}_{\mathfrak{G}}$, and so $\mathfrak{F}_{\mathfrak{G}} = \langle \mathfrak{T}_{\mathfrak{G}}, f_{\mathfrak{G}} \rangle$ is a DTS. Consider now the DTM $\mathfrak{M}_{\mathfrak{G}} = \langle \mathfrak{F}_{\mathfrak{G}}, \mathfrak{V}_{\mathfrak{G}} \rangle$, where the valuation $\mathfrak{V}_{\mathfrak{G}}$ is defined by taking, for every variable p_i,

$$\mathfrak{V}_{\mathfrak{G}}(p_) \;=\; \{ \boldsymbol{x} \in Uf(T) \mid \mathfrak{V}(p_i) \in \boldsymbol{x} \}.$$

(A) First we show by induction on the construction of the formula that, for every ψ of the form (2) and every $\boldsymbol{x} \in Uf(T)$,

$$\mathfrak{V}(\psi) \in \boldsymbol{x} \qquad \text{iff} \qquad (\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x}) \models \psi, \tag{25}$$

where $\mathfrak{V}(\psi) = \{ w \in T \mid (\mathfrak{M}, w) \models \psi \}$.

The basis of induction is trivial. Let us consider compound formulas.

- $\mathfrak{V}(\neg\psi) \in \boldsymbol{x}$ iff $T \setminus \mathfrak{V}(\psi) \in \boldsymbol{x}$ iff $\mathfrak{V}(\psi) \notin \boldsymbol{x}$ iff (by IH) $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x}) \not\models \psi$ iff $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x}) \models \neg\psi$.

[3] Alternatively, $f_{\mathfrak{G}}(\boldsymbol{x}) = \{ f(A) \mid A \in \boldsymbol{x} \}$.

- $\mathfrak{V}(\psi_1 \wedge \psi_2) \in \boldsymbol{x}$ iff $\mathfrak{V}(\psi_1) \cap \mathfrak{B}(\psi_2) \in \boldsymbol{x}$ iff $\mathfrak{B}(\psi_1) \in \boldsymbol{x}$ and $\mathfrak{V}(\psi_2) \in \boldsymbol{x}$ iff (by IH) $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x}) \models \psi_1$ and $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x}) \models \psi_2$ iff $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x}) \models \psi_1 \wedge \psi_2$.

- Suppose $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x}) \models \mathbf{I}\psi$ but $\mathfrak{V}(\mathbf{I}\psi) \notin \boldsymbol{x}$. Then $\mathbb{I}\mathfrak{V}(\psi) \notin \boldsymbol{x}$ and, by the definition of ultrafilter, $\mathbb{C}(T \setminus \mathfrak{V}(\psi)) \in \boldsymbol{x}$. Note now that the set

  $$\boldsymbol{y_0} \;=\; \{T \setminus \mathfrak{V}(\psi)\} \cup \{A \subseteq T \mid \mathbb{I}A \in \boldsymbol{x}\} \tag{26}$$

  has the *finite intersection property.* Indeed, otherwise there exist sets $A_1, \dots, A_n \subseteq T$ such that

  $$(T \setminus \mathfrak{V}(\psi)) \cap A_1 \cap \dots \cap A_n = \emptyset.$$

  But then $\mathbb{C}(T \setminus \mathfrak{V}(\psi)) \cap \mathbb{I}(A_1 \cap \dots \cap A_n) = \emptyset$ and, as the second set is $\mathbb{I}A_1 \cap \dots \cap \mathbb{I}A_n$ and all the sets in the left-hand side belong to $\boldsymbol{x}$, we must have $\emptyset \in \boldsymbol{x}$, contrary to $\boldsymbol{x}$ being an ultrafilter.

  Now, as $\boldsymbol{y_0}$ has the finite intersection property, there is $\boldsymbol{y} \in \mathit{Uf}(T)$ such that $\boldsymbol{y_0} \subseteq \boldsymbol{y}$. From (26), $\boldsymbol{xRy}$ and so $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{y}) \models \psi$. By IH, $\mathfrak{V}(\psi) \in \boldsymbol{y}$, contrary to $T \setminus \mathfrak{V}(\psi) \in \boldsymbol{y}$ and $\boldsymbol{y}$ being an ultrafilter.

