On dynamic topological and metric logics

B. Konev[1], R. Kontchakov[2], F. Wolter[1], and M. Zakharyaschev[2]

[1]Department of Computer Science, University of Liverpool,
Liverpool L69 3BX, U.K.
{b.konev, frank}@csc.liv.ac.uk,

[2]Department of Computer Science, King's College London,
Strand, London WC2R 2LS, U.K.
{romanvk, mz}@dcs.kcl.ac.uk

Abstract

We investigate computational properties of propositional logics for dynamical systems. First, we consider logics for dynamic topological systems $\langle W, f\rangle$, where W is a topological space and f a homeomorphism on W. The logics come with 'modal' operators interpreted by the topological closure and interior, and temporal operators interpreted along the orbits $\{w, f(w), f^2(w), \dots\}$ of points $w \in W$. We show that for various classes of topological spaces the resulting logics are not recursively enumerable (and so not recursively axiomatisable). This gives a 'negative' solution to a conjecture of Kremer and Mints. Second, we consider logics for dynamical systems $\langle W, f\rangle$, where W is a metric space and f an isometric function. The operators for topological interior/closure are replaced by distance operators of the form 'everywhere/somewhere in the ball of radius a,' for $a \in \mathbb{Q}^+$. In contrast to the topological case, the resulting logic turns out to be decidable, but not in time bounded by any elementary function.

1 Introduction

Dynamic topological logics were first introduced in 1997 (see [12, 13, 15, 2, 14]) as a logical formalism for describing the behaviour of *dynamical systems*, e.g., in order to specify liveness and safety properties of hybrid systems [6]. Dynamical systems [4, 11] are usually represented by some 'mathematical' space W (modelling possible system states) and a function f on W (modelling the evolution of the system), with one of the main research problems being the study of iterations of f, in particular, the orbits $O(w) = \{w, f(w), f^2(w), \dots\}$ of states $w \in W$.

A natural logical formalism for speaking about such iterations is a variant of temporal logic. For example, given a subset V of W, we can introduce the standard temporal operators $\bigcirc$ ('at the next moment'), $\Box_F$ ('always in the future') and its dual $\Diamond_F$ ('eventually') by taking

$$\bigcirc V = f^{-1}(V), \qquad \Box_F V = \bigcap_{0<n<\omega} f^{-n}(V) \qquad \text{and} \qquad \Diamond_F V = \bigcup_{0<n<\omega} f^{-n}(V).$$

Using this language we can describe in a succinct and transparent way properties like

- starting from a state in some region P, one will never leave a region Q: $P \to \Box_F Q$;
- starting from a state in a region P, one will eventually reach a state in Q: $P \to \Diamond_F Q$;
- every point of P 'visits' P ever and ever again: $P \to \Box_F \Diamond_F P$.

To speak about the structure of the underlying space W—important examples are (subspaces of) the Euclidean spaces $\mathbb{R}^n$, general topological spaces, metric spaces, and measure spaces—as well as about the type of the intended functions f, one may require different non-temporal operators. So far, research has mainly been focused on topological spaces with continuous mappings. The corresponding logical constructors are those of modal logic **S4** which can also be regarded as the topological closure and interior operators—we denote them by $\mathbf{C}$ and $\mathbf{I}$, respectively. For example, a property similar to Poincaré's recurrence theorem corresponds in this language to the validity of the formula $\mathbf{I}P \to \mathbf{C}\Diamond_F \mathbf{I}P$ in spaces based on the unit disc with measure preserving continuous mappings.

The resulting combinations of temporal and topological logics are of clear 'two-dimensional character,' which makes it very difficult to analyse their computational properties (see, e.g., [7]). Perhaps this is the main reason why in the field of dynamic topological systems no (un)decidability or axiomatisability results for the full language containing both $\bigcirc$ and the infinitary $\Box_F$ have been obtained yet.

This paper provides answers to some of the open problems. First, we show that the dynamic topological logics introduced in [14] and interpreted in

- the Euclidean space $\mathbb{R}^n$ with homeomorphisms, for $n \geq 1$,
- arbitrary topological spaces with homeomorphisms,
- arbitrary Aleksandrov spaces with homeomorphisms, and
- the n-dimensional unit ball with measure preserving homeomorphisms, for $n \geq 2$,

are not recursively enumerable (and so are not recursively axiomatisable). Actually, we begin with a much stronger result. Instead of interpreting $\Box_F$ and $\Diamond_F$ along *infinite* orbits $\{w, f(w), f^2(w), \dots\}$, we only consider their arbitrarily long *finite* initial segments. The resulting 'finitary' logics are enough to specify safety conditions stating that some 'bad' region can never be reached. We prove that even these finitary versions are not recursively enumerable, and then reduce them to the corresponding infinitary logics thereby showing that the latter cannot be recursively enumerable either.

Having obtained these 'negative' results, we are facing the question whether there exist at all decidable (or at least axiomatisable) logics for reasoning about dynamical systems. One way of designing such logics is to restrict the language. For instance, if topological operators cannot be applied to formulas containing a temporal operator, then the resulting fragment is axiomatisable [14] (similar combinations of spatial and temporal logics can be found in [9]). Another way is to relax the constraints on the functions f, e.g., to interpret our full language in topological spaces with continuous mappings. We have analysed the computational properties of such logics in [8].

In this paper we take a different approach and consider a possibility of modelling dynamical systems by means of *metric* spaces with isometric functions. Instead of the topological closure and interior which are used to say that some point x is infinitely close to some set A, we introduce distance operators of the form $\exists^{\leq a}$ and $\forall^{\leq a}$ that are defined in terms of 'a point

x is in the a-neighbourhood of A,' where a is a positive rational number. For example, the formula $P \to \Box_F \neg \exists^{\leq a} Q$ states that, having started from a point in P, one can never reach the a-neighbourhood of some 'unsafe' area Q. Metric logics with such operators were first considered in [16]; a discussion of their use to model dynamical systems was started in [18].

This seemingly insignificant change results in a formalism with completely different computational properties. Namely, in the second half of the paper we prove that the resulting logic (interpreted over arbitrary metric spaces with isometric functions) turns out to be decidable (and, therefore, axiomatisable). However, we also show that it is of very high computational complexity, in fact, not decidable in time bounded by any elementary function.

Although numerous problems remain open, the obtained results and their proofs clearly indicate that the logics for dynamical systems are very sensitive to the available operators (say, topological vs. distance) as well as the constraints imposed on the spaces $\langle W, f\rangle$ (e.g., the proof of the undecidability result mentioned above does not go through for continuous functions [8], while the decidability proof works only for *arbitrary* metric spaces, but not for, say, compact ones).

The paper is structured as follows. Having introduced basic definitions in Section 2, we present in Section 3 some known results and prove auxiliary lemmata. In Section 4 we show that dynamic topological logics with finite iterations are not recursively enumerable by a reduction of the Post correspondence problem. Then, in Section 5, this result is transferred to logics with infinitely many iterations. In Section 6, we prove decidability of the dynamic metric logic and establish its computational complexity.

2 Definitions

Syntax. The language $\mathcal{DTL}$ of *dynamic topological logic* (or *dynamic topo-logic*, for short) [2, 14] is constructed in the standard way from a countably infinite set of 'propositional' variables using the Booleans $\neg$ and $\wedge$, the 'modal' operators $\mathbf{I}$ and $\mathbf{C}$ (for the topological interior and closure), and the temporal operators $\bigcirc$ (for 'next'), $\Box_F$ and $\Diamond_F$ (for 'always' and 'eventually'):

- if p is a variable, then p is an (atomic) $\mathcal{DTL}$-formula;
- if φ and ψ are $\mathcal{DTL}$-formulas, then so are $\neg\varphi$, $\varphi \wedge \psi$, $\mathbf{I}\varphi$, $\mathbf{C}\varphi$, $\Box_F\varphi$, $\Diamond_F\varphi$ and $\bigcirc\varphi$.

Other Boolean operators such as $\vee$ and $\to$ are defined as abbreviation in the usual way. We also write $\Box_F^+\varphi$ for $\varphi \wedge \Box_F\varphi$ and set dually $\Diamond_F^+\varphi = \varphi \vee \Diamond_F\varphi$, for every $\mathcal{DTL}$-formula φ.

Semantics. In this paper, by a *dynamic topological structure* (or DTS, for short) we understand a pair of the form $\mathfrak{F} = \langle \mathfrak{T}, f\rangle$, where $\mathfrak{T}$ is a topology on a space T (that is, $\emptyset, T \in \mathfrak{T}$ and $\mathfrak{T}$ is closed under arbitrary unions and finite intersections) and f is a homeomorphism[1] (i.e., a bijective continuous and open mapping) on $\mathfrak{T}$. A *dynamic topological model with N, $N \leq \omega$, iterations* (or DTM_N, for short) is a triple $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}, N\rangle$, where $\mathfrak{F}$ is a DTS and $\mathfrak{V}$, a *valuation*, associates with each variable p a subset $\mathfrak{V}(p)$ of T. The truth of a formula at $w \in T$ depends on how many iterations N of f we consider and, if N is finite, on the iteration step m where the formula is evaluated. For every DTM_N $\mathfrak{M}$ and every *finite* $m \leq N$ define

[1] In a more general setting, f can be a continuous mapping.

the *truth relation* $(\mathfrak{M}, w) \models_m \varphi$ ('in model $\mathfrak{M}$, formula φ is true in w after m iterations of f') as follows:

$$
\begin{array}{lll}
(\mathfrak{M}, w) \models_m p & \text{iff} & w \in \mathfrak{V}(p), \\
(\mathfrak{M}, w) \models_m \neg\varphi & \text{iff} & (\mathfrak{M}, w) \not\models_m \varphi, \\
(\mathfrak{M}, w) \models_m \varphi \wedge \psi & \text{iff} & (\mathfrak{M}, w) \models_m \varphi \text{ and } (\mathfrak{M}, w) \models_m \psi, \\
(\mathfrak{M}, w) \models_m \mathbf{I}\varphi & \text{iff} & w \in \mathbb{I}\{v \in T \mid (\mathfrak{M}, v) \models_m \varphi\}, \\
(\mathfrak{M}, w) \models_m \mathbf{C}\varphi & \text{iff} & w \in \mathbb{C}\{v \in T \mid (\mathfrak{M}, v) \models_m \varphi\}, \\
(\mathfrak{M}, w) \models_m \bigcirc\varphi & \text{iff} & (\mathfrak{M}, f(w)) \models_{m+1} \varphi \text{ and } m < N, \\
(\mathfrak{M}, w) \models_m \Box_F\varphi & \text{iff} & (\mathfrak{M}, f^n(w)) \models_{m+n} \varphi \text{ for every } n > 0 \text{ such that } m + n \leq N, \\
(\mathfrak{M}, w) \models_m \Diamond_F\varphi & \text{iff} & (\mathfrak{M}, f^n(w)) \models_{m+n} \varphi \text{ for some } n > 0 \text{ such that } m + n \leq N.
\end{array}
$$

Here $f^n(w) = \overbrace{f \dots f}^{n}(w)$, and $\mathbb{I}$ and $\mathbb{C}$ are the *interior* and *closure operators* determined by the topology $\mathfrak{T}$, that is, for all $A \subseteq T$,

$$
\mathbb{I}(A) = \bigcup\{U \in \mathfrak{T} \mid U \subseteq A\} \qquad \text{and} \qquad \mathbb{C}(A) = T - \mathbb{I}(T - A).
$$

Note that if a formula φ contains no temporal operators or if $N = \omega$, then the truth relation $(\mathfrak{M}, w) \models_m \varphi$ does not depend on the current iteration m. In these cases we will simply write $(\mathfrak{M}, w) \models \varphi$. If $(\mathfrak{M}, w) \models_0 \varphi$ for some $w \in T$, then we say that φ is *satisfied* in $\mathfrak{M}$. A $\mathcal{DTL}$-formula φ is said to be *finitely satisfiable in* a dynamic topological structure $\mathfrak{F}$ if φ is satisfied in some DTM_N based on $\mathfrak{F}$ and $N < \omega$; φ is called *ω-satisfiable in* $\mathfrak{F}$ if it is satisfied in a DTM_ω based on $\mathfrak{F}$.

Given a class $\mathcal{K}$ of dynamic topological structures, we define *dynamic topo-logic* $\mathsf{Log}_\omega \mathcal{K}$ and *finite iteration dynamic topo-logic* $\mathsf{Log}_{\mathsf{fin}} \mathcal{K}$ *of* $\mathcal{K}$ as follows:

- $\mathsf{Log}_\omega \mathcal{K}$ is the set of all $\mathcal{DTL}$-formulas φ such that $(\mathfrak{M}, w) \models_0 \varphi$ holds for every DTM_ω $\mathfrak{M}$ based on a DTS from $\mathcal{K}$ and every point w in $\mathfrak{M}$;
- $\mathsf{Log}_{\mathsf{fin}} \mathcal{K}$ is the set of all $\mathcal{DTL}$-formulas φ such that $(\mathfrak{M}, w) \models_0 \varphi$ holds for every DTM_N $\mathfrak{M}$, $N < \omega$, based on a DTS from $\mathcal{K}$ and every point w in $\mathfrak{M}$.

Let $\mathcal{DTL}_\bigcirc$ be the fragment of $\mathcal{DTL}$ which does not use $\Box_F$ and $\Diamond_F$. Then

- $\mathsf{Log}_\bigcirc \mathcal{K}$ is the fragment of $\mathsf{Log}_\omega \mathcal{K}$ in the language $\mathcal{DTL}_\bigcirc$.

Note that $\mathcal{DTL}_\bigcirc$ does not 'feel' the difference between models with infinite and (arbitrarily large) finite iterations. More precisely, a $\mathcal{DTL}_\bigcirc$-formula is satisfied in a DTM_ω iff it is satisfied in a DTM_N, where N is the maximal number of nestings of $\bigcirc$ in the formula or exceeds this number.

