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DECIDABLE FRAGMENTS OF FIRST-ORDER MODAL LOGICS

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Abstract. The paper considers the set $\mathcal{ML}_1$ of first-order polymodal formulas the modal operators in which can be applied to subformulas of at most one free variable. Using a mosaic technique, we prove a general satisfiability criterion for formulas in $\mathcal{ML}_1$, which reduces the modal satisfiability to the classical one. The criterion is then used to single out a number of new, in a sense optimal, decidable fragments of various modal predicate logics.

§1. Introduction. The classical decision problem—to single out expressive and decidable fragments of first-order logic—has a long history and hardly needs any justification: after all, classical first-order logic was and still remains in the very center of logical studies, both in mathematics and applications. Here are only three examples (out of dozens) of such fragments (the choice is not accidental—we shall use these results later on):

- the fragment containing only monadic predicate symbols [27, 5];
- the fragment with only two individual variables [36, 31];[1]
- the guarded fragment with quantification of the form $\exists\overline{y}(G(\overline{x},\overline{y})\wedge\varphi(\overline{x},\overline{y}))$, where the guard $G(\overline{x},\overline{y})$ is atomic[2] [1].

(The current state of art in this field is presented in the recent monograph [6]; see also [1, 43, 44, 13, 14, 18, 29, 32].)

For modal logicians the decision problem in first-order *modal* logics seemed almost hopeless. The following list covers basically all known results and leaves not too much space for maneuver:

- the monadic fragment of practically all modal predicate logics is undecidable [25] (see also [24]);
- the two-variable fragment of practically all modal predicate logics with constant domains (without equality and function symbols) is undecidable

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[1]The fragment with binary predicates and three variables is undecidable [42].

[2]For a precise definition see Section 5.

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[11, 21]; for example, such is the two-variable fragment of the predicate logics based on **K**, **K4**, **S4**, **S5**, etc.;
- only the one-variable fragment of various standard modal predicate logics turns out to be decidable (see [37, 10, 38, 2]).

The expressive capabilities of the one-variable fragment are rather weak: actually, predicates and quantifiers in its formulas are not more expressive than propositional variables and the modal operators of **S5**, respectively.[3] Two variables used without any restrictions immediately lead to undecidability (at least for logics with constant domains). But what if we somehow restrict their use?

All the existing proofs of undecidability for fragments of modal predicate logics exploit formulas of the form $\Box\psi(x, y)$ in which the necessity operator has at least two free variables in its scope; in fact, such formulas play an essential role in the reduction of undecidable problems to those fragments (see, for example, the proof of Theorem 2.3 below). So it seems natural to assume that it is the possibility of quantifying into modal contexts with two or more free variables that is 'responsible' for the high complexity of modal predicate logics.

The main aim of this paper is to show that this is indeed the case. We consider the fragment $\mathcal{ML}_1$ of the polymodal first-order language (without function symbols and equality) consisting of all those formulas that do not contain subformulas of the form $\Box\psi(x_1, \ldots, x_n)$ for $n \geq 2$. (In other words, we restrict modalities *de re* to formulas of one free variable and impose no restrictions whatsoever on modalities *de dicto*.) The fragment itself is certainly undecidable: it still contains full classical predicate logic. But many restrictions of the pure predicate (i.e., non-modal) part of $\mathcal{ML}_1$ to decidable fragments provide us with decidable fragments of modal predicate logics! For instance, we obtain the decidability of the one-variable fragment of modal predicate logics simply by using the fact that the corresponding non-modal fragment is decidable. Similarly, by restricting to two the number of distinct individual variables in $\mathcal{ML}_1$-formulas or by allowing only the guarded quantification we obtain decidable two-variable and guarded fragments of $\mathcal{ML}_1$ for various modal predicate logics. Of course, the resulting decidable fragments are not able to say that a binary relation $\psi(x, y)$ holds in all (or some) possible worlds for some objects x and y. As in the one-variable fragment, we can do this only for properties $\psi(x)$. The major difference, however, is that now we get means to work with much more complex properties $\psi(x)$ by choosing suitable decidable fragments of classical predicate logic. Thus, the obtained decidable fragments are in a sense optimal.