  Conversely, assume $\mathfrak{V}(\mathbf{I}\psi) \in \boldsymbol{x}$. Then $\mathbb{I}\mathfrak{V}(\psi) \in \boldsymbol{x}$. Take an arbitrary $\boldsymbol{y} \in \mathit{Uf}(T)$ with $\boldsymbol{xRy}$. Then we have $\mathfrak{V}(\psi) \in \boldsymbol{y}$, by IH, $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{y}) \models \psi$. Therefore, $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x}) \models \mathbf{I}\psi$.

- $\mathfrak{V}(\bigcirc\psi) \in \boldsymbol{x}$ iff $f^{-1}(\mathfrak{V}(\psi)) \in \boldsymbol{x}$ iff $\mathfrak{V}(\psi) \in f_{\mathfrak{G}}(\boldsymbol{x})$ iff $(\mathfrak{M}_{\mathfrak{G}}, f_{\mathfrak{G}}(\boldsymbol{x})) \models \psi$ iff $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x}) \models \bigcirc\psi$.

(B) Second, we show that, for every φ of the form (3) and every *principal* ultrafilter $\boldsymbol{x_w} = \{A \subseteq T \mid w \in A\}$ (the ultrafilter generated by w),

$$\mathfrak{V}(\varphi) \in \boldsymbol{x_w} \qquad \text{iff} \qquad (\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x_w}) \models \varphi, \tag{27}$$

In order to prove this, we first show that, for every $w \in T$,

$$\boldsymbol{x_w} \text{ is principal} \quad \text{iff} \quad f_{\mathfrak{G}}(\boldsymbol{x_w}) \text{ is principal}. \tag{28}$$

Indeed, $f_{\mathfrak{G}}(\boldsymbol{x_w}) = \boldsymbol{x_{f(w)}}$: for every $A \subseteq T$,

$$\begin{aligned}
A \in f_{\mathfrak{G}}(\boldsymbol{x_w}) &\quad \text{iff (by the definition of } f_{\mathfrak{G}})\\
f^{-1}(A) \in \boldsymbol{x_w} &\quad \text{iff (by the definition of } \boldsymbol{x_w})\\
w \in f^{-1}(A) &\quad \text{iff (by the definition of } f^{-1})\\
f(w) \in A &\quad \text{iff (by the definition of } \boldsymbol{x_{f(w)}})\\
A \in \boldsymbol{x_{f(w)}}. &
\end{aligned}$$

We are now in a position to show (27). The basis of induction follows from (25).

- The cases of $\neg\varphi$ and $\varphi_1 \wedge \varphi_2$ are completely analogous to those in the proof of (25).

- In view of (28), the case of $\bigcirc\varphi$ is also similar to the one in (25).

- Let $\mathfrak{V}(\Box_F\varphi) \in \boldsymbol{x_w}$. Then $\bigcap_{n>0} f^{-n}(\mathfrak{V}(\varphi)) \in \boldsymbol{x_w}$ and since $\boldsymbol{x_w}$ is an ultrafilter, $f^{-n}(\mathfrak{V}(\varphi)) \in \boldsymbol{x_w}$, for all $n > 0$. Therefore, for all $n > 0$, we have $\mathfrak{V}(\varphi) \in f^n_{\mathfrak{G}}(\boldsymbol{x_w})$ and, by IH and (28), $(\mathfrak{M}_{\mathfrak{G}}, f^n_{\mathfrak{G}}(\boldsymbol{x_w})) \models \varphi$, which implies $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x_w}) \models \Box_F\varphi$.