We will be dealing with the following classes of dynamic topological structures: the class of all DTSs, the class of DTSs that are based on the standard n-dimensional Euclidean space $\mathbb{R}^n$, for $n \geq 1$ (in particular, the real line $\mathbb{R}$), those based on *Aleksandrov spaces*, and the *n-dimensional unit balls*, for $n \geq 2$.

Aleksandrov spaces. We remind the reader that every quasi-order $\mathfrak{G} = \langle W, R\rangle$ (R is a reflexive and transitive relation on W) gives rise to the topological space $\mathfrak{T}_{\mathfrak{G}}$ over W consisting of all R-closed subsets of W. In other words, the interior operator $\mathbb{I}_{\mathfrak{G}}$ on $\mathfrak{T}_{\mathfrak{G}}$ can be defined by taking, for every $X \subseteq W$,

$$\mathbb{I}_{\mathfrak{G}} X = \{w \in X \mid \forall v \in W \ (wRv \to v \in X)\}.$$

Such spaces are known as *Aleksandrov spaces.* Alternatively they can be defined as topological spaces where arbitrary (not only finite) intersections of open sets are open; for details see [1, 3]. Clearly, for $\mathfrak{M} = \langle\langle \mathfrak{T}_{\mathfrak{G}}, f\rangle, \mathfrak{V}, N\rangle$, $N \leq \omega$ and finite $m \leq N$, we have

$$\begin{array}{lll} (\mathfrak{M}, w) \models_m \mathbf{I}\varphi & \text{iff} & (\mathfrak{M}, v) \models_m \varphi \text{ for every } v \in W \text{ with } wRv, \\ (\mathfrak{M}, w) \models_m \mathbf{C}\varphi & \text{iff} & \text{there is } v \in W \text{ such that } wRv \text{ and } (\mathfrak{M}, v) \models_m \varphi. \end{array}$$

It should be also clear that a function $f \colon W \to W$ is a *continuous mapping* on $\mathfrak{T}_{\mathfrak{G}}$ iff for all $w, v \in W$,

$$wRv \qquad \text{implies} \qquad f(w)Rf(v).$$

A bijection f is a *homeomorphism* on $\mathfrak{T}_{\mathfrak{G}}$ iff both the above implication and its converse hold.

Unit balls. For $n \geq 1$, let $B_r^n(c)$ be the *closed n-ball of radius* $r > 0$ with centre $c \in \mathbb{R}^n$, that is

$$B_r^n(c) = \{x \in \mathbb{R}^n \mid \|x - c\| \leq r\},$$

where $\|x - c\|$ is the distance between points x and c in $\mathbb{R}^n$. Similarly, the *open n-ball of radius* $r > 0$ with centre $c \in \mathbb{R}^n$ is the set

$$O_r^n(c) = \{x \in \mathbb{R}^n \mid \|x - c\| < r\}.$$

Often, we will not specify the centre of closed n-balls and write, slightly abusing notation, B^n to denote a *closed unit n-ball* (i.e., a closed n-ball of radius 1).

Let μ be the Lebesgue measure on B^n. A function $f : B^n \to B^n$ is said to be *measure preserving* if, for every measurable set $S \subseteq B^n$, we have $\mu(f^{-1}(S)) = \mu(S)$.

By a *unit n-ball DTS* $\mathfrak{B}^n$ we will mean the pair $\langle B^n, f\rangle$, where B^n is the closed unit n-ball and f a measure preserving homeomorphism on B^n. The *logic of closed unit n-balls*,

$$\mathsf{Log}_\omega \{\mathfrak{B}^n \mid \mathfrak{B}^n \text{ a unit } n\text{-ball DTS}\},$$

for $n \geq 1$, is of particular interest to dynamical systems because it is expressive enough to formalise a version of the Poincaré recurrence theorem (see, for example, [21]), which states that for a measure preserving function on a unit ball, almost all orbits starting from a set of positive measure will come back infinitely many times. It was observed in [14] that the formula $\mathbf{I}p \to \mathbf{C}\Diamond_F \mathbf{I}p$ of this logic expresses the phenomenon of recurrence.

3 Some known results

The aim of this section is to prove the following theorem which will be required in Section 4.

Theorem 1. *The dynamic topo-logics listed below coincide, have the finite model property, are finitely axiomatisable, and so decidable:*

1. $\mathsf{Log}_{\bigcirc}\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS*$\}$,

2. $\mathsf{Log}_{\bigcirc}\{\langle\mathbb{R}^n, f\rangle \mid \langle\mathbb{R}^n, f\rangle$ *a DTS*$\}$, *for* $n \geq 1$,

3. $\mathsf{Log}_{\bigcirc}\{\langle\mathbb{R}^n, (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n + 1)\rangle\}$, *for* $n \geq 1$,

4. $\mathsf{Log}_{\bigcirc}\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS*, $\mathfrak{T}$ *an Aleksandrov space*$\}$,

5. $\mathsf{Log}_{\bigcirc}\{\mathfrak{B}^n \mid \mathfrak{B}^n$ *a unit* n*-ball DTS*$\}$, *for* $n \geq 2$.

Remark 2. Note that the logic $\mathsf{Log}_{\bigcirc}\{\mathfrak{B}^1 \mid \mathfrak{B}^1$ a unit 1-ball DTS$\}$ *properly contains* all the logics mentioned in the formulation of Theorem 1. To show this we first observe that there exist exactly two measure preserving homeomorphisms on the unit interval, namely, $x \mapsto x$ and $x \mapsto (1-x)$. Indeed, as any measure preserving homeomorphism f on $[0,1]$ must be monotone, either $f(0) = 0, f(1) = 1$ or $f(0) = 1, f(1) = 0$. We only consider the former case (because the latter one is analogous). Take an arbitrary interval $[0,t]$, for $t \in [0,1]$. Since f^{-1} is continuous and f is monotone, $f^{-1}([0,t])$ is an interval $[0,s]$, for some $s \in [0,1]$. As f is measure preserving, we have $s = t$. Thus, $f^{-1}(t) = t$, and so $f(t) = t$ for every $t \in [0,1]$. It follows that $\bigcirc\bigcirc p \leftrightarrow p$ is valid in any unit 1-ball DTS. On the other hand, this formula does not belong to any of the logics from the theorem.

The results of Theorem 1 were explicitly proved in or easily follow from [2, 15, 14]. That the logics listed under items 1, 2, 4, and $\mathsf{Log}_{\bigcirc}\{\langle\mathbb{R}, x \mapsto x+1\rangle\}$ coincide is proved in [14]. Although the logic of unit balls was also introduced in [14], it was not shown there that it coincides with the other four. The proof of the missing equalities can be obtained by a rather straightforward adaptation of proofs given in [14].

We require the following two lemmata.

Lemma 3. *Let* $n \geq 2$. *For every finite* $N > 0$, *there exist a measure preserving homeomorphism* g *on* B^n, *an open ball* $E \subset B^n$ *and a point* $e \in E$ *such that*

(1) E, $g(E)$, $g^2(E)$, $\ldots$, $g^N(E)$ *are pairwise disjoint open balls,*

(2) $g^j(e) = g^N(e)$ *iff* $j = N$, *for every* $j < \omega$.

Proof. Suppose first that $n = 2$ and consider the unit disc DTS $\mathfrak{B}^2 = \langle B^2, g\rangle$, where g is the clockwise *rotation* of B^2 by some *rational* angle α, $0 < \alpha < \frac{2\pi}{N+1}$ (see Fig. 1). Obviously, g is a measure preserving homeomorphism.

Let E be a smaller open disc contained in the sector $[-\alpha/2, \alpha/2]$ of B^2 and e the centre of E. Clearly, $E, g(E), g^2(E), \ldots, g^N(E)$ are pairwise disjoint sets. As α is rational, we have $g^j(e) = g^N(e)$ iff $j = N$.

For $n > 2$ the construction is similar: we rotate the n-ball around a fixed axis. ❑

Lemma 4. *Let* $x \in \mathbb{R}^n$ *and let* φ *be a* $\mathcal{DTL}$*-formula without occurrences of temporal operators. Suppose also that* $\mathfrak{V}$ *and* $\mathfrak{V}'$ *are valuations in* $\mathbb{R}^n$ *for which there exists some* $\varepsilon > 0$ *such that*

$$\mathfrak{V}(p) \cap O^n_\varepsilon(x) \quad = \quad \mathfrak{V}'(p) \cap O^n_\varepsilon(x),$$

for every variable p. *Then, for every* $N \leq \omega$ *and every DTS* $\mathfrak{F} = \langle\mathbb{R}^n, f\rangle$,

$$(\langle\mathfrak{F}, \mathfrak{V}, N\rangle, x) \models \varphi \qquad \textit{iff} \qquad (\langle\mathfrak{F}, \mathfrak{V}', N\rangle, x) \models \varphi.$$

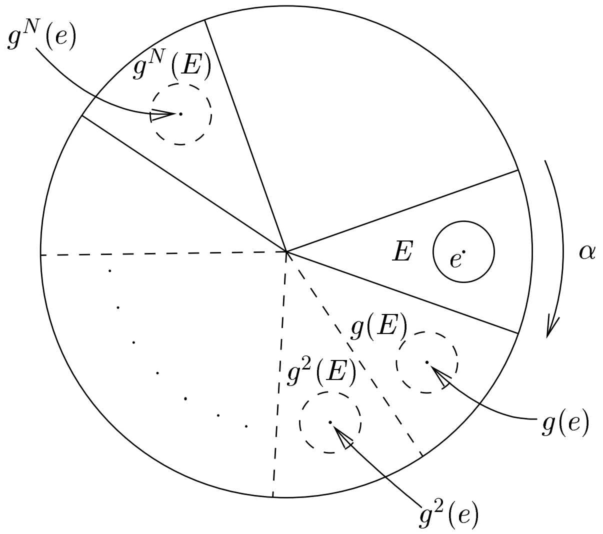

Figure 1: Rotation of the unit ball.

Proof. The proof is easy and left to the reader. ❑

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let $L_n = \mathsf{Log}_{\bigcirc}\{\langle\mathbb{R}^n, (x_1,\dots,x_{n-1},x_n) \mapsto (x_1,\dots,x_{n-1},x_n+1)\rangle\}$, for $n \geq 1$. First we show that $L_1 = L_n$. As $L_1 = \mathsf{Log}_{\bigcirc}\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle \text{ a DTS}\}$ (see [14]), we have $L_1 \subseteq L_n$. To show the converse inclusion, suppose that a formula φ is satisfied in a DTM$_\omega$ $\langle\langle\mathbb{R}, x \mapsto x+1\rangle, \mathfrak{V}, \omega\rangle$. Define a valuation $\mathfrak{V}'$ on $\mathbb{R}^n$ by taking

$$\mathfrak{V}'(p) \quad = \quad \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n \in \mathfrak{V}(p)\}.$$

It is readily seen that φ is satisfied in $\langle\langle\mathbb{R}^n, (x_1,\dots,x_n) \mapsto (x_1,\dots,x_{n-1},x_n+1)\rangle, \mathfrak{V}', \omega\rangle$.

We can now show that, for $n \geq 2$,

$$\mathsf{Log}_{\bigcirc}\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle \text{ a DTS}\} \quad = \quad \mathsf{Log}_{\bigcirc}\{\mathfrak{B}^n \mid \mathfrak{B}^n \text{ a unit } n\text{-ball DTS}\}.$$

The inclusion $\subseteq$ is clear. To prove the converse, suppose that φ is satisfiable in a DTS $\langle\mathfrak{T}, f\rangle$. As $\mathsf{Log}_{\bigcirc}\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle \text{ a DTS}\} = L_1 = L_n$, there exists a valuation $\mathfrak{V}$ such that

$$(\langle\langle\mathbb{R}^n, (x_1,\dots,x_n) \mapsto (x_1,\dots,x_{n-1},x_n+1\rangle, \mathfrak{V}, \omega\rangle, (0,\dots,0)) \models \varphi.$$

Denote by N the maximal number of nestings of the operator $\bigcirc$ in φ. Let $E \subset B^n$ and a function g be as specified by Lemma 3. Let h_0 be some homeomorphism from the 'stripe' $\mathbb{R}^{n-1} \times \left(-\frac{1}{2}, +\frac{1}{2}\right)$ onto E. Note that, for every $i < \omega$, the function h_i defined by taking

$$h_i(x) \quad = \quad h_0 \circ \{(x_0,\dots,x_n) \mapsto (x_0,\dots,x_{n-1},x_n - i)\}$$

is a homeomorphism from $\mathbb{R}^{n-1} \times \left(i-\frac{1}{2}, i+\frac{1}{2}\right)$ onto E. Then $g^i \circ h_i$ is a homeomorphism from $H_i = \mathbb{R}^{n-1} \times \left(i-\frac{1}{2}, i+\frac{1}{2}\right)$ onto $g^i(E)$. Define a valuation $\mathfrak{V}'$ in B^n by taking

$$\mathfrak{V}'(p) \quad = \quad \bigcup_{i=0}^{N} g^i(h_i(H_i \cap \mathfrak{V}(p)))$$

for every variable p. Using Lemma 4, it can be readily checked by induction on the construction of φ that $(\mathfrak{M}', h_0(0,\dots,0)) \models \varphi$, where $\mathfrak{M}' = \langle \langle B^n, g \rangle, \mathfrak{V}', \omega \rangle$. ❑

We shall see now that the computational behaviour of dynamic topo-logics becomes completely different if we allow the use of the infinitary operators $\Box_F$ and $\Diamond_F$.

4 Topo-logics with finitely many iterations

Theorem 5. *The topo-logics listed below coincide and are not recursively enumerable:*

1. $\mathsf{Log}_{\mathsf{fin}}\, \{\langle \mathfrak{T}, f\rangle \mid \langle \mathfrak{T}, f\rangle$ *a DTS*$\}$,
2. $\mathsf{Log}_{\mathsf{fin}}\, \{\langle \mathbb{R}^n, f\rangle \mid \langle \mathbb{R}^n, f\rangle$ *a DTS*$\}$, *for* $n \geq 1$,
3. $\mathsf{Log}_{\mathsf{fin}}\, \{\langle \mathfrak{T}, f\rangle \mid \langle \mathfrak{T}, f\rangle$ *a DTS*, $\mathfrak{T}$ *an Aleksandrov space*$\}$,
4. $\mathsf{Log}_{\mathsf{fin}}\, \{\mathfrak{B}^n \mid \mathfrak{B}^n$ *a unit ball DTS*$\}$, *for* $n \geq 2$.