As in the case of the guarded fragment of classical first-order logic [1], the idea to consider the $\mathcal{ML}_1$-fragment of modal predicate logics came from *propositional modal logic*. In the series of papers [34, 26, 4, 3, 46, 49, 47, 50] a number of expressive and decidable concept description languages with modal operators were designed and investigated with the aim of representing and processing knowledge in dynamic application domains (where knowledge may depend on interacting agents, time, actions, etc.). Although those formalisms appeared

[3] We remind the reader that the one-variable fragment of classical first-order logic is equivalent to the propositional modal logic **S5** [45]. More generally, as was observed in [12], the one-variable fragment of a modal predicate logic with constant domains and the propositional part L is equivalent to the Cartesian product $L \times \textbf{S5}$; see Section 6 for details.

in the framework of knowledge representation in artificial intelligence, from the modal logic point of view they can be regarded as multi-dimensional polymodal systems whose semantical structures resemble Kripke models for modal predicate logics (see Section 6 for more details). And in fact those systems turn out to be embeddable in the fragments constructed in this paper, which may serve as an evidence of that these fragments really possess enough expressive power to be practically applicable.

It is not our aim in this paper to prove the decidability of fragments of as many modal predicate logics as possible (actually, that might be quite boring; modal logic has already passed through this experience). Instead, we deal with only one sufficiently representative polymodal logic $\mathbf{QK}^*$ with two kinds of necessity operators: $\Box_1, \dots, \Box_l$ interpreted in models (with constant and expanding domains) by arbitrary binary relations $R_1, \dots, R_l$, and $\Box_1^*, \dots, \Box_l^*$ interpreted by the transitive and reflexive closures of $R_1, \dots, R_l$. The reason for selecting $\mathbf{QK}^*$ is purely technical. First, it is complex enough to show how our method works. On the other hand, it is not too sophisticated to involve too many technical details and special cases (for example, we could take quantified **PDL**, but this would require to explain details that are not necessary to understand the method; the reader can restore them following [49]). Finally, as an easy consequence we obtain the decidability of the corresponding fragments of some standard modal predicate logics such as **QK**, **QT**, **QK4**, **QS4** simply by embedding them into $\mathbf{QK}^*$.

The paper is organized in the following way. The next section defines the syntax and semantics of the first-order modal logics we consider in this paper. It contains also two technical results: we show that the two-variable monadic fragment of $\mathbf{QK}^*$ is not recursively enumerable, and that in many cases satisfiability in models with varying and expanding domains can be reduced to satisfiability in models with constant domains. In Section 3 we codify Kripke models satisfying a given formula $\varphi \in \mathcal{ML}_1$ in certain structures called quasimodels and intended to 'take care' of the modal contexts in φ. In Section 4 we use a mosaic technique to prove a rather general satisfiability criterion for such formulas, and in Section 5 we apply it to single out several decidable fragments of first-order modal logics. Section 6 shows how the obtained results can be used to prove decidability of description logics with modal operators and the Cartesian products of various propositional modal logics with **S5**. Some open problems arising from this work are listed in Section 7.

§2. First-order modal logic.

In this paper, we deal with the *first-order polymodal language* $\mathcal{ML}$ constructed in the standard way from the following alphabet:

- *predicate symbols* $P_0, P_1, \dots,$
- *individual variables* $x_0, x_1, \dots,$
- *individual constants* $c_0, c_1, \dots,$
- the *booleans* $\wedge$, $\neg$,
- the *universal quantifier* $\forall x$ for each individual variable x,
- the *necessity operators* $\Box_1, \dots, \Box_l$, $l \geq 1$.

As usual, we assume that the set of predicate symbols in $\mathcal{ML}$ is non-empty and that each of them is of some fixed arity; 0-ary predicates are called *propositional variables* and denoted by $p_0, p_1, \dots$. It will be assumed that we have a sufficient supply of these variables, unary predicate symbols, and an infinite set *var* of individual variables. Formulas in $\mathcal{ML}$ will be denoted by small Greek letters φ, ψ, etc. Finally, by $\mathcal{L}$ we mean the classical (non-modal) first-order language that results from $\mathcal{ML}$ by omitting all formulas containing occurrences of the necessity operators.

