# On the relation between intuitionistic and classical modal logics

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Intuitionistic propositional logic **Int** and its extensions, known as intermediate or superintuitionistic logics, in many respects can be regarded just as fragments of classical modal logics containing **S4**. At the syntactical level, the Gödel translation $t$ embeds every intermediate logic $L = \mathbf{Int} + \Gamma$ into modal logics in the interval $\rho^{-1}L = [\tau L = \mathbf{S4} \oplus t(\Gamma), \sigma L = \mathbf{Grz} \oplus t(\Gamma)]$. Semantically this is reflected by the fact that Heyting algebras are precisely the algebras of open elements of topological Boolean algebras. From the lattice-theoretic standpoint the map $\rho$ is a homomorphism of the lattice of logics containing **S4** onto the lattice of intermediate logics, while $\sigma$, according to the Blok–Esakia theorem, is an isomorphism of the latter onto the lattice of extensions of the Grzegorczyk system **Grz**. At the philosophical level the Gödel translation provides a classical interpretation of the intuitionistic connectives. And from the technical point of view this embedding is a powerful tool for transferring various kinds of results from intermediate logics to modal ones and back via preservation theorems. (For details and references consult [6].) Both classical modal logic and the theory of intermediate logics have gained from this correspondence.

The main aim of this paper is to construct a similar correspondence between intermediate logics enriched with modal operators—we call them intuitionistic modal logics—and classical polymodal logics. That the Gödel translation can be extended to an embedding of at least a few particular intuitionistic modal systems into some classical polymodal logics was observed by several authors (cf. [13], [25], [26], [5]). Fischer Servi [13], [15] used a variant of the translation to define "true" intuitionistic analogues of a number of classical modal systems. In [27] we exploited the translation proposed by Shehtman [25] to embed intuitionistic modal logics with the single necessity operator $\Box$ of **K** into bimodal logics

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above $\mathbf{S4} \otimes \mathbf{K}$. However, like $\forall$ and $\exists$, the necessity and possibility operators $\Box$ and $\Diamond$ are not supposed to be dual under the intuitionistic laws.

Here we consider the much more extensive class of intuitionistic polymodal logics (first brought in sight by Sotirov [26]) in which modal operators satisfy only the congruence rules and so may be regarded as various sorts of independent $\Box$ and $\Diamond$. These logics are defined in Section 1. Section 2 introduces algebraic and (quasi-) relational semantics for them and develops duality theory and a little bit of correspondence theory for logics with normal $\Box$-like and $\Diamond$-like operators. In Section 3 we bridge the semantics for intuitionistic and classical modal logics and show that the translation prefixing the $\mathbf{S4}$-necessity to all subformulas of intuitionistic modal formulas embeds intuitionistic modal logics under consideration into classical polymodal logics. Moreover, we prove an analog of the Blok–Esakia theorem by establishing that the lattice of intuitionistic modal logics is isomorphic to a principal filter in the lattice of classical modal logics. We show that the embedding reflects decidability, the finite model property and tabularity and use this result along with preservation theorems of [28] and [12] to prove that the finite model property of an intermediate logic in inherited under adding to it a modal operator satisfying some simple axioms and inference rules. In the final Section 4 we study the embedding of normal intuitionistic modal logics.

# 1 Logics

All the logics considered in this paper are formulated in the propositional modal language $\mathcal{LM}_n$ with the standard connectives $\to$, $\wedge$, $\vee$, $\bot$ ($\neg\varphi$ is defined as $\varphi \to \bot$ and $\top$ as $\bot \to \bot$) and the modal operators $\bigcirc_i$, for $i = 1, \dots, n$. An *intuitionistic modal logic* in the language $\mathcal{LM}_n$ (*IM-logic*, for short) is a set of $\mathcal{LM}_n$-formulas containing intuitionistic logic $\mathbf{Int}$ in the language $\mathcal{LM}_0$ (with only the first four connectives above) and closed under substitution, modus ponens and the congruence rules $\varphi \leftrightarrow \psi / \bigcirc_i \varphi \leftrightarrow \bigcirc_i \psi$, for all $i = 1, \dots, n$. The smallest monomodal IM-logic is denoted by $\mathbf{IntC}$ ($\mathbf{C}$ stands for "congruential" in accord with Segerberg's nomenclature in [24]). For a set of formulas $\Gamma$ and an IM-logic $L$, we denote by $L \oplus \Gamma$ the smallest IM-logic to contain $\Gamma$ and $L$. Several different types of IM-logics have been considered in the literature, and all of them are covered by our definition, which is similar to Sotirov's one in [26]. Here are a few basic monomodal and bimodal systems.

A monomodal IM-logic $L$ (in the language $\mathcal{LM}_1$ with $\bigcirc = \bigcirc_1$) is said to be *regular* if it is closed under the regularity rule $\varphi \to \psi / \bigcirc \varphi \to \bigcirc \psi$. Equivalently, $L$ is regular iff it contains $\bigcirc(p \wedge q) \to \bigcirc p$. The smallest regular IM-logic is denoted by $\mathbf{IntR}$. A regular IM-logic $L$ is said to be $\Box$-*normal* if it contains $\bigcirc(p \wedge q) \leftrightarrow \bigcirc p \wedge \bigcirc q$ and $\bigcirc\top$. In such a case we write $\Box$ instead of $\bigcirc$ and call it the *necessity operator*. Every $\Box$-normal logic is closed under necessitation $\varphi / \Box\varphi$. The smallest $\Box$-normal IM-logic is denoted by $\mathbf{IntK}_\Box$. A regular IM-logic $L$ is called $\Diamond$-*normal* if $\bigcirc(p \vee q) \leftrightarrow \bigcirc p \vee \bigcirc q$ and $\neg \bigcirc \bot$ belong to it. In this case we write $\Diamond$ instead of $\bigcirc$ and call it the *possibility operator*. Every $\Diamond$-normal logic is closed under $\neg\varphi/\neg\Diamond\varphi$. The smallest $\Diamond$-normal logic is denoted by $\mathbf{IntK}_\Diamond$. Some particular $\Box$-normal IM-systems have been investigated in [5], [21] and [26]; general results on the finite model property of such logics can be found in [27]. $\Diamond$-normal systems have been considered in [5] and [26].

As in classical modal logic, given a $\Box$-normal IM-logic $L$, we can define the dual operator $\Diamond$ by taking $\Diamond\varphi = \neg\Box\neg\varphi$. Likewise, in a $\Diamond$-normal logic $L'$ we can take $\neg\Diamond\neg\varphi$ as a definition of $\Box\varphi$. However, $L$ and $L'$ are not necessarily $\Diamond$-normal and $\Box$-normal with respect to the defined operators. (This will certainly be the case if their underlying non-modal logic is classical.) On the other hand, the dual definition of $\Box$ and $\Diamond$ is not consistent with intuitionistic principles (according to which $\forall$ and $\exists$ are not dual).

To construct IM-logics with absolutely independent modal operators, we can take IM-logics $L_1$ and $L_2$, formulated in languages with disjoint sets of modal operators, and then form their *fusion* $L_1 \otimes L_2$, the smallest IM-logic in the joined language containing $L_1 \cup L_2$. In this way we can define the bimodal logic $\mathbf{IntK}_{\Box\Diamond} = \mathbf{IntK}_\Box \otimes \mathbf{IntK}_\Diamond$. Its extensions are called $\Box\Diamond$-*IM-logics*. There is no connection between $\Diamond$ and $\Box$ in $\mathbf{IntK}_{\Box\Diamond}$ and it is both $\Box$- and $\Diamond$-normal. Extensions of $\mathbf{IntK}_\Box$ ($\mathbf{IntK}_\Diamond$) can clearly be identified with extensions of $\mathbf{IntK}_{\Box\Diamond} \oplus \Diamond p \leftrightarrow p$ (respectively, $\mathbf{IntK}_{\Box\Diamond} \oplus \Box p \leftrightarrow p$).

A well motivated $\Box\Diamond$-IM-logic containing $\mathbf{IntK}_{\Box\Diamond}$ was introduced by Fischer Servi [14], [15]:

$$\mathbf{FS} = \mathbf{IntK}_{\Box\Diamond} \oplus \Diamond(p \to q) \to (\Box p \to \Diamond q) \oplus (\Diamond p \to \Box q) \to \Box(p \to q).$$

Extensions of $\mathbf{FS}$ will be called $\mathbf{FS}$-*logics*. Some of them were studied in [15], [1] and [10].

By adding to a consistent IM-logic in the language $\mathcal{LM}_n$ the Law of the Excluded Middle $p \vee \neg p$, we obtain a classical logic with $n$ modal operators. We denote by $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{K}$ the monomodal logics $\mathbf{IntC} \oplus p \vee \neg p$, $\mathbf{IntR} \oplus p \vee \neg p$ and $\mathbf{IntK}_\Box \oplus p \vee \neg p$, respectively.

## 2 Semantics and Duality

The logics introduced above correspond to varieties (equational classes) of Heyting (or pseudo-Boolean) algebras with operators. More precisely, given a language $\mathcal{LM}_n$, we consider algebras of the form

$$\mathfrak{A} = \langle A, \to, \wedge, \vee, \top, \bigcirc_1, \ldots, \bigcirc_n \rangle$$

where $\langle A, \to, \wedge, \vee, \top \rangle$ is a Heyting algebra with unit element $\top$ and $\bigcirc_i$, for $1 \le i \le n$, are unary operators on $A$. Such algebras will be called *IM-algebras*. A *valuation* $\mathfrak{V}$ of $\mathcal{LM}_n$ in $\mathfrak{A}$ is a homomorphism of the algebra of $\mathcal{LM}_n$-formulas into $\mathfrak{A}$. A formula $\varphi$ is *true* in $\mathfrak{A}$ under $\mathfrak{V}$ if $\mathfrak{V}(\varphi) = \top$; $\varphi$ is *valid* in $\mathfrak{A}$, $\mathfrak{A} \models \varphi$ in symbols, if it is true under any valuation.

An IM-logic $L$ is *characterized* by a class $\mathcal{C}$ of IM-algebras if

$$L = \{\varphi : \ \forall \mathfrak{A} \in \mathcal{C} \ \mathfrak{A} \models \varphi\}.$$

In the standard way one can show that the class of IM-algebras, validating all the formulas in an IM-logic $L$, forms a variety characterizing $L$.

The relational semantics is usually derived from the algebraic one using the Stone-Jónsson-Tarski representation of Heyting and modal algebras. Since the logics under consideration are rather weak, we introduce first some intermediate structures combining a relational intuitionistic component and an algebraic modal one.