Proof. That the four logics above coincide follows immediately from Theorem 1. It suffices to observe that, for every $N < \omega$ and every $\mathcal{DTL}$-formula φ, there exists a $\mathcal{DTL}_{\bigcirc}$-formula ψ such that, for every DTM_N $\mathfrak{M}$ and every point w in it, we have $(\mathfrak{M}, w) \models \varphi$ iff $(\mathfrak{M}, w) \models \psi$.

The fact that the logics in question are not recursively enumerable can be proved by reduction of *Post's correspondence problem* or PCP, for short [19]. The idea of the proof is taken from [7]. Let A be a finite alphabet and P a finite set of pairs $\langle \mathbf{v}_1, \mathbf{u}_1\rangle, \dots, \langle \mathbf{v}_k, \mathbf{u}_k\rangle$ of nonempty finite words

$$\mathbf{v}_i = \left\langle b^i_1, \dots, b^i_{l_i}\right\rangle \qquad \text{and} \qquad \mathbf{u}_i = \left\langle c^i_1, \dots, c^i_{r_i}\right\rangle \qquad (i = 1, \dots, k)$$

over A. For a sequence of indices $i_1, \dots, i_N$ ranging over $1, \dots, k$, let

$$m_j = l_{i_1} + \cdots + l_{i_j} \qquad \text{and} \qquad n_j = r_{i_1} + \cdots + r_{i_j},$$

for $1 \leq j \leq N$. The following problem is known to be undecidable (for a proof see, e.g., [10]): *given a set P of pairs of words as above (a PCP instance), decide whether there exist an $N \geq 1$ and a sequence $i_1, \dots, i_N$ of indices such that*

$$\mathbf{v}_{i_1} * \cdots * \mathbf{v}_{i_N} = \mathbf{u}_{i_1} * \cdots * \mathbf{u}_{i_N}, \tag{1}$$

where $*$ is the concatenation operation. If the condition above holds for P then we say that it *has a solution.* Later, we will use a consequence of this result, namely, that the set of PCP instances without solutions *is not recursively enumerable.*

The reduction formula $\varphi_{A,P}$ is constructed using the propositional variables s, *left* and *right*, left_a and right_a, for every $a \in A$, as well as pair_i, for every pair $\langle \mathbf{v}_i, \mathbf{u}_i\rangle$ in P, $1 \leq i \leq k$.

The variable s is used to introduce a new operator $\mathbf{S}$ (which can be interpreted as an almost 'strict diamond' in Kripke quasi-ordered frames; its precise semantical meaning will be given later on). Namely, for every $\mathcal{DTL}$-formula ψ, we put

$$\mathbf{S}\psi = \big(s \to \mathbf{C}(\neg s \wedge \mathbf{C}\psi)\big) \wedge \big(\neg s \to \mathbf{C}(s \wedge \mathbf{C}\psi)\big).$$

Denote by $\mathbf{S}^m$ a string of m operators $\mathbf{S}$ (so that $\mathbf{S}^0\psi = \psi$). Now $\varphi_{A,P}$ is defined as the conjunction

$$\varphi_{A,P} = \varphi_{eq} \wedge \varphi_{pair} \wedge \varphi_{stripe} \wedge \varphi_{left} \wedge \varphi_{right},$$

where

$$\varphi_{eq} = \Diamond_F\big(\neg\bigcirc\top \wedge \bigwedge_{a\in A} \mathbf{I}(\mathit{left}_a \leftrightarrow \mathit{right}_a)\big),$$

$$\varphi_{pair} = \Box_F^+\Big(\bigcirc\top \to \big(\bigvee_{1\le i\le k} \mathit{pair}_i \wedge \bigwedge_{1\le i<j\le k} \neg(\mathit{pair}_i \wedge \mathit{pair}_j)\big)\Big),$$

$$\varphi_{stripe} = \Box_F^+\big(\bigcirc\top \to \mathbf{I}(s \leftrightarrow \bigcirc s)\big),$$

φ_{left} is the conjunction of (2)–(8), for $1 \le i \le k$,

$$\bigwedge_{\substack{a\neq b\\ a,b\in A}} \Box_F^+\mathbf{I}\neg\big(\mathit{left}_a \wedge \mathit{left}_b\big) \ \wedge\ \Box_F^+\mathbf{I}\big(\mathit{left} \leftrightarrow \bigvee_{a\in A} \mathit{left}_a\big), \tag{2}$$

$$\bigwedge_{a\in A} \Box_F^+\big(\bigcirc\top \to \mathbf{I}(\mathit{left}_a \to \bigcirc \mathit{left}_a)\big), \tag{3}$$

$$\mathbf{I}\neg\mathit{left} \ \wedge\ \Box_F^+\big(\bigcirc\top \to \mathbf{I}(\neg\mathit{left} \to \neg\mathbf{S}\mathit{left})\big), \tag{4}$$

$$\Box_F^+\big(\mathit{pair}_i \to \mathbf{I}(\neg\mathit{left} \to \bigcirc\neg\mathbf{S}^{l_i}\mathit{left})\big), \tag{5}$$

$$\Box_F^+\big(\mathit{pair}_i \to \bigwedge_{j<l_i} \bigcirc\mathbf{I}(\mathbf{S}^j\mathit{left} \wedge \neg\mathbf{S}^{j+1}\mathit{left} \to \mathit{left}_{b^i_{l_i-j}})\big), \tag{6}$$

$$\mathit{pair}_i \to \bigcirc\lambda_i^{\mathit{left}}, \tag{7}$$

$$\Box_F^+\big(\mathit{pair}_i \to \mathbf{I}(\mathit{left} \wedge \neg\mathbf{S}\mathit{left} \to \mathbf{S}\bigcirc\lambda_i^{\mathit{left}})\big), \tag{8}$$

where

$$\lambda_i^{\mathit{left}} = \mathit{left}_{b^i_1} \wedge \mathbf{S}\big(\mathit{left}_{b^i_2} \wedge \mathbf{S}(\mathit{left}_{b^i_3} \wedge \cdots \wedge \mathbf{S}\mathit{left}_{b^i_{l_i}})\dots\big)$$

(remember that l_i is the length of the word $\mathbf{v}_i = \langle b^i_1, \dots, b^i_{l_i}\rangle$), and the conjunct φ_{right}—the 'right counterpart' of φ_{left}—is defined by replacing in φ_{left} all occurrences of *left* with *right*, left_a with right_a, l_i with r_i, etc.

Lemma 6. *If P has a solution, then $\varphi_{A,P}$ is satisfiable in a DTM$_M$, $M < \omega$, based on some DTS $\langle\mathfrak{T}, f\rangle$, where $\mathfrak{T}$ is an Aleksandrov space.*

Proof. Suppose that P has a solution, that is, there is an N and a sequence of indices $i_1, \dots, i_N$ such that

$$\mathbf{v}_{i_1} * \cdots * \mathbf{v}_{i_N} = \langle a_0, \dots, a_{m_N-1}\rangle = \mathbf{u}_{i_1} * \cdots * \mathbf{u}_{i_N}. \tag{9}$$

Define a quasi-order $\mathfrak{G} = \langle W, R\rangle$ by taking $W = \{0, \dots, 2m_N\} \times \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, and $(x,y)R(x',y')$ iff $x \le x'$ and $y = y'$. Define $f : W \to W$ by taking $f(x,y) = (x, y+1)$. Clearly, f is a homeomorphism on $\mathfrak{T}_{\mathfrak{G}}$. Finally, define $\mathfrak{V}$ by taking

$$\begin{array}{rcll}
\mathfrak{V}(s) &=& \{(2k,j) \mid 0 \le 2k \le 2m_N,\ j \le N\}, & \\
\mathfrak{V}(\mathit{pair}_i) &=& \{(0,j-1) \mid i = i_j, \text{ for some } j \le N\}, & \text{for } 1 \le i \le k,\\
\mathfrak{V}(\mathit{left}_a) &=& \{(2k,j) \mid 0 < j \le N,\ k < m_j,\ a_k = a\}, & \text{for } a \in A,\\
\mathfrak{V}(\mathit{right}_a) &=& \{(2k,j) \mid 0 < j \le N,\ k < n_j,\ a_k = a\}, & \text{for } a \in A,\\
\mathfrak{V}(\mathit{left}) &=& \bigcup_{a\in A} \mathfrak{V}(\mathit{left}_a), & \\
\mathfrak{V}(\mathit{right}) &=& \bigcup_{a\in A} \mathfrak{V}(\mathit{right}_a). &
\end{array}$$

Let $\mathfrak{M} = \langle\mathfrak{T}_{\mathfrak{G}}, \mathfrak{V}, N\rangle$. It is an easy exercise to show that $(\mathfrak{M}, (0,0)) \models_0 \varphi_{A,P}$. ❑

Lemma 7. *If $\varphi_{A,P}$ is satisfiable in a DTM_N based on an Aleksandrov space and $N < \omega$, then P has a solution.*

Proof. Suppose that $(\mathfrak{M}, w_1^0) \models_0 \varphi_{A,P}$ for some DTM_N $\mathfrak{M} = \langle \langle \mathfrak{T}_{\mathfrak{G}}, f \rangle, \mathfrak{V}, N \rangle$, where $N < \omega$, $\mathfrak{G} = \langle W, R \rangle$ is a quasi-order, f a homeomorphism on $\mathfrak{T}_{\mathfrak{G}}$ and $w_1^0 \in W$. For $j \leq N$, let

$$W_j = \{w \in W \mid f^j(w_1^0) R w\}.$$

Note that, as f is a homeomorphism, we have $W_j = f^j(W_0)$ for every $j \leq N$.

As $(\mathfrak{M}, w_1^0) \models \Diamond_F(\neg \bigcirc \top \wedge \bigwedge_{a \in A} \mathbf{I}(\mathit{left}_a \leftrightarrow \mathit{right}_a))$, we have

$$(\mathfrak{M}, f^N(w_1^0)) \models \bigwedge_{a \in A} \mathbf{I}(\mathit{left}_a \leftrightarrow \mathit{right}_a). \tag{10}$$

Since $(\mathfrak{M}, w_1^0) \models_0 \varphi_{stripe}$, for each $w \in W_0$ and each $j \leq N$, we have $(\mathfrak{M}, w) \models s$ iff $(\mathfrak{M}, f^j(w)) \models s$ (in other words, $W_j \cap \mathfrak{V}(s) = f^j(W_0 \cap \mathfrak{V}(s))$ for every $j \leq N$).

Denote by S_j, $j \leq N$, the transitive binary relation on W_j defined by taking $w S_j v$ iff there is $u \in W_j$ such that $wRuRv$ and $(\mathfrak{M}, w) \models s$ iff $(\mathfrak{M}, u) \not\models s$. Then we clearly have that, for every $m, j \leq N$ and every $w \in W_j$,

$$(\mathfrak{M}, w) \models_m \mathbf{S}\psi \quad \text{iff} \quad \text{there is } v \in W_j \text{ such that } w S_j v \text{ and } (\mathfrak{M}, v) \models_m \psi.$$

Note that, since f is a homeomorphism and in view of $(\mathfrak{M}, w_1^0) \models_0 \varphi_{stripe}$, for all $w, v \in W_0$ and $i \leq N$, we have $w S_0 v$ iff $f^i(w) S_i f^i(v)$.

Let $i_1, \ldots, i_N$ be the sequence of indices such that $(\mathfrak{M}, f^{j-1}(w_1^0)) \models \mathit{pair}_{i_j}$, for all j, $1 \leq j \leq N$ (φ_{pair} ensures that there is a unique sequence of this sort). We claim that (1) holds for this sequence.

For every j with $1 \leq j \leq N$, let

$$W_j^L = \{w \in W_j \mid (\mathfrak{M}, w) \models \mathit{left}\}.$$

Call a sequence $\langle w_1, \ldots, w_l \rangle$ of (not necessarily distinct) points from W_j^L an *S_j-path in W_j^L of length l* if $w_1 S_j w_2 S_j \ldots S_j w_l$, and set

$$\mathit{leftword}_j(w_1, \ldots, w_l) = \langle a_1, \ldots, a_l \rangle,$$

where the a_i are the (uniquely determined by (2)) symbols from A such that $(\mathfrak{M}, w_i) \models \mathit{left}_{a_i}$.

We show now that there is a sequence $\pi_1, \ldots, \pi_N$ such that, for every j with $1 \leq j \leq N$, the following hold:

(a) $\pi_j = \langle w_1^j, \ldots, w_{m_j}^j \rangle$ is an S_j-path in W_j^L of length m_j, and there is no S_j-path in W_j^L of length $> m_j$;

(b) $f(w_1^0) = w_1^1$ and if $j > 1$ then $w_m^j = f(w_m^{j-1})$, for all m, $1 \leq m \leq m_{j-1}$;

(c) $\mathit{leftword}_j(w_1^j, \ldots, w_{m_j}^j) = \mathbf{v}_{i_1} * \ldots * \mathbf{v}_{i_j}$;

(d) for every S_j-path $\langle w_1, \ldots, w_{m_j} \rangle$ in W_j^L, we have $\mathit{leftword}_j(w_1, \ldots, w_{m_j}) = \mathbf{v}_{i_1} * \ldots * \mathbf{v}_{i_j}$.

Indeed, by $(\mathfrak{M}, w_1^0) \models pair_{i_1}$, (7), (4) and (5), there exists an S_1-path π_1 in W_1^L such that (a)–(c) hold. Condition (d) follows from (6).