$\mathcal{ML}$ is interpreted in *first-order Kripke models*, which are structures of the form $\mathfrak{M} = \langle \mathfrak{F}, D, I \rangle$, where $\mathfrak{F} = \langle W, R_1, \dots, R_l \rangle$ is the underlying (propositional) *polymodal Kripke frame* (R_i a binary relation on a non-empty set of worlds W), D is a non-empty set, the *domain* of $\mathfrak{M}$, and I is a function associating with every world $w \in W$ a first-order $\mathcal{L}$-structure

$$I(w) = \left\langle D, P_0^{I(w)}, \dots, c_0^{I(w)}, \dots \right\rangle$$

in which $P_i^{I(w)}$, for each i, is a predicate on D of the same arity as P_i (for a propositional variable p_i, the predicate $p_i^{I(w)}$ is simply one of the propositional constants $\top$, 'truth', or $\bot$, 'falsehood'), and $c_i^{I(w)}$ is an element in D such that $c_i^{I(u)} = c_i^{I(v)}$ for any $u, v \in W$. To simplify notation we will omit I and write P_i^w, p_i^w, c_i^w, etc., if this does not involve ambiguity.

An *assignment* in D is a function $\mathfrak{a}$ from *var* to D. The *value* $\tau^{\mathfrak{M},\mathfrak{a}}$ (or simply $\tau^{\mathfrak{a}}$, if understood) of a term—a variable or a constant—τ under $\mathfrak{a}$ in $\mathfrak{M}$ is $\mathfrak{a}(\tau)$ if τ is a variable, and $\tau^{I(w)}$ otherwise, where w is some (any) world in W. The *truth-relation* $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ (or simply $w \models^{\mathfrak{a}} \varphi$, if $\mathfrak{M}$ is understood) in the model $\mathfrak{M}$ under the assignment $\mathfrak{a}$ is defined inductively in the standard manner:

- $w \models^{\mathfrak{a}} P_i(\tau_1, \dots, \tau_n)$ iff $P_i^w(\tau_1^{\mathfrak{a}}, \dots, \tau_n^{\mathfrak{a}})$ is true in $I(w)$ (this fact will also be written as $I(w) \models^{\mathfrak{a}} P_i(\tau_1, \dots, \tau_n)$ or $I(w) \models P_i[\tau_1^{\mathfrak{a}}, \dots, \tau_n^{\mathfrak{a}}]$);
- $w \models^{\mathfrak{a}} \psi \wedge \chi$ iff $w \models^{\mathfrak{a}} \psi$ and $w \models^{\mathfrak{a}} \chi$;
- $w \models^{\mathfrak{a}} \neg\psi$ iff $w \not\models^{\mathfrak{a}} \psi$;
- $w \models^{\mathfrak{a}} \forall x \psi$ iff $w \models^{\mathfrak{b}} \psi$ for every assignment $\mathfrak{b}$ in D that may differ from $\mathfrak{a}$ only on x;
- $w \models^{\mathfrak{a}} \Box_i \varphi$ iff $v \models^{\mathfrak{a}} \varphi$ for all $v \in W$ such that wR_iv.

The set of $\mathcal{ML}$-formulas that are true in all models under all assignments will be denoted by $\mathbf{QK}$. (As the number of necessity operators is always clear from the context, we don't include it in the names of logics.)

Remark 2.1. (i) The models introduced above are known as *models with constant domains*. To define *models with varying* (or *changing*) *domains* we need one more function $\mathfrak{d} : W \mapsto 2^D$ such that $\mathfrak{d}(w) \neq \emptyset$ and $D = \bigcup_{w \in W} \mathfrak{d}(w)$. The only difference from the models with constant domains is in the truth-condition for $\forall x \psi$, which now looks as follows:

- $w \models^{\mathfrak{a}} \forall x \psi$ iff $w \models^{\mathfrak{b}} \psi$ for every assignment $\mathfrak{b}$ that may differ from $\mathfrak{a}$ only on x, provided that $\mathfrak{b}(x) \in \mathfrak{d}(w)$.

