Speaking about transitive frames in propositional languages

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Abstract

This paper is a comparative study of the propositional intuitionistic (non-modal) and classical modal languages interpreted in the standard way on transitive frames. It shows that, when talking about these frames rather than conventional quasi-orders, the intuitionistic language displays some unusual features: its expressive power becomes weaker than that of the modal language, the induced consequence relation does not have a deduction theorem and is not protoalgebraic. Nevertheless, the paper develops a manageable model theory for this consequence and its extensions which also reveals some unexpected phenomena. The balance between the intuitionistic and modal languages is restored by adding to the former one more implication.

1 Both modal and intuitionistic propositional languages may be regarded as talking about *quasi-orders* $\mathfrak{F} = \langle W, R \rangle$, R a reflexive and transitive binary relation on a set W. The modal language $\mathcal{ML}$ with the primitive connectives, say, $\wedge$, $\vee$, $\rightarrow$, $\bot$ and $\Box$ is interpreted in $\mathfrak{F}$ by means of valuations $\mathfrak{V}$ of propositional variables in the power-set 2^W of W and the truth-relation $\models$ in the following way:

$$
\begin{aligned}
x \models p \quad &\text{iff} \quad x \in \mathfrak{V}(p),\ p \text{ a variable}; && (1)\\
x \not\models \bot; & && (2)\\
x \models \varphi \wedge \psi \quad &\text{iff} \quad x \models \varphi \text{ and } x \models \psi; && (3)\\
x \models \varphi \vee \psi \quad &\text{iff} \quad x \models \varphi \text{ or } x \models \psi; && (4)\\
x \models \varphi \rightarrow \psi \quad &\text{iff} \quad x \models \varphi \text{ implies } x \models \psi; && (5)\\
x \models \Box\varphi \quad &\text{iff} \quad \forall y \in W\ (xRy \Rightarrow y \models \varphi). && (6)
\end{aligned}
$$

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The intended interpretation of the intuitionistic language $\mathcal{L} = \mathcal{ML} - \Box$ differs from that above in two respects: variables are evaluated in the set

$$\mathrm{Up}W = \{X \subseteq W : \forall x, y \ (x \in X \wedge xRy \Rightarrow y \in X)\}$$

of *cones* (or *upward closed sets*) and (5) is replaced by

$$x \models \varphi \to \psi \quad \text{iff} \quad \forall y \in W \ (xRy \wedge y \models \varphi \Rightarrow y \models \psi). \tag{7}$$

By the definition, the truth-sets $\mathfrak{V}(\varphi) = \{x \in W : x \models \varphi\}$ of modal formulas may be arbitrary subsets of W, while those of intuitionistic ones are restricted to $\mathrm{Up}W$; in particular, intuitionistic formulas cannot distinguish between points in the same cluster $C(x) = \{x\} \cup \{y \in W : xRy \wedge yRx\}$. However, as far as only cones are concerned, the modal and intuitionistic languages are of the same expressive power at both functional (local) and axiomatic (global) levels. To make this statement more precise, we require some definitions.

With every $\mathcal{L}$-formula $\varphi(p_1, \dots, p_n)$ of variables occurring in the list $p_1, \dots, p_n$ we associate an operator $\varphi_{\mathfrak{F}}(X_1, \dots, X_n)$ such that, for any fixed quasi-order $\mathfrak{F} = \langle W, R \rangle$, $\varphi_{\mathfrak{F}}$ is the n-ary function on $\mathrm{Up}W$ determined by φ, i.e., for every intuitionistic valuation $\mathfrak{V}$ in $\mathfrak{F}$,

$$\varphi_{\mathfrak{F}}(\mathfrak{V}(p_1), \dots, \mathfrak{V}(p_n)) = \mathfrak{V}(\varphi).$$

For a modal formula $\varphi(p_1, \dots, p_n)$ we first construct an operator $\varphi^*_{\mathfrak{F}}(X_1, \dots, X_n)$ such that, for each $\mathfrak{F} = \langle W, R \rangle$, $\varphi^*_{\mathfrak{F}}$ is the n-ary function on 2^W determined by φ (under arbitrary modal valuations), and then define $\varphi^{\triangle}_{\mathfrak{F}}$ to be the restriction of $\varphi^*_{\mathfrak{F}}$ to cones when $\varphi^*_{\mathfrak{F}}(X_1, \dots, X_n) \in \mathrm{Up}W$ for every $\mathfrak{F} = \langle W, R \rangle$ and all $X_1, \dots, X_n \in \mathrm{Up}W$, and $\varphi^{\triangle}_{\mathfrak{F}} = \bot^*_{\mathfrak{F}}$ otherwise.

Proposition 1 $\{\varphi_{\mathfrak{F}} : \varphi \in \mathcal{L}\} = \{\varphi^{\triangle}_{\mathfrak{F}} : \varphi \in \mathcal{ML}\}$.

Proof As is well known (see e.g. [20], [8]), for every $\mathcal{L}$-formula φ, we have $\varphi_{\mathfrak{F}} = (\mathsf{T}\varphi)^{\triangle}_{\mathfrak{F}}$, where T is the *Gödel translation* prefixing $\Box$ to all subformulas of φ (save conjunctions and disjunctions). Thus $\{\varphi_{\mathfrak{F}} : \varphi \in \mathcal{L}\} \subseteq \{\varphi^{\triangle}_{\mathfrak{F}} : \varphi \in \mathcal{ML}\}$. The converse inclusion follows from the fact (see e.g. [20] or Lemmas 8.32 and 8.33 in [8]) that every cone constructed from cones $X_1, \dots, X_n$ using the Boolean operations and $\Box$ can be also obtained from $X_1, \dots, X_n$ in a uniform way with the help of intuitionistic operations. In other words, given an $\mathcal{ML}$-formula $\varphi(p_1, \dots, p_n)$, one can construct an $\mathcal{L}$-formula $\psi(p_1, \dots, p_n)$ such that $(\Box\varphi)^{\triangle}_{\mathfrak{F}} = \psi_{\mathfrak{F}}$. ❑

A class $\mathcal{C}$ of quasi-orders is said to be $\mathcal{L}$- (or $\mathcal{ML}$-) *axiomatic* if there is a set Γ of $\mathcal{L}$- (respectively, $\mathcal{ML}$-) formulas such that, for every quasi-order $\mathfrak{F}$,

$$\mathfrak{F} \models \Gamma \text{ iff } \mathfrak{F} \in \mathcal{C}.$$

($\mathfrak{F} \models \Gamma$ means that all formulas in Γ are true at all points in $\mathfrak{F}$ under all possible valuations.) Since $\mathcal{L}$-formulas do not distinguish between points in one cluster, when comparing the axiomatic power of modal and intuitionistic formulas we should consider frame classes modulo clusters. More precisely, say that a class of quasi-orders is *skeleton-closed* if with every $\mathfrak{F}$ it contains also all the quasi-orders whose skeletons are isomorphic to the skeleton of $\mathfrak{F}$. Here by the *skeleton* of $\mathfrak{F} = \langle W, R\rangle$ we mean the partial order $\mathfrak{F}^\circ = \langle W^\circ, R^\circ\rangle$ in which W° is the set $\{C(x) : x \in W\}$ of clusters in $\mathfrak{F}$ and $C(x)R^\circ C(y)$ iff xRy.

Proposition 2 *A skeleton-closed class $\mathcal{C}$ of quasi-orders is $\mathcal{L}$-axiomatic iff it is $\mathcal{ML}$-axiomatic.*

Proof If $\mathcal{C}$ is axiomatized by a set Γ of $\mathcal{L}$-formulas then it is clearly axiomatizable by the set $\{\mathsf{T}\varphi : \varphi \in \Gamma\}$ of $\mathcal{ML}$-formulas. Conversely, suppose $\mathcal{C}$ is $\mathcal{ML}$-axiomatic. Then, as follows from [30], it can be axiomatized by a set of modal canonical formulas $\{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\}$ built on quasi-orders (see also Section 5 below). By the refutability criterion for canonical formulas, $\mathfrak{F} \models \alpha(\mathfrak{F}_i^\circ, \mathfrak{D}_i, \bot)$ implies $\mathfrak{F} \models \alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$, and $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i^\circ, \mathfrak{D}_i, \bot)$ implies $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$, for some quasi-order $\mathfrak{G}$ with $\mathfrak{G}^\circ \cong \mathfrak{F}^\circ$. Since $\mathcal{C}$ is skeleton-closed, it follows that it is axiomatizable by the set $\{\alpha(\mathfrak{F}_i^\circ, \mathfrak{D}_i, \bot) : i \in I\}$ and so, in view of Corollary 9.60 of [8], $\mathcal{C}$ is axiomatizable by the set $\{\beta(\mathfrak{F}_i^\circ, \mathfrak{D}_i, \bot) : i \in I\}$ of intuitionistic canonical formulas (see also Corollary 24 below). ❑

Example 3 The class of all partial orders without infinite strictly ascending chains is $\mathcal{ML}$-axiomatic; it is axiomatizable by the Grzegorczyk formula

$$\Box(\Box(p \to \Box p) \to p) \to p$$

but not $\mathcal{L}$-axiomatic; it is not skeleton-closed.

Another manifestation of this connection between $\mathcal{ML}$ and $\mathcal{L}$ is the fact that the Gödel translation T embeds extensions of intuitionistic logic **Int** into extensions of classical modal logic **S4**. We remind the reader that **Int** and **S4** are the sets of $\mathcal{L}$- and, respectively, $\mathcal{ML}$-formulas that are valid in all quasi-orders. Ext**Int**, the class of extensions of **Int** known as *superintuitionistic* or *intermediate logics* (si-logics, for short), consists of all sets $L \subseteq \mathcal{L}$ that contain **Int** and are closed under substitution and modus ponens

MP: if $\varphi \in L$ and $\varphi \to \psi \in L$ then $\psi \in L$.