We remind the reader that an *intuitionistic frame* (or **Int**-frame, for short) is a structure of the form $\mathfrak{F} = \langle W, R, P\rangle$ where $R$ is a partial order on a non-empty set $W$ and $P$ a collection of cones (= upward closed sets) in $W$ with respect to $R$ containing $\emptyset$ and closed under $\cap$, $\cup$ and the operation

$$X \supset Y = \{x \in W : \ \forall y \in W \ (xRy \wedge y \in X \Rightarrow y \in Y)\}.$$

If $P$ contains all the cones in $W$ then we call $\mathfrak{F}$ a *full* (or *Kripke*) frame and write $\langle W, R\rangle$ instead of $\langle W, R, P\rangle$. The underlying full frame of $\mathfrak{F}$ is denoted by $\kappa\mathfrak{F}$.

Now we define a *quasi-IM-frame* as a structure

$$\mathfrak{F} = \langle W, R, \bigcirc_1, \dots, \bigcirc_n, P\rangle$$

such that $\langle W, R, P\rangle$ is an **Int**-frame and $\bigcirc_i$, for $i = 1, \dots, n$, are just operations on $P$. Every quasi-IM-frame gives rise to the IM-algebra

$$\mathfrak{F}^\dagger = \langle P, \supset, \cap, \cup, W, \bigcirc_1, \dots, \bigcirc_n\rangle\,,$$

called the *dual* of $\mathfrak{F}$. We write $\mathfrak{F} \models \varphi$ to mean that $\mathfrak{F}^\dagger \models \varphi$. All the other semantic notions above can be translated to quasi-frames in the same way. A model on $\mathfrak{F}$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ where $\mathfrak{V}$ is a valuation in $\mathfrak{F}$ (= in $\mathfrak{F}^\dagger$). If $x \in \mathfrak{V}(\varphi)$ then we write $(\mathfrak{M}, x) \models \varphi$ or simply $x \models \varphi$, if understood, and say that $\varphi$ is *true* at $x$ (under $\mathfrak{V}$). It should be clear that $\mathfrak{V}(\varphi)$ is a cone for every formula $\varphi$.

Conversely, with each IM-algebra $\mathfrak{A} = \langle A, \rightarrow, \wedge, \vee, \top, \bigcirc_1, \dots, \bigcirc_n\rangle$ we can associate its *dual* — the quasi-IM-frame

$$\mathfrak{A}_\dagger = \langle W, R, \bigcirc'_1, \dots, \bigcirc'_n, P\rangle$$

in which $W$ is the set of prime filters in $\mathfrak{A}$ and, for every $x, y \in W$ and $a \in A$,

$$xRy \text{ iff } x \subseteq y,$$

$$P(a) = \{x \in W : \ a \in x\},$$
$$P = \{P(a) : \ a \in A\},$$
$$\bigcirc_i'(P(a)) = P(\bigcirc_i(a)), \ 1 \le i \le n.$$

Using the well known correspondence between **Int**-frames and Heyting algebras (see, e.g. [7]), one can readily see that every IM-algebra $\mathfrak{A}$ is isomorphic to its bidual, $\mathfrak{A} \simeq (\mathfrak{A}_\dagger)^\dagger$ in symbols. A quasi-IM-frame $\mathfrak{F}$ is called *descriptive* if $\mathfrak{F} \simeq (\mathfrak{F}^\dagger)_\dagger$. Every quasi-IM-frame of the form $\mathfrak{A}_\dagger$ is clearly descriptive. Hence we have

**Proposition 1** *Each IM-logic is characterized by a suitable class of descriptive quasi-IM-frames.*

Another sort of adequate relational semantics for IM-logics — neighborhood frames — was introduced in [26].

For normal IM-logics the algebraic modal component in quasi-IM-frames can be also replaced with a relational one.

Say that an IM-algebra $\mathfrak{A} = \langle A, \to, \wedge, \vee, \top, \Box, \Diamond\rangle$ is a $\Box\Diamond$-*IM-algebra* if the following identities hold in it:

$$\Box\top = \top, \ \Box(a \wedge b) = \Box a \wedge \Box b, \ \neg\Diamond\bot = \top, \ \Diamond(a \vee b) = \Diamond a \vee \Diamond b.$$

All $\Box\Diamond$-IM-logics are clearly characterized by varieties of $\Box\Diamond$-IM-algebras.

Given a $\Box\Diamond$-IM-algebra $\mathfrak{A} = \langle A, \to, \wedge, \vee, \top, \Box, \Diamond\rangle$, we define its *dual* $\mathfrak{A}_+$ to be the structure $\langle W, R, R_\Box, R_\Diamond, P\rangle$, where $\langle W, R, P\rangle$ is the dual of the Heyting algebra $\langle A, \to, \wedge, \vee, \top\rangle$ and, for every $x, y \in W$,

$$xR_\Box y \text{ iff } \forall a \in A \ (\Box a \in x \Rightarrow a \in y),$$

$$xR_\Diamond y \text{ iff } \forall a \in A \ (a \in y \Rightarrow \Diamond a \in x).$$

It follows immediately from the definition that, for all $x, u, v, y \in W$,

$$xRu \wedge uR_\Box v \wedge vRy \Rightarrow xR_\Box y,$$

$$xRu \wedge vR_\Diamond u \wedge vRy \Rightarrow yR_\Diamond x,$$

or, equivalently,

$$R \circ R_\Box \circ R \subseteq R_\Box, \tag{1}$$
$$R \circ R_\Diamond^{-1} \circ R \subseteq R_\Diamond^{-1}. \tag{2}$$

(Here $\circ$ denotes the composition of relations.)

Structures of the form $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$, where $\langle W, R, P\rangle$ is an **Int**-frame, $R_\Box$, $R_\Diamond$ are binary relations on $W$ satisfying (1) and (2) and $P$ is closed under the operations $\Box$ and $\Diamond$ defined by

$$\Box X = \{x \in W : \ \forall y \in X \ (xR_\Box y \Rightarrow y \in X)\},$$

$$\Diamond X = \{x \in W : \exists y \in X \ xR_\Diamond y\},$$

will be called *□◇-IM-frames*. The dual of a □◇-IM-frame $\mathfrak{F}$ is then the □◇-IM-algebra $\mathfrak{F}^+ = \langle P, \supset, \cap, \cup, W, \Box, \Diamond \rangle$.

It is not hard to check that $\mathfrak{F}^+$ is a □◇-IM-algebra and that again $\mathfrak{A} \simeq (\mathfrak{A}_+)^+$, for every □◇-IM-algebra $\mathfrak{A}$. Say that a □◇-IM-frame $\mathfrak{F}$ is *descriptive* if $\mathfrak{F} \simeq (\mathfrak{F}^+)_+$. Since the frames of the form $\mathfrak{A}_+$ are descriptive, we have

**Proposition 2** *Every □◇-IM-logic is characterized by a suitable class of descriptive □◇-IM-frames.*

The following internal characterization of descriptive □◇-IM-frames is obtained by a straightforward combination of the corresponding characterizations of descriptive modal and intuitionistic frames. For details consult [17] and [7].

**Proposition 3** *A □◇-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P \rangle$ is descriptive iff $\mathfrak{F}$ is tight$_R$, tight$_{R_\Box}$ and tight$_{R_\Diamond}$, i.e.,*

$$xRy \text{ iff } \forall X \in P \ (x \in X \Rightarrow y \in X),$$

$$xR_\Box y \text{ iff } \forall X \in P \ (x \in \Box X \Rightarrow y \in X),$$

$$xR_\Diamond y \text{ iff } \forall X \in P \ (y \in X \Rightarrow x \in \Diamond X),$$

*and compact, i.e., for every $\mathcal{X} \subseteq P$ and $\mathcal{Y} \subseteq \{W - X : X \in P\}$, if $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.*

A □◇-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P \rangle$ is a *full* (or *Kripke*) □◇-IM-frame if $\langle W, R, P \rangle$ is a full **Int**-frame. A □◇-IM-logic is called *complete* if it is characterized by a class of full □◇-IM-frames. The underlying full □◇-IM-frame of a □◇-IM-frame $\mathfrak{F}$ is denoted by $\kappa\mathfrak{F}$. A □◇-logic $L$ is said to be *d-persistent* if $\kappa\mathfrak{F} \models L$ whenever $\mathfrak{F}$ is a descriptive frame validating $L$. All d-persistent logics are clearly complete. Another useful property of d-persistence is that it is preserved under sums, i.e., if logics $L_1$ and $L_2$ are d-persistent then so is $L_1 \oplus L_2$. (However, completeness as well as many other important properties are not in general preserved under sums of logics.) We show some examples of d-persistent □◇-logics. To this end we require the following well known lemma on the existence of prime filters (see [22]).

**Lemma 4** *Suppose that $\mathfrak{A} = \langle A, \to, \wedge, \vee, \top \rangle$ is a Heyting algebra, $B$ and $C$ are non-empty subsets of $A$ such that (i) $b_1 \wedge \ldots \wedge b_n \not\leq c$, for any $b_1, \ldots, b_n \in B$, $c \in C$, and (ii) for every $c_1, c_2 \in C$ there is $c \in C$ for which $c_1 \vee c_2 \leq c$. Then there exists a prime filter $\nabla$ in $\mathfrak{A}$ such that $B \subseteq \nabla$ and $C \cap \nabla = \emptyset$.*

(Here $\leq$ is the lattice partial order on $A$ defined by $a \leq b$ iff $a \wedge b = a$.)

**Proposition 5** **FS** *is d-persistent.*

**Proof.** It suffices to show that any $\Box\Diamond$-IM-frame satisfying the conditions

$$xR_\Diamond y \Rightarrow \exists z\ (yRz \wedge xR_\Box z \wedge xR_\Diamond z) \tag{3}$$

$$xR_\Box y \Rightarrow \exists z\ (xRz \wedge zR_\Box y \wedge zR_\Diamond y) \tag{4}$$

validates **FS** and that (3) and (4) hold in any descriptive frame for **FS**.

To prove the former claim, suppose that a $\Box\Diamond$-IM-frame $\mathfrak{F}$ satisfies (3) but $\Diamond(p \to q) \to (\Box p \to \Diamond q)$ is refuted in $\mathfrak{F}$ under some valuation. Then $x \models \Diamond(p \to q)$, $x \models \Box p$, $x \not\models \Diamond q$, for some $x$ in $\mathfrak{F}$, and so there is $y$ such that $xR_\Diamond y$ and $y \models p \to q$. By (3), we have $yRz$ (since the truth-set of any formula is a cone), $xR_\Box z$ and $xR_\Diamond z$ for some point $z$. But then $z \models p \to q$, $z \models p$ and $z \not\models q$, which is impossible. The second axiom of **FS** is considered analogously with the help of (4).