Thus, $\mathfrak{d}(w)$ is regarded as the true domain of $I(w)$. (For more details and a discussion of different approaches consult [24].)

(ii) Another important class of models consists of *models with expanding domains*, i.e., models with varying domains in which $\mathfrak{d}(u) \subseteq \mathfrak{d}(v)$ whenever uR_iv. Actually, later on in this section we will show that both varying and expanding domains can be reduced to constant ones, at least as far as the decidability of the fragments we are interested in is concerned. For that reason we will mostly be considering models with constant domains.

(iii) We deal with models having *rigid designators* in the sense that for any $u, v \in W$, we have $c_i^u = c_i^v$. But again, everything we prove in this paper holds for the non-rigid case as well.

Syntactically the logic **QK** can be represented by the calculus QK containing all the axiom schemata and inference rules of classical predicate calculus, the Barcan formulas

$$\forall x \Box_i \varphi \leftrightarrow \Box_i \forall x \varphi,$$

the modal axiom schemata

$$\Box_i(\varphi \to \psi) \to (\Box_i \varphi \to \Box_i \psi),$$

and the necessitation rules $\varphi / \Box_i \varphi$, for all $i = 1, \dots, l$. (Here $\chi_1 \to \chi_2$ can be regarded as an abbreviation for $\neg(\chi_1 \wedge \neg \chi_2)$.)

By adding to QK the standard modal axiom schemata of **T** (i.e., $\Box_i \varphi \to \varphi$), **K4** (i.e., $\Box_i \varphi \to \Box_i \Box_i \varphi$), **S4** (i.e., those of **T** and **K4**) we obtain modal predicate calculi QT, $QK4$, $QS4$ representing the corresponding logics **QT**, **QK4**, **QS4** that are known to be characterized by the classes of all reflexive, transitive, and quasi-ordered models with constant domains, respectively (see e.g. [24]).

One more modal predicate logic, $\mathbf{QK}^*$, we consider in this paper is formulated in the language $\mathcal{ML}$ with $2l$ necessity operators $\Box_i$, $\Box_i^*$, for $i = 1, \dots, l$. As before, $\Box_i$ is interpreted in models by an arbitrary accessibility relation R_i, while $\Box_i^*$ is interpreted by the reflexive and transitive closure R_i^* of R_i. Since R_i^* is uniquely determined by R_i, we may denote frames for this language as before: $\mathfrak{F} = \langle W, R_1, \dots, R_l \rangle$. To simplify notation we will assume that $\circ$ is either blank or $*$, so that $\Box_i^\circ$ ranges over $\{\Box_i, \Box_i^*\}$ and R_i° over $\{R_i, R_i^*\}$.

The logic $\mathbf{QK}^*$ ($\mathbf{QK}^*_e$) is the set of all $\mathcal{ML}$-formulas that are true in all models with constant (respectively, expanding) domains for this language. It is easy to see that **QK**, **QT**, **QK4**, **QS4** are embeddable into $\mathbf{QK}^*$ (for details consult Section 5). So from now on we will be concentrating mainly on $\mathbf{QK}^*$.

It is worth noting that the logic $\mathbf{QK}^*$ is very complex. Even its two-variable monadic fragment turns out to be not recursively enumerable. One can show this by reducing the recurrent tiling problem for $\mathbb{N} \times \mathbb{N}$ (which is known to be Σ_1^1-complete; see [20]) to the satisfiability problem for formulas in that fragment. Here is a sketch of the proof based actually on the same idea as the reductions of [39, 30].