Now assume inductively that conditions (a)–(d) hold for some $j-1$ with $1 \leq j-1 < N$. Let $\pi_{j-1} = \langle w_1^{j-1}, \ldots, w_{m_{j-1}}^{j-1} \rangle$ be an S_{j-1}-path in W_{j-1}^L for which (a)–(d) hold. By (3), the sequence $\langle f(w_1^{j-1}), \ldots, f(w_{m_{j-1}}^{j-1}) \rangle$ is an S_j-path in W_j^L. Since $(\mathfrak{M}, w_{m_{j-1}}^{j-1}) \models \mathit{left} \wedge \neg \mathbf{S}\mathit{left}$ and $(\mathfrak{M}, w_1^{j-1}) \models pair_{i_j}$, (8) means that there exists a sequence $w_{m_{j-1}+1}^j, \ldots, w_{m_{j-1}+l_{i_j}}^j$ of points in W_j^L such that

$$\pi_j = \langle f(w_1^{j-1}), \ldots, f(w_{m_{j-1}}^{j-1}), w_{m_{j-1}+1}^j, \ldots, w_{m_{j-1}+l_{i_j}}^j \rangle$$

is an S_j-path in W_j^L of length $m_j = m_{j-1} + l_{i_j}$ with $\mathit{leftword}_j(w_{n_{j-1}+1}^j, \ldots, w_{m_{j-1}+l_{i_j}}^j) = \mathbf{v}_{i_j}$. By (5) and the induction hypothesis, there is no S_j-path in W_j^L of length $> m_j$. Thus, (a) and (b) hold for π_j, (c) follows from (3), and (d) from (6) and the induction hypothesis.

Now define sets W_j^R in the same way as W_j^L, but with *left* replaced by *right*, introduce the notion of an S_j-path in W_j^R, and, for every sequence $w_1, \ldots, w_l$ of points from W_j^R, set

$$\mathit{rightword}_j(w_1, \ldots, w_l) = \langle a_1, \ldots, a_l \rangle,$$

where the a_i are the uniquely determined (by the 'right analogue' of (2)) elements of A such that $(\mathfrak{M}, w_i) \models \mathit{right}_{a_i}$. In precisely the same way as above we show now that there is a sequence $\pi_1', \ldots, \pi_N'$ such that, for every j with $1 \leq j \leq N$,

(a′) $\pi_j' = \langle w_1^j, \ldots, w_{n_j}^j \rangle$ is an S_j-path in W_j^R of length n_j, and there is no S_j-path in W_j^R of length $> n_j$;

(b′) $f(w_1^0) = w_1^1$ and if $j > 1$ then $w_n^j = f(w_n^{j-1})$, for all n with $1 \leq n \leq n_{j-1}$;

(c′) $\mathit{rightword}_j(w_1^j, \ldots, w_{n_j}^j) = \mathbf{u}_{i_1} * \ldots * \mathbf{u}_{i_j}$;

(d′) for every S_j-path $\langle w_1, \ldots, w_{n_j} \rangle$ in W_j^R, we have $\mathit{rightword}_j(w_1, \ldots, w_{n_j}) = \mathbf{u}_{i_1} * \ldots * \mathbf{u}_{i_j}$.

Now it is easy to see that (10) means that

$$\mathbf{v}_{i_1} * \ldots * \mathbf{v}_{i_N} = \mathit{leftword}_N(w_1^N, \ldots, w_{m_N}^N) = \mathit{rightword}_N(w_1^N, \ldots, w_{n_N}^N) = \mathbf{u}_{i_1} * \ldots * \mathbf{u}_{i_N},$$

as required. ❏

Thus, we have proved that $\varphi_{A,P}$ is finitely satisfiable iff P has a solution. It follows that the logic $\mathsf{Log}_{\mathsf{fin}}\,\{\langle \mathfrak{T}, f \rangle \mid \langle \mathfrak{T}, f \rangle$ a DTS$\}$ in not recursively enumerable, for otherwise we could enumerate all instances of PCP without solutions. ❏

It is worth noting that in fact we have proved a slightly stronger statement. Say that a $\mathcal{DTL}$-formula ψ is in *temporal normal form* if

$$\psi = \psi_1 \ \wedge \ \Box_F^+ \psi_2 \ \wedge \ \Diamond_F(\neg \bigcirc \top \wedge \psi_3),$$

where ψ_3 does not contain occurrences of temporal operators and both ψ_1 and ψ_2 are $\mathcal{DTL}_{\bigcirc}$-formulas without nested $\bigcirc$. The formula $\varphi_{A,P}$ defined above is clearly equivalent to a formula in temporal normal form. So, we obtain:

Corollary 8. *The set of formula of the form* $\neg\psi \in \mathsf{Log}_{\mathsf{fin}}\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS*$\}$, *where* ψ *is in temporal normal form, is not recursively enumerable.*

We will use this corollary in order to extend the non-axiomatisability result of this section to dynamic topological logics with infinitely many iterations.

5 Dynamic topo-logics with infinitely many iterations

Theorem 9. *No logic from the list below is recursively enumerable:*

1. $\mathsf{Log}_\omega\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS*$\}$,
2. $\mathsf{Log}_\omega\{\langle\mathbb{R}^n, f\rangle \mid \langle\mathbb{R}^n, f\rangle$ *a DTS*$\}$, *for* $n \geq 1$,
3. $\mathsf{Log}_\omega\{\langle\mathfrak{T}, f\rangle \mid \langle\mathfrak{T}, f\rangle$ *a DTS,* $\mathfrak{T}$ *an Aleksandrov space*$\}$,
4. $\mathsf{Log}_\omega\{\mathfrak{B}^n \mid \mathfrak{B}^n$ *a unit ball DTS*$\}$, *for* $n \geq 2$.

Remark 10. Before proceeding to the proof, we note that all logics mentioned in the formulation of this theorem are different. As was shown in [20], the formula $\mathbf{I}\Diamond_F(p \wedge \mathbf{CI}\neg p)$ is not satisfiable in any DTM$_\omega$ based on $\langle\mathbb{R}^n, f\rangle$, while it is clearly satisfiable. According to [14], the formula $\mathbf{I}p \to \mathbf{C}\Diamond_F\mathbf{I}p$ is valid in all unit balls, but refuted in some DTM$_\omega$s based on Aleksandrov spaces and $\langle\mathbb{R}^n, (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n + 1)\rangle$. Finally, the formula $\Diamond_F\mathbf{C}p \leftrightarrow \mathbf{C}\Diamond_F p$ is valid in DTM$_\omega$ based on Aleksandrov spaces, but refuted in both $\langle\mathbb{R}^n, f\rangle$ and all unit balls.

Proof of Theorem 9. Let $\psi = \psi_1 \wedge \Box_F^+\psi_2 \wedge \Diamond_F(\neg\bigcirc\top \wedge \psi_3)$ be a $\mathcal{DTL}$-formula in temporal normal form. Take a fresh variable r and set

$$\psi^r = \psi_1 \ \wedge\ \Box_F(r \to \Box_F\neg r) \ \wedge\ \Box_F^+(\Diamond_F r \to \psi_2) \ \wedge\ \Box_F^+(r \to \psi_2[\bigcirc/\bot]) \ \wedge\ \Diamond_F(r \wedge \psi_3),$$

where $\psi_2[\bigcirc/\bot]$ is the result of replacing every subformula $\bigcirc\chi$ of ψ_2 with $\bot$.

Lemma 11. *Let* ψ *be a* $\mathcal{DTL}$*-formula in temporal normal form. If* ψ^r *is satisfiable in a DTM*$_\omega$ $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{V}, \omega\rangle$, *then there exists some* $N < \omega$ *such that* ψ *is satisfiable in the DTM*$_N$ $\mathfrak{M}_N = \langle\mathfrak{F}, \mathfrak{V}, N\rangle$.

Proof. Let $(\mathfrak{M}, w) \models_0 \psi^r$ for some point w in $\mathfrak{M}$. Let N be such that $(\mathfrak{M}, w) \models_N r$ (such an N always exists, since $(\mathfrak{M}, w) \models_0 \Diamond_F r$, and is uniquely defined because $(\mathfrak{M}, w) \models_m r$ implies $(\mathfrak{M}, w) \not\models_n r$ for all $n > m$). Now it can be easily seen that $(\mathfrak{M}_N, w) \models \psi$. ❑

Lemma 12. *Let* ψ *be a* $\mathcal{DTL}$*-formula in temporal normal form. If* ψ *is satisfiable in a DTM*$_N$, $N < \omega$, *based on an Aleksandrov space (equivalently, some DTM*$_N$, $N < \omega$, *based on* $\langle\mathbb{R}^n, (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n + 1)\rangle$*), then*

(i) ψ^r *is satisfiable in a DTM*$_\omega$ *based on a DTS* $\langle\mathfrak{T}, f\rangle$, *where* $\mathfrak{T}$ *is an Aleksandrov space;*

(ii) ψ^r *is satisfiable in a DTM*$_\omega$;

(iii) ψ^r *is satisfiable in a DTM*$_\omega$ *based on* $\langle\mathbb{R}^n, f\rangle$, $n > 0$, *where* f *is a homeomorphism;*

(iv) ψ^r *is satisfiable in a DTM*$_\omega$ *based on* $\mathfrak{B}^n$, $n \geq 2$.

Proof. (i) Assume that ψ is satisfiable in some DTM_N, $N < \omega$, based on an Aleksandrov space $\mathfrak{T}_\mathfrak{G}$, that is, $(\langle\langle\mathfrak{T}_\mathfrak{G}, f\rangle, \mathfrak{V}, N\rangle, x) \models \psi$, where f is a homeomorphism on $\mathfrak{T}_\mathfrak{G}$, $\mathfrak{V}$ is a valuation, and x is a point of $\mathfrak{T}_\mathfrak{G}$. Consider the Cartesian product $\mathfrak{T}_* = \mathfrak{T}_\mathfrak{G} \times \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, with the product topology (the topology on $\mathbb{Z}$ is assumed to be discrete). Clearly, $\mathfrak{T}_*$ is an Aleksandrov space. Then the function $f_* : (x, n) \mapsto (f(x), n+1)$ is a homeomorphism on $\mathfrak{T}_*$. Let $\mathfrak{V}_*(p) = \bigcup_{i\in\mathbb{N}}(\mathfrak{V}(p) \times \{i\})$. It should be clear that $(\langle\langle\mathfrak{T}_*, f_*\rangle, \mathfrak{V}_*, N\rangle, (x, 0)) \models \psi$. Note that for every point y of $\mathfrak{T}_\mathfrak{G} \times \mathbb{Z}$ and all $i, j < \omega$, if $f_*^i(y) = f_*^j(y)$ then $i = j$. We define a new valuation $\mathfrak{V}'_*$ which coincides with $\mathfrak{V}_*$ except for only now we set $\mathfrak{V}'_*(r) = \{f_*^N((x, 0))\}$. (Note that r does not occur in ψ.) Then $(\langle\langle\mathfrak{T}_*, f_*\rangle, \mathfrak{V}'_*, N\rangle, (x, 0)) \models \psi^r$.

(ii) follows from (i).

(iii) Suppose that ψ is true at $(0, \dots, 0)$ in a DTM_N, $N < \omega$, based on

$$\langle\mathbb{R}^n, (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, x_n + 1)\rangle$$

under some valuation $\mathfrak{V}$. As r does not occur in ψ, we may assume that $\mathfrak{V}(r) = \{(0, \dots, 0, N)\}$. It is easy to see that ψ^r is true at $(0, \dots, 0)$ in the resulting DTM_ω.

(iv) Suppose that ψ is true at $(0, \dots, 0)$ in a DTM_N, $N < \omega$, based on

$$\langle\mathbb{R}^n, (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, x_n + 1)\rangle$$

under some valuation $\mathfrak{V}$. As in the proof of Theorem 1, let a set $E \subset B^n$, function g, and point e be as specified by Lemma 3. Let h_0 be a homeomorphism from E onto the 'stripe' $\mathbb{R}^{n-1} \times (-\frac{1}{2}, +\frac{1}{2})$, and let $h_0(e) = (0, \dots, 0)$. Define H_i and the homeomorphisms h_i from H_i onto E as in the proof of Theorem 1. Define a valuation $\mathfrak{V}'$ in B^n as before by taking for every variable p *with the exception of* r,

$$\mathfrak{V}'(p) = \bigcup_{i=0}^{N} g^i(h_i(H_i \cap \mathfrak{V}(p))).$$

Set

$$\mathfrak{V}'(r) = \{g^N(e)\}.$$

Using Lemmata 4 and 3, it can be readily checked by induction on the structure of ψ^r that ψ^r is satisfied at e in the model $\langle B^n, g\rangle$ under the valuation $\mathfrak{V}'$. ❑

Theorem 9 is an immediate consequence of Theorem 8 and Lemmata 11, 12. ❑

6 Dynamic metric logic

The language $\mathcal{DML}$ of *dynamic metric logic* is defined in the same way as $\mathcal{DTL}$ with the exception that the topological operators are replaced by the *metric operators* $\exists^{\leq a}$ and $\forall^{\leq a}$, for $a \in \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the set of positive rational numbers. These numbers will be called *numerical parameters* of $\mathcal{DML}$-formulas. The intended semantics of this logic is defined as follows.

A *dynamic metric structure* (DMS, for short) is a pair $\mathfrak{F} = \langle\langle W, d\rangle, f\rangle$, where $\langle W, d\rangle$ is a metric space (with a metric d) and $f : W \to W$ is an *isometric function*, i.e., a bijection on W such that $d(x, y) = d(f(x), f(y))$ for all $x, y \in W$. For example, the map $x \mapsto x + 1$ on

$\mathbb{R}$ and the rotation g on B^2 considered above are isometric automorphisms on the respective spaces with the Euclidean metric.