The smallest si-logic containing a set of $\mathcal{L}$-formulas Γ is denoted by $\mathbf{Int} + \Gamma$. NExt**S4** is the class of normal extensions of **S4** which are sets of $\mathcal{ML}$-formulas containing **S4** and closed under substitution, modus ponens and necessitation $\varphi/\Box\varphi$. $\mathbf{S4} \oplus \Gamma$ is the smallest normal extension of **S4** to contain $\Gamma \subseteq \mathcal{ML}$.

Define three maps $\boldsymbol{\rho} : \mathrm{NExt}\mathbf{S4} \mapsto \mathrm{Ext}\mathbf{Int}$ and $\boldsymbol{\tau}, \boldsymbol{\sigma} : \mathrm{Ext}\mathbf{Int} \mapsto \mathrm{NExt}\mathbf{S4}$ by taking, for any $M \in \mathrm{NExt}\mathbf{S4}$ and $L \in \mathrm{Ext}\mathbf{Int}$,

$$\begin{aligned} \boldsymbol{\rho} M &= \{\varphi \in \mathcal{L} : \mathsf{T}\varphi \in M\}, && (8)\\ \boldsymbol{\tau} L &= \mathbf{S4} \oplus \{\mathsf{T}\varphi : \varphi \in L\}, && (9)\\ \boldsymbol{\sigma} L &= \boldsymbol{\tau} L \oplus \Box(\Box(p \to \Box p) \to p) \to p. && (10) \end{aligned}$$

As was shown in [18, 6, 11] (see also [8]), $\boldsymbol{\rho}$ is a surjective lattice homomorphism, while $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ are lattice isomorphisms "into", with $\boldsymbol{\sigma}$ being an isomorphism of $\langle \mathrm{Ext}\mathbf{Int}, \subseteq\rangle$ onto $\langle \mathrm{NExt}\mathbf{Grz}, \subseteq\rangle$, where $\mathbf{Grz} = \boldsymbol{\sigma}\mathbf{Int}$;

$$\boldsymbol{\rho}^{-1}L = \{M \in \mathrm{NExt}\mathbf{S4} : \boldsymbol{\tau}L \subseteq M \subseteq \boldsymbol{\sigma}L\}.$$

It is relevant here to recall that the translation T was introduced by Orlov [19] and Gödel [12] in order to obtain a classical interpretation of the intuitionistic connectives via the necessity operator $\Box$ of **S4** understood as "it is provable"—a sort of refinement of the Brouwer–Heyting–Kolmogorov proof interpretation. Thus **S4** can be regarded as a logic of informal provability, even in a very precise sense, as has been recently shown by Artemov [3].

An embedding of **Int** into the logic of formal provability (in Peano arithmetic) **GL** was constructed by Boolos [7], Goldblatt [14] and Kuznetsov and Muravitskij [16]. Here we need the map T^+ which first takes the Gödel translation $\mathsf{T}\varphi$ of an $\mathcal{L}$-formula φ and then—to simulate reflexivity in irreflexive frames for **GL**—replaces every $\Box\psi$ in $\mathsf{T}\varphi$ by $\Box^+\psi = \psi \wedge \Box\psi$. T alone is not able to embed **Int** into **GL**; for instance,

$$\mathsf{T}(p \wedge (p \to q) \to q) \notin \mathbf{GL}.$$

What is $\boldsymbol{\rho}\mathbf{GL}$, i.e., what is the logic in the language $\mathcal{L}$ having the formal provability interpretation? Or, in other words, what is the set of $\mathcal{L}$-formulas that are valid in strict orders without infinite ascending chains under the standard intuitionistic valuations? This problem was raised and solved by Visser [25] who described $\boldsymbol{\rho}\mathbf{GL}$ and $\boldsymbol{\rho}\mathbf{K4}$ (**K4** is the modal logic of all transitive frames) in the form of natural deduction systems.

To analyze the behavior of $\mathcal{L}$-formulas on different variations of conventional quasi-orders, say, by giving up the requirement of reflexivity or transitivity, is interesting not only from the technical point of view. For instance, Wansing [26] claims that when $\mathcal{L}$ is used for "talking about the development of information stages, one might want to dispense with the assumption that such states always possibly develop into themselves. There might be information states which in practice simply 'must' be changed, say, in the light of overwhelming and undeniable evidence. In other words, it may make sense not to require reflexivity". Ruitenburg [22], criticizing the BHK interpretation of **Int** for not explaining the logical connectives in simpler terms, proposed to interpret implication in the following way:

- a proof of $\varphi \to \psi$ is a construction that uses the assumption φ to produce a proof of ψ.[1]

"Using assumption φ, rather than a proof of φ, to produce a proof of ψ avoids the need for converting proofs as in the BHK interpretation. It also makes it harder to prove ψ, since less information is provided." Under Ruitenburg's

[1] The standard BHK interpretation looks like this: a proof of $\varphi \to \psi$ is a construction that converts proofs of φ into proofs of ψ.

interpretation, the formula $(\top \to \varphi) \to \varphi$ is not valid, "for a proof of it is a construction p that uses the assumption that there is a construction q that produces a proof of φ to produce a construction that produces a proof of φ. However, the construction q is assumed to exist rather than explicitly given. So we may not be able to deliver p without the assumption $\top \to \varphi$ being 'satisfied', that is, being proven." Actually, Ruitenburg shows that his proof interpretation gives rise not to **Int** but to a weaker logic which is characterized by the class of arbitrary transitive (not necessarily reflexive) frames.

The aim of this paper is to clarify how far the relationship between $\mathcal{L}$ and $\mathcal{ML}$ considered above can be extended on the class of frames $\mathfrak{F} = \langle W, R\rangle$ with arbitrary transitive relations R. More generally, our concern is to find a suitable non-modal propositional language which could talk about transitive frames as fluently as $\mathcal{L}$ can talk about quasi-orders.

It is worth emphasizing here once again that in this paper we consider only *upward closed* valuations of $\mathcal{L}$ in transitive frames. Axiomatizations of the sets of $\mathcal{L}$ formulas valid in various classes of frames without this restriction can be found in [9], [10], [26].

2 From now on by a (*Kripke*) *frame* we mean a pair $\mathfrak{F} = \langle W, R\rangle$ in which R is a transitive binary relation on a set $W \neq \emptyset$. A *model* of the language $\mathcal{L}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$, where $\mathfrak{F}$ is a frame and $\mathfrak{V}$ maps propositional variables into $\mathrm{Up}W$; such valuations will be called *intuitionistic*. The truth-relation $\models$ in $\mathfrak{M}$ is defined by (1)–(4) and (7). $\mathfrak{M} \models \varphi$, φ is *true* in $\mathfrak{M}$, means that $x \models \varphi$ for every $x \in W$ (or $\mathfrak{V}(\varphi) = W$), and $\mathfrak{F} \models \varphi$, φ is *valid* in $\mathfrak{F}$, that φ is true in every model on $\mathfrak{F}$. Needless to say that $\mathfrak{V}(\varphi) \in \mathrm{Up}W$ for every formula φ.

Example 4 Every implication, in particular, $\top \to \bot$ (where $\top$, the constant "truth", is $\bot \to \bot$), is true at every dead end, a final irreflexive point, in any model.

Analogous semantical notions for the modal language $\mathcal{ML}$ are introduced in the standard way (no restrictions on modal valuations; $\models$ is defined by (1)–(6)).

We begin our analysis of the behavior of $\mathcal{L}$-formulas on transitive frames by observing that now they define less functions on cones than $\mathcal{ML}$-formulas.

Proposition 5 $\{\varphi_{\mathfrak{F}} : \varphi \in \mathcal{L}\} \subset \{\varphi_{\mathfrak{F}}^{\triangle} : \varphi \in \mathcal{ML}\}$, *where* $\mathfrak{F}$ *ranges over the class of all transitive frames.*

Proof It is not hard to show by induction on the construction of an $\mathcal{L}$-formula φ that $\varphi_{\mathfrak{F}} = (\mathsf{T}'\varphi)_{\mathfrak{F}}^{\triangle}$, where T' prefixes $\Box$ to every subformula of φ of the form $\psi \to \chi$. On the other hand, for no $\mathcal{L}$-formula φ do we have $\varphi_{\mathfrak{F}} = (\Box^+\neg p)_{\mathfrak{F}}^{\triangle}$ (as usual, $\neg p$ abbreviates $p \to \bot$). Indeed, consider the frame $\mathfrak{F} = \langle\{a, b\}, \emptyset\rangle$. Clearly, $(\Box^+\neg p)_{\mathfrak{F}}^{\triangle}(\{a\}) = \{b\}$. However, for every $\mathcal{L}$-formula $\varphi(p)$, $\varphi_{\mathfrak{F}}(\{a\}) \neq \{b\}$ because $\mathfrak{F}$ validates all implications $\psi \to \chi$. ❑

Also, not all $\mathcal{ML}$-axiomatic classes of transitive frames are $\mathcal{L}$-axiomatic even if they are skeleton-closed. (The definitions of clusters and skeleton-closed classes remain the same as in Section 1; the only difference is that now $\mathfrak{F}^\circ = \langle W^\circ, R^\circ \rangle$ is not necessarily a partial order, for it may contain degenerate clusters, i.e., irreflexive points.)