Suppose now that $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond\rangle$ is a descriptive frame for **FS** and show that it satisfies (4). Without loss of generality we may assume that $\mathfrak{F} \simeq \mathfrak{A}_+$ for some $\Box\Diamond$-IM-algebra $\mathfrak{A} \models$ **FS**. Thus, points in $\mathfrak{F}$ are prime filters in $\mathfrak{A}$. Let $x, y \in W$ and $xR_\Box y$. Put

$$B = x \cup \{\Diamond b : \ b \in y\}, \ C = \{\Box c : \ c \notin y\}$$

and show that $B$ and $C$ satisfy (i) and (ii) in Lemma 4. Suppose

$$a \wedge \Diamond b_1 \wedge \ldots \wedge \Diamond b_n \leq \Box c$$

for some $a \in x$ ($x$ is closed under $\wedge$), $b_1, \ldots, b_n \in y$ and $c \notin y$. Then

$$a \wedge \Diamond b_1 \wedge \ldots \wedge \Diamond b_n \to \Box c = \top$$

in $\mathfrak{A}$, from which, by the second axiom of **FS**, we obtain

$$a \to \Box(b_1 \wedge \ldots \wedge b_n \to c) = \top.$$

It follows that $\Box(b \to c) \in x$ for some $b \in y$ and $c \notin y$. Since $xR_\Box y$, we then have $b \to c \in y$ and $c \in y$, which is a contradiction. Therefore, (i) holds. To show (ii), suppose $c_1, c_2 \notin y$. Since $y$ is prime, $c_1 \vee c_2 \notin y$ and so $\Box(c_1 \vee c_2) \in C$ and $\Box c_1 \vee \Box c_2 \leq \Box(c_1 \vee c_2)$.

By Lemma 4, there is a prime filter $z \in W$ such that $B \subseteq z$ and $C \cap z = \emptyset$. This means that $xRz$, $zR_\Box y$ and $zR_\Diamond y$, as required by (4).

In the same way, using Lemma 4 and the first axiom of **FS**, one can show that $\mathfrak{F}$ satisfies (3). ❑

Using the same sort of technique it is not hard to prove the following proposition in which $\Box^n$ and $\Diamond^n$ are strings of $n$ boxes and diamonds, respectively.

**Proposition 6** *For every $k, l, m, n \geq 0$, the logic*

$$\mathbf{L}(k, l, m, n) = \mathbf{IntK}_{\Box\Diamond} \oplus \Diamond^k \Box^l p \to \Box^m \Diamond^n p$$

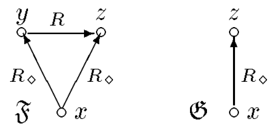


Figure 1:

*is d-persistent, with every descriptive $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ for* $\mathbf{L}(k,l,m,n)$ *satisfying the condition*

$$xR_\Diamond^k y \wedge xR_\Box^m z \Rightarrow \exists u\ (yR_\Box^l u \wedge zR_\Diamond^n u).$$

We remind the reader that it is this kind of result, obtained by Lemmon and Scott [18], that was the starting point of the project in correspondence theory in classical modal logic which finally led to Sahlqvist's Theorem [23].

Having established the correspondence between $\Box\Diamond$-IM-algebras and $\Box\Diamond$-IM-frames, let us extend it to the algebraic operators of forming subalgebras, homomorphic images and direct products. As to the latter operator, its relational analog is the standard *disjoint union* of frames defined in exactly the same way as in the purely intuitionistic or classical modal case (see [17], [7]). However, the duals of the notions of homomorphism and subalgebra of $\Box\Diamond$-IM-algebras are not direct translations of the standard definitions.

Let $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ be a $\Box\Diamond$-IM-frame and $V$ a non-empty subset of $W$ satisfying the following two conditions:

$$\forall x \in V \forall y \in W\ (xRy \vee xR_\Box y \Rightarrow y \in V), \tag{5}$$

$$\forall x \in V \forall y \in W\ (xR_\Diamond y \Rightarrow \exists z \in V\ (xR_\Diamond z \wedge yRz)). \tag{6}$$

Then it is easy to see that the structure

$$\mathfrak{G} = \langle V, R{\restriction}V, R_\Box{\restriction}V, R_\Diamond{\restriction}V, \{X \cap V :\ X \in P\}\rangle$$

is also a $\Box\Diamond$-IM-frame. It is called a *generated subframe* of $\mathfrak{F}$. The condition (5) is standard: it requires $V$ to be upward closed with respect to both $R$ and $R_\Box$. However, according to (6), $V$ is not necessarily upward closed with respect to $R_\Diamond$. This is illustrated by Fig. 1 in which $\mathfrak{G}$ is a generated subframe of $\mathfrak{F}$, although the set $\{x, z\}$ is not upward closed in $\mathfrak{F}$ with respect to $R_\Diamond$.

**Theorem 7** (i) *If $\mathfrak{G} = \langle V, S, S_\Box, S_\Diamond, Q\rangle$ is a generated subframe of a $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ then the map $h$ defined by*

$$h(X) = X \cap V,\ \textit{for every } X \in P,$$

*is a homomorphism of $\mathfrak{F}^+$ onto $\mathfrak{G}^+$.*

(ii) *If $h$ is a homomorphism of a $\Box\Diamond$-IM-algebra $\mathfrak{A}$ onto a $\Box\Diamond$-IM-algebra $\mathfrak{B}$ then the map $h_+$ defined by*

$$h_+(\nabla) = h^{-1}(\nabla), \text{ for every prime filter } \nabla \text{ in } \mathfrak{B},$$

*is an isomorphism of $\mathfrak{B}_+$ onto a generated subframe of $\mathfrak{A}_+$.*

**Proof.** (i) That $h$ is a surjection preserving $\supset$, $\cap$, $\cup$ and $\Box$ is proved in the usual way (see [17], [7]). We show that $h$ preserves $\Diamond$, i.e., that

$$h(\Diamond X) = \Diamond h(X), \text{ for every } X \in P.$$

Suppose $x \in \Diamond X \cap V$ in $\mathfrak{F}$. Then there is $y \in X$ such that $xR_\Diamond y$ and so, by (6), we have $z \in V$ with $xR_\Diamond z$ and $yRz$. Since $X$ is a cone, it follows that $z \in X$, and hence $x \in \Diamond(X \cap V)$ in $\mathfrak{G}$. Thus, $h(\Diamond X) \subseteq \Diamond h(X)$. The converse inclusion is trivial.

(ii) Let $\mathfrak{A}_+ = \langle W, R, R_\Box, R_\Diamond, P\rangle$ and $\mathfrak{B}_+ = \langle U, S, S_\Box, S_\Diamond, Q\rangle$. Put

$$V = \{\nabla \in W : \ h^{-1}(\top) \subseteq \nabla\}.$$

It is shown in [17], [7] that $V$ is upward closed in $\mathfrak{A}_+$ with respect to $R$ and $R_\Box$, $h$ is a bijection of $V$ onto $U$ and that $h_+$ is an isomorphism of $\mathfrak{B}_+$ onto the subframe of $\mathfrak{A}_+$ generated by $V$ as far as $R$ and $R_\Box$ are concerned. So it remains to show that $V$ satisfies (6) and that $\nabla_1 S_\Diamond \nabla_2$ iff $h_+(\nabla_1) R_\Diamond h_+(\nabla_2)$, for every $\nabla_1, \nabla_2 \in U$.

Suppose that $h^{-1}(\top) \subseteq \nabla$ (i.e. $\nabla \in V$) and $\nabla R_\Diamond \nabla'$ for some $\nabla' \in W$. Put

$$B = \nabla' \cup h^{-1}(\top), \ C = \{a \in \mathfrak{A} : \ \Diamond a \notin \nabla\}$$

and show that $B$, $C$ satisfy the conditions of Lemma 4. Suppose $b \wedge c \leq a$ for some $b \in \nabla'$, $c \in h^{-1}(\top)$ and $\Diamond a \notin \nabla$. Then $h(b \wedge c) = h(b) \leq h(a)$ and so $h(\Diamond b) \leq h(\Diamond a)$. Since $\Diamond b \in \nabla$, $h(\nabla)$ is a filter in $\mathfrak{B}$ and $h^{-1}(h(\nabla)) = \nabla$, we must have $\Diamond a \in \nabla$, which is a contradiction. Now suppose $\Diamond a_1, \Diamond a_2 \notin \nabla$. Since $\nabla$ is prime, $\Diamond a_1 \vee \Diamond a_2 \notin \nabla$, from which $\Diamond(a_1 \vee a_2) = \Diamond a_1 \vee \Diamond a_2 \notin \nabla$.

Let $\nabla_1$ be a prime filter in $\mathfrak{A}$ such that $B \subseteq \nabla_1$ and $C \cap \nabla_1 = \emptyset$. Then clearly $\nabla_1 \in V$, $\nabla R_\Diamond \nabla_1$ and $\nabla' \subseteq \nabla_1$. Thus $V$ satisfies (6).

Suppose that $\nabla_1 S_\Diamond \nabla_2$, i.e., $\Diamond b \in \nabla_1$ whenever $b \in \nabla_2$, and that $a \in h_+(\nabla_2)$ for some $a$ in $\mathfrak{A}$. Then $h(a) \in \nabla_2$, $h(\Diamond a) = \Diamond h(a) \in \nabla_1$ and so $\Diamond a \in h_+(\nabla_1)$. Conversely, assume that $h_+(\nabla_1) R_\Diamond h_+(\nabla_2)$. Then for all $a$ in $\mathfrak{A}$, $a \in h_+(\nabla_2)$ implies $\Diamond a \in h_+(\nabla_1)$. Since $h$ is a bijection of $V$ onto $U$, if $b \in \nabla_2$ then $b = h(a)$ for some $a \in h_+(\nabla_2)$. So $\Diamond a \in h_+(\nabla_1)$ and $\Diamond b = \Diamond h(a) = h(\Diamond a) \in \nabla_1$. ❑

Given $\Box\Diamond$-IM-frames $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ and $\mathfrak{G} = \langle V, S, S_\Box, S_\Diamond, Q\rangle$, we say a map $f$ from $W$ onto $V$ is a *reduction* (or *p-morphism*) of $\mathfrak{F}$ to $\mathfrak{G}$ if, for all $x \in W$, $y \in V$ and $X \in Q$, the following three conditions hold:

$$xR_\bullet y \Rightarrow f(x) S_\bullet f(y), \ \bullet \in \{\ , \Box, \Diamond\}, \tag{7}$$

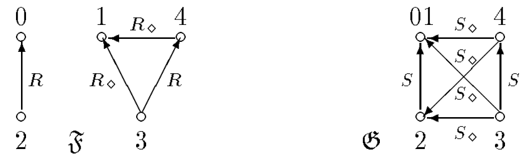


Figure 2:

$$f(x)S_\bullet y \Rightarrow \exists z \in f^{-1}(y)\ xR_\bullet z,\ \bullet \in \{\ , \Box\}, \tag{8}$$

$$f(x)S_\Diamond y \Rightarrow \exists z \in W\ (xR_\Diamond z \wedge ySf(z)), \tag{9}$$

$$f^{-1}(X) \in P. \tag{10}$$

For example, the map gluing the points 0 and 1 in the frame $\mathfrak{F}$ in Fig. 2 is a reduction of $\mathfrak{F}$ to $\mathfrak{G}$ in Fig. 2. Notice that if we consider these frames as classical bimodal frames, then $\mathfrak{F}$ is not reducible to $\mathfrak{G}$ because the points 2 and 3 as well as 2 and 4 are connected by $S_\Diamond$-arrows. If we remove these arrows then the modified $\mathfrak{G}$ will not be a $\Box\Diamond$-IM-frame, since the condition (2) will not hold.