We remind the reader that a tile $\mathfrak{t}$ is a 1×1 square with fixed orientation and colored edges $right(\mathfrak{t})$, $left(\mathfrak{t})$, $up(\mathfrak{t})$, and $down(\mathfrak{t})$. The $\mathbb{N} \times \mathbb{N}$ recurrent tiling problem is formulated as follows: given a finite set $\mathcal{T}$ of tiles and a tile $\mathfrak{t}_0 \in \mathcal{T}$, to determine whether there is a tiling of $\mathbb{N} \times \mathbb{N}$ by $\mathcal{T}$ such that $\mathfrak{t}_0$ occurs infinitely often in the first column. More precisely, the problem is to find out whether there exists a function f from $\mathbb{N} \times \mathbb{N}$ into $\mathcal{T}$ such that, for all $m, n \in \mathbb{N}$,

- $right(f(n, m)) = left(f(n + 1, m))$,

- $up(f(n,m)) = down(f(n,m+1))$,
- the set $\{n \in \mathbb{N} : f(0,n) = \mathfrak{t}_0\}$ is infinite.

Suppose we are given a set $\mathcal{T} = \{\mathfrak{t}_0, \dots, \mathfrak{t}_n\}$ of tiles. Let R be a binary predicate and $P_0, \dots, P_n$ unary predicates. Define a first-order modal formula $\varphi_{\mathcal{T}}$ in this language as the conjunction of the following formulas:

$$\begin{aligned}
&\exists x(P_0(x) \wedge \Box^*\Diamond^*(P_0(x) \wedge \Diamond\top)),\\
&\Box^*\forall x\exists y R(x,y),\\
&\Box^*\forall x,y((R(x,y) \to \Box^* R(x,y)) \wedge (\neg R(x,y) \to \Box^*\neg R(x,y))),\\
&\Box^*\forall x(\bigvee_{i=0}^{n} P_i(x) \wedge \bigwedge_{i\neq j}(P_i(x) \to \neg P_j(x))),\\
&\Box^*\forall x(P_i(x) \to \forall y(R(x,y) \to \bigvee_{right(\mathfrak{t}_i)=left(\mathfrak{t}_j)} P_j(y))),\\
&\Box^*\forall x(P_i(x) \to \Box \bigvee_{up(\mathfrak{t}_i)=down(\mathfrak{t}_j)} P_j(x)).
\end{aligned}$$

(Here $\Diamond^\circ\psi = \neg\Box^\circ\neg\psi$ and $\exists x\psi = \neg\forall x\neg\psi$.)

LEMMA 2.2. *The formula $\varphi_{\mathcal{T}}$ is satisfiable in a model with constant or expanding domains iff there is a recurrent tiling of $\mathbb{N} \times \mathbb{N}$ by $\mathcal{T}$.*

PROOF. It should be clear how to satisfy $\varphi_{\mathcal{T}}$ if $\mathcal{T}$ recurrently tiles $\mathbb{N} \times \mathbb{N}$. Conversely, suppose $\varphi_{\mathcal{T}}$ is satisfied at the root w_0 in a model $\langle \mathfrak{F}, D, \mathfrak{d}, I\rangle$ (with constant or expanding domains), where $\mathfrak{F} = \langle W, S\rangle$ and

$$I(w) = \left\langle D, R^{I(w)}, P_0^{I(w)}, \dots, P_n^{I(w)} \right\rangle,$$

for $w \in W$. By the first conjunct of $\varphi_{\mathcal{T}}$, we have an element $a_0 \in \mathfrak{d}(w_0)$, for which $w_0 \models P_0[a_0]$, and an infinite sequence of (not necessarily distinct) worlds $w_0 S w_1 S w_2 S \dots$ such that the set $\{w_i : w_i \models P_0[a_0]\}$ is infinite. According to the second conjunct, we have an infinite sequence $a_0 R^{I(w_0)} a_1 R^{I(w_0)} a_2 \dots$ of (not necessarily distinct) elements in $\mathfrak{d}(w_0)$. By the third conjunct, for all $w \in W$ and $i, j \in \mathbb{N}$, we have $a_i R^{I(w)} a_j$ iff $a_i R^{I(w_0)} a_j$. Now define a function f by taking, for all $i, j \in \mathbb{N}$, $f(i,j) = \mathfrak{t}_k$, whenever $I(w_j) \models P_k[a_i]$. It is straightforward to check that f is a recurrent tiling of $\mathbb{N} \times \mathbb{N}$. $\dashv$