Proposition 6 *The class $\mathcal{Q}$ of all quasi-orders is $\mathcal{ML}$-axiomatic but not $\mathcal{L}$-axiomatic.*

Proof As is well known, $\mathfrak{F} \in \mathcal{Q}$ iff $\mathfrak{F} \models \Box p \to p$. On the other hand, every $\mathcal{L}$-formula $\varphi \in \mathbf{Int}$ (and even $\varphi \in \mathbf{Cl}$) is valid also in the frame $\langle \{a\}, \emptyset \rangle$, as is easily shown by induction on the construction of φ. So if $\mathcal{Q}$ would be axiomatizable by a set of $\mathcal{L}$-formulas Γ then $\Gamma \subseteq \mathbf{Int}$ and consequently $\langle \{a\}, \emptyset \rangle \in \mathcal{Q}$, which is a contradiction. ❑

Let us consider now the set

$$\mathbf{V} = \{\varphi \in \mathcal{L} : \forall \mathfrak{F}\ \mathfrak{F} \models \varphi\}.$$

According to the completeness theorem of Visser [25], $\mathbf{V}$ coincides with the set of formulas derivable in the basic propositional logic BPL[2] represented by Visser in the form of a natural deduction system. A Gentzen-style system axiomatizing $\mathbf{V}$ can be found in [2]. Quite recently H. Ono (personal communication) has observed that a Hilbert-style representation of $\mathbf{V}$ can be easily extracted from the completeness proof of Corsi [9]. We formulate this observation as

Proposition 7 $\mathbf{V}$ *coincides with the closure under substitution and modus ponens of the following set of 12 axioms:*

- $p \to p, \quad p \wedge q \to p, \quad p \wedge q \to q,$
- $(r \to p) \wedge (r \to q) \to (r \to p \wedge q),$
- $p \to p \vee q, \quad q \to p \vee q,$
- $(p \to r) \wedge (q \to r) \to (p \vee q \to r),$
- $\top, \quad \bot \to p, \quad p \to (q \to p),$
- $p \wedge (q \vee r) \to (p \wedge q) \vee (p \wedge r),$
- $(p \to q) \wedge (q \to r) \to (p \to r).$

Proof We show only that $\mathbf{V}$ is closed under MP. Suppose otherwise, i.e., there are formulas φ and ψ such that $\varphi, \varphi \to \psi \in \mathbf{V}$ but $\psi \notin \mathbf{V}$. This means that there is a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ in which ψ is refuted at some point y. Add to $\mathfrak{F}$ a new root x and denote the resulting frame by $\mathfrak{G}$. Let $\mathfrak{U}$ be the valuation in $\mathfrak{G}$ such that $\mathfrak{U}(p) = \mathfrak{V}(p)$ for every variable p, and $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$. Clearly, $(\mathfrak{N}, y) \not\models \psi$. On the other hand, we have $(\mathfrak{N}, y) \models \varphi$ and so $(\mathfrak{N}, x) \not\models \varphi \to \psi$, contrary to $\varphi \to \psi \in \mathbf{V}$. ❑

[2] Visser gave this name to the logic in view of that $\mathbf{K4}$ is sometimes called the basic modal logic (cf. [24]).

Note that only the axiom

$$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$$

in the standard axiomatization of **Int**, say in [8], does not belong to **V**.

Semantically the consequence relation $\vdash_{\mathbf{Int}}$ in intuitionistic logic can be defined as

$$\Gamma \vdash_{\mathbf{Int}} \varphi \text{ iff } \forall \mathfrak{M} \forall x\ ((\mathfrak{M}, x) \models \Gamma \Rightarrow (\mathfrak{M}, x) \models \varphi),$$

where $\mathfrak{M}$ ranges over intuitionistic models and x over points in $\mathfrak{M}$. As was shown by Visser [25], the relation $\vdash_{\mathbf{V}}$ defined by

$$\Gamma \vdash_{\mathbf{V}} \varphi \text{ iff } \forall \mathfrak{M} \forall x\ ((\mathfrak{M}, x) \models \Gamma \Rightarrow (\mathfrak{M}, x) \models \varphi),$$

where $\mathfrak{M}$ ranges over all transitive models of $\mathcal{L}$, is the consequence relation of his natural deduction system for **V**.

Now, considering $\langle \mathcal{L}, \vdash_{\mathbf{V}} \rangle$ as a deductive system, we see that modus ponens is not a derivable rule in it: $p, p \to q \vdash_{\mathbf{V}} q$ is refuted by the model based on a single irreflexive point at which only p is true. Moreover, the deduction theorem does not hold in $\langle \mathcal{L}, \vdash_{\mathbf{V}} \rangle$ either.

Proposition 8 *There exists no formula $\chi(p, q)$ such that, for all Γ, φ, ψ,*

$$\Gamma, \psi \vdash_{\mathbf{V}} \varphi \textit{ iff } \Gamma \vdash_{\mathbf{V}} \chi(\psi, \varphi).$$

Proof Suppose on the contrary that such a formula $\chi(p, q)$ exists. Then we have

$$\top \to \bot, \psi \vdash_{\mathbf{V}} \varphi \text{ iff } \top \to \bot \vdash_{\mathbf{V}} \chi(\psi, \varphi). \tag{11}$$

It should be clear that $\top \to \bot, p \not\vdash_{\mathbf{V}} q$. On the other hand, if $\top \to \bot$ holds in a model $\mathfrak{M}$ then $\mathfrak{M}$ is based on the disjoint union of irreflexive points, and so every implication $\alpha \to \beta$ holds in it. It follows that all formulas are monotone in $\mathfrak{M}$. Consequently,

$$\top \to \bot, \chi(p \wedge q, q) \vdash_{\mathbf{V}} \chi(p, q).$$

By (11) and $\top \to \bot, p \wedge q \vdash_{\mathbf{V}} q$, we must have $\top \to \bot \vdash_{\mathbf{V}} \chi(p \wedge q, q)$, from which $\top \to \bot \vdash_{\mathbf{V}} \chi(p, q)$ and so, again by (11), $\top \to \bot, p \vdash_{\mathbf{V}} q$, which is a contradiction. ❑

Unfortunately, as far as we know, no finite Hilbert-style axiomatization of $\vdash_{\mathbf{V}}$ has been found yet.

3 The Kripke semantics we considered in the previous section is not enough for dealing with extensions of **V**. An algebraic semantics for **V** was introduced by Ardeshir and Ruitenburg [2]. The aim of this section is to define a notion of a general frame for **V** and develop to some extent duality theory for the algebraic and relational semantics.

We can get an impression how algebras for **V** may look like by representing transitive frames $\mathfrak{F} = \langle W, R \rangle$ as the algebras of cones $\mathfrak{F}^+ = \langle \mathrm{Up}W, \cap, \cup, \to, \emptyset, W \rangle$ in which

$$X \to Y = \{x \in W : \forall y \ (xRy \wedge y \in X \Rightarrow y \in Y)\} \tag{12}$$

(the logical connectives $\wedge$, $\vee$, $\to$, $\bot$, $\top$ are interpreted in $\mathfrak{F}^+$ by the operations $\cap$, $\cup$, $\to$, $\emptyset$, W, respectively). Every such algebra is clearly a bounded (i.e., with top and bottom) distributive lattice satisfying the following equations ($a \leq b$ means $a \wedge b = a$):

$$a \to b \wedge c = (a \to b) \wedge (a \to c);$$

$$b \vee c \to a = (b \to a) \wedge (c \to a);$$

$$a \to a = \top \text{ and } a \leq \top \to a;$$

$$(a \to b) \wedge (b \to c) \leq a \to c.$$

Let us take these properties as a definition and call a bounded distributive lattice $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, \top \rangle$ satisfying the equations above a **V**-*algebra.* Our goal now is to show that all **V**-algebras are induced by frames, are subalgebras of the corresponding algebras of cones, to be more exact. To this end we require the following lemma on the existence of prime filters in **V**-algebras.

Lemma 9 *Suppose $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, \top \rangle$ is a* **V**-*algebra, ∇ a prime filter in $\mathfrak{A}$ and let C and D be subsets of A such that*

$$\forall c_1, \ldots, c_m \in C \ \forall d_1, \ldots, d_n \in D \ c_1 \wedge \ldots \wedge c_m \to d_1 \vee \ldots \vee d_n \notin \nabla. \tag{13}$$

Then there exists a prime filter ∇' in $\mathfrak{A}$ such that $C \subseteq \nabla'$, $\nabla' \cap D = \emptyset$ and $\nabla R \nabla'$, where

$$\nabla R \nabla' \text{ iff } \forall a, b \in A \ (a \to b \in \nabla \wedge a \in \nabla' \Rightarrow b \in \nabla'). \tag{14}$$

Proof By Zorn's lemma, there is a maximal set $\nabla' \supseteq C$ satisfying (13). We show that ∇' is the prime filter we need. First, it is easily checked that ∇' is a filter such that $b \in \nabla'$ whenever $a \in \nabla'$ and $a \to b \in \nabla$. Since $a \to a = \top \in \nabla$ for every $a \in A$, we have $\nabla' \cap D = \emptyset$. So it remains to show that ∇' is prime, i.e., $b_1 \vee b_2 \in \nabla'$ implies $b_1 \in \nabla'$ or $b_2 \in \nabla'$. Suppose $b_1, b_2 \notin \nabla'$. Then there are $a \in \nabla'$ and $d_1, \ldots, d_n \in D$ such that, for $i = 1, 2$,

$$a \wedge b_i \to d_1 \vee \ldots \vee d_n \in \nabla.$$

Hence

$$(a \wedge b_1) \vee (a \wedge b_2) \to d_1 \vee \ldots \vee d_n \in \nabla$$

and so, by distributivity,

$$a \wedge (b_1 \vee b_2) \to d_1 \vee \ldots \vee d_n \in \nabla,$$

which means that $b_1 \vee b_2 \notin \nabla'$. ❑

Theorem 10 *All subalgebras of algebras of the form $\mathfrak{F}^+$, $\mathfrak{F}$ a transitive frame, comprise (up to isomorphism) the variety (equational class) of* **V***-algebras.*

Proof Each subalgebra of $\mathfrak{F}^+$ is clearly a **V**-algebra. Conversely, with each **V**-algebra $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot, \top \rangle$ we can associate the frame $\mathfrak{A}_\dagger = \langle W, R \rangle$, where W is the set of prime filters in $\mathfrak{A}$ and R is defined by (14). It is easily seen that R is transitive, and using Lemma 9 one can prove that the map $s : \mathfrak{A} \mapsto (\mathfrak{A}_\dagger)^+$ defined by $s(a) = \{\nabla \in W : a \in \nabla\}$ is an embedding of $\mathfrak{A}$ into $(\mathfrak{A}_\dagger)^+$. Here we show only that $s(a) \rightarrow s(b) \subseteq s(a \rightarrow b)$. Suppose $\nabla \notin s(a \rightarrow b)$. Then $a \rightarrow b \notin \nabla$. Put $C = \{a\}$ and $D = \{b\}$. By Lemma 9, we have a prime filter ∇' in $\mathfrak{A}$ such that $\nabla R \nabla'$, $a \in \nabla'$ and $b \notin \nabla'$, from which $\nabla \notin s(a) \rightarrow s(b)$.