**Theorem 8** (i) *If $f$ is a reduction of a $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ to a $\Box\Diamond$-IM-frame $\mathfrak{G} = \langle V, S, S_\Box, S_\Diamond, Q\rangle$ then the map $f^+$ defined by*

$$f^+(X) = f^{-1}(X),\ \textit{for every } X \in Q,$$

*is an embedding of $\mathfrak{G}^+$ into $\mathfrak{F}^+$.*

(ii) *If $\mathfrak{B}$ is a subalgebra of a $\Box\Diamond$-IM-algebra $\mathfrak{A}$ then the map $f$ defined by*

$$f(\nabla) = \nabla \cap \mathfrak{B},\ \textit{for every prime filter } \nabla \textit{ in } \mathfrak{A},$$

*is a reduction of $\mathfrak{A}_+$ to $\mathfrak{B}_+$.*

**Proof.** (i) It is known (cf. [17], [7]) that $f^+$ is an injection preserving $\rightarrow$, $\wedge$, $\vee$, $\Box$. So it remains to show that

$$f^{-1}(\Diamond X) = \Diamond f^{-1}(X),$$

for every $X \in Q$. Suppose that $x \in f^{-1}(\Diamond X)$, i.e., $f(x)S_\Diamond y$ for some $y \in X$. By (9), we then have a $z \in W$ such that $xR_\Diamond z$ and $ySf(z)$. Since $X$ is a cone, $f(z) \in X$ and so $x \in \Diamond f^{-1}(X)$. Thus, $f^{-1}(\Diamond X) \subseteq \Diamond f^{-1}(X)$. The converse inclusion follows from (7).

(ii) It was proved in [17] and [7] that $f$ satisfies (7), (8) and (10). To show (9), suppose $(\nabla_1 \cap \mathfrak{B})S_\Diamond \nabla_2$ for some prime filters $\nabla_1$ in $\mathfrak{A}$ and $\nabla_2$ in $\mathfrak{B}$. Put

$$B = \nabla_2,\ C = \{a \in \mathfrak{A} :\ \Diamond a \notin \nabla_1\}$$

and show that $B$ and $C$ satisfy the conditions of Lemma 4. That (ii) holds was established in the proof of Theorem 7. Suppose that (i) does not hold. Then there exist $b \in \nabla_2$ and $\Diamond a \notin \nabla_1$ such that $b \leq a$. It follows that $\Diamond b \leq \Diamond a$. Since $(\nabla_1 \cap \mathfrak{B}) S_\Diamond \nabla_2$, we have $\Diamond \mathfrak{B} \in \nabla_1 \cap B$ and so $\Diamond a \in \nabla_1$, which is a contradiction. Let $\nabla$ be a prime filter in $\mathfrak{A}$ such that $B \subseteq \nabla$ and $C \cap \nabla = \emptyset$. Then clearly $\nabla_1 R_\Diamond \nabla$ and $\nabla_2 (R\nabla \cap \mathfrak{B})$. ❑

In exactly the same way as in classical modal logic (cf. [17], [2], [3]) one can use the duality results above to prove the following definability theorems.

**Theorem 9** *A class $\mathcal{C}$ of $\Box\Diamond$-IM-frames is definable by $\mathcal{LM}_2$-formulas (in the sense that there exists a set $\Gamma$ of $\mathcal{LM}_2$-formulas such that $\mathcal{C} = \{\mathfrak{F} : \mathfrak{F} \models \Gamma\}$) iff $\mathcal{C}$ is closed under the formation of generated subframes, reducts, disjoint unions, and both $\mathcal{C}$ and its complement (in the class of all $\Box\Diamond$-IM-frames) are closed under the operator $\mathfrak{F} \mapsto (\mathfrak{F}^+)_+$.*

For a full $\Box\Diamond$-IM-frame $\mathfrak{F}$, the frame $\kappa(\mathfrak{F}^+)_+$ is called the *prime filter extension* of $\mathfrak{F}$. This notion is the intuitionistic counterpart of the notion of ultrafilter extension in classical modal logic introduced in [2].

**Theorem 10** *A class $\mathcal{C}$ of full $\Box\Diamond$-IM-frames coincides with the class of all full $\Box\Diamond$-IM-frames validating a d-persistent $\Box\Diamond$-IM-logic $L$ iff $\mathcal{C}$ is closed under the formation of generated subframes, reducts, disjoint unions, and both $\mathcal{C}$ and its complement (in the class of all full $\Box\Diamond$-IM-frames) are closed under forming prime filter extensions.*

**Theorem 11** *If a $\Box\Diamond$-IM-logic $L$ is characterized by a class of full $\Box\Diamond$-IM-frames which is closed under elementary equivalence (in the first order language with the predicates $=$, $R$, $R_\Box$ and $R_\Diamond$), then $L$ is d-persistent.*

We conclude this section with a few remarks concerning other semantics for $\Box\Diamond$-IM-logics. Observe first that the conditions (1) and (2) can be considerably weakened. Say that a structure $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ is a *weak $\Box\Diamond$-IM-frame* if $\langle W, R, P\rangle$ is an **Int**-frame, $R_\Box$ an arbitrary binary relation, $R_\Diamond$ a binary relation such that, for every $x, y \in W$,

$$xRy \wedge xR_\Diamond z \Rightarrow \exists u\ (yR_\Diamond u \wedge zRu) \tag{11}$$

and $P$ is closed under the operations

$$\Box X = \{x \in W : \forall y, z\ (xRyR_\Box z \Rightarrow z \in X)\},$$

$$\Diamond X = \{x \in W : \exists y \in X\ xR_\Diamond y\}.$$

One can readily check that if $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ is a weak $\Box\Diamond$-IM-frame then the structure $\mathfrak{G} = \langle W, R, \Box, \Diamond, P\rangle$ is a quasi-IM-frame validating $\mathbf{IntK}_{\Box\Diamond}$. It follows that $\mathfrak{G}^\dagger$ is a $\Box\Diamond$-IM-algebra. We denote it by $\mathfrak{F}^+$. It is not hard to see also that the set of cones (with respect to $R$) in $\mathfrak{F}$ is closed under the operations $\Box$ and $\Diamond$. If $P$ contains all such cones then $\mathfrak{F}$ is called *full.* A $\Box\Diamond$-IM-logic is *weakly complete* if it is characterized by a class of full weak $\Box\Diamond$-IM-frames.

One can argue as to which conditions on $R_\Box$ and $R_\Diamond$ are more natural: (11) or (1) and (2) or something between them, like Ono's frames from [20] or those of Božić and Došen [5]. However, from the technical point of view this gives us nothing new. Indeed, with every weak $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ we can associate the $\Box\Diamond$-IM-frame

$$\mathfrak{F}^\circ = \left\langle W, R, R \circ R_\Box \circ R, R^{-1} \circ R_\Diamond \circ R^{-1}, P \right\rangle.$$

And then we have

**Proposition 12** *For every weak $\Box\Diamond$-IM-frame $\mathfrak{F}$ and every formula $\varphi$, $\mathfrak{F} \models \varphi$ iff $\mathfrak{F}^\circ \models \varphi$.*

**Proof.** One can either show that $\mathfrak{F}^+ = (\mathfrak{F}^\circ)^+$ or simply conduct straightforward induction on the construction of $\varphi$. ❑

In particular, we obtain

**Corollary 13** *A $\Box\Diamond$-IM-logic is complete iff it is weakly complete.*

Fischer Servi's birelational frames for **FS**, introduced in [14], can also be derived from weak $\Box\Diamond$-IM-frames. Say that a $\Box\Diamond$-IM-frame is an **FS**-*frame* if it satisfies the conditions (3) and (4). Given an **FS**-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$, define the relation $S = R_\Box \cap R_\Diamond$. It follows from (1) – (4) that $S$ satisfies (11), i.e., $xRy$ and $xSz$ imply $ySu$ and $zRu$, for some $u \in W$, and

$$xSyRz \Rightarrow \exists u\ xRuSz. \tag{12}$$

Denote the weak $\Box\Diamond$-IM-frame $\langle W, R, S, S, P\rangle$ by $\mathfrak{F}^\bullet$.

Say that a weak $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, S, S, P\rangle$ is a *birelational* **FS**-*frame* if it satisfies (12). One can easily verify that every birelational **FS**-frame validates **FS** and that $\mathfrak{F}^+ = (\mathfrak{F}^\bullet)^+$, for every **FS**-frame $\mathfrak{F}$. Therefore, we have

**Proposition 14** *Every* **FS**-*logic is characterized by a class of birelational* **FS**-*frames.*

Since $\mathfrak{F}^\circ$ is an **FS**-frame whenever $\mathfrak{F}$ is a birelational **FS**-frame, we have also

**Proposition 15** *An* **FS**-*logic is complete iff it is characterized by full birelational* **FS**-*frames.*

## 3 Embedding

Gödel [16] embedded **Int** into **S4** via the translation $t$ prefixing $\Box$ to all subformulas of intuitionistic formulas[1]. Dummett and Lemmon [8] extended Gödel's embedding to all intermediate logics, and Maksimova and Rybakov [19], Blok [4], and Esakia [9] started the systematic investigation into the structure of "modal companions" of intermediate logics.

In [27] we used the natural generalization of Gödel's translation (first introduced by Fischer Servi [13] and Shehtman [25]), which embeds extensions of $\mathbf{IntK}_\Box$ into classical bimodal logics containing $\mathbf{S4} \otimes \mathbf{K}$, to obtain a number of general completeness results for intuitionistic modal logics. Our aim here is to study the embedding of (not necessarily normal or regular) IM-logics in an arbitrary language $\mathcal{LM}_n$ into classical logics with $n+1$ modal operators.