A *dynamic metric model* (or DMM) is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, where $\mathfrak{F}$ is a DMS and $\mathfrak{V}$ is a valuation defined in precisely the same way as in the topological case. The truth relation is defined similarly to the case of DTMs with ω iterations: the only difference is that the truth conditions for the topological operators $\mathbf{I}$ and $\mathbf{C}$ are replaced by

$$(\mathfrak{M}, x) \models \forall^{\leq a}\varphi \quad \text{iff} \quad (\mathfrak{M}, y) \models \varphi \text{ for every } y \in W \text{ such that } d(x, y) \leq a;$$
$$(\mathfrak{M}, x) \models \exists^{\leq a}\varphi \quad \text{iff} \quad \text{there exists } y \in W \text{ such that } d(x, y) \leq a \text{ and } (\mathfrak{M}, y) \models \varphi.$$

In contrast to dynamic topological logics, now we have the following:

Theorem 13. *The set of $\mathcal{DML}$-formulas that are valid in all DMSs is decidable. However, the decision problem is not elementary.*

Roughly, the idea of the decidability proof is similar to that of Theorem 13.6 from [7]: first we represents DMMs in the form of *quasimodels* and then show that quasimodels can be encoded in monadic second-order logic. The main novelty of this proof is the rather involved notion of a quasimodel.

Given a $\mathcal{DML}$-formula φ, denote by γ_φ the maximal numerical parameter occurring in φ and by $M[\varphi] \subseteq \mathbb{Q}^+$ the smallest set containing all numerical parameters in φ and closed under the rule

(+) if $a, b \in M[\varphi]$ and $a + b \leq \gamma_\varphi$, then $a + b \in M[\varphi]$.

Clearly, $M[\varphi]$ is finite. Let $M^+[\varphi] = M[\varphi] \cup \{2 \cdot \gamma_\varphi\}$.

Define the *metric depth* $mtd(\varphi)$ of φ inductively by taking:

$$\begin{aligned} mtd(p) &= 0, & mtd(\exists^{\leq a}\varphi) &= mtd(\varphi) + a, \\ mtd(\neg\varphi) &= mtd(\varphi), & mtd(\bigcirc\varphi) &= mtd(\varphi), \\ mtd(\varphi_1 \land \varphi_2) &= \max(mtd(\varphi_1), mtd(\varphi_2)), & mtd(\Box_F\varphi) &= mtd(\varphi). \end{aligned}$$

In what follows we consider $\Diamond_F\varphi$ and $\forall^{\leq a}\varphi$ as abbreviations for $\neg\Box_F\neg\varphi$ and $\neg\exists^{\leq a}\neg\varphi$, respectively.

Our first observation is that every satisfiable $\mathcal{DML}$-formula φ can be satisfied in a DMS which is based on the metric space generated by some intransitive labelled tree.

Given an intransitive tree $\langle T, < \rangle$ and a function δ labelling the edges of $\langle T, < \rangle$ with positive real numbers, we denote by $\langle T, \delta^* \rangle$ the *tree metric space induced by* $\langle T, < \rangle$ and δ, that is, for any $x \neq y$ in T, $\delta^*(x, y)$ is the sum of labels on the edges occurring in the shortest path from x to y in $\langle T, < \rangle$, and $\delta^*(x, x) = 0$. If $\delta^*(x, y)$ is bounded, then the number

$$\max\{\delta^*(r, x) \mid r \text{ the root and } x \in T\}$$

is called the *radius* of the tree metric space $\langle T, \delta^* \rangle$.

Lemma 14. *A $\mathcal{DML}$-formula φ (with $mtd(\varphi) > 0$) is satisfiable iff it is satisfiable in a DMS of the form $\mathfrak{F}' = \langle \langle T \times \mathbb{Z}, d' \rangle, f' \rangle$, where*

- *$\langle \langle T, < \rangle, \delta \rangle$ is a labelled intransitive tree such that $\delta(x, y) \in M^+[\varphi]$, for all $x, y \in T$ with $x < y$, and $\langle T, \delta^* \rangle$ is of radius $\leq mtd(\varphi)$,*

- $\langle T \times \mathbb{Z}, d' \rangle$ *is the metric space with*

$$d'(\langle x, i\rangle, \langle y, j\rangle) = \begin{cases} \delta^*(x,y), & \textit{if } \ i = j, \\ 3 \cdot \textit{mtd}(\varphi), & \textit{otherwise}, \end{cases}$$

- $f'(\langle x, i\rangle) = \langle x, i+1\rangle$, *for all* $\langle x, i\rangle \in T \times \mathbb{Z}$.

Proof. Suppose that $(\mathfrak{M}, u_0) \models \varphi$ for some model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ with $\mathfrak{F} = \langle \langle W, d\rangle, f\rangle$ and some point $u_0 \in W$. For any $u, v \in W$, put

$$d_0(u,v) \quad = \quad \min\bigl(\{2 \cdot \gamma_\varphi\} \cup \{a \in M[\varphi] \mid d(u,v) \leq a\}\bigr) \in M^+[\varphi].$$

Now define the required labelled tree $\langle \langle T, <\rangle, \delta\rangle$ by taking

$$\begin{array}{l} T = \{\langle u_0, u_1, \dots, u_n\rangle \mid \sum_{i=1}^n d_0(u_{i-1}, u_i) \leq \textit{mtd}(\varphi), \ u_1, \dots, u_n \in W\}, \\ x < y \quad \text{iff} \quad x = \langle u_0, \dots, u_n\rangle \text{ and } y = \langle u_0, \dots, u_n, u_{n+1}\rangle, \\ \delta(\langle u_0, \dots, u_n\rangle, \langle u_0, \dots, u_n, u_{n+1}\rangle) = d_0(u_n, u_{n+1}). \end{array}$$

Clearly, the radius of $\langle T, \delta^*\rangle$ is $\leq \textit{mtd}(\varphi)$.

Let $\mathfrak{F}' = \langle \langle T \times \mathbb{Z}, d'\rangle, f'\rangle$, where d' and f' are defined as in the formulation of the lemma. It is easy to see that $\langle T \times \mathbb{Z}, d'\rangle$ is indeed a metric space and $f' \colon T \times \mathbb{Z} \to T \times \mathbb{Z}$ is a metric automorphism. Put, for every variable p,

$$\mathfrak{V}'(p) = \{\langle \langle u_0, \dots, u_n\rangle, i\rangle \in T \times \mathbb{Z} \mid (\mathfrak{M}, f^i(u_n)) \models p\}$$

and $\mathfrak{M}' = \langle \mathfrak{F}', \mathfrak{V}'\rangle$.

Denote by x_0 the root $\langle u_0\rangle$ of $\langle T, <\rangle$. We show now by induction that, for every subformula ψ of φ, every $i \in \mathbb{Z}$ and every $x = \langle u_0, \dots, u_n\rangle \in T$ such that

$$\delta^*(x_0, x) \leq \textit{mtd}(\varphi) - \textit{mtd}(\psi), \tag{11}$$

we have

$$(\mathfrak{M}, f^i(u_n)) \models \psi \quad \text{iff} \quad (\mathfrak{M}', \langle x, i\rangle) \models \psi. \tag{12}$$

The basis of induction follows immediately from the definition of $\mathfrak{M}'$. The case of the Booleans is trivial.

Case $\psi = \exists^{\leq a}\psi'$. Let $(\mathfrak{M}, f^i(u_n)) \models \exists^{\leq a}\psi'$. Since f is a metric automorphism, there is $u_{n+1} \in W$ such that $d(u_n, u_{n+1}) = d(f^i(u_n), f^i(u_{n+1})) \leq a$ and $(\mathfrak{M}, f^i(u_{n+1})) \models \psi'$. Take $y = \langle u_0, \dots, u_n, u_{n+1}\rangle$. By (11), $y \in T$. By the definition of δ, we have $\delta^*(x,y) \leq a$. By the triangle inequality,

$$\delta^*(x_0, y) \quad \leq \quad \delta^*(x_0, x) + \delta^*(x,y) \quad \leq \quad \textit{mtd}(\varphi) - \textit{mtd}(\psi').$$

By IH, $(\mathfrak{M}', \langle y, i\rangle) \models \psi'$ and, since $d'(\langle x, i\rangle, \langle y, i\rangle) \leq a$, we finally obtain $(\mathfrak{M}', \langle x, i\rangle) \models \exists^{\leq a}\psi'$.

Conversely, suppose $(\mathfrak{M}', \langle x, i\rangle) \models \exists^{\leq a}\psi'$. Then there is a point $\langle y, j\rangle \in T \times \mathbb{Z}$ with $d'(\langle x, i\rangle, \langle y, j\rangle) \leq a$ and $(\mathfrak{M}', \langle y, j\rangle) \models \psi'$. By the definition of d', we must have $j = i$, and so $\delta^*(x,y) \leq a$. By the triangle inequality,

$$\delta^*(x_0, y) \quad \leq \quad \delta^*(x_0, x) + \delta^*(x,y) \quad \leq \quad \textit{mtd}(\varphi) - \textit{mtd}(\psi').$$

By IH, $(\mathfrak{M}, f^i(v_m)) \models \psi'$ holds for $\langle u_0, v_1, \dots, v_m\rangle = y$. As f is a metric automorphism, $d(f^i(u_n), f^i(v_m)) = d(u_n, v_m)$ and, by the definition of δ^*, we have $d(u_n, v_m) \leq \delta^*(x, y)$. Therefore, $d(f^i(u_n), f^i(v_m)) \leq a$ and $(\mathfrak{M}, f^i(u_n)) \models \exists^{\leq a}\psi'$.

Case $\psi = \bigcirc\psi'$. We have

$$
\begin{array}{lll}
(\mathfrak{M}, f^i(u_n)) \models \bigcirc\psi' & \text{iff} & (\mathfrak{M}, f^{i+1}(u_n)) \models \psi' \\
& \text{iff} & (\mathfrak{M}', \langle x, i+1\rangle) \models \psi' \qquad \text{[by IH]} \\
& \text{iff} & (\mathfrak{M}', \langle x, i\rangle) \models \bigcirc\psi' \qquad \text{[by definition of } f'].
\end{array}
$$

Case $\psi = \Box_F\psi'$ is similar.

It follows that $(\mathfrak{M}', \langle x_0, 0\rangle) \models \varphi$. ❑

Define the *closure* $cl\,\varphi$ of φ to be the set

$$\{\psi, \neg\psi \mid \psi \in sub\,\varphi\} \cup \{\exists^{\leq a}\psi, \neg\exists^{\leq a}\psi \mid \psi \in sub\,\varphi \text{ and } a \in M[\varphi]\},$$

where $sub\,\varphi$ is the set of all subformulas of φ. A *type* for φ is a Boolean-saturated subset $\boldsymbol{t}$ of $cl\,\varphi$, i.e.,

- for every $\neg\psi \in cl\,\varphi$, $\quad \psi \in \boldsymbol{t} \quad$ iff $\quad \neg\psi \notin \boldsymbol{t}$;
- for every $\psi_1 \wedge \psi_2 \in cl\,\varphi$, $\quad \psi_1 \wedge \psi_2 \in \boldsymbol{t} \quad$ iff $\quad \psi_1 \in \boldsymbol{t}$ and $\psi_2 \in \boldsymbol{t}$.

We are now in a position to define the notion of a quasimodel for a given formula φ as a set of quasistates connected by runs.

A *quasistate* $\boldsymbol{q}$ for φ is a triple $\boldsymbol{q} = \langle\langle T_{\boldsymbol{q}}, <_{\boldsymbol{q}}\rangle, \delta_{\boldsymbol{q}}, \boldsymbol{t}_{\boldsymbol{q}}\rangle$, where

- $\langle T_{\boldsymbol{q}}, <_{\boldsymbol{q}}\rangle$ is an intransitive tree with a labelling function

  $$\delta_{\boldsymbol{q}} : \{(u, v) \in T_{\boldsymbol{q}} \times T_{\boldsymbol{q}} \mid u <_{\boldsymbol{q}} v\} \to M^+[\varphi]$$

  such that the radius of $\langle T_{\boldsymbol{q}}, \delta^*_{\boldsymbol{q}}\rangle$ is bounded by $mtd(\varphi)$;

- $\boldsymbol{t}_{\boldsymbol{q}}$ is a function associating with each $u \in T_{\boldsymbol{q}}$ a type $\boldsymbol{t}_{\boldsymbol{q}}(u)$ for φ, and

  (qs1) for every $u \in T_{\boldsymbol{q}}$ and every $\exists^{\leq a}\psi \in cl\,\varphi$, we have $\exists^{\leq a}\psi \in \boldsymbol{t}_{\boldsymbol{q}}(u)$ iff there is $v \in T_{\boldsymbol{q}}$ such that $\delta^*_{\boldsymbol{q}}(u, v) \leq a$ and $\psi \in \boldsymbol{t}_{\boldsymbol{q}}(v)$;

  (qs2) for no $u \in T_{\boldsymbol{q}}$, there exist two isomorphic substructures generated by $<_{\boldsymbol{q}}$-successors v_1 and v_2 of u and such that $\delta_{\boldsymbol{q}}(u, v_1) = \delta_{\boldsymbol{q}}(u, v_2)$.

We say that a point $u \in T_{\boldsymbol{q}}$ is of *index* $\langle a_1, \dots, a_n\rangle$ if $u_0 <_{\boldsymbol{q}} u_1 <_{\boldsymbol{q}} \cdots <_{\boldsymbol{q}} u_n = u$, where u_0 is the root of $\langle T_{\boldsymbol{q}}, <_{\boldsymbol{q}}\rangle$ and $a_i = \delta_{\boldsymbol{q}}(u_{i-1}, u_i)$, for all i, $1 \leq i \leq n$. The index of the root, u_0, is $\langle\rangle$.

Let $\mathbf{q}$ be a function associating with each $i \in \mathbb{Z}$ a quasistate $\mathbf{q}(i) = \langle\langle T_i, <_i\rangle, \delta_i, \boldsymbol{t}_i\rangle$ for φ. A *run of index* $\langle a_1, \dots, a_n\rangle$ through $\mathbf{q}$ is a function r mapping each $i \in \mathbb{Z}$ to a point $r(i) \in T_i$ of index $\langle a_1, \dots, a_n\rangle$ such that, for every $i \in \mathbb{Z}$

- and every $\bigcirc\psi \in cl\,\varphi$, $\quad \bigcirc\psi \in \boldsymbol{t}_i(r(i)) \quad$ iff $\quad \psi \in \boldsymbol{t}_{i+1}(r(i+1))$;
- and every $\Box_F\psi \in cl\,\varphi$, $\quad \Box_F\psi \in \boldsymbol{t}_i(r(i)) \quad$ iff $\quad \psi \in \boldsymbol{t}_j(r(j))$ for all $j > i$.