Thus, $\mathfrak{A}$ is isomorphic to a subalgebra of $\langle W, R \rangle^+$. ❑

Following the standard model-theoretic terminology of modal logic, we call a *general* **V***-frame* any structure $\mathfrak{F} = \langle W, R, P \rangle$ in which $\langle W, R \rangle$ is a transitive Kripke frame and P a set of R-cones containing $\emptyset$ and closed under $\cap$, $\cup$, and the operation $\rightarrow$ defined by (12). If $P = \mathrm{Up}W$ then, as before, we call $\mathfrak{F}$ a Kripke frame and may not mention P explicitly. The *dual* of $\mathfrak{F}$, denoted by $\mathfrak{F}^+$, is the subalgebra of $\langle W, R \rangle^+$ with domain P.

The proof of Theorem 10 shows that every **V**-algebra $\mathfrak{A}$ is isomorphic to its *bidual* $(\mathfrak{A}_+)^+$, where $\mathfrak{A}_+ = \langle W, R, P \rangle$, $\langle W, R \rangle = \mathfrak{A}_\dagger$ and $P = \{s(a) : a \in A\}$. On the other hand, it is easy to construct a general **V**-frame $\mathfrak{F}$ which is not isomorphic to its bidual $(\mathfrak{F}^+)_+$. The following theorem gives an intrinsic characterization of those frames that are isomorphic to their biduals.

Theorem 11 *A general* **V***-frame $\mathfrak{F} = \langle W, R, P \rangle$ is isomorphic to $(\mathfrak{F}^+)_+$ iff $\mathfrak{F}$ is descriptive in the sense that*

- $x = y$ *iff* $\forall X \in P \ (x \in X \Leftrightarrow y \in X)$*;*
- xRy *iff* $\forall X, Y \in P \ (x \in X \rightarrow Y \wedge y \in X \Rightarrow y \in Y)$*;*
- $\langle W, P \rangle$ *is compact, i.e., for all $\mathcal{X} \subseteq P$ and $\mathcal{Y} \subseteq \{W - X : X \in P\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.*

Proof Similar to the proof of Theorem 8.51 in [8]. ❑

Example 12 Two examples of descriptive **V**-frames are shown in Fig. 1. The frame $\mathfrak{F} = \langle W, R, P \rangle$ on the left consists of two irreflexive points (represented by $\bullet$) which do not see each other; all the cones in the set P of possible values, save W and $\emptyset$, are indicated explicitly by curve lines. Arrows in the second frame $\mathfrak{G}$ define its accessibility relation. It may be of interest to notice that although these frames are finite, they are not Kripke frames (i.e., their sets of possible values do not contain all cones), which contrasts with the standard case of frames for modal and intuitionistic logics.

Given a general **V**-frame $\mathfrak{F} = \langle W, R, P \rangle$ and a cone V in it, the structure $\mathfrak{G} = \langle V, R \cap (V \times V), \{X \cap V : X \in P\} \rangle$ turns out to be a **V**-frame as well; it is called a *generated subframe* of $\mathfrak{F}$.

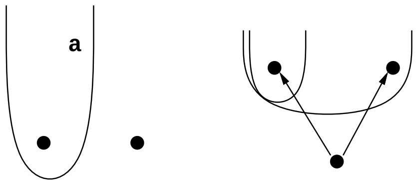

Figure 1:

A map f from W onto V is said to be a *reduction* (or *p-morphism*) of a frame $\mathfrak{F} = \langle W, R, P\rangle$ to a frame $\mathfrak{G} = \langle V, S, Q\rangle$ if, for all $x, y \in W$ and $X \in Q$,

$xRy \Rightarrow f(x)Sf(y)$,
$f(x)Sf(y) \Rightarrow \exists z \in f^{-1}(y)\ xRz$,
$f^{-1}(X) \in P$.

The following duality theorem is proved in the standard way using Lemma 9 (cf. [8]).

Theorem 13 (i) *If $\mathfrak{G} = \langle V, S, Q\rangle$ is a generated subframe of $\mathfrak{F} = \langle W, R, P\rangle$ then the map f defined by $f(X) = X \cap V$, for $X \in P$, is a homomorphism from $\mathfrak{F}^+$ onto $\mathfrak{G}^+$.*

(ii) *If f is a homomorphism from a $\mathbf{V}$-algebra $\mathfrak{A}$ onto a $\mathbf{V}$-algebra $\mathfrak{B}$ then f_+ defined by $f_+(\nabla) = f^{-1}(\nabla)$, for ∇ a prime filter in $\mathfrak{B}$, is an isomorphism from $\mathfrak{B}_+$ onto a generated subframe of $\mathfrak{A}_+$.*

(iii) *If h is a reduction of $\mathfrak{F} = \langle W, R, P\rangle$ to $\mathfrak{G} = \langle V, S, Q\rangle$ then h^+ defined by $h^+(X) = h^{-1}(X)$, for $X \in Q$, is an embedding of $\mathfrak{G}^+$ into $\mathfrak{F}^+$.*

(iv) *If $\mathfrak{B}$ is a subalgebra of a $\mathbf{V}$-algebra $\mathfrak{A}$ then the map h defined by $h(\nabla) = \nabla \cap B$, ∇ a prime filter in $\mathfrak{A}$ and B the universe of $\mathfrak{B}$, is a reduction of $\mathfrak{A}_+$ to $\mathfrak{B}_+$.*

Proof We show only (iv); the other items are proved analogously. Suppose $\mathfrak{A}_+ = \langle W, R, P\rangle$ and $\mathfrak{B}_+ = \langle V, S, Q\rangle$. It is straightforward to check that h is well-defined, surjective, and that $h^{-1}(X) \in P$, for every $X \in Q$. Let $\nabla_1 R \nabla_2$. Then for all $a, b \in B$ with $a \to b \in \nabla_1$ and $a \in \nabla_2$, we have $b \in \nabla_2$. Hence $h(\nabla_1)Sh(\nabla_2)$. Suppose now that $h(\nabla_1)S\nabla_2$, where $\nabla_1 \in W$ and $\nabla_2 \in V$. Then condition (13) of Lemma 9 holds for $\nabla = \nabla_1$, $C = \nabla_2$ and $D = B - \nabla_2$. So there exists $\nabla_2' \in W$ such that $\nabla_1 R \nabla_2'$, $C \subseteq \nabla_2'$ and $\nabla_2' \cap (B - \nabla_2) = \emptyset$. But this means that $h(\nabla_2') = \nabla_2$. ❑

Although $\mathbf{V}$ and $\vdash_{\mathbf{V}}$ are characterized by the variety of $\mathbf{V}$-algebras, the connection between algebraic properties of this variety and the consequence relation $\vdash_{\mathbf{V}}$ is not as close as it is between, say, intuitionistic logic and Heyting algebras or normal modal logics and modal algebras. For instance, almost all non-pathological propositional logics are protoalgebraic in the sense of Blok and Pigozzi [4]. However, as we show below, this is not the case for $\vdash_{\mathbf{V}}$.

Roughly speaking, a consequence relation $\vdash$ is protoalgebraic if there is a close connection between designated elements and congruences in matrices for $\vdash$. A syntactic definition looks like this. $\vdash$ is called protoalgebraic iff there exists a set of formulas $\Delta = \{\varphi(p,q) : \varphi(p,q) \in \Delta\}$ in two variables p and q such that

- $\vdash \varphi(p,p)$, for all $\varphi(p,q) \in \Delta$,
- $p, \{\varphi(p,q) : \varphi(p,q) \in \Delta\} \vdash q$.

Theorem 14 $\vdash_{\mathbf{V}}$ *is not protoalgebraic.*

Proof We use the following algebraic characterization of protoalgebraic consequence relations. Consider a matrix $\mathsf{M} = (\mathfrak{A}, D)$, i.e., an algebra $\mathfrak{A}$ together with a subset D of the domain A of $\mathfrak{A}$. M is a *matrix for* a consequence relation $\vdash$ if $\mathfrak{V}(\Gamma) \subseteq D$ implies $\mathfrak{V}(\varphi) \in D$ whenever $\Gamma \vdash \varphi$ and $\mathfrak{V}$ is a valuation in $\mathfrak{A}$. By ΩD we denote the largest congruence relation in $\mathfrak{A}$ which respects D, i.e., such that $(a,b) \in \Omega D$ implies $a \in D$ iff $b \in D$. Blok and Pigozzi [4] showed that a consequence relation $\vdash$ is protoalgebraic iff $D_1 \subseteq D_2$ implies $\Omega D_1 \subseteq \Omega D_2$ whenever $(\mathfrak{A}, D_1)$ and $(\mathfrak{A}, D_2)$ are matrices for $\vdash$. Consider now the matrices $(\mathfrak{F}^+, \{\top\})$ and $(\mathfrak{F}^+, \{\top, a\})$, where $\mathfrak{F}$ is the frame defined in Example 12. Clearly, both of them are matrices for $\vdash_{\mathbf{V}}$. It is easily verified that $\Omega\{\top\}$ identifies only a and $\bot$ and $\Omega\{\top, a\}$ only $\top$ and a. Hence $\Omega\{\top\} \not\subseteq \Omega\{\top, a\}$, and so $\vdash_{\mathbf{V}}$ is not protoalgebraic. ❑

4 Let us now have a quick look at extensions of $\mathbf{V}$. The first problem we encounter with is what kind of extensions are worth considering. Of course, as in the case of Ext**Int**, we can define a *formula-extension* of $\mathbf{V}$ as a set of formulas L that contains $\mathbf{V}$ and is closed under substitutions and $\vdash_{\mathbf{V}}$ (in the sense that $\psi \in L$ whenever $\Gamma \subseteq L$ and $\Gamma \vdash_{\mathbf{V}} \psi$). **Int** as well as classical logic are certainly formula-extensions of $\mathbf{V}$. However, as we observed above, the class $\mathcal{Q}$ of quasi-orders is not $\mathcal{L}$-axiomatic. In other words, there is no formula-extension of $\mathbf{V}$ whose frames are precisely all the quasi-orders. Many other natural classes of frames, e.g. frames with the diagonal accessibility relations, are not definable by means of formula-extensions.