  Conversely, suppose $(\mathfrak{M}_{\mathfrak{G}}, \boldsymbol{x_w}) \models \Box_F\varphi$. Then, for each $n > 0$, we have $(\mathfrak{M}_{\mathfrak{G}}, f^n_{\mathfrak{G}}(\boldsymbol{x_w})) \models \varphi$. By IH and (28), $\mathfrak{V}(\varphi) \in f^n_{\mathfrak{G}}(\boldsymbol{x_w})$ and thus $f^{-n}(\mathfrak{V}(\varphi)) \in \boldsymbol{x_w}$, for all $n > 0$. The latter means $w \in f^{-n}(\mathfrak{V}(\varphi))$, for all $n > 0$, and therefore, $w \in \bigcap_{n>0} f^{-n}(\mathfrak{V}(\varphi)) = \mathfrak{V}(\Box_F\varphi)$, which implies $\mathfrak{V}(\Box_F\varphi) \in \boldsymbol{x_w}$.

Finally, it follows from (27) that if a $\mathcal{DTL}_1$-formula φ is satisfied in a DTM $\mathfrak{M}$ at w then φ is satisfied in the induced Aleksandrov model $\mathfrak{M}_{\mathfrak{G}}$ at $\boldsymbol{x_w}$.

The existence of R-maximal points in sets of the form $A_{\boldsymbol{x},\psi}$, in $\mathfrak{M}_{\mathfrak{G}}$ follows from a well-known result of Fine [6]. Here is a sketch of the proof. Consider the family

$$\mathcal{X} \;=\; \{\boldsymbol{X} \subseteq A_{\boldsymbol{x},\psi} \mid R \cap (\boldsymbol{X} \times \boldsymbol{X}) \text{ is linear, with smallest element } \boldsymbol{x}\}.$$

Let $\boldsymbol{C}$ be a $\subseteq$-maximal set in $\mathcal{X}$ (i.e., for every $\boldsymbol{C'} \in \mathcal{X}$, $\boldsymbol{C} \subseteq \boldsymbol{C'}$ implies $\boldsymbol{C'} = \boldsymbol{C}$); its existence can be readily proved with the help of Zorn's lemma. Now take the set

$$\boldsymbol{y_0} \;=\; \{A \subseteq T \mid \exists \boldsymbol{z} \in \boldsymbol{C}\; \forall \boldsymbol{z'} \in \boldsymbol{C}\; (\boldsymbol{z}R\boldsymbol{z'} \to A \in \boldsymbol{z'})\}.$$

This set is not empty, since $\mathfrak{V}(\psi) \in \boldsymbol{y_0}$, and clearly $\boldsymbol{y_0}$ has the finite intersection property. Hence we can find an ultrafilter $\boldsymbol{y} \in \mathit{Uf}(T)$ containing $\boldsymbol{y_0}$. Then it is easy to see, using the definition of R, that

$$\forall \boldsymbol{z} \in \boldsymbol{C}\; \boldsymbol{z}R\boldsymbol{y}. \tag{29}$$

We claim that $\boldsymbol{y}$ is R-maximal in $A_{\boldsymbol{x},\psi}$. Indeed, take some $\boldsymbol{y'} \in A_{\boldsymbol{x},\psi}$ such that $\boldsymbol{y}R\boldsymbol{y'}$. If $\boldsymbol{y'} \in \boldsymbol{C}$ then $\boldsymbol{y'}R\boldsymbol{y}$ holds by (29). If $\boldsymbol{y'} \notin \boldsymbol{C}$ then, by the $\subseteq$-maximality of $\boldsymbol{C}$ in $\mathcal{X}$, R is not linear on $\boldsymbol{C} \cup \{\boldsymbol{y'}\}$. Since by (29) and $\boldsymbol{y}R\boldsymbol{y'}$, we have $\boldsymbol{z}R\boldsymbol{y'}$ for all $\boldsymbol{z} \in \boldsymbol{C}$, there exists a $\boldsymbol{z'} \in \boldsymbol{C}$ such that $\boldsymbol{y'}R\boldsymbol{z'}$, and so, again by (29), $\boldsymbol{y'}R\boldsymbol{y}$ as required. ❑

In the proof of Lemma 2, step (B) does not go through for arbitrary ultrafilters and formulas of the form $\Box_F\varphi$: one cannot guarantee that the infinite sequence $\bigcap_{k=1}^{n} f^{-k}(\mathfrak{V}(\varphi))$ of elements of an ultrafilter converges to an element of the same ultrafilter (as it happens in principal ultrafilters).