Given a language $\mathcal{LM}_n$, we define its extension $\mathcal{LM}'_n$ with one more modal operator $\Box_I$ and consider classical $n+1$-modal logics in $\mathcal{LM}'_n$ (*CM-logics*, for short) containing the **S4**-axioms for $\Box_I$:

$$\Box_I(p \wedge q) \leftrightarrow \Box_I p \wedge \Box_I q, \quad \Box_I \top, \quad \Box_I p \to p, \quad \Box_I p \to \Box_I \Box_I p.$$

These logics can be interpreted by *quasi-CM-frames* which are structures of the form $\mathfrak{F} = \langle W, R_I, \bigcirc_1, \dots, \bigcirc_n, P\rangle$, where $R_I$ is a quasi-order on $W \neq \emptyset$, $\bigcirc_i$ an arbitrary operation on $P$ and $P \subseteq 2^W$ contains $\emptyset$ and is closed under the Boolean operations and the operation $\Box_I$ defined by

$$\Box_I X = \{x \in W : \ \forall y \ (xR_I y \Rightarrow y \in X)\}.$$

The dual of $\mathfrak{F}$, i.e., the modal algebra $\langle P, \cap, -, \top, \Box_I, \bigcirc_1, \dots, \bigcirc_n\rangle$, is denoted by $\mathfrak{F}^\dagger$. Conversely, for a topological Boolean algebra with $n$ operators $\mathfrak{A} = \langle A, \wedge, -, \top, \Box_I, \bigcirc_1, \dots, \bigcirc_n\rangle$ (which validates the **S4**-axioms), we define its dual $\mathfrak{A}_\dagger = \langle W, R_I, \bigcirc'_1, \dots, \bigcirc'_n, P\rangle$ in almost the same way as in Section 2: the only difference is that now

$$xR_I y \text{ iff } \forall a \in A \ (\Box_I a \in x \Rightarrow a \in y).$$

Again we have $\mathfrak{A} \simeq (\mathfrak{A}_\dagger)^\dagger$ and call a quasi-CM-frame $\mathfrak{F}$ *descriptive* if $\mathfrak{F} \simeq (\mathfrak{F}^\dagger)_\dagger$.

Let $t$ be the translation of $\mathcal{LM}_n$ into $\mathcal{LM}'_n$ which prefixes $\Box_I$ to every subformula of a given $\mathcal{LM}_n$-formula. To show that $t$ is an embedding of IM-logics (in $\mathcal{LM}_n$) into CM-logics (in $\mathcal{LM}'_n$), we need operators transforming quasi-IM-frames to quasi-CM-frames and back. They are generalizations of the operators $\sigma$ and $\rho$ defined in [27]. Since the number of modal operators is not essential, we will be considering for simplicity the monomodal language $\mathcal{LM}$ with the operator $\bigcirc$.

Given a quasi-IM-frame $\mathfrak{F} = \langle W, R, \bigcirc, P\rangle$, we construct a quasi-CM-frame $\sigma\mathfrak{F} = \langle W, R_I, \sigma\bigcirc, \sigma P\rangle$ by taking $R_I = R$, $\sigma P$ the Boolean closure of $P$ and

[1] Actually, Gödel used a somewhat different translation, but it is equivalent to $t$ as far as only **S4** and its normal extensions are concerned

$\sigma \bigcirc X = \bigcirc \Box_I X$, for every $X \in \sigma P$. As was shown in [27] (see the proof of Lemma 2), $\Box_I X \in P$ for every $X \in \sigma P$. Therefore, $\sigma P$ is closed under $\sigma\bigcirc$ and so $\sigma\mathfrak{F}$ is a quasi-CM-frame indeed. Observe also that

$$\Box_I \sigma \bigcirc \Box_I X = \Box_I \bigcirc \Box_I \Box_I X = \bigcirc \Box_I X = \sigma \bigcirc X,$$

for all $X \in \sigma P$. It follows that the formula

$$Mix = \Box_I \bigcirc \Box_I p \leftrightarrow \bigcirc p$$

is valid in $\sigma\mathfrak{F}$. Too, it is well known that $\langle W, R_I, \sigma P\rangle$ validates the monomodal Grzegorczyk logic $\mathbf{Grz} = \mathbf{S4} \oplus \Box_I(\Box_I(p \to \Box_I p) \to p) \to p$. To sum up, we obtain

**Lemma 16** *If $\mathfrak{F}$ is a quasi-IM-frame then $\sigma\mathfrak{F}$ is a quasi-CM-frame validating $Mix$ and* **Grz**.

Conversely, let $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ be a quasi-CM-frame. We construct from it a quasi-IM-frame $\rho\mathfrak{F}$ first by modifying $\bigcirc$ so that the resulting frame $\mathfrak{F}^*$ would validate $Mix$ (and the same $t$-translations of IM-formula as $\mathfrak{F}$) and then by collapsing clusters in $\mathfrak{F}^*$ into single points and converting the result to a quasi-IM-frame in the standard way (see [22]).

Define an operation $\bigcirc^*$ on $P$ by taking, for every $X \in P$,

$$\bigcirc^* X = \Box_I \bigcirc \Box_I X$$

and put $\mathfrak{F}^* = \langle W, R_I, \bigcirc^*, P\rangle$.

**Lemma 17** *If $\mathfrak{F}$ is a quasi-CM-frame then*
(i) *$\mathfrak{F}^*$ is a quasi-CM-frame too;*
(ii) *$\mathfrak{F}^* \models Mix$;*
(iii) *for every $\mathcal{LM}$-formula $\varphi$, $\mathfrak{F}^* \models t(\varphi)$ iff $\mathfrak{F} \models t(\varphi)$.*

**Proof.** (i) and (ii) are trivial and (iii) is proved by straightforward induction on the construction of $\varphi$. ❑

Assume now that a quasi-CM-frame $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ validates $Mix$. Denote by $[x]$ the cluster containing $x$, i.e., $[x] = \{y \in W : xR_I y \text{ and } yR_I x\}$ and put

$$[X] = \{[x] : x \in X\},$$

$$[x][R_I][y] \text{ iff } xR_I y,$$

$$[P] = \{[X] : \bigcup[X] \in P\},$$

$$[\bigcirc][X] = \{[x] : x \in \bigcirc(\bigcup[X]\}.$$

The structure $[\mathfrak{F}] = \langle [W], [R_I], [\bigcirc], [P]\rangle$ is called the *skeleton* of $\mathfrak{F}$.

**Lemma 18** *If $\mathfrak{F}$ is a quasi-CM-frame validating $Mix$ then*

(i) $[\mathfrak{F}]$ *is also a quasi-CM-frame, with $[\mathfrak{F}]^\dagger$ being a subalgebra of $\mathfrak{F}^\dagger$;*

(ii) $[R_I]$ *is a partial order on $[W]$;*

(iii) *for every $\mathcal{LM}$-formula $\varphi$, $\mathfrak{F} \models t(\varphi)$ iff $[\mathfrak{F}] \models t(\varphi)$.*

**Proof.** (i) It is well known that the map $x \mapsto [x]$ is a p-morphism of the **S4**-frame $\langle W, R_I, P\rangle$ onto $\langle [W], [R_I], [P]\rangle$. This means that the map $f : [X] \mapsto \bigcup[X]$ is an injection of $[P]$ into $P$ preserving $\Box_I$. So it remains to show that $f$ preserves the second modal operator. We clearly have

$$f([\bigcirc][X]) = \bigcup\{[x] : \ x \in \bigcirc(\bigcup[X])\} \supseteq \bigcirc(\bigcup[X]) = \bigcirc(f([X])).$$

And the converse inclusion follows from $Mix$. Indeed,

$$\bigcirc(\bigcup[X]) = \Box_I \bigcirc \Box_I(\bigcup[X])$$

and so the whole cluster $[x]$ is in $\bigcirc(\bigcup[X])$ whenever one of its points is there.

(ii) is clear and (iii) is established by induction. ❑

Finally, given an arbitrary quasi-CM-frame $\mathfrak{F}$, we first form the frame $[\mathfrak{F}^*] = \langle W, R_I, \bigcirc_1, P\rangle$ and then transform it to a quasi-IM-frame $\rho\mathfrak{F} = \langle W, R, \bigcirc, \rho P\rangle$ by taking $R = R_I$ and $\rho P = \{\Box_I X : \ X \in P\}$. If we drop $\bigcirc$, $\rho$ will be just the standard operator converting **S4**-frames to **Int**-frames. And since, by $Mix$, $\bigcirc$ maps cones to cones, $\rho\mathfrak{F}$ is an quasi-IM-frame indeed. By induction on the construction of $\varphi$ and using Lemmas 17 and 18 one can readily prove the following

**Lemma 19** *For every $\mathcal{LM}$-formula $\varphi$ and every quasi-CM-frame $\mathfrak{F}$,*

$$\mathfrak{F} \models t(\varphi) \text{ iff } \rho\mathfrak{F} \models \varphi.$$

Clearly we also have

**Lemma 20** *$\mathfrak{F} \simeq \rho\sigma\mathfrak{F}$, for every quasi-IM-frame $\mathfrak{F}$.*

Everything is ready now to embed IM-logics $L$ into extensions of $\mathbf{S4} \otimes \mathbf{C}$, **S4** in the language with $\Box_I$ and **C** in that with $\bigcirc$ (with the modal operators of $L$, to be more exact). Say that a CM-logic $M$ is a *CM-companion* of $L$ and $L$ the *IM-fragment* of $M$ if, for all $\mathcal{LM}$-formulas $\varphi$,

$$\varphi \in L \text{ iff } t(\varphi) \in M.$$

It is easy to see that, for every extension $M$ of $\mathbf{S4} \otimes \mathbf{C}$ (in $\mathcal{LM}'$), the set

$$\rho M = \{\varphi \in \mathcal{LM} : \ t(\varphi) \in M\}$$

is the IM-fragment (in $\mathcal{LM}$) of $M$ and that $\rho$ is a homomorphism of the lattice of CM-logics onto that of IM-logics.

**Proposition 21** *If a CM-logic $M$ is characterized by a class $\mathcal{C}$ of quasi-CM-frames the $\rho M$ is characterized by the class $\rho\mathcal{C} = \{\rho\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$.*

**Proof.** Follows from Lemma 19. ❑

The following theorem describes an (infinite) family of CM-companions of each consistent IM-logic.

**Theorem 22** *Every logic $M$ in the interval*

$$[(\mathbf{S4} \otimes \mathbf{C}) \oplus t(\Gamma), (\mathbf{Grz} \otimes \mathbf{C}) \oplus t(\Gamma) \oplus Mix]$$

*is a CM-companion of the IM-logic $L = \mathbf{IntC} \oplus \Gamma$, where $\Gamma$ is a set of $\mathcal{LM}$-formulas.*

**Proof.** Suppose $\varphi \notin L$. Then there is a quasi-IM-frame $\mathfrak{F}$ for $L$ refuting $\varphi$. By Lemmas 19 and 20, we have $\sigma\mathfrak{F} \not\models t(\varphi)$ and $\sigma\mathfrak{F} \models t(\Gamma)$. By Lemma 16, $\sigma\mathcal{F} \models \mathbf{Grz}$ and $\sigma\mathfrak{F} \models Mix$. Thus, we obtain $\sigma\mathfrak{F} \models M$ and $\sigma\mathfrak{F} \not\models t(\varphi)$, from which $\varphi \notin \rho M$.