The formula $\varphi_{\mathcal{T}}$ contains only two variables, and the binary predicate R can be simulated by means of two unary predicates Q_1, Q_2 (see e.g. [24]):

$$R(x,y) = \Diamond^*(Q_1(x) \wedge Q_2(y)).$$

This yields us

THEOREM 2.3. *The two-variable monadic fragments of $\mathbf{QK}^*$ and $\mathbf{QK}_e^*$ are not recursively enumerable.*

Let us see now how satisfiability in models with varying and expanding domains can be reduced to satisfiability in models with constant domains. Denote by $md(\varphi)$ the *modal depth* of φ, i.e., the maximal number of nested necessity operators in φ. Put $\Box^{\leq 0}\varphi = \varphi$ and, for $m \geq 0$,

$$\Box^{\leq m+1}\varphi = \Box^{\leq m}\varphi \wedge \bigwedge_{i=1}^{l} \Box_i^* \Box^{\leq m}\varphi.$$

Let $E(x)$ be a unary predicate which does not occur in φ. By induction on the construction of φ we define its *relativization* φ^E:

$$\begin{aligned}
P_i^E(\tau_1, \dots, \tau_n) &= P_i(\tau_1, \dots, \tau_n), \\
(\psi \wedge \chi)^E &= \psi^E \wedge \chi^E, \\
(\neg\psi)^E &= \neg\psi^E, \\
(\forall x \psi)^E &= \forall x (E(x) \rightarrow \psi^E), \\
(\Box_i^\circ \psi)^E &= \Box_i^\circ \psi^E.
\end{aligned}$$

PROPOSITION 2.4. *For every Kripke frame $\mathfrak{F} = \langle W, R_1, \dots, R_l \rangle$ the following holds:*

(i) *φ is satisfied in a model based on $\mathfrak{F}$ and having varying domains iff the formula φ^E is satisfiable in a model with constant domain;*

(ii) *φ is satisfied in a model based on $\mathfrak{F}$ and having expanding domains iff the formula*

$$\varphi' = \varphi^E \wedge \Box^{\leq md(\varphi)} \forall x (E(x) \rightarrow \bigwedge_{i=1}^{l} \Box_i^* E(x))$$

is satisfied in a model on $\mathfrak{F}$ with constant domain.

PROOF. We prove only (ii), leaving the simpler case (i) to the reader. Assuming that φ is satisfied in a model $\mathfrak{M} = \langle \mathfrak{F}, D, \mathfrak{d}, I \rangle$ with expanding domains and that

$$I(w) = \left\langle D, P_0^{I(w)}, \dots, c_0^{I(w)}, \dots \right\rangle$$

for $w \in W$, we construct a model $\mathfrak{N} = \langle \mathfrak{F}, D, J \rangle$ with constant domain D by taking

$$J(w) = \left\langle D, E^{J(w)}, P_0^{I(w)}, \dots, c_0^{I(w)}, \dots \right\rangle,$$

where $E^{J(w)} = \mathfrak{d}(w)$. It is readily checked by induction that $(\mathfrak{M}, w) \models^{\mathfrak{a}} \psi$ iff $(\mathfrak{N}, w) \models^{\mathfrak{a}} \psi^E$, for every $w \in W$, every subformula ψ of φ, and every assignment $\mathfrak{a} : var \mapsto D$. It follows that φ' is satisfied in $\mathfrak{N}$.