E Proof of Lemma 5

Proof. Suppose that $\mathfrak{M} = \langle\langle\mathfrak{T}, f\rangle, \mathfrak{V}\rangle$ and that $\mathfrak{T}$ is induced by $\langle W, R\rangle$. Assume also that $(\mathfrak{M}, x_0) \models \varphi$ for some x_0. We construct a new DTM $\mathfrak{M}^{ep}$ based on a subset of $\mathbb{N} \times \mathbb{N} \times W$ (where the first component is used to represent the iteration number, the second one to keep track of the 'age' of a point to ensure that the function is injective, and the last component is for points from W). The universe W^{ep} of $\mathfrak{M}^{ep}$ is constructed in a step-by-step manner. Let $cl\varphi$ be the set of all subformulas of φ and their negations. Let $V_0 = \{(0, 0, x_0)\}$. Suppose now that we have already constructed V_k. Define W_k^{ep} to be the closure of V_k under the following rule:

- if $(\mathfrak{M}, f^n(x)) \models \mathbf{C}\psi$, for some $(k, n, x) \in V_k$ and $\mathbf{C}\psi \in cl\varphi$, and there is no $(k, m, y) \in V_k$ such that $f^n(x)Rf^m(y)$ and $(\mathfrak{M}, f^m(y)) \models \psi$, then, by (max), we choose an R-maximal point z in W such that $f^n(x)Rz$ and $(\mathfrak{M}, z) \models \psi$, and set $V_k := V_k \cup \{(k, 0, z)\}$.

Then put $V_{k+1} = \{(k + 1, n + 1, x) \mid (k, n, x) \in W_k^{ep}\}$ and repeat the above process of taking closure of V_{k+1} to obtain W_{k+1}^{ep}, and so on. It is easy to see that $|W_n^{ep}| \leq ((1 + \ell(\varphi))!)^{n+1}$, for all $n \geq 0$.

Let $W^{ep} = \bigcup_{n=0}^{\infty} W_n^{ep}$. Define R^{ep} by taking

$$(k, n, x)R^{ep}(k', n', x') \quad \text{iff} \quad k = k' \text{ and } f^n(x)Rf^{n'}(x'),$$

for all $(k, n, x), (k', n', x') \in W^{ep}$ (R_n^{ep} is the restriction of R^{ep} to W_n^{ep}). It is readily seen that R^{ep} is a quasi-order. Let $\mathfrak{T}^{ep}$ be the Aleksandrov space induced by $\langle W^{ep}, R^{ep}\rangle$. Define f^{ep} by taking, for all $(k, n, x) \in W^{ep}$,

$$f^{ep}((k, n, x)) = (k + 1, n + 1, x).$$

By definition, **(root)** holds. As f is monotonous on $\langle W, R\rangle$, f^{ep} is monotonous on $\langle W^{ep}, R^{ep}\rangle$ as well. It is also easily seen that f^{ep} is injective.

Finally, define a valuation $\mathfrak{V}^{ep}$ by taking, every variable p,

$$\mathfrak{V}^{ep}(p) = \{(k, n, x) \in W^{ep} \mid f^n(x) \in \mathfrak{V}(p)\}.$$

Let $\mathfrak{M}^{ep} = \langle\langle\mathfrak{T}^{ep}, f^{ep}\rangle, \mathfrak{V}^{ep}\rangle$ and $r_0 = (0, 0, x_0)$. It is routinely checked that $(\mathfrak{M}^{ep}, r_0) \models \varphi$. ❑

F Proof of Lemma 9

Proof. The construction of [3] is inductive and is based on the existence of a continuous and open mapping of I onto a finite tree of depth 2. In our case it will be a mapping of I onto an ω-tree of depth 2. Here we show the existence of such an embedding and provide a sketch of the rest of the proof leaving the details for the reader.