Conversely, if $\varphi \notin \rho M$ then $t(\varphi) \notin M$ and so there is a quasi-CM-frame $\mathfrak{F}$ for $M$ refuting $t(\varphi)$. By Lemma 19, $\rho\mathfrak{F} \not\models \varphi$ and $\rho\mathfrak{F} \models \Gamma$. So, $\varphi \notin L$. ❑

**Example 23**

1. If an extension $M$ of $\mathbf{S4}$ is a modal companion of an intermediate logic $\mathbf{Int} + \Gamma^2$ then $M \otimes \mathbf{C}$ is a CM-companion of $\mathbf{IntC} \oplus \Gamma$. (For we have $M = M' \oplus t(\Gamma)$, for some $M'$ in the interval $[\mathbf{S4}, \mathbf{Grz}]$, and so $M \otimes \mathbf{C} = (M' \otimes \mathbf{C}) \oplus t(\Gamma)$.) In particular, $\mathbf{S4} \otimes \mathbf{C}$ and $\mathbf{Grz} \otimes \mathbf{C}$ are CM-companions of $\mathbf{IntC}$.

2. $\mathbf{S4} \otimes (\mathbf{C} \oplus \bigcirc\top)$ is a CM-companion of $\mathbf{IntC} \oplus \bigcirc\top$. This follows from the inclusions

$$(\mathbf{S4} \otimes \mathbf{C}) \oplus t(\bigcirc\top) \subseteq \mathbf{S4} \otimes (\mathbf{C} \oplus \bigcirc\top) \subseteq (\mathbf{S4} \oplus \mathbf{C}) \oplus Mix \oplus t(\bigcirc\top).$$

3. $\mathbf{S4} \otimes (\mathbf{C} \oplus \bigcirc p \to p)$ is a CM-companion of $\mathbf{IntC} \oplus \bigcirc p \to p$. The proof is analogous.

4. Each IM-logic $L = \mathbf{IntR} \oplus \Gamma$ is embeddable by $t$ in any logic in the interval $[(\mathbf{S4} \otimes \mathbf{R}) \oplus t(\Gamma), (\mathbf{Grz} \otimes \mathbf{R}) \oplus Mix \oplus t(\Gamma)]$. Indeed, let $\phi = \bigcirc(p \wedge q) \to \bigcirc p$. Then the claim follows from the inclusions

$$(\mathbf{S4} \otimes \mathbf{C}) \oplus t(\Gamma) \oplus t(\phi) \subseteq (\mathbf{S4} \otimes \mathbf{R}) \oplus t(\Gamma) \subseteq (\mathbf{S4} \oplus \mathbf{C}) \oplus Mix \oplus t(\Gamma) \oplus t(\phi)$$

which are established by a simple syntactical argument.

5. Each IM-logic $L = \mathbf{IntK} \oplus \Gamma$ is embeddable by $t$ in any logic in the interval $[(\mathbf{S4} \otimes \mathbf{K}) \oplus t(\Gamma), (\mathbf{Grz} \otimes \mathbf{K}) \oplus Mix \oplus t(\Gamma)]$. The proof is similar (for details consult [27]).

[2] $+$ presupposes taking the closure only under modus ponens and substitution.

It is worth noting that every CM-companion $M$ of an IM-logic $L$ can be reduced, in a sense, to a CM-companion of $L$ containing $Mix$. Say that a CM-logic $M'$ is a *$Mix$-reduct* of a CM-logic $M$ if $Mix \in M'$ and, for every formula $\varphi$, $\varphi \in M'$ iff $r(\varphi) \in M$, where $r$ replaces each occurrence of $\bigcirc$ in $\varphi$ with $\Box_I \bigcirc \Box_I$. Then, by Lemma 17, for each CM-companion $M$ of an IM-logic $L$, there exists a $Mix$-reduct $M'$ of $M$ such that $\rho M' = L$ (if $M$ is characterized by a frame $\mathfrak{F}$ then $M'$ can be defined as the logic of $\mathfrak{F}^*$).

As far as CM-companions with $Mix$ are concerned, we can get a correspondence similar to that between intermediate logics and their modal companions above **S4** (see [6]). Indeed, the logic $(\mathbf{S4} \otimes \mathbf{C}) \oplus t(\Gamma) \oplus Mix$ is clearly the smallest CM-companion with $Mix$ of an IM-logic $L = \mathbf{IntC} \oplus \Gamma$; we denote it by $\tau L$. And now we are going to show that the logic $\sigma L = (\mathbf{Grz} \otimes \mathbf{C}) \oplus t(\Gamma) \oplus Mix$ is the greatest CM-companion of $L$ containing $Mix$. To this end we require a lemma concerning monomodal frames for **Grz** in the language with $\Box$.

**Lemma 24** *Suppose $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is a model based upon a partially ordered frame $\mathfrak{F} = \langle W, R, P \rangle$ for* **Grz** *and $\Gamma$ a finite set of formulas closed under subformulas. Then there is a model $\mathfrak{N} = \langle \sigma\rho\mathfrak{F}, \mathfrak{U} \rangle$ (based upon the frame $\sigma\rho\mathfrak{F} = \langle W, R, \sigma\rho P \rangle$) such that, for every $\varphi \in \Gamma$,*

$$\mathfrak{V}(\Box\varphi) = \mathfrak{U}(\Box\varphi).$$

**Proof.** Clearly it is enough to show that there exists a valuation $\mathfrak{U}$ in $\sigma\rho\mathfrak{F}$ such that $\mathfrak{V}(\Diamond\varphi) = \mathfrak{U}(\Diamond\varphi)$, for all $\varphi \in \Gamma$. To construct it, we first apply to $\mathfrak{M}$ and $\Gamma$ the Selection Procedure introduced in [29]. As a result we obtain a finite model $\mathfrak{M}^* = \langle \mathfrak{F}^*, \mathfrak{V}^* \rangle$ and a cofinal subreduction $f$ of $\mathfrak{F}$ to $\mathfrak{F}^* = \langle W^*, R^* \rangle$ satisfying the following properties:

(i) $\mathfrak{F}^*$ is a partial order (since $\mathfrak{F} \models \mathbf{Grz}$);

(ii) $\forall x \in \mathrm{dom} f\ \forall \varphi \in \Gamma\ (x \in \mathfrak{V}(\varphi) \Leftrightarrow f(x) \in \mathfrak{V}^*(\varphi))$;

(iii) $\forall x \in W - \mathrm{dom} f\ \exists y \in \mathrm{dom} f\ (xRy \wedge x \sim_\Gamma y)$[3] (from which it follows that $f$ satisfies the closed domain condition for the set $\mathfrak{D}^*$ of closed domains in $\mathfrak{M}^*$).

With each point $v \in W^*$ we associate the set

$$X_v = \Diamond f^{-1}(v) - \bigcup_{\neg v R^* u} \Diamond f^{-1}(u).$$

Since $f^{-1}(v) \in P$, it follows immediately from the definition that $X_v \in \sigma\rho P$, $f^{-1}(v) \subseteq X_v$ and $f^{-1}(v)$ is a cover for $X_v$. Then for every $x \in W$, we put

$$g(x) = \begin{cases} v & \text{if } x \in X_v,\ v \in W^* \\ \text{undefined} & \text{otherwise.} \end{cases}$$

One can readily check (consult [28] or [7]) that $g$ is a cofinal subreduction of $\sigma\rho\mathfrak{F}$ to $\mathfrak{F}^*$ satisfying the closed domain condition for $\mathfrak{D}^*$.

[3] $x \sim_\Gamma y$ means that the same formulas in $\Gamma$ are true at $x$ and $y$ in $\mathfrak{M}$.

Now we define a valuation $\mathfrak{U}$ in $\sigma\rho\mathfrak{F}$ in the same way as it was done in the proof of Proposition 9 in [29]. Namely, for every $x \in \mathrm{dom}g$ and every variable $p$, we put

$$x \in \mathfrak{U}(p) \text{ iff } g(x) \in \mathfrak{V}^*(p).$$

And if $x \notin \mathrm{dom}g$ then, by (iii), there is $y \in \mathrm{dom}g$ such that $xRy$ and $x \sim_\Gamma y$. Then for every $z \notin \mathrm{dom}g$ such that $g(\{u : xRu\}) = g(\{u : zRu\})$, we put

$$x \in \mathfrak{U}(p) \text{ iff } g(y) \in \mathfrak{V}^*(p).$$

Let $\mathfrak{N} = \langle \sigma\rho\mathfrak{F}, \mathfrak{U}\rangle$. By the proof of Proposition 9 in [29], for every $\varphi \in \Gamma$, we have

- if $x \in \mathrm{dom}g$, then $x \in \mathfrak{U}(\varphi)$ iff $g(x) \in \mathfrak{V}^*(\varphi)$;
- if $x \notin \mathrm{dom}g$, then there is $y \in \mathrm{dom}g$ such that $xRy$ and $x \in \mathfrak{U}(\varphi)$ iff $y \in \mathfrak{U}(\varphi)$.

The claim of our lemma follows immediately from these properties, (ii) and (iii). ❑

We use Lemma 24 to prove the following

**Lemma 25** *Let $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ be a quasi-CM-frame such that $R_I$ a partial order and $\mathfrak{F} \models Mix$. Then for all $\varphi \in \mathcal{LM}'$,*

$$\mathfrak{F} \models \varphi \textit{ iff } \sigma\rho\mathfrak{F} \models \varphi.$$

**Proof.** The implication $\mathfrak{F} \models \varphi \Rightarrow \sigma\rho\mathfrak{F} \models \varphi$ follows from $\sigma\rho P \subseteq P$.