A possible solution to this problem is to consider extensions not of the logic $\mathbf{V}$ but of the consequence relation $\vdash_{\mathbf{V}}$. The most general class of such extensions consists of arbitrary finitary (i.e., if $\Gamma \vdash \varphi$ then $\Delta \vdash \varphi$ for some finite $\Delta \subseteq \Gamma$) structural (i.e., closed under substitution) consequence relations containing $\vdash_{\mathbf{V}}$. Each of them can be looked at as the result of adding to $\vdash_{\mathbf{V}}$ a set Ξ of inference rules. Let $\vdash_{\mathbf{V}} + \Xi$ denote the smallest finitary structural consequence relation containing $\vdash_{\mathbf{V}}$ and respecting the rules in Ξ.

Example 15 Here are three consequence relations considered in [25].

$$\vdash_{\mathbf{FPC}} = \vdash_{\mathbf{V}} + \frac{(\top \to p) \to p}{\top \to p}$$

is the consequence relation whose set of tautologies coincides with $\rho\mathbf{GL}$ (**FPC** stands for "formal propositional calculus").

$$\vdash_{\mathbf{Int}} = \vdash_{\mathbf{V}} + \frac{p, p \to q}{q}.$$

Formulas, as we know, are not enough to axiomatize $\vdash_{\mathbf{Int}}$ over $\vdash_{\mathbf{V}}$, for they are not able to separate the irreflexive point from the class of quasi-orders. However, it is not hard to see that

$$\vdash_{\mathbf{FPC}} = \vdash_{\mathbf{V}} + ((\top \to p) \to p) \to (\top \to p).$$

The consequence relation

$$\vdash_{\mathbf{VL}} = \vdash_{\mathbf{V}} + (p \to q) \vee ((p \to q) \to p).$$

is determined by the class of linear frames.

The semantic equivalents of the consequence relations introduced above are the finitary consequences $\models^*_{\mathcal{F}}$ determined by classes $\mathcal{F}$ of general $\mathbf{V}$-frames in the following way: $\Gamma \models^*_{\mathcal{F}} \varphi$ iff for any model $\mathfrak{M}$ based on a frame in $\mathcal{F}$,

$$\mathfrak{M} \models \Gamma \Rightarrow \mathfrak{M} \models \varphi.$$

Note, however, that in general the frame classes definable by such consequence relations are not closed under the formation of generated subframes. To see this, one can consider the consequence relation determined by the canonical frame for **Int**. It is generated by the set of admissible rules in **Int** whose class of frames is not closed under generated subframes simply because there are admissible but not derivable rules in **Int** (for details see [23]). Although such consequences are of definite interest from both logical and algebraic points of view,[3] in this paper we confine ourselves to considering only those of them that are definable by classes of frames closed under the formation of generated subframes. They can be obtained by "localizing" the definition above.

Namely, we say that a consequence relation $\vdash$ is a $\mathbf{V}$-*consequence* if it is finitary and characterized by a class $\mathcal{F}$ of general $\mathbf{V}$-frames in the sense that $\vdash$ coincides with the relation $\models_{\mathcal{F}}$ such that $\Gamma \models_{\mathcal{F}} \varphi$ iff for any model $\mathfrak{M}$ based on a frame in $\mathcal{F}$ and any point x in $\mathfrak{M}$,

$$(\mathfrak{M}, x) \models \Gamma \Rightarrow (\mathfrak{M}, x) \models \varphi.$$

The class $\{\mathfrak{F} : \vdash \subseteq \models_{\mathfrak{F}}\}$ of frames for $\vdash$ will be denoted by $\mathrm{Fr} \vdash$.

The corresponding notions for $\mathbf{V}$-algebras can be defined as follows. For a class $\mathcal{A}$ of $\mathbf{V}$-algebras we write $\Gamma \models_{\mathcal{A}} \varphi$ iff there exists a finite subset Γ' of Γ such that the equation $\bigwedge \Gamma' \leq \varphi$ is valid in all members of $\mathcal{A}$. The class $\{\mathfrak{A} : \vdash \subseteq \models_{\mathfrak{A}}\}$ of algebras for $\vdash$ is denoted by $\mathrm{Alg} \vdash$.

As follows from [25], all the consequence relations considered in Example 15 are $\mathbf{V}$-consequences (the "finitary part" of $\vdash_{\mathbf{FPC}}$ is characterized by the class of all irreflexive Kripke frames without infinite ascending chains).

[3] Their algebraic equivalents are quasi-varieties of $\mathbf{V}$-algebras.

Theorem 16 (i) *A class of* $\mathbf{V}$*-algebras is of the form* $\mathrm{Alg} \vdash$ *for a* $\mathbf{V}$*-consequence* $\vdash$ *iff it is a subvariety of the variety of all* $\mathbf{V}$*-algebras.*

(ii) *A class of general* $\mathbf{V}$*-frames is of the form* $\mathrm{Fr} \vdash$ *for a* $\mathbf{V}$*-consequence* $\vdash$ *iff it is closed under generated subframes, reductions, disjoint unions and it as well as its complement are closed under the formation of biduals.*

Proof (i) Let $\mathcal{A} = \mathrm{Alg} \vdash$, for a $\mathbf{V}$-consequence $\vdash$. Then $\mathcal{A}$ is the class of $\mathbf{V}$-algebras defined by the equations

$$\{\bigwedge \Gamma \leq \varphi : \Gamma \vdash \varphi,\ \Gamma \text{ is finite}\}$$

and so $\mathcal{A}$ is a variety. Conversely, given a variety $\mathcal{A}$ contained in the variety of $\mathbf{V}$-algebras, one can easily check that $\mathrm{Alg} \models_{\mathcal{A}}$ coincides with $\mathcal{A}$.

(ii) The closure conditions for classes of the form $\mathrm{Fr} \vdash$ are clear. Conversely, assume that $\mathcal{F}$ is a class of general $\mathbf{V}$-frames closed under generated subframes, reductions, disjoint unions and it as well as its complement are closed under the formation of biduals. First we show that $\models_{\mathcal{A}}$ is finitary. To this end suppose that $\Gamma' \not\models_{\mathcal{A}} \varphi$, for every finite subset Γ' of a set of formulas Γ. Take for each such Γ' a frame $\mathfrak{F} \in \mathcal{F}$ refuting $\Gamma' \vdash \varphi$ and form the disjoint union $\mathfrak{G}$ of all those $\mathfrak{F}$. Then in view of the compactness of the descriptive frame $(\mathfrak{G}^+)_+ \in \mathcal{F}$, it must refute $\Gamma \vdash \varphi$. It follows that $\models_{\mathcal{F}}$ is a $\mathbf{V}$-consequence. So it remains to show that $\mathcal{F} = \mathrm{Fr} \models_{\mathcal{F}}$. But this is obtained from (i) by using the results on duality between general $\mathbf{V}$-frames and $\mathbf{V}$-algebras (see [13] for a similar argument). ❑

The consequence relations in Example 15 are *complete* in the sense that, for any *finite* set of formulas Γ and formula φ, if $\Gamma \not\vdash \varphi$ then there exists a Kripke frame $\mathfrak{F} \in \mathrm{Fr} \vdash$ such that $\Gamma \not\models_{\mathfrak{F}} \varphi$. Moreover, $\vdash_{\mathbf{Int}}$ and $\vdash_{\mathbf{VL}}$ are even strongly complete (i.e., Γ in the previous definition may be infinite), but this is not so for $\vdash_{\mathbf{FPC}}$ (the proof is similar to the proof that $\mathbf{GL}$ is not strongly complete). In contrast with superintuitionistic logics it is almost trivial to construct incomplete $\mathbf{V}$-consequences.

Proposition 17 (i) *The consequence relation* $\models_{\mathfrak{G}}$*, where* $\mathfrak{G}$ *is the frame defined in Example 12, is not complete.*

(ii) $\vdash_{\mathbf{V}} + (p \to q) \vee (q \to p)$ *is not complete.*

Proof Let

$$\varphi_1 = (p \to q) \vee ((p \to q) \to p),\ \varphi_2 = (p \to q) \vee (q \to p).$$

One can easily show that a Kripke frame validates φ_1 iff it validates φ_2 iff it is linear. However, $\mathfrak{G}$ refutes φ_1 but validates φ_2. The claims of the proposition follow immediately. ❑

The class of all $\mathbf{V}$-consequences ordered by inclusion forms a complete lattice; we denote it by $\mathrm{Ext} \vdash_{\mathbf{V}}$. Having the isomorphism $\boldsymbol{\sigma}$ between $\mathrm{Ext}\mathbf{Int}$ and

NExt**Grz**, it is natural to conjecture that there exists an isomorphism also between Ext $\vdash_{\mathbf{V}}$ and NExt$L$, for some $L \in$ NExt**K4**. A natural candidate for L would be the logic $\mathbf{Grz}'$ determined by all transitive frames without proper clusters and infinite strictly ascending chains. However, this is not the case. Here we only illustrate the proof by the following weaker statement.