Without loss of generality we may assume that I is a bounded interval (a, b). Recall that the Cantor set on I is constructed by taking out infinitely many open subintervals of I. More precisely, at step 1, the open interval

$I_1^1 = \left(a + \frac{b-a}{3}, a + \frac{2b-a}{3}\right)$ is taken out. At step 2, two open intervals, $I_1^2 = \left(a + \frac{b-a}{3^2}, a + \frac{2b-a}{3^2}\right)$ and $I_2^2 = \left(a + \frac{7(b-a)}{3^2}, a + \frac{8(b-a)}{3^2}\right)$ are taken out. In general, at step m, the open intervals $I_1^m, \dots, I_{2^{m-1}}^m$ are taken out of I. Denote $I^m = \bigcup_{i=1}^{2^{m-1}} I_i^m$. Then the Cantor set on I is $I \setminus \bigcup_{m=1}^{\infty} I^m$.

Let $W = \{r\} \cup \{t_0, t_1, \dots, \}$ and R be the reflexive closure of $\{(r, t_i)\}$. Fix some enumeration of all pairs of natural numbers $\alpha\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Define

$$f_I^{\mathfrak{G}}(z) = \begin{cases} t_k, & \text{if } z \in \bigcup_{m=1}^{\infty}\{I^m \mid \alpha(m) = (k, j), \text{ for some } j \in \mathbb{N}\}, \\ r, & \text{otherwise.} \end{cases}$$

It is a routine exercise to check that $f_I^{\mathfrak{G}}$ is a well-defined continuous and open function from I onto $\mathfrak{G}$, for details see [3]. It should be noted that r is the image of the Cantor set on I while each t_k is the image of a countable union of open intervals. The above embedding can easily be extended for the case of arbitrary ω-trees of clusters of depth 2 (see Lemma 15 in [3]).

Having a mapping of I onto ω-trees of depth 2 at hand one can extend it by induction to ω-trees of arbitrary finite depth. This part of the proof goes along the lines of Theorem 16 in [3]: every ω-tree $\mathfrak{G}_{d+1}$ of depth $d+1$ can be regarded as an ω-tree $\mathfrak{G}_d$ of depth d with ω-trees of depth 2 attached to its final clusters; suppose that one has an open and continuous mapping $f_I^{\mathfrak{G}_d}$; it follows from **(final)** that each final cluster of $\mathfrak{G}_d$ is an images of (countably many) open intervals which in their turn can be mapped onto the respective ω-tree of depth 2 'attached' to that final cluster. This results in a required mapping $f_I^{\mathfrak{G}_{d+1}}$. For further details see [3]. ❑

G Proof of Lemma 11

Proof. The proof proceeds by reduction of the undecidable *space-boundedness* problem for lossy counter machines [15, Theorem 12]: given a 4-counter lossy counter machine M with *any* lossiness relation, decide whether the space used by M is bounded by a constant. Lossy counter machines can be simulated by lossy channel systems, as demonstrated in [15], so it is undecidable to determine, for a 4-channel lossy channel system S with any lossiness relation $\to$ satisfying (6), whether the set of configurations of S reachable from $\langle q_0, \epsilon\rangle$ is *finite*. Finally, using the reduction of systems with k channels to systems with one channel from [16], one can demonstrate that the space-boundedness problem for a single channel lossy channel system with *arbitrary* lossiness relation, satisfying the conditions of the lemma, is undecidable.

Now, given a channel system S, we construct a system S' in such a way that S is bounded, that is, S has only finitely many configurations reachable from $\langle q_0, \epsilon\rangle$, iff S' does not have the ω-reachability property. The set of states of S' extends that of S with one additional state q_{rec}, and the set of transitions of S' is that of S plus non-deterministic transitions from every state of S into q_{rec}. Being in q_{rec}, the system reads one symbol from the channel and stays in q_{rec}. It should be clear now that there exists a (lossy) computation of S' starting with $\langle q_0, \epsilon \rangle$ and reaching q_{rec} arbitrary many times iff S is unbounded. ❑