Given a set $\mathfrak{R}$ of runs, we denote by $\mathfrak{R}_{\langle a_1,\dots,a_n\rangle}$ its subset of all runs of index $\langle a_1,\dots,a_n\rangle$.

A *quasimodel* for φ is a pair $\langle \mathbf{q},\mathfrak{R}\rangle$, where for every $i\in\mathbb{Z}$, $\mathbf{q}(i)=\langle\langle T_i,<_i\rangle,\delta_i,\boldsymbol{t}_i\rangle$ is a quasistate for φ such that

(qm2) $\varphi\in\boldsymbol{t}_0(u_0)$, where u_0 is the root of $\langle T_0,<_0\rangle$,

and $\mathfrak{R}$ is a set of runs through $\mathbf{q}$ satisfying the following condition

(qm3) $\mathfrak{R}_{\langle\rangle}\neq\emptyset$ and, for all $r\in\mathfrak{R}_{\langle a_1,\dots,a_n\rangle}$, $i\in\mathbb{Z}$ and $u\in T_i$, if $r(i)<_i u$ and $\delta_i(r(i),u)=a_{n+1}$ then there is a run $r'\in\mathfrak{R}_{\langle a_1,\dots,a_n,a_{n+1}\rangle}$ such that $r'(i)=u$ and $r(i)<_i r'(i)$ for all $i\in\mathbb{Z}$.

Lemma 15. *A $\mathcal{DML}$-formula φ (with $mtd(\varphi)>0$) is satisfiable in a DMM iff there is a quasimodel for φ.*

Proof. ($\Rightarrow$) Suppose φ is satisfiable. Take a model $\mathfrak{M}'=\langle\mathfrak{F}',\mathfrak{V}'\rangle$ with $\mathfrak{F}'=\langle\langle T\times\mathbb{Z},d'\rangle,f'\rangle$ provided by Lemma 14. Let $(\mathfrak{M}',\langle x_0,0\rangle)\models\varphi$, where x_0 is the root of $\langle T,<\rangle$.

For every pair $\langle x,i\rangle\in T\times\mathbb{Z}$, we define a type $\boldsymbol{t}(x,i)$ for φ by taking

$$\boldsymbol{t}(x,i)=\{\psi\in cl\varphi\mid(\mathfrak{M}',\langle x,i\rangle)\models\psi\}.$$

Fix now $i\in\mathbb{Z}$ and define a binary relation $\sim_i$ on T by taking

$$\begin{aligned}
x\sim_i y\quad\text{iff}\quad & \boldsymbol{t}(x,i)=\boldsymbol{t}(y,i)\\
&\wedge\ \forall z\in T\big(x<z\to\exists z'\in T\big(y<z'\ \wedge\ \delta(x,z)=\delta(y,z')\ \wedge\ z\sim_i z'\big)\big)\\
&\wedge\ \forall z\in T\big(y<z'\to\exists z\in T\big(x<z\ \wedge\ \delta(x,z)=\delta(y,z')\ \wedge\ z\sim_i z'\big)\big).
\end{aligned}$$

Clearly, $\sim_i$ is an equivalence relation on T. Denote by $[x]_i$ the $\sim_i$-equivalence class of $x\in T$ and put, for $a\in M^+[\varphi]$,

$$[x]_i S_i^a[y]_i\quad\text{iff}\quad\exists y'\in[y]_i\ \big(x<y'\ \wedge\ \delta(x,y')=a\big).$$

By the definition of $\sim_i$, the S_i^a are well-defined. Let

$$\begin{aligned}
&T_i=\{\langle[x_0]_i,a_1,[x_1]_i,a_2,\dots,a_n,[x_n]_i\rangle\ \mid\ [x_0]_iS_i^{a_1}[x_1]_iS_i^{a_2}\dots S_i^{a_n}[x_n]_i\},\\
&u<_i v\quad\text{iff}\quad u=\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i\rangle,\quad v=\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i,a_{n+1},[x_{n+1}]_i\rangle\quad\text{and}\\
&\qquad\qquad\quad [x_n]_iS_i^{a_{n+1}}[x_{n+1}]_i,\\
&\delta_i(\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i\rangle,\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i,a_{n+1},[x_{n+1}]_i\rangle)=\delta(x_n,x_{n+1})=a_{n+1},\\
&\boldsymbol{t}_i(\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i\rangle)=\boldsymbol{t}(x_n,i).
\end{aligned}$$

We show that $\langle\langle T_i,<_i\rangle,\delta_i,\boldsymbol{t}\rangle$ is a quasistate for φ. It is easy to see that $\langle T_i,<_i\rangle$ is an intransitive tree, the tree metric space $\langle T_i,\delta_i^*\rangle$ induced by $\langle\langle T_i,<_i\rangle,\delta_i\rangle$ is of radius $\le mtd(\varphi)$ and that **(qs2)** holds.

To show **(qs1)**, suppose first that $\exists^{\le a}\psi\in\boldsymbol{t}_i(u)$, for $u=\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i\rangle\in T_i$. By the definition of $\boldsymbol{t}_i$, we have $\exists^{\le a}\psi\in\boldsymbol{t}(x_n,i)$, and so $(\mathfrak{M}',\langle x_n,i\rangle)\models\exists^{\le a}\psi$. Then there is $\langle y_m,j\rangle\in T\times\mathbb{Z}$ such that $d'(\langle x_n,i\rangle,\langle y_m,j\rangle)\le a$ and $(\mathfrak{M}',\langle y_m,j\rangle)\models\psi$. By the definition of d', $j=i$ and $\delta^*(x_n,y_m)\le a$. Then $\psi\in\boldsymbol{t}(y_m,i)$. Take the sequence $x_0=y_0<y_1<\dots<y_m$. Let $v=\langle[y_0]_i,b_1,\dots,b_m,[y_m]_i\rangle\in T_i$. By the definition of $\boldsymbol{t}_i$, $\psi\in\boldsymbol{t}_i(v)$ and clearly we have $\delta_i^*(u,v)\le\delta^*(x_n,y_m)\le a$.

Conversely, consider $u=\langle[x_0]_i,a_1,\dots,a_n,[x_n]_i\rangle\in T_i$ and $v=\langle[y_0]_i,b_1,\dots,b_m,[y_m]_i\rangle\in T_i$ such that $\delta_i^*(u,v)\le a$ and $\psi\in\boldsymbol{t}_i(v)$. Clearly, $x_0=y_0$. Let $k\ge0$ be the maximal number such that $a_j = b_j$ and $x_j \sim_i y_j$, for all j, $1 \leq j \leq k$. Denote by w the common prefix $\langle [x_0]_i, a_1, \ldots, a_k, [x_k]_i \rangle$ of u and v. By the definition of δ_i^*, we have $\delta_i^*(w,u) = \delta^*(x_k, x_n)$ and $\delta_i^*(w,v) = \delta^*(y_k, y_m)$. Let $b = \delta^*(y_k, y_m)$. Since $\langle T_i, \delta_i^* \rangle$ is a tree metric space, $\delta_i^*(u,v) = \delta_i^*(w,u) + \delta_i^*(w,v)$. Then $b \leq a$ follows from $\delta_i^*(u,v) \leq a$ and, as b is a sum of elements of $M^+[\varphi]$ not exceeding γ_φ, $b \in M[\varphi]$.

By the definition of $\boldsymbol{t}_i$, $\psi \in \boldsymbol{t}(y_m, i)$, and so $(\mathfrak{M}', \langle y_m, i \rangle) \models \psi$. Then $(\mathfrak{M}', \langle y_k, i \rangle) \models \exists^{\leq b}\psi$ and, as $\exists^{\leq b}\psi \in cl\varphi$, $\exists^{\leq b}\psi \in \boldsymbol{t}(y_k, i)$. Since $\boldsymbol{t}(x_k, i) = \boldsymbol{t}(y_k, i)$, we obtain $(\mathfrak{M}', \langle x_k, i \rangle) \models \exists^{\leq b}\psi$. Then there is $z \in T$ such that $\delta^*(x_k, z) \leq b$ and $(\mathfrak{M}', \langle z, i \rangle) \models \psi$. Now, by the triangle inequality,

$$\delta^*(x_n, z) \;\leq\; \underbrace{\delta^*(x_n, x_k)}_{\leq a-b} + \underbrace{\delta^*(x_k, z)}_{\leq b}.$$

Then $\delta^*(x_n, z) \leq a$ and so $(\mathfrak{M}', \langle x_n, i \rangle) \models \exists^{\leq a}\psi$. Therefore, $\exists^{\leq a}\psi \in \boldsymbol{t}(x_n, i)$ and $\exists^{\leq a}\psi \in \boldsymbol{t}_i(u)$.

Thus, the $\langle \langle T_i, <_i \rangle, \delta_i, \boldsymbol{t}_i \rangle$ are quasistates for φ. Let $\mathbf{q}(i) = \langle \langle T_i, <_i \rangle, \delta_i, \boldsymbol{t}_i \rangle$, for $i \in \mathbb{Z}$. As $\varphi \in \boldsymbol{t}_0([x_0]_0)$, **(qm2)** holds for $\mathbf{q}$. It remains to define runs through $\mathbf{q}$. For each sequence $\langle x_0, x_1, \ldots, x_n \rangle$ of points in T such that $x_0 < x_1 < \cdots < x_n$, take the map

$$r \colon i \mapsto \langle [x_0]_i, a_1, [x_1]_i, a_2, \ldots, a_n, [x_n]_i \rangle,$$

where $a_j = \delta(x_{j-1}, x_j)$, for all j, $1 \leq j \leq n$. It is easy to see that r is a run of index $\langle a_1, \ldots, a_n \rangle$. Let $\mathfrak{R}$ be the set of all such runs. Observe that $r_0 \in \mathfrak{R}_{\langle\rangle}$, where $r_0 \colon i \mapsto \langle [x_0]_i \rangle$. Take $r \in \mathfrak{R}_{\langle a_1, \ldots, a_n \rangle}$, $i \in \mathbb{Z}$ and $u \in T_i$ such that $r(i) <_i u$ and $\delta_i(r(i), u) = a_{n+1}$. Clearly, r is of the form $r \colon i \mapsto \langle [x_0]_i, a_1, \ldots, a_n, [x_n]_i \rangle$, for $x_1, \ldots, x_n \in T$ such that $x_0 < \cdots < x_n$ and $a_j = \delta(x_{j-1}, x_j)$, for all j, $1 \leq j \leq n$, and u is of the form $\langle [x_0]_i, a_1, \ldots, a_n, [x_n]_i, a_{n+1}, [x_{n+1}]_i \rangle$, for $x_{n+1} \in T$ such that $x_n < x_{n+1}$ and $a_{n+1} = \delta(x_n, x_{n+1})$. It is easy to check that the run

$$r' \colon i \mapsto \langle [x_0]_i, a_1, \ldots, a_n, [x_n]_i, a_{n+1}, [x_{n+1}]_i \rangle$$

is as required by **(qm3)**.

$(\Leftarrow)$ Suppose that $\langle \mathbf{q}, \mathfrak{R} \rangle$ is a quasimodel for φ. First we construct a tree metric space $\langle T_{\mathfrak{R}}, \delta_{\mathfrak{R}}^* \rangle$ whose elements are sequences of runs from $\mathfrak{R}$. For each $a \in M[\varphi]$, define a binary relation $\lhd_a$ on $\mathfrak{R}$ as follows: for $r_1, r_2 \in \mathfrak{R}$,

$$r_1 \lhd_a r_2 \quad \text{iff} \quad r_1(i) <_i r_2(i) \text{ and } \delta_i(r_1(i), r_2(i)) = a, \text{ for all } i \in \mathbb{Z}.$$

Let $r_0 \in \mathfrak{R}_{\langle\rangle}$ (by **(qm3)** such a run exists). Set

$$\begin{aligned}
&T_{\mathfrak{R}} = \{\langle r_0, r_1, \ldots, r_n \rangle \mid r_0 \lhd_{a_1} r_1 \lhd_{a_2} \cdots \lhd_{a_n} r_n, \quad a_1, \ldots, a_n \in M[\varphi]\};\\
&u <_{\mathfrak{R}} v \quad \text{iff} \quad u = \langle r_0, \ldots, r_n \rangle, v = \langle r_0, \ldots, r_n, r_{n+1} \rangle \text{ and } r_n \lhd_a r_{n+1};\\
&\delta_{\mathfrak{R}}(\langle r_0, \ldots, r_n \rangle, \langle r_0, \ldots, r_n, r_{n+1} \rangle) = a \quad \text{iff} \quad r_n \lhd_a r_{n+1}.
\end{aligned}$$

Clearly, $\langle \langle T_{\mathfrak{R}}, <_{\mathfrak{R}} \rangle, \delta_{\mathfrak{R}} \rangle$ is a labelled intransitive tree inducing a tree metric space $\langle T_{\mathfrak{R}}, \delta_{\mathfrak{R}}^* \rangle$ of radius $\leq mtd(\varphi)$. Now we construct a model $\mathfrak{M}'' = \langle \mathfrak{F}'', \mathfrak{V}'' \rangle$ with $\mathfrak{F}'' = \langle \langle T_{\mathfrak{R}} \times \mathbb{Z}, d'' \rangle, f'' \rangle$ by taking

$$d''(\langle u, i \rangle, \langle v, j \rangle) = \begin{cases} \delta_{\mathfrak{R}}^*(u, v), & \text{if } i = j,\\ 3 \cdot mtd(\varphi), & \text{otherwise,} \end{cases}$$

$f''(\langle u, i\rangle) = \langle u, i+1\rangle$, for all $\langle u, i\rangle \in T_{\mathfrak{R}} \times \mathbb{Z}$, and

$$\mathfrak{V}''(p) = \{\langle\langle r_0, \dots, r_n\rangle, i\rangle \in T_{\mathfrak{R}} \times \mathbb{Z} \mid p \in \boldsymbol{t}_i(r_n(i))\},$$

for every variable p. By induction on the construction of $\psi \in cl\varphi$ we show that, for every point $\langle\langle r_0, \dots, r_n\rangle, i\rangle$ in $\mathfrak{M}''$, we have

$$(\mathfrak{M}'', \langle\langle r_0, \dots, r_n\rangle, i\rangle) \models \psi \qquad \text{iff} \qquad \psi \in \boldsymbol{t}_i(r_n(i)).$$

The basis of induction follows from the definition of $\mathfrak{V}''$. The case of the Booleans follows from the definition of type.