Theorem 18 *The lattice of* $\mathbf{V}$*-consequences containing* $\vdash_{\mathbf{FPC}}$ *is not isomorphic to the lattice* NExt**GL**.

Proof The *codimension* of an element c in a lattice $\mathfrak{D}$ is the length of a longest chain from c to the top element of $\mathfrak{D}$. For each frame $\mathfrak{F}$ in Fig. 2, construct the $\mathbf{V}$-consequence characterized by $\mathfrak{F}$. One can check (by a straightforward but rather tedious proof following, for example, [21]) that the consequence relations associated with the frames in the first, second and third rows in Fig. 2 comprise all the $\bigcap$-irreducible consequences in Ext $\vdash_{\mathbf{FPC}}$ of codimensions 2, 3, and 4, respectively. However, one can show that NExt**GL** contains only 5 $\bigcap$-irreducible logics of codimension 4. Thus, the lattices under consideration are not isomorphic. ❑

5 From the semantical point of view, all the "peculiarities" of the language $\mathcal{L}$ interpreted on transitive frames as well as of the logic $\mathbf{V}$ and its extensions we observed in the three previous sections are explained by the fact that being in an irreflexive world x, we can talk about x using only $\wedge$ and $\vee$; $\to$ is for talking about successors of x. A way of improving the expressive power of $\mathcal{L}$ must be clear: we just can add to it one more implication—let us denote it by $\hookrightarrow$—whose intended meaning in a transitive frame $\langle W, R\rangle$ is the same as the meaning of the standard intuitionistic implication in the quasi-order $\langle W, R^r\rangle$, where R^r is the reflexive closure of R:

$$x \models \varphi \hookrightarrow \psi \text{ iff } \forall y \in W\ ((x = y \vee xRy) \wedge y \models \varphi \Rightarrow y \models \psi). \qquad (15)$$

The resulting "biarrow" language is denoted by $\mathcal{L}_2$.

Notice now that the original implication $\to$ can be defined via $\hookrightarrow$ and the standard necessity operator $\Box$ on transitive frames:

$$x \models \varphi \to \psi \text{ iff } x \models \Box(\varphi \hookrightarrow \psi).$$

And conversely, $\Box$ is definable via $\to$ and $\top$:

$$x \models \Box\varphi \text{ iff } x \models \top \to \varphi.$$

So instead of the biarrow language $\mathcal{L}_2$ we may consider the modal language $\mathcal{ML}_{\hookrightarrow}$ which results from $\mathcal{ML}$ by replacing $\to$ with $\hookrightarrow$. $\mathcal{ML}_{\hookrightarrow}$ is interpreted in transitive frames $\mathfrak{F} = \langle W, R\rangle$ by means of valuations $\mathfrak{V}$ of propositional variables in UpW and the truth-relation defined by (1)–(4), (15) and (6).

Let $\mathbf{U}$ be the set of $\mathcal{ML}_{\hookrightarrow}$-formulas that are valid in all transitive frames and let

$$\Gamma \vdash_{\mathbf{U}} \varphi \text{ iff } \forall\mathfrak{M}\forall x\ ((\mathfrak{M}, x) \models \Gamma \Rightarrow (\mathfrak{M}, x) \models \varphi).$$

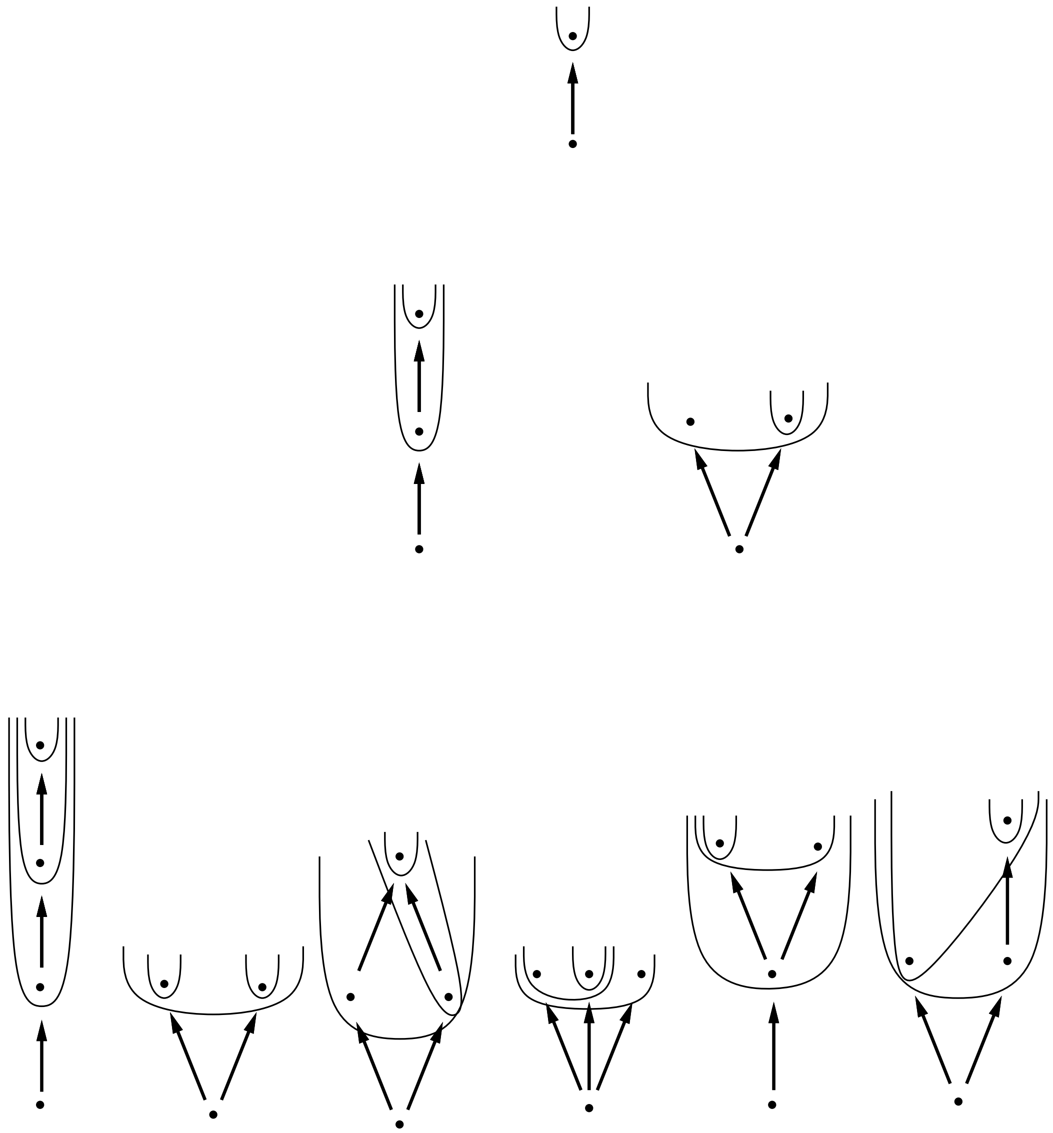

Figure 2:

It should be clear that the deduction theorem holds for $\vdash_{\mathbf{U}}$ and $\hookrightarrow$:

$$\Gamma, \varphi \vdash_{\mathbf{U}} \psi \text{ iff } \Gamma \vdash_{\mathbf{U}} \varphi \hookrightarrow \psi.$$

One can easily check also that $\vdash_{\mathbf{U}}$ is algebraizable in the sense of [5] and so it is protoalgebraic.

U can be considered as a normal modal logic on the intuitionistic basis. This observation and completeness results of [28] provide a Hilbert-style axiomatization for **U** and $\vdash_{\mathbf{U}}$.

Theorem 19 *The calculus U in the language $\mathcal{ML}_{\hookrightarrow}$ with modus ponens and substitution as its inference rules and the axioms*

1. *those of* **Int**,
2. $\Box(p \hookrightarrow q) \hookrightarrow (\Box p \hookrightarrow \Box q)$, $\Box p \hookrightarrow \Box\Box p$, $p \hookrightarrow \Box p$,
3. $\Box p \hookrightarrow (q \vee (q \hookrightarrow p))$

is strongly complete with respect to the class of transitive frames, i.e.,

$$\Gamma \vdash_U \varphi \textit{ iff } \Gamma \vdash_{\mathbf{U}} \varphi.$$

Proof According to [28], any normal intuitionistic modal logic in the language $\mathcal{ML}_{\hookrightarrow}$, in particular that axiomatized by U, is characterized by a class of descriptive IM-frames $\mathfrak{F} = \langle W, R_{\hookrightarrow}, R, P\rangle$ in which $\langle W, R_{\hookrightarrow}, P\rangle$ is a descriptive (quasi-ordered) frame for **Int** (i.e., $\hookrightarrow$ is interpreted via $R_{\hookrightarrow}$), P is closed under the standard $\Box$ interpreted via R, $\mathfrak{F}$ is tight with respect to R, i.e.,

$$xRy \text{ iff } \forall X \in P\ (x \in \Box X \Rightarrow y \in X),$$

and $R_{\hookrightarrow} \circ R \circ R_{\hookrightarrow} = R$. The logic $\mathbf{U}'$, axiomatizable by 1–2 in the formulation of our theorem, is characterized by the class of all descriptive IM-frames such that R is transitive and $R \subseteq R_{\hookrightarrow}$, and moreover, it is d-persistent in the sense that for every descriptive frame $\langle W, R_{\hookrightarrow}, R, P\rangle$ validating $\mathbf{U}'$, its underlying Kripke frame $\langle W, R_{\hookrightarrow}, R\rangle$ also validates $\mathbf{U}'$.