Conversely, suppose φ' is satisfied at a root v of a model $\mathfrak{N} = \langle \mathfrak{F}, D, J \rangle$ (with constant domain D) under an assignment $\mathfrak{a}$, and

$$J(w) = \left\langle D, E^{J(w)}, P_0^{J(w)}, \dots, c_0^{J(w)}, \dots \right\rangle,$$

for $w \in W$. Consider the model $\mathfrak{M} = \langle \mathfrak{F}, D, \mathfrak{d}, I \rangle$ such that

$$I(w) = \left\langle D, P_0^{J(w)}, \dots, c_0^{J(w)}, \dots \right\rangle$$

for all $w \in W$, $\mathfrak{d}(w) = E^{J(w)}$ whenever w is accessible in $\leq md(\varphi)$ steps from v (via the R_i^*) and $\mathfrak{d}(w) = D$ otherwise. By the second conjunct of φ', $\mathfrak{M}$ has expanding domains. Now, using the fact that the truth-value of φ at v depends only on the worlds that are accessible in $\leq md(\varphi)$ steps from v, one can easily show that $(\mathfrak{M}, v) \models^{\mathfrak{a}} \varphi$. ⊣

Remark 2.5. It is to be noted that φ and φ' are of different forms. So when proving decidability of a fragment of, say, $\mathbf{QK}^*$ and $\mathbf{QK}^*_e$, we should always check that φ and φ' belong to it simultaneously. Fortunately this will always be the case for all the fragments we deal with in this paper. So from now on we will be considering only models with constant domains.

Given an $\mathcal{ML}$-formula φ, denote by $sub_n\varphi$ the closure under negation of the set of all subformulas in φ containing $\leq n$ free variables; $sub\varphi$ is the set of all subformulas in φ, and $con\varphi$ the set of all constants in φ. Without loss of generality we may identify ψ and $\neg\neg\psi$, so $sub_n\varphi$ is always finite.

DEFINITION 2.6. Denote by $\mathcal{ML}_1$ the set of all $\mathcal{ML}$-formulas φ such that every subformula of φ of the form $\Box_i^\circ\psi$ contains at most one free variable.[4] In other words, $\mathcal{ML}_1$ allows quantification into modal contexts only with one free variable. From now on we will be assuming that *all our formulas are in* $\mathcal{ML}_1$.

Remark 2.7. It is worth noting that, unlike the one-variable fragment of first-order modal logic, $\mathcal{ML}_1$ imposes restrictions only on modalities *de re*; modalities *de dicto* are arbitrary.

For each formula $\psi(x) = \Box_i^\circ\varphi(x)$ with one free variable x, we reserve a unary predicate and denote it by $P_\psi(x)$. Likewise, for every sentence $\psi = \Box_i^\circ\varphi$ we fix a propositional variable p_ψ. $P_\psi(x)$ and p_ψ will be called the *surrogates* for $\psi(x)$ and ψ, respectively.

Given a formula φ, denote by $\overline{\varphi}$ the formula that results from φ by replacing all subformulas of the form $\Box_i^\circ\psi(x)$ and $\Box_i^\circ\chi$, which are not within the scope of another box, with their surrogates. Thus, $\overline{\varphi}$ contains no occurrences of modal operators, i.e., it is an $\mathcal{L}$-formula; we will call $\overline{\varphi}$ the *$\mathcal{L}$-reduct* of φ. For a set Γ of $\mathcal{ML}_1$-formulas, we put $\overline{\Gamma} = \{\overline{\psi} : \psi \in \Gamma\}$. Quite often we will not distinguish between a finite set Γ of formulas and the conjunction $\bigwedge\Gamma$ of formulas in it.

§3. Quasimodels. Suppose we are given an $\mathcal{ML}_1$-sentence φ and asked whether it is satisfiable. Being only modal logicians, we would like to assume, at least for a moment, that we have an access to a classical oracle that is capable of providing us with an $\mathcal{L}$-structure satisfying the $\mathcal{L}$-reduct of φ, if it exists. Suppose we have got such a structure. Our task then is to try to expand it to a Kripke model satisfying φ. This means, in particular, that we should take care of only the 'boxed' subformulas of φ which, remember, have at most one free variable.

The aim of this section is to show that modulo those subformulas every such Kripke model can be codified in a structure, called a *quasimodel*, which may be thought of as a model whose worlds have disjoint domains with a bounded number of elements (depending on φ), each element satisfies some specified set of subformulas of φ