Case $\psi = \exists^{\leq a}\psi'$. Let $u = \langle r_0, r_1, \dots, r_n\rangle$ and $(\mathfrak{M}'', \langle u, i\rangle) \models \exists^{\leq a}\psi'$. Then there is $\langle v, j\rangle \in T_{\mathfrak{R}} \times \mathbb{Z}$, $v = \langle r_0, r'_1, \dots, r'_m\rangle$, such that $d''(\langle u, i\rangle, \langle v, j\rangle) \leq a$ and $(\mathfrak{M}'', \langle v, j\rangle) \models \psi$. By the definition of d'', $j = i$ and $\delta^*_{\mathfrak{R}}(u, v) \leq a$. By IH, $\psi \in \boldsymbol{t}_i(r'_m(i))$. Let w be the longest common prefix of u and v, and k the maximal number such that $r_j = r'_j$, for all j, $1 \leq j \leq k$. As $\langle T_{\mathfrak{R}}, \delta^*_{\mathfrak{R}}\rangle$ is a tree metric space, we have $\delta^*_{\mathfrak{R}}(u, v) = \delta^*_{\mathfrak{R}}(w, u) + \delta^*_{\mathfrak{R}}(w, v)$. By the definition of $\delta_{\mathfrak{R}}$, $\delta^*_{\mathfrak{R}}(w, u) = \delta_i(r_k(i), r_n(i))$ and $\delta^*_{\mathfrak{R}}(w, v) = \delta_i(r'_k(i), r'_m(i))$. As $r_k = r'_k$, by the triangle inequality, $\delta_i(r_n(i), r'_m(i)) \leq a$ and therefore, by **(qs1)**, $\exists^{\leq a}\psi \in \boldsymbol{t}_i(r_n(i))$.

Conversely, let $\exists^{\leq a}\psi \in \boldsymbol{t}_i(r_n(i))$ for $u = \langle r_0, \dots, r_n\rangle \in T_{\mathfrak{R}}$. Then, by **(qs1)**, there is $x \in T_i$ such that $\delta^*_i(r_n(i), x) \leq a$ and $\psi \in \boldsymbol{t}_i(x)$. Let k be the maximal number such that $r_k(i) = x_k$,

$$x_k <_i x_{k+1} <_i \cdots <_i x_m$$

and $x_m = x$. Clearly, such a $k \geq 0$ exists. As $\langle T_i, \delta^*_i\rangle$ is a tree metric space, we have $\delta^*_i(r_n(i), x) = \delta^*_i(r_n(i), r_k(i)) + \delta^*_i(x_k, x_m)$. Then, by applying **(qm3)** sufficiently many times, one can construct a sequence of runs $\langle r'_k, r'_{k+1}, \dots, r'_m\rangle$ such that $r'_k = r_k$, $r'_j(i) = x_j$ and $r'_{j-1} \lhd_{a_j} r'_j$, for all j, $k < j \leq m$. Consider $v = \langle r_0, \dots r_k, r'_{k+1}, \dots, r'_m\rangle \in T_{\mathfrak{R}}$. Since $\delta_{\mathfrak{R}}(\langle r_0, \dots, r_k\rangle, v) = \delta^*_i(x_k, x_m)$ and $\delta_{\mathfrak{R}}(\langle r_0, \dots, r_k\rangle, u) = \delta^*_i(r_k(i), r_n(i))$, we obtain $\delta^*_{\mathfrak{R}}(u, v) \leq a$. By IH, $(\mathfrak{M}'', \langle v, i\rangle) \models \psi$, and so $(\mathfrak{M}'', \langle u, i\rangle) \models \exists^{\leq a}\psi$.

Case $\psi = \bigcirc\psi'$. We have:

$$\begin{array}{lll}
(\mathfrak{M}'', \langle\langle r_0, \dots, r_n\rangle, i\rangle) \models \bigcirc\psi' & \text{iff} & (\mathfrak{M}, \langle\langle r_0, \dots, r_n\rangle, i+1\rangle) \models \psi \\
& \text{iff} & \psi \in \boldsymbol{t}_{i+1}(r_n(i+1)) \quad [\text{by IH}] \\
& \text{iff} & \bigcirc\psi \in \boldsymbol{t}_i(r_n(i)) \quad [r_n \text{ is a run}].
\end{array}$$

Case $\psi = \Box_F\psi'$ is similar.

It follows from **(qm2)** that $(\mathfrak{M}'', \langle\langle r_0\rangle, 0\rangle) \models \varphi$. ❑

We can now deduce the decidability of the satisfiability problem for $\mathcal{DML}$-formulas by translating into monadic second-order logic the statement that there exists a quasimodel for a given formula φ. We require a number of auxiliary formulas. Denote by Σ the set of all quasistates for φ. Given a quasistate $\boldsymbol{q} = \langle\langle T_{\boldsymbol{q}}, <_{\boldsymbol{q}}\rangle, \delta_{\boldsymbol{q}}, \boldsymbol{t}_{\boldsymbol{q}}\rangle$ from Σ and a point u in $T_{\boldsymbol{q}}$ we denote its index by $idx_{\boldsymbol{q}}(u)$. Further, let

$$\Xi = \left\{\langle a_1, \dots, a_n\rangle \quad \middle| \quad a_i \in M[\varphi] \text{ and } \textstyle\sum_{i=1}^n a_i \leq mtd(\varphi)\right\}.$$

Introduce a unary predicate variable $P_{\boldsymbol{q}}$ for each quasistate $\boldsymbol{q} \in \Sigma$ and a unary predicate variable $R^{\boldsymbol{a}}_\psi$ for each $\psi \in cl\varphi$ and index $\boldsymbol{a} \in \Xi$. Given a type $\boldsymbol{t}$ for φ and such an index $\boldsymbol{a} \in \Xi$, let

$$\chi_{\boldsymbol{t}}\big(\overline{R^{\boldsymbol{a}}}(x)\big) \;=\; \bigwedge_{\psi \in \boldsymbol{t}} R^{\boldsymbol{a}}_\psi(x) \;\wedge\; \bigwedge_{\neg\psi \in \boldsymbol{t}} \neg R^{\boldsymbol{a}}_\psi(x),$$

saying that the type $\boldsymbol{t}$ at 'moment' x of index $\boldsymbol{a}$ is defined with the help of the set of predicates

$$\overline{R^{\boldsymbol{a}}}(x) = \big\langle R^{\boldsymbol{a}}_{\psi}(x) \;\;\big|\;\; \psi \in cl\,\varphi \big\rangle\,.$$

For each index $\boldsymbol{a} \in \Xi$, let $\mathsf{run}_0(\overline{P}(x), \overline{R^{\boldsymbol{a}}}(x))$ denote the following conjunction

$$\bigwedge_{\boldsymbol{q}\in\Sigma} \forall x \Big[P_{\boldsymbol{q}}(x) \to \bigvee_{\substack{u\in T_{\boldsymbol{q}}\\ idx(u)=\boldsymbol{a}}} \chi_{\boldsymbol{t_q}(u)}\big(\overline{R^{\boldsymbol{a}}}(x)\big)\Big] \quad\wedge$$

$$\bigwedge_{\Box_F\psi\in cl\,\varphi} \forall x \Big[R^{\boldsymbol{a}}_{\Box_F\psi}(x) \leftrightarrow \forall y\big(x<y \to R^{\boldsymbol{a}}_{\psi}(y)\big)\Big] \quad\wedge\quad \bigwedge_{\bigcirc\psi\in cl\,\varphi} \forall x \Big[R^{\boldsymbol{a}}_{\bigcirc\psi}(x) \leftrightarrow R^{\boldsymbol{a}}_{\psi}(S(x))\Big]$$

—this is intended to say that $\overline{R^{\boldsymbol{a}}}(x)$ defines a run of index $\boldsymbol{a}$ through a sequence of quasistates defined with the help of

$$\overline{P}(x) = \big\langle P_{\boldsymbol{q}}(x) \;\;\big|\;\; \boldsymbol{q} \in \Sigma\big\rangle.$$

However, we have to refine this definition in order to ensure that condition **(qm3)** holds. To this end, we define, by 'backwards' induction on the length of the index, another formula $\mathsf{run}(\overline{P}(x), \overline{R^{\boldsymbol{a}}}(x))$ as follows. If $\boldsymbol{a}$ is maximal in Ξ (that is, we are at the 'leaf-level') then take

$$\mathsf{run}(\overline{P}(x),\ \overline{R^{\boldsymbol{a}}}(x)) = \mathsf{run}_0(\overline{P}(x),\ \overline{R^{\boldsymbol{a}}}(x)).$$

Suppose, inductively, that for all proper extensions $\boldsymbol{a}'$ of $\boldsymbol{a}$ (i.e., $\boldsymbol{a}$ is a proper prefix of $\boldsymbol{a}'$) we have already defined $\mathsf{run}(\overline{P}(x),\ \overline{R^{\boldsymbol{a}'}}(x))$. Then $\mathsf{run}(\overline{P}(x),\ \overline{R^{\boldsymbol{a}}}(x))$ is the following formula:

$$\begin{aligned}
&\mathsf{run}_0\big(\overline{P}(x),\ \overline{R^{\boldsymbol{a}}}(x)\big) \quad\wedge\\
&\bigwedge_{\boldsymbol{q}\in\Sigma}\ \bigwedge_{\substack{u\in T_{\boldsymbol{q}}\\ idx(u)=\boldsymbol{a}}} \forall x \Big[P_{\boldsymbol{q}}(x) \;\wedge\; \chi_{\boldsymbol{t_q}(u)}\big(\overline{R^{\boldsymbol{a}}}(x)\big) \quad\to\\
&\qquad \bigwedge_{\substack{v\in T_q\\ \delta_{\boldsymbol{q}}(u,v)=b}}\ \exists_{\psi\in cl\,\varphi} R^{\boldsymbol{a}\cdot b}_{\psi}\ \Big(\mathsf{run}\big(\overline{P}(x),\ \overline{R^{\boldsymbol{a}\cdot b}}(x)\big) \;\wedge\; \chi_{\boldsymbol{t_q}(v)}\big(\overline{R^{\boldsymbol{a}\cdot b}}(x)\big) \quad\wedge\\
&\qquad\qquad \bigwedge_{\boldsymbol{s}\in\Sigma}\ \bigwedge_{\substack{u'\in T_{\boldsymbol{s}}\\ idx(u')=\boldsymbol{a}}} \forall z \Big(P_{\boldsymbol{s}}(z) \;\wedge\; \chi_{\boldsymbol{t_s}(u')}\big(\overline{R^{\boldsymbol{a}}}(z)\big) \quad\to \bigvee_{\substack{v'\in T_{\boldsymbol{s}}\\ \delta_{\boldsymbol{s}}(u',v')=b}} \chi_{\boldsymbol{t_s}(v')}\big(\overline{R^{\boldsymbol{a}\cdot b}}(z)\big)\Big)\Big)\Big],
\end{aligned}$$

where $\boldsymbol{a}\cdot b = \langle a_1,\dots,a_n,b\rangle$ whenever $\boldsymbol{a} = \langle a_1,\dots,a_n\rangle$.

Finally, we define a monadic second-order sentence $\varphi^\sharp$ to be the following formula

$$\begin{aligned}
&\exists_{\boldsymbol{q}\in\Sigma} P_{\boldsymbol{q}}\ \Big[\forall x \bigvee_{\boldsymbol{q}\in\Sigma}\Big(P_{\boldsymbol{q}}(x) \;\wedge \bigwedge_{\substack{\boldsymbol{q}'\in\Sigma\\ \boldsymbol{q}\neq\boldsymbol{q}'}} \neg P_{\boldsymbol{q}'}(x)\Big) \quad\wedge\\
&\qquad \bigvee_{\boldsymbol{s}\in\Sigma}\ \bigvee_{\substack{u\in T_{\boldsymbol{s}}\\ idx(u)=\langle\rangle\\ \varphi\in\boldsymbol{t_s}(u)}} \exists x\ \Big(P_{\boldsymbol{s}}(x) \;\wedge\; \exists_{\psi\in cl\,\varphi} R^{\langle\rangle}_{\psi}\ \Big(\mathsf{run}\big(\overline{P}(x),\ \overline{R^{\langle\rangle}}(x)\big) \;\wedge\; \chi_{\boldsymbol{t_s}(u)}\big(\overline{R^{\langle\rangle}}(x)\big)\Big)\Big)\Big].
\end{aligned}$$

Evaluated in $\langle \mathbb{Z}, < \rangle$, the first line of $\varphi^\sharp$ says that the sets $P_{\boldsymbol{q}} \subseteq \mathbb{Z}$ ($\boldsymbol{q} \in \Sigma$) form a partition of $\mathbb{Z}$. By defining the map $\mathbf{q} \colon \mathbb{Z} \to \Sigma$ as

$$\mathbf{q}(i) = \boldsymbol{q} \qquad \text{iff} \qquad i \in P_{\boldsymbol{q}}$$

we obtain a quasimodel $\langle \mathbf{q}, \mathfrak{R} \rangle$ for φ: the second line of $\varphi^\sharp$ states condition **(qm2)**; condition **(qm3)** is satisfied by the formulas $\mathsf{run}(\overline{P}(x),\ \overline{R^{\boldsymbol{a}}}(x))$. Therefore, it is easy to see that the following holds:

Lemma 16. *For every $\mathcal{DML}$-formula φ with $mtd(\varphi) > 0$, we have $\langle \mathbb{Z}, < \rangle \models \varphi^\sharp$ iff there is a quasimodel for φ.*

Clearly, Σ can be constructed from φ by an algorithm. So we can now apply the result of Büchi [5] on the decidability of monadic second-order logic over $\langle \mathbb{Z}, < \rangle$.