Let us observe now that if a descriptive IM-frame $\langle W, R_{\hookrightarrow}, R, P\rangle$ for $\mathbf{U}'$ validates $\Box p \hookrightarrow (q \vee (q \hookrightarrow p))$ then

$$R^r = R_{\hookrightarrow}. \tag{16}$$

Conversely, every (not necessarily descriptive) IM-frame for $\mathbf{U}'$ satisfying (16) validates this axiom. It follows that U is d-persistent and so canonical and strongly complete for IM-frames with (16) or, in other terms, for the class of transitive Kripke frames we are dealing with. ❑

Remark. Not every general frame for **V** can be regarded as an IM-frame because it is not necessarily closed under $\hookrightarrow$; examples of that sort are presented in Fig. 1. IM-frames for **U** defined in the proof above will be called **U**-*frames* (it will be always clear from the context whether the **U**-frames under consideration are general or Kripke frames). Since $R_{\hookrightarrow}$ is uniquely determined by R, we may omit $R_{\hookrightarrow}$ and denote these frames by $\mathfrak{F} = \langle W, R, P\rangle$.

Let us consider now the class NExt**U** of normal extensions of **U**, that is sets of $\mathcal{ML}_{\hookrightarrow}$-formulas containing **U** and closed under modus ponens and substitution (the closure under necessitation is ensured by the axiom $p \hookrightarrow \Box p$). The following completeness result is an immediate consequence of the considerations above.

Proposition 20 *Every logic in* NExt**U** *is characterized by a class of (descriptive)* **U***-frames. Conversely, every class of general* **U***-frames determines a logic in* NExt**U**.

Here is a couple of examples of logics in NExt**U**. Kuznetsov [15] (see also [17]) introduced an intuitionistic analog of classical provability logic **GL**:

$$\mathbf{I}^{\triangle} = \mathbf{U} \oplus (\Box p \hookrightarrow p) \hookrightarrow p.$$

The logic $\mathbf{I}^{\triangle}$ is characterized by the class of irreflexive Kripke frames without infinite ascending chains, i.e., by the same frames as **GL** but with a different interpretation of the implication.

Another example is due to Yashin [29] who showed that the logic

$$\mathbf{Y} = \mathbf{I}^{\triangle} \oplus \Box\Box p \hookrightarrow (p \hookrightarrow q) \vee (q \hookrightarrow p)$$

is Novikov-complete in the sense that $\Box$ is not definable in **Int** (it is "a new intuitionistic connective") and **Y** is a maximal logic in NExt**U** which is conservative over **Int**.

Kuznetsov and Muravitskij [17] proved that there is a lattice isomorphism between NExt$\mathbf{I}^{\triangle}$ and NExt**GL**. Moreover, it turns out that in general the relationship between the lattices NExt**U** and NExt**K4** is similar to that between Ext**Int** and NExt**S4** discussed in Section 1. To show this, we take advantage of the results on embeddings of intuitionistic modal logics into classical polymodal logics obtained in [27, 28].

Let $\mathcal{ML}_2$ be the language with two necessity operators $\Box_I$ and $\Box$ (and the implication $\to$), and let T'' be the translation from $\mathcal{ML}_{\hookrightarrow}$ into $\mathcal{ML}_2$ prefixing $\Box_I$ to all subformulas and replacing $\hookrightarrow$ with $\to$. Given logics L_1 and L_2 in the unimodal languages $\mathcal{ML}_2 - \Box$ and $\mathcal{ML}_2 - \Box_I$, respectively, denote by $L_1 \otimes L_2$ their *fusion*, the smallest bimodal logic in $\mathcal{ML}_2$ to contain $L_1 \cup L_2$. By **IntK** we mean the minimal normal intuitionistic modal logic in the language $\mathcal{ML}_{\hookrightarrow}$ (i.e., the smallest set of formulas containing **Int**, the modal axiom of **K** and closed under modus ponens, substitution and necessitation). As is shown in [27],

(i) the map

$$\boldsymbol{\rho} M = \{\varphi \in \mathcal{ML}_{\hookrightarrow} : \mathsf{T}''(\varphi) \in M\},$$

is a lattice homomorphism from NExt($\mathbf{S4} \otimes \mathbf{K}$) onto NExt**IntK** (preserving the finite model property and decidability);

(ii) each logic $\mathbf{IntK} \oplus \Gamma$ is embedded by T'' into any logic M in the interval

$$(\mathbf{S4} \otimes \mathbf{K}) \oplus \mathsf{T}''(\Gamma) \subseteq M \subseteq (\mathbf{Grz} \otimes \mathbf{K}) \oplus \boldsymbol{mix} \oplus \mathsf{T}''(\Gamma),$$

where $\boldsymbol{mix} = \Box_I \Box \Box_I p \leftrightarrow \Box p$, and

(iii) the map

$$\boldsymbol{\sigma}(\mathbf{IntK} \oplus \Gamma) = (\mathbf{Grz} \otimes \mathbf{K}) \oplus \boldsymbol{mix} \oplus \mathsf{T}''(\Gamma)$$

is a lattice isomorphism from NExt$\mathbf{IntK}$ onto NExt$(\mathbf{Grz} \otimes \mathbf{K}) \oplus \boldsymbol{mix}$. (As before, the operation $\oplus$ means "take the union and close it under the postulated inference rules".)

If we consider now $\mathbf{K4}$ as a bimodal logic in $\mathcal{ML}_2$ by defining $\Box_I \varphi = \varphi \wedge \Box \varphi$, then we may assume $\mathbf{K4}$ to be in the class NExt$(\mathbf{S4} \otimes \mathbf{K4})$. Since this "bimodal" $\mathbf{K4}$ is characterized by the class of frames of the form $\langle W, R^r, R \rangle$ and in view of Proposition 21 in [28], $\boldsymbol{\rho}\mathbf{K4} = \mathbf{U}$. Therefore, $\mathbf{U}$ has the finite model property and $\boldsymbol{\rho}$ is a lattice homomorphism from NExt$\mathbf{K4}$ onto NExt$\mathbf{U}$. The logic

$$\mathbf{Grz}' = \mathbf{K4} \oplus \Box(\Box(p \to \Box p) \to p) \to \Box p$$

is known to be determined by the class of finite Kripke frames without proper (i.e., containing ≥ 2 points) clusters (see e.g. [1]). $\mathbf{U}$ is characterized by this class too. It follows that $\boldsymbol{\rho}\mathbf{Grz}'$ is also $\mathbf{U}$. And since $\boldsymbol{mix} \in \mathbf{K4}$ and the "bimodal" $\mathbf{Grz}'$ is in NExt$(\mathbf{Grz} \otimes \mathbf{K4})$, we finally obtain

Theorem 21 *The map $\boldsymbol{\sigma}$ is an isomorphism from* NExt$\mathbf{U}$ *onto* NExt$\mathbf{Grz}'$.

It is not hard to see also that modulo clusters the languages $\mathcal{ML}_{\hookrightarrow}$ and $\mathcal{ML}$ have the same functional power on the class of transitive frames. For an $\mathcal{ML}_{\hookrightarrow}$-formula φ, let $\varphi_{\mathfrak{F}}$ be the operator induced by φ on the class of transitive frames (under valuations of $\mathcal{ML}_{\hookrightarrow}$) in the same way as in the intuitionistic case.

Proposition 22 $\{\varphi_{\mathfrak{F}} : \varphi \in \mathcal{ML}_{\hookrightarrow}\} = \{\varphi_{\mathfrak{F}}^{\triangle} : \varphi \in \mathcal{ML}\}$, *where $\mathfrak{F}$ ranges over the class of all transitive frames.*

Proof Similar to the proof of Proposition 1. ❑

To prove that the languages under consideration have the same axiomatic power we require frame-based $\mathcal{ML}_{\hookrightarrow}$-formulas simulating canonical formulas for $\mathbf{K4}$ of [30]. Namely, with every finite rooted transitive frame $\mathfrak{F} = \langle W, R \rangle$ without proper clusters—let $a_0, \ldots, a_n$ be all its points and a_0 the root—and a set $\mathfrak{D}$ of antichains in $\mathfrak{F}$ we associate a formula $\gamma(\mathfrak{F}, \mathfrak{D}, \perp)$ which is the implication ($\hookrightarrow$) whose consequent is p_0 and the antecedent is the conjunction of all formulas of the form

$$
\begin{array}{rcll}
 & & \Box p_0 & \text{if} \quad \neg a_0 R a_0, \\
 & & \Box p_i \hookrightarrow p_i & \text{if} \quad a_i R a_i, \\
\gamma_{ij} & = & (\bigwedge \Gamma_j \hookrightarrow p_j) \hookrightarrow p_i & \text{if} \quad a_i R a_j, \\
\gamma_{\mathfrak{d}} & = & \bigwedge\limits_{a_j \in W - \mathfrak{d}\underline{\uparrow}} (\bigwedge \Gamma_j \hookrightarrow p_j) \hookrightarrow \bigvee\limits_{a_i \in \mathfrak{d}} p_i & \text{if} \quad \mathfrak{d} \in \mathfrak{D}, \\
\gamma_{\bot} & = & \bigwedge\limits_{i=0}^{n} (\bigwedge \Gamma_j \hookrightarrow p_j) \hookrightarrow \bot, &
\end{array}
$$

where

$$
\Gamma_j = \begin{cases} \{p_k : a_k \notin a_j\underline{\uparrow}\} & \text{if } a_j R a_j \\ \{\Box p_j, p_k : a_k \notin a_j\underline{\uparrow}\} & \text{if } \neg a_j R a_j, \end{cases}
$$

and

$$
X\uparrow = \{y \in W : \exists x \in X \ xRy\}, \ X\underline{\uparrow} = X \cup X\uparrow,
$$

$$
X\downarrow = \{y \in W : \exists x \in X \ yRx\}, \ X\overline{\downarrow} = X \cup X\downarrow .
$$

Given a frame $\mathfrak{G} = \langle V, S \rangle$, a partial map f from V onto W is called a *subreduction* of $\mathfrak{G}$ to $\mathfrak{F}$ if, for all $x, y \in \mathrm{dom} f$,

(R1) xSy implies $f(x)Rf(y)$;

(R2) $f(x)Rf(y)$ implies $\exists z \in x\uparrow \ f(z) = f(y)$.