Conversely, suppose $\mathfrak{F}$ refutes $\varphi$. For each subformula $\bigcirc\psi$ of $\varphi$ we fix a new variable $q(\bigcirc\psi)$ and put

$$\begin{array}{rcl} \chi^q &=& \chi,\ \chi \text{ atomic},\\ (\chi_1 \to \chi_2)^q &=& \chi_1^q \to \chi_2^q,\\ (\chi_1 \wedge \chi_2)^q &=& \chi_1^q \wedge \chi_2^q,\\ (\chi_1 \vee \chi_2)^q &=& \chi_1^q \vee \chi_2^q,\\ (\Box_I\chi)^q &=& \Box_I\chi^q,\\ (\bigcirc\chi)^q &=& \Box_I q(\bigcirc\chi). \end{array}$$

Let $\Gamma = \{\psi^q : \psi \in \mathbf{Sub}\varphi\}$, where $\mathbf{Sub}\varphi$ is the set of $\varphi$'s subformulas, and let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ be a model refuting $\varphi$. Define a valuation $\mathfrak{U}$ of the extended language in $\mathfrak{F}$ by taking

$$\mathfrak{U}(p) = \mathfrak{V}(p),\ \text{for } p \in \mathbf{Sub}\varphi,$$

$$\mathfrak{U}(q(\bigcirc\psi)) = \mathfrak{V}(\bigcirc\psi),\ \text{for } \bigcirc\psi \in \mathbf{Sub}\varphi.$$

Then for $\psi \in \mathbf{Sub}\varphi$ and $x \in W$ we clearly have

$$\mathfrak{V}(\psi) = \mathfrak{U}(\psi) = \mathfrak{U}(\psi^q). \tag{13}$$

By Lemma 24, there exists a valuation $\mathfrak{U}'$ in $\sigma\rho\mathfrak{F}$ such that, for all $\psi \in \Gamma$,

$$\mathfrak{U}'(\Box_I \psi) = \mathfrak{U}(\Box_I \psi). \tag{14}$$

Now we prove by induction that, for all $\psi \in \mathbf{Sub}\varphi$,

$$\mathfrak{U}'(\psi^q) = \mathfrak{U}'(\psi). \tag{15}$$

The only non-trivial case is that for $\bigcirc\psi$. We have:

$$\begin{array}{rcll}
\mathfrak{U}'((\bigcirc\psi)^q) & = & \mathfrak{U}((\bigcirc\psi)^q) & \text{(by (14))} \\
& = & \mathfrak{U}(\Box_I \bigcirc \Box_I \psi^q) & \text{(by (13) and } Mix) \\
& = & \mathfrak{U}'(\Box_I \bigcirc \Box_I \psi^q) & \text{(by (14))} \\
& = & \mathfrak{U}'(\Box_I \bigcirc \Box_I \psi) & \text{(by IH)} \\
& = & \mathfrak{U}'(\bigcirc\psi) & \text{(by } Mix).
\end{array}$$

Now it follows from (13), (14) and (15) that $\varphi$ is refuted in $\sigma\rho\mathfrak{F}$. ❏

**Lemma 26** *Each CM-logic $L$ containing $(\mathbf{Grz} \otimes \mathbf{C}) \oplus Mix$ is characterized by a frame $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ such that $R_I$ is a partial order.*

**Proof.** Consider a descriptive quasi-CM-frame $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ which determines $L$. Say that a point $x \in W$ is *eliminable* in $\mathfrak{F}$ if it has a proper $R_I$-successor in every set $X \in P$ containing $x$. Put $W' = \{x \in W :\ x \text{ is noneliminable in } \mathfrak{F}\}$ and $P' = \{X \cap W' :\ X \in P\}$. One can readily check now that the structure $\mathfrak{F}' = \langle W', R_I{\restriction}W', \bigcirc', P'\rangle$, where $\bigcirc'(X \cap W') = W' \cap \bigcirc X$, is a quasi-CM-frame such that $\mathfrak{F}^\dagger \simeq \mathfrak{F}'^\dagger$ and $R_I{\restriction}W'$ is a partial order (for details consult [11] or [7]). ❏

We are in a position now to prove an analog of Blok-Esakia's theorem for IM-logics and their CM-companions containing $Mix$.

**Theorem 27** *A CM-logic $M$ containing $Mix$ is a CM-companion of an IM-logic $L$ iff $\tau L \subseteq M \subseteq \sigma L$.*

**Proof.** ($\Leftarrow$) follows from Theorem 22.

($\Rightarrow$) It suffices to show that $M \subseteq \sigma L$. Let us first prove that

$$\{\rho\mathfrak{F} :\ \mathfrak{F} \models M\} = \{\mathfrak{G} :\ \mathfrak{G} \models L\}. \tag{16}$$

(Of course, we do not distinguish between isomorphic frames.) To establish this we require Birkhoff's characterization of varieties; so it will be more convenient for us to consider frames as algebras, that is to establish the equality

$$\{(\rho\mathfrak{F})^\dagger :\ \mathfrak{F} \models M\} = \{\mathfrak{G}^\dagger :\ \mathfrak{G} \models L\}.$$

By Proposition 21, $L$ is characterized by the class $\mathcal{C} = \{(\rho\mathfrak{F})^\dagger : \mathfrak{F} \models M\}$ and so it suffices to show that $\mathcal{C}$ is closed under forming direct products, subalgebras and homomorphic images. That $\mathcal{C}$ is closed under the first two operations is shown in the same way as in [19]. To prove the closure under homomorphisms, suppose that $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ is a quasi-CM-frame for $M$ and $h$ a homomorphism from $(\rho\mathfrak{F})^\dagger$ onto $\mathfrak{H}^\dagger$. Since $(\sigma\rho\mathfrak{F})^\dagger$ is a subalgebra of $\mathfrak{F}^\dagger$, $(\sigma\rho\mathfrak{F})^\dagger \models M$. Besides, by Lemma 20, we have $(\rho\sigma\mathfrak{H})^\dagger \simeq \mathfrak{H}^\dagger$. So it is sufficient to construct a homomorphism $g$ from $(\sigma\rho\mathfrak{F})^\dagger$ onto $(\sigma\mathfrak{H})^\dagger$, for then we shall have $\sigma\mathfrak{H} \models M$. Every set $X \in \sigma\rho P$ can be represented as

$$X = \bigcap_{i=1}^{n}(-Y_i \cup Z_i),$$

for some $Y_i, Z_i \in \rho P$. Define $g$ by taking

$$g(X) = \bigcap_{i=1}^{n}(-h(Y_i) \cup h(Z_i)).$$

Clearly, $g(X)$ is an element in $(\sigma\mathfrak{H})^\dagger$ coinciding with $h(X)$ for every $X \in \rho P$. It was shown in [19] that $g$ is a surjection and preserves the Boolean operations and $\Box_I$. Let us show that it preserves $\bigcirc$ as well. Using $Mix$ we have:

$$\begin{aligned} g(\bigcirc X) &= g(\bigcirc\Box_I \bigcap_{i=1}^{n}(-Y_i \cup Z_i)) \\ &= g(\bigcirc \bigcap_{i=1}^{n} \Box_I(-Y_i \cup Z_i)) \\ &= g(\bigcirc \bigcap_{i=1}^{n}(Y_i \supset Z_i)) \\ &= h(\bigcirc \bigcap_{i=1}^{n}(Y_i \supset Z_i)) \\ &= \bigcirc \bigcap_{i=1}^{n}(h(Y_i) \supset h(Z_i)) \\ &= \bigcirc g(X). \end{aligned}$$

Now, to prove that $M \subseteq \sigma L$ it suffices to show that a characteristic frame $\mathfrak{F} = \langle W, R_I, \bigcirc, P\rangle$ for $\sigma L$ is also a frame for $M$. By Lemma 26, without loss of generality we may assume that $R_I$ is a partial order. Since $\rho\mathfrak{F} \models L$, there is, by (16), a frame $\mathfrak{F}'$ for $M$ such that $\rho\mathfrak{F} \simeq \rho\mathfrak{F}'$ and so $\sigma\rho\mathfrak{F} \simeq \sigma\rho\mathfrak{F}'$. Clearly, $\sigma\rho\mathfrak{F}' \models M$. Therefore, $\sigma\rho\mathfrak{F} \models M$ and, by Lemma 25, $\mathfrak{F} \models M$. ❑

**Corollary 28** *The map $\sigma$ is an isomorphism from the lattice of IM-logics onto the lattice of CM-logics containing* $(\mathbf{Grz} \otimes \mathbf{C}) \oplus Mix$.

**Remark.** It is worth noting that the analogy with Blok-Esakia's Theorem is not perfect if we consider CM-logics without $Mix$. For, as has been shown by C. Grefe, there is an IM-logic $L$ and its CM-companion $M$ (without $Mix$) such that $L \neq \rho(M \oplus Mix)$.

**Proposition 29** *If an IM-logic $L$ is characterized by a class $\mathcal{C}$ of quasi-IM-frames then $\sigma L$ is characterized by the class $\sigma\mathcal{C} = \{\sigma\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$.*

**Proof.** If $\mathfrak{F} \models L$ then, by Lemmas 19 and 20, $\sigma\mathfrak{F} \models t(L)$. And, as we know, $\sigma\mathfrak{F}$ validates the Grzegorczyk formula in the monomodal language with $\Box_I$. Hence $\mathfrak{F} \models \sigma L$. Suppose now that $\varphi \notin \sigma L$ and consider the logic $\sigma L \oplus \varphi$. By Theorem 27, $\rho(\sigma L \oplus \varphi)$ is a proper extension of $L$ and so there is a formula $\psi \notin L$ such that $\sigma L \oplus \varphi = \sigma L \oplus t(\psi)$. Take any frame $\mathfrak{F} \in \mathcal{C}$ separating $\psi$ from $L$. Then, by Lemmas 19 and 20, $\sigma\mathfrak{F}$ will separate $t(\psi)$ and so $\varphi$ from $\sigma L$. ❑

**Theorem 30** *The map $\rho$ preserves decidability, the finite model property and tabularity. The map $\sigma$ preserves the finite model property and tabularity.*

**Proof.** That $\rho$ preserves decidability follows directly from the definition of $\rho$ and the rest from Propositions 21, 29 and the fact that $\rho\mathfrak{F}$ is a finite IM-frame whenever $\mathfrak{F}$ is a finite CM-frame and $\sigma\mathfrak{F}$ is finite whenever $\mathfrak{F}$ is finite. ❑

This preservation result provides us with a tool for establishing the finite model property (FMP, for short) of IM-logics by means of proving it for suitable CM-logics. For example, we have

**Theorem 31** *Suppose that an intermediate logic* $\mathbf{Int} + \Gamma$ *has FMP. Then the following IM-logics possess FMP too:*

- $\mathbf{IntC} \oplus \Gamma$, $\mathbf{IntC} \oplus \Gamma \oplus \bigcirc\top$, $\mathbf{IntC} \oplus \Gamma \oplus \bigcirc p \to p$;
- $\mathbf{IntR} \oplus \Gamma$, $\mathbf{IntR} \oplus \Gamma \oplus \bigcirc\top$, $\mathbf{IntR} \oplus \Gamma \oplus \bigcirc p \to p$;
- $\mathbf{IntK}_\Box \oplus \Gamma$, $\mathbf{IntK}_\Box \oplus \Gamma \oplus \bigcirc p \to p$.

**Proof.** By Theorem 30, it suffices to present CM-companions of these logics having FMP. The examples given above show that the logics under consideration have CM-companions of the form $(\mathbf{S4} \oplus t(\Gamma)) \otimes L$, where $L$ is a monomodal classical logic in the list

$$\{\mathbf{C}, \mathbf{C} \oplus \bigcirc\top, \mathbf{C} \oplus \bigcirc p \to p, \mathbf{R}, \mathbf{R} \oplus \bigcirc\top, \mathbf{R} \oplus \bigcirc p \to p, \mathbf{K}, \mathbf{K} \oplus \bigcirc p \to p\}.$$

All the listed logics are known to have the global FMP in the sense that for every formulas $\varphi$ and $\psi$, if there is a frame $\mathfrak{F}$ for $L$ such that $\mathfrak{F} \models \varphi$ and $\mathfrak{F} \not\models \psi$ then there is a finite frame for $L$ validating $\varphi$ and refuting $\psi$. The claim of the theorem follows now from the two preservation results of [28] and [12], namely: (i) if $\mathbf{Int} + \Gamma$ has FMP then $\mathbf{S4} \oplus \Gamma$ has global FMP and (ii) if two classical monomodal logics $L_1$ and $L_2$ have global FMP then $L_1 \otimes L_2$ has it as well. ❑

For more results on the finite model property of IM-logics see [27].