Lemma 17. *The satisfiability problem for $\mathcal{DML}$-formulas in DMSs is not decidable in time bounded by any elementary function.*

Proof. Denote by $\mathbf{PTL}_{\Box\bigcirc}$ the fragment of standard linear time temporal logic without the operator 'until,' but with $\Box_F$, $\Diamond_F$, and $\bigcirc$. Our proof is by a polynomial reduction of the satisfiability problem for the product modal logic $\mathbf{PTL}_{\Box\bigcirc} \times \mathbf{K}$ (which is non-elementary by Theorem 6.37 and Claim 6.25 of [7]) to the satisfiability problem for $\mathcal{DML}$-formulas in DMSs.

The language of $\mathbf{PTL}_{\Box\bigcirc} \times \mathbf{K}$ extends the language of propositional logic by means of the temporal operators $\Box_F$, $\Diamond_F$ and $\bigcirc$ and the modal operators $\Box$ and $\Diamond$ (here we consider $\Box\psi$ and $\Box_F\psi$ as abbreviations for $\neg\Diamond\neg\psi$ and $\neg\Diamond_F\neg\psi$, respectively). For the purposes of our proof, we may only consider *product frames* for $\mathbf{PTL}_{\Box\bigcirc} \times \mathbf{K}$, that is, 3-frames of the form $\langle \mathbb{Z} \times W, R_{\Diamond_F}, R_{\bigcirc}, R_{\Diamond} \rangle$, where, for all $\langle i, x \rangle, \langle j, y \rangle \in \mathbb{Z} \times W$,

$$\begin{aligned}
\langle i, x \rangle \, R_{\Diamond_F} \, \langle j, y \rangle \quad & \text{iff} \quad i < j \quad \text{and} \quad x = y, \\
\langle i, x \rangle \, R_{\bigcirc} \, \langle j, y \rangle \quad & \text{iff} \quad i + 1 = j \quad \text{and} \quad x = y, \\
\langle i, x \rangle \, R_{\Diamond} \, \langle j, y \rangle \quad & \text{iff} \quad i = j \quad \text{and} \quad xRy.
\end{aligned}$$

In what follows such a 3-frame will be denoted by $\langle \mathbb{Z}, <, +1 \rangle \times \langle W, R \rangle$.

Now we define a translation from $\mathbf{PTL}_{\Box\bigcirc} \times \mathbf{K}$-formulas into $\mathcal{DML}$-formulas. The main difficulty in the translation is that although, by Lemma 14, we can restrict ourselves to DMMs induced by tree metric spaces (which essentially look like product frames for $\mathbf{PTL}_{\Box\bigcirc} \times \mathbf{K}$), the metric operators $\exists^{\leq a}$ are interpreted there by symmetric relations, whereas the modal operator $\Diamond$ of $\mathbf{PTL}_{\Box\bigcirc} \times \mathbf{K}$ is not necessarily 'symmetric.' In order to solve this problem, we take a $\mathbf{PTL}_{\Box\bigcirc} \times \mathbf{K}$ model based on a product frame $\langle \mathbb{Z}, <, +1 \rangle \times \langle W, R \rangle$ as a basis for a DMM and between any pair $\langle x, 0 \rangle$, $\langle y, 0 \rangle$ of points of the DMM such that xRy ($x, y \in W$), we insert two intermediate points $\langle x, 1 \rangle$ and $\langle x, 2 \rangle$ which define the 'traverse order' in the sense that the translation of a formula of the form $\Diamond\psi$ can access $\langle y, 0 \rangle$ from $\langle x, 0 \rangle$ (in three steps through $\langle x, 1 \rangle$ and $\langle x, 2 \rangle$) but not the other way round. More precisely, take fresh variables q_0, q_1 and define $\tau_0 = q_0 \wedge q_1$, $\tau_1 = q_0 \wedge \neg q_1$ and $\tau_2 = \neg q_0 \wedge \neg q_1$. Now we set inductively

$$\begin{aligned}
p_i^\sharp &= p_i, & (\bigcirc\psi)^\sharp &= \bigcirc\psi^\sharp, \\
(\neg\psi)^\sharp &= \neg\psi^\sharp, & (\Diamond_F\psi)^\sharp &= \Diamond_F\psi^\sharp, \\
(\psi_1 \wedge \psi_2)^\sharp &= \psi_1^\sharp \wedge \psi_2^\sharp, & (\Diamond\psi)^\sharp &= \exists^{\leq 1/3}\big(\tau_1 \wedge \exists^{\leq 1/3}\big(\tau_2 \wedge \exists^{\leq 1/3}(\tau_0 \wedge \psi^\sharp)\big)\big).
\end{aligned}$$

Let φ be a $\mathbf{PTL}_{\Box\bigcirc} \times \mathbf{K}$-formula. We show now that φ is satisfiable iff

$$(\tau_0 \wedge \varphi^\sharp) \quad \wedge \quad \forall^{\leq mtd(\varphi^\sharp)} \Box_F^+ \big((q_0 \leftrightarrow \bigcirc q_0) \wedge (q_1 \leftrightarrow \bigcirc q_1)\big) \tag{13}$$

is satisfiable in a DMS.

($\Rightarrow$) If φ is satisfiable then it is satisfiable in a model $\mathfrak{M}$ based on a product frame of the form $\langle \mathbb{Z}, <, +1\rangle \times \langle T, <\rangle$, where $\langle T, <\rangle$ is an intransitive irreflexive tree of depth $\leq mtd(\varphi^\sharp)$ (note that the maximal number of 'nested' $\Diamond$ operators in φ equals $mtd(\varphi^\sharp)$). We construct a labelled intransitive tree $\langle\langle T', <'\rangle, \delta\rangle$ by taking $T' = T \times \{0, 1, 2\}$,

$$\begin{array}{lll} \langle x, 0\rangle <' \langle x, 1\rangle <' \langle x, 2\rangle \quad \text{and} \quad \delta(\langle x, 0\rangle, \langle x, 1\rangle) = \delta(\langle x, 1\rangle, \langle x, 2\rangle) = {}^1\!/_3, & & \text{for every } x \in T, \\ \langle x, 2\rangle <' \langle y, 0\rangle \quad \text{and} \quad \delta(\langle x, 2\rangle, \langle y, 0\rangle) = {}^1\!/_3, & & \text{for every } x, y \in T \text{ with } x < y. \end{array}$$

Clearly, the distance from the root of $\langle\langle T', <'\rangle, \delta\rangle$ to its leaves does not exceed $mtd(\varphi^\sharp)$ and this labelled intransitive tree induces a DMS $\mathfrak{F}' = \langle\langle T' \times \mathbb{Z}, d\rangle, f\rangle$ by taking, for all $\langle x', i\rangle, \langle y', j\rangle \in T' \times \mathbb{Z}$,

$$\begin{aligned} d(\langle x', i\rangle, \langle y', j\rangle) &= \begin{cases} \delta(x', y'), & \text{if } i = j, \\ 3 \cdot mtd(\varphi^\sharp), & \text{otherwise,} \end{cases} \\ f(\langle x', i\rangle) &= \langle x', i+1\rangle. \end{aligned}$$

It follows that (13) is satisfied in a DMM $\langle \mathfrak{F}', \mathfrak{V}'\rangle$, where

$$\begin{aligned} &\mathfrak{V}'(p_i) = \{\langle\langle x, 0\rangle, w\rangle \in T' \times \mathbb{Z} \mid (\mathfrak{M}, \langle w, x\rangle) \models p_i\}, \\ &\mathfrak{V}'(q_0) = (T \times \{0, 1\}) \times \mathbb{Z}, \\ &\mathfrak{V}'(q_1) = (T \times \{0\}) \times \mathbb{Z}. \end{aligned}$$

($\Leftarrow$) Suppose that (13) is satisfiable in a DMS. Then, by Lemma 14, it is also satisfiable in a model $\mathfrak{M}$ based on a DMS $\mathfrak{F} = \langle\langle W, d\rangle, f\rangle$ induced by a tree metric space of radius $\leq mtd(\varphi^\sharp)$. Let

$$W_j = \{x \in W \mid (\mathfrak{M}, x) \models \tau_j\}, \qquad \text{for } j = 0, 1, 2.$$

Define a binary relation R on W_0 by taking, for every $x, y \in W_0$,

$$xRy \quad \text{iff} \quad \exists z_1 \in W_1, z_2 \in W_2 \;\text{ such that }\; d(x, z_1) \leq {}^1\!/_3, \;\; d(z_1, z_2) \leq {}^1\!/_3, \;\; d(z_2, y) \leq {}^1\!/_3.$$

Since f is a metric automorphism and $\mathfrak{F}$ is induced by a metric space of radius $\leq mtd(\varphi^\sharp)$, by the second conjunct of (13), we have

- $x \in W_j$ iff $f^i(x) \in W_j$, for every $x \in W$, $i \in \mathbb{Z}$ and each $j = 0, 1, 2$;
- xRy iff $f^i(x) R f^i(y)$, for every $x, y \in W_0$ and $i \in \mathbb{Z}$.

Now we construct a product model $\mathfrak{M}' = \langle \mathfrak{F}', \mathfrak{V}'\rangle$ to satisfy φ. Let $\mathfrak{F}' = \langle \mathbb{Z}, <, +1\rangle \times \langle W_0, R\rangle$ and

$$\mathfrak{V}'(p_j) = \{\langle x, i\rangle \in \mathfrak{F}' \mid (\mathfrak{M}, f^i(x)) \models p_j\}.$$

It is not difficult to see that φ is satisfied in $\mathfrak{M}'$. ❑

7 Discussion

In this paper we have considered two logical approaches to modelling dynamical systems.

The first one uses dynamic topological logics interpreted in various natural classes of topological spaces with homeomorphisms. Our main result in the framework of this approach shows that such dynamic topological logics cannot be recursively enumerable—even if we assume that the orbits are finite—which gives a solution to a conjecture of Kremer and Mints.

An interesting and challenging open problem is the decidability of dynamic topological logics interpreted in topological spaces with *continuous* mappings. A rather unexpected partial result in this direction has recently been obtained in [8]: it is shown that the dynamic topological logic interpreted in models with *finite* iterations (based on arbitrary topological spaces with continuous mappings) is decidable, but not in time/space bounded by any primitive recursive function. Although we still do not know whether the corresponding logic with infinite iterations is decidable, a reduction similar to that in Section 5 shows that this logic cannot be decidable in primitive recursive time.

Our second approach was to represent dynamical systems by means of the dynamic metric logic interpreted in metric spaces with isometric functions. Unlike dynamic topo-logics, the dynamic metric logic turns out to be decidable (and so recursively enumerable), but not in time/space bounded by an elementary function.

Note that this result holds only for the class of arbitrary metric spaces. It would be of interest to consider metric logics interpreted in more restricted classes of spaces, for example, in various *compact* metric spaces (for a justification and more details see, e.g., [14]).

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References

[1] P. S. Alexandroff. Diskrete Räume. *Matematicheskii Sbornik*, 2 (44):501–518, 1937.

[2] S. Artemov, J. Davoren, and A. Nerode. Modal logics and topological semantics for hybrid systems. Technical Report MSI 97-05, Cornell University, 1997.

[3] N. Bourbaki. *General Topology, Part 1.* Hermann, Paris and Addison-Wesley, 1966.

[4] J. R. Brown. *Ergodic Theory and Topological Dynamics.* Academic Press, 1976.

[5] J. R. Büchi. On a decision method in restricted second order arithmetic. In *Logic, Methodology and Philosophy of Science: Proceedings of the 1960 International Congress,* pages 1–11. Stanford University Press, 1962.

[6] J. Davoren and A. Nerode. Logics for hybrid systems. *Proceedings of the IEEE*, 88:985–1010, 2000.

[7] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications*, volume 148 of *Studies in Logic*. Elsevier, 2003.

[8] D. Gabelaia, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Non-primitive recursive decidability of products of modal logics with expanding domains*. Manuscript, 2004. Available at `http://www.dcs.kcl.ac.uk/staff/mz`.

[9] D. Gabelaia, R. Kontchakov, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Combining spatial and temporal logics: expressiveness vs. complexity*. *Journal of Artificial Intelligence Research*, 23, 2005.

[10] J. Hopcroft, R. Motwani, and J. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison–Wesley, 2001.

[11] A. Katok and B. Hasselblatt. *Introduction to Modern Theory of Dynamical Systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Elsevier, 1995.

[12] P. Kremer. Temporal logic over **S4**: an axiomatizable fragment of dynamic topological logic. *Bulletin of Symbolic Logic*, 3:375–376, 1997.

[13] P. Kremer and G. Mints. Dynamic topological logic. *Bulletin of Symbolic Logic*, 3:371–372, 1997.

[14] P. Kremer and G. Mints. Dynamic topological logic. *Annals of Pure and Applied logic*, 2005. To appear.

[15] P. Kremer, G. Mints, and V. Rybakov. Axiomatizing the next-interior fragment of dynamic topological logic. *Bulletin of Symbolic Logic*, 3:376–377, 1997.

[16] O. Kutz, H. Sturm, N.-Y. Suzuki, F. Wolter, and M. Zakharyaschev. Logics of metric spaces. *ACM Transactions on Computational Logic*, 4:260–294, 2003.

[17] J.C.C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[18] T. Moor and J. Davoren. Robust controller synthesis for hybrid systems using modal logics. In M.D. Di Benedetto and A. Sangiovanni-Vincentelli, editors, *Hybrid Systems: Computation and Control* (*HSCC'01*), volume 2034 of *LNCS*, pages 433–446. Springer, 2001.

[19] E. Post. A variant of a recursively unsolvable problem. *Bulletin of the AMS*, 52:264–268, 1946.

[20] S. Slavnov. Two counterexamples in the logic of dynamic topological systems. Technical Report TR–2003015, Cornell University, 2003.

[21] P. Walters. *An Introduction to Ergodic Theory*, Springer-Verlag, Berlin, 1982.