A subreduction f is said to be *cofinal* if $\mathrm{dom} f\uparrow \subseteq \mathrm{dom} f\overline{\downarrow}$.

Proposition 23 *For any transitive frame $\mathfrak{G} = \langle V, S \rangle$, $\mathfrak{G} \not\models \gamma(\mathfrak{F}, \mathfrak{D}, \bot)$ iff there is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying the following (closed domain) condition*

(CDC) $\neg\exists x \in \mathrm{dom} f\uparrow - \mathrm{dom} f \ \exists \mathfrak{d} \in \mathfrak{D} \ f(x\uparrow) = \mathfrak{d}\underline{\uparrow}$.

Proof ($\Rightarrow$) Suppose $\mathfrak{G}$ refutes $\gamma(\mathfrak{F}, \mathfrak{D}, \bot)$ under some valuation (in $\mathrm{Up}V$) and π is the premise of $\gamma(\mathfrak{F}, \mathfrak{D}, \bot)$. Define a partial map from V onto W by taking, for $x \in V$,

$$
f(x) = \begin{cases} a_i & \text{if } x \not\models p_i, \ x \models \Gamma_i, \ x \models \pi \\ \text{undefined} & \text{otherwise} \end{cases}
$$

and show that it is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC). Notice first that f is a partial function. Indeed, since $\mathfrak{F}$ contains no proper clusters, if $a_i \neq a_j$ then either $\neg a_i R a_j$ or $\neg a_j R a_i$; in the former case $p_j \in \Gamma_i$ and in the latter $p_i \in \Gamma_j$.

Let xSy, $f(x) = a_i$ and $f(y) = a_j$. Then (since the valuation is intuitionistic) $x \not\models p_j$ from which $p_j \notin \Gamma_i$ and so $a_j \in a_i\underline{\uparrow}$, i.e., either $a_i R a_j$ or $a_i = a_j$. Now, if $a_i = a_j$ and $\neg a_i R a_i$ then $\Box p_i \in \Gamma_i$, so $x \models \Box p_i$ and $y \models p_i$, which is a contradiction. Thus, f satisfies (R1). To show that it satisfies (R2) suppose $f(x) = a_i$ and $a_i R a_j$. If $a_i \neq a_j$ then $x \not\models p_i$, $x \models \gamma_{ij}$, and so there is $y \in x\uparrow$

such that $y \models \Gamma_j$ and $y \not\models p_j$, i.e., $f(y) = a_j$. And if $a_i = a_j$ then, since $x \not\models p_i$ and $x \models \Box p_i \hookrightarrow p_i$, we have $x \not\models \Box p_i$, i.e., there is $y \in x{\uparrow}$ such that $y \not\models p_i$, and again $f(y) = a_i$.

Since, by the definition, $f(x) = a_0$ whenever $x \not\models \gamma(\mathfrak{F}, \mathfrak{D}, \bot)$, the map f is a surjection. The fact that f is cofinal is clearly ensured by the conjunct $\gamma_\bot$ and that it satisfies (CDC) by $\gamma_\mathfrak{d}$.

($\Leftarrow$) Let f be a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC). Define a valuation in $\mathfrak{G}$ by taking

$$x \models p_i \text{ iff } x \notin f^{-1}(a_i)\overline{\downarrow}.$$

By a straightforward inspection one can easily verify that under this valuation $x \not\models \gamma(\mathfrak{F}, \mathfrak{D}, \bot)$ for every $x \in f^{-1}(a_0)$. ❑

Corollary 24 *For every Kripke frame $\mathfrak{G}$, every finite rooted frame $\mathfrak{F}$ without proper clusters and every set $\mathfrak{D}$ of antichains in $\mathfrak{F}$,*

$$\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot) \textit{ iff } \mathfrak{G} \not\models \gamma(\mathfrak{F}, \mathfrak{D}, \bot).$$

Proof Follows from Proposition 23 and the refutability criterion for canonical formulas in [30]. ❑

Remark. Actually, it is not hard to show that Proposition 23 holds for any general **U**-frame $\mathfrak{G}$. It follows that the formulas of the form $\gamma(\mathfrak{F}, \mathfrak{G}, \bot)$ are enough to axiomatize all logics in NExt**U**.

Proposition 25 *A skeleton-closed class $\mathcal{C}$ of transitive frames is $\mathcal{ML}_\hookrightarrow$-axiomatic iff it is $\mathcal{ML}$-axiomatic.*

Proof If $\mathcal{C}$ is axiomatized by a set Γ of $\mathcal{ML}_\hookrightarrow$-formulas then it is also axiomatizable by the set $\mathsf{T}''(\Gamma)$. Suppose now that L is the logic in $\mathcal{ML}$ characterized by $\mathcal{C}$. Since $\mathcal{C}$ is skeleton-closed, it is axiomatizable by a set Γ of canonical formulas for **K4** built on frames without proper clusters. The logic $\boldsymbol{\rho}L \in$ NExt**U** is also characterized by $\mathcal{C}$. It follows that $\gamma(\mathfrak{F}, \mathfrak{D}, \bot) \in \boldsymbol{\rho}L$ whenever $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \Gamma$. Now, if $\mathfrak{G} \notin \mathcal{C}$ then $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, for some $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \Gamma$ and so $\mathfrak{G} \not\models \gamma(\mathfrak{F}, \mathfrak{D}, \bot)$. Thus, $\mathcal{C}$ is axiomatized by $\boldsymbol{\rho}L$ (or by the $\mathcal{ML}_\hookrightarrow$-formulas $\gamma(\mathfrak{F}, \mathfrak{D}, \bot)$ such that $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \Gamma$). ❑

As we saw in Section 2, not all $\mathcal{ML}$-definable skeleton-closed classes of transitive frames are $\mathcal{L}$-definable. The situation changes drastically, however, when we consider frame classes definable by rules. Call a class of general **V**-frames *$\mathcal{L}$-rule definable* if it is of the form Fr $\vdash$, for some **V**-consequence $\vdash$. A class of transitive Kripke frames is *$\mathcal{L}$-rule definable* if it coincides with the subclass of all Kripke frames in some $\mathcal{L}$-rule definable class of general **V**-frames.

Theorem 26 *(i) Let $\mathcal{C}$ be an $\mathcal{L}_2$-definable class of general* **U***-frames. Then there exists an $\mathcal{L}$-rule definable class $\mathcal{C}'$ of general* **V***-frames such that $\mathcal{C}$ coincides with the subclass of all* **U***-frames in $\mathcal{C}'$.*

(ii) A class of Kripke frames is $\mathcal{L}_2$-definable iff it is $\mathcal{L}$-rule definable.

Proof Clearly, (ii) follows from (i), and to prove (i) it suffices to show that for any $\mathcal{L}_2$-definable class of descr iptive **U**-frames, there exists an $\mathcal{L}$-rule definable class $\mathcal{C}'$ of descriptive **V**-frames such that $\mathcal{C}$ consists of precisely the **U**-frames in $\mathcal{C}'$ (for a **V**-frame $\mathfrak{F}$ is a **U**-frame iff $(\mathfrak{F}^+)_+$ is a **U**-frame). To this end consider the variety $\mathcal{V}$ of **V**-algebras generated by $\mathcal{C}^+ = \{\mathfrak{F}^+ : \mathfrak{F} \in \mathcal{C}\}$. $\mathcal{V} = HSP\mathcal{C}^+$, where H denotes the operation of taking homomorphic images, S the operation of taking subalgebras, and P the operation of forming direct products. It is enough to show that for any $\mathfrak{A} \in \mathcal{V}$ such that $\mathfrak{A}_+$ is a **U**-frame, we have $\mathfrak{A}_+ \in \mathcal{C}$. Suppose that $\mathfrak{A} \in HSP\mathcal{C}^+$ and $\mathfrak{A}_+$ is a **U**-frame. Then $\mathfrak{A} \in HS\mathcal{C}^+$, since $\mathcal{C}^+$ is closed under products. By Theorem 13, there are descriptive frames $\mathfrak{H}$ and $\mathfrak{G}$ such that $\mathfrak{G} \in \mathcal{C}$, $\mathfrak{A}_+$ is a generated subframe of $\mathfrak{H}$ and $\mathfrak{G}$ is reducible to $\mathfrak{H}$ by some f. For a frame $\mathfrak{F} = \langle W, R, P\rangle$, denote by P^b the smallest set of cones containing P and such that $\mathfrak{F}^b = \langle W, R, P^b\rangle$ is a **U**-frame. In other words, P^b is the closure of P under the operations $\hookrightarrow$, $\rightarrow$, $\cap$ and $\cup$. One can easily show that $\mathfrak{H}^b$ is a reduct of $\mathfrak{G}^b = \mathfrak{G}$ (since $f^{-1}(X \odot Y) = f^{-1}(X) \odot f^{-1}(Y)$, for $\odot \in \{\hookrightarrow, \rightarrow, \cap, \cup\}$) and that $\mathfrak{A}_+ = (\mathfrak{A}_+)^b$ is a generated subframe of $\mathfrak{H}^b$. And since $\mathcal{C}$ is closed under generated subframes, which are **U**-frames, and reducts, which are also **U**-frames, we finally obtain $\mathfrak{A}_+ \in \mathcal{C}$. ❑

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