## 4 CM-companions of $\Box\Diamond$-IM-logics

We now focus attention on embeddings of extensions of $\mathbf{IntK}_{\Box\Diamond}$. According to Example 23, all extensions of $\mathbf{IntK}_{\Box}$ are embedded by $t$ into normal bimodal logics. However, nothing guarantees that extensions of $\mathbf{Int}_{\Diamond}$ and, more generally, arbitrary $\Box\Diamond$-IM-logics can be embedded into normal CM-logics. The reason is that although the $t$-translation of $\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$ is deductively equal to itself in $(\mathbf{S4} \otimes \mathbf{C}) \oplus Mix$, this is not the case for the $t$-translation of $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$, wich will be denoted by $t\Diamond$: modulo $Mix$ it is deductively equal only to $\Diamond(\Box_I p \vee \Box_I q) \leftrightarrow \Diamond p \vee \Diamond q$. This is another important difference between $\Diamond$-like and $\Box$-like operators in intuitionistic modal logic, which reflects the nonstandard behavior of generated subframes and p-morphisms.

We formulate now a variant of Blok-Esakia's Theorem for $\Box\Diamond$-IM-logics. Put

$$\Phi_1 = \{\Diamond(\Box_I p \vee \Box_I q) \leftrightarrow \Diamond p \vee \Diamond q, \neg\Diamond\bot\},$$

$$\Phi_2 = \{\Box p \leftrightarrow \Box_I \Box \Box_I p, \Diamond p \leftrightarrow \Box_I \Diamond \Box_I p\},$$

$$\Phi = \Phi_1 \cup \Phi_2.$$

As a consequence of Theorem 22, Example 23 and Corollary 28 we obtain

**Theorem 32** *Each $\Box\Diamond$-IM-logic $\mathbf{IntK}_{\Box\Diamond} \oplus \Gamma$ is embeddable by $t$ into any logic in the interval*

$$[(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus t\Diamond \oplus t(\neg\Diamond\bot) \oplus t(\Gamma), (\mathbf{Grz} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus \Phi \oplus t(\Gamma)].$$

*The map $\sigma$, restricted to the lattice of $\Box\Diamond$-IM-logics, is an isomorphism of that lattice onto the lattice of extensions of $(\mathbf{Grz} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus \Phi$.*

**Example 33** Using $\Phi_2$, one can easily show that for every $k, l, m, n \geq 0$ the logic

$$(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus \Phi \oplus \Diamond^k \Box^l p \rightarrow \Box^m \Diamond^n p$$

is a CM-companion of $\mathbf{IntK}_{\Box\Diamond} \oplus \Diamond^k \Box^l p \rightarrow \Box^m \Diamond^n p$.

It is worth noting that although logics containing $(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus \Phi$ are not necessarily normal (in fact, they are normal only if $\Diamond$ is almost trivial), they have a rather natural Kripke-type semantics with a nonstandard truth-condition for $\Diamond$, viz., frames of the form $\mathfrak{F} = \langle W, R_I, R_{\Box}, R_{\Diamond}, P\rangle$ such that $\langle W, R_I, P\rangle$ is an $\mathbf{S4}$-frame, $R_{\Box}$ and $R_{\Diamond}$ satisfy the conditions (1) and (2), respectively, and $P$ is closed under the usual $\Box$ and the unusual $\Diamond$:

$$\Diamond X = \{x \in W : \exists y \in \Box_I X \;\; x R_{\Diamond} y\}.$$

By adapting the Stone-Jónsson-Tarski argument to this case, one can show that the defined semantics is adequate for logics containing $(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{R}) \oplus \Phi$.

We do not know whether all $\Box\Diamond$-IM-logics have normal (with respect to $\Diamond$) CM-companions. (We conjecture that this is not the case.) But those that are complete do have them.

Let $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond \rangle$ be a full weak $\Box\Diamond$-IM-frame. We can consider it also as a frame for the language with three modal operators $\Box_I$, $\Box$ and $\Diamond$ and the classical truth-conditions for them. The classical modal logic with those operators, characterized by a class $\mathcal{C}$ of such frames, is denoted by $\mathsf{Th}\mathcal{C}$.

By (11), every full weak $\Box\Diamond$-IM-frame $\mathfrak{F}$ validates $\Diamond\Box_I p \to \Box_I \Diamond p$. Too, we have

**Lemma 34** *For every full weak $\Box\Diamond$-IM-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond \rangle$ and every $\varphi$ in the language with $\Box$ and $\Diamond$,*

$$\mathfrak{F} \models \varphi \textit{ iff } \mathfrak{F} \models t(\varphi),$$

*where the former $\models$ is intuitionistic, while the latter one is classical.*

**Proof.** ($\Rightarrow$) Let $\mathfrak{V}$ be an intuitionistic valuation in $\mathfrak{F}$. We can consider it also as a classical valuation which will be denoted by $\mathfrak{U}$. Let us show by induction on the construction of $\varphi$ that

$$\mathfrak{V}(\varphi) = \mathfrak{U}(t(\varphi)).$$

The only non-trivial cases are $\varphi = \Box\psi$ and $\varphi = \Diamond\psi$. Suppose $x \notin \mathfrak{V}(\Box\psi)$. Then there are $y \in W$ and $z \notin \mathfrak{V}(\psi)$ such that $xRyR_\Box z$. By the induction hypothesis, $z \notin \mathfrak{U}(t(\psi))$ and so $x \notin \mathfrak{U}(t(\Box\psi))$, because $t(\Box\psi) = \Box_I \Box t(\psi)$. Let $x \notin \mathfrak{U}(\Box_I \Box t(\psi))$. Then $xRyR_\Box z$, for some $y \in W$ and $z \notin \mathfrak{U}(t(\psi))$, from which $x \notin \mathfrak{V}(\Box\psi)$.

Assume now that $x \in \mathfrak{V}(\Diamond\psi)$, i.e., $xR_\Diamond y$ for some $y \in \mathfrak{V}(\psi)$, and let $x \notin \mathfrak{U}(\Box_I \Diamond t(\psi))$, i.e., there is $z$ such that $xRz$ and no $R_\Diamond$-successor of $z$ is in $\mathfrak{U}(t(\psi))$. By (11) we have a point $u$ such that $yRu$ and $zR_\Diamond u$. But then $u \in \mathfrak{V}(\psi)$ (since $\mathfrak{V}(\psi)$ is a cone) and $u \notin \mathfrak{U}(t(\psi))$, contrary to the induction hypothesis. Conversely, if $x \in \mathfrak{U}(\Box_I \Diamond t(\psi))$ then, since $xR_I x$, there is $y \in \mathfrak{U}(t(\psi))$ such that $xR_\Diamond y$, from which $x \in \mathfrak{V}(\Diamond\psi)$.

($\Leftarrow$) Given a classical valuation $\mathfrak{U}$ in $\mathfrak{F}$, we define an intuitionistic valuation $\mathfrak{V}$ by taking $\mathfrak{V}(p) = \mathfrak{U}(\Box_I p)$ and in exactly the same way as above prove that $\mathfrak{V}(\varphi) = \mathfrak{U}(t(\varphi))$. ❑

**Theorem 35** *Suppose that a $\Box\Diamond$-IM-logic $L = \mathbf{IntK}_{\Box\Diamond} \oplus \Gamma$ is characterized by a class $\mathcal{C}$ of full weak $\Box\Diamond$-IM-frames. Then $L$ is embedded by $t$ into any logic $M$ in the interval*

$$[(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{K}) \oplus \Diamond\Box_I p \to \Box_I \Diamond p \oplus t(\Gamma), \mathsf{Th}\mathcal{C}].$$

**Proof.** Let $\varphi \notin L$. Then there is $\mathfrak{F} \in \mathcal{C}$ separating $\varphi$ from $L$. By Lemma 34, $\mathfrak{F} \not\models t(\varphi)$ and so $t(\varphi) \notin \mathsf{Th}\mathcal{C}$. On the other hand, it is readily checked that the

$t$-translations of the axioms of $\mathbf{IntK}_{\Box\Diamond}$ are in $(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{K}) \oplus \Diamond\Box_I p \to \Box_I \Diamond p$ and so $t(\varphi) \in M$ whenever $\varphi \in L$. ❑

Some consequences of Theorem 35 for **FS**-logics are worth noting. Fischer Servi [14], [15] proposed a somewhat different embedding $t'$ of a few complete **FS**-logics into bimodal classical logics (in the language with $\Box_I$ and $\Box$) containing $\Box_I \Box p \to \Box\Box_I p$ and $\Diamond\Box_I p \to \Box_I \Diamond p$, where $\Diamond$ is dual to $\Box$. Namely, she defined

$$t'(\Box\varphi) = \Box_I \Box t'(\varphi),$$
$$t'(\Diamond\varphi) = \Diamond t'(\varphi).$$

It turns out, however, that in fact $t'$ is a special case of $t$ in the framework of complete **FS**-logics.

Indeed, let $L = \mathbf{FS} \oplus \Gamma$ be a complete **FS**-logic. By Proposition 15, it is characterized by a class of full birelational frames (in which the relations for $\Box$ and $\Diamond$ coincide). It follows from Theorem 35 and (12) that $L$ is embedded by $t$ into the logic

$$(\mathbf{S4} \otimes \mathbf{K} \otimes \mathbf{K}) \oplus \Box_I \Box p \to \Box\Box_I p \oplus \Diamond\Box_I p \to \Box_I \Diamond p \oplus \Diamond' p \leftrightarrow \Diamond p \oplus t(\Gamma),$$

where $\Diamond'$ is dual to $\Box$. Identifying $\Diamond'$ and $\Diamond$, we conclude then that $t$ embeds $L$ into

$$(\mathbf{S4} \otimes \mathbf{K}) \oplus \Box_I \Box p \to \Box\Box_I p \oplus \Diamond\Box_I p \to \Box_I \Diamond p \oplus t(\Gamma).$$

By induction on the construction of $\varphi$ it is not hard to show that all frames for this logic validate $t(\varphi) \leftrightarrow t'(\varphi)$, and so we obtain

**Corollary 36** *Each complete* **FS**-*logic* $L = \mathbf{FS} \oplus \Gamma$ *is embedded by* $t'$ *into*

$$(\mathbf{S4} \otimes \mathbf{K}) \oplus \Box_I \Box p \to \Box\Box_I p \oplus \Diamond\Box_I p \to \Box_I \Diamond p \oplus t'(\Gamma).$$

It is not clear, however, whether all **FS**-logics are embedded by $t'$ into bimodal logics of this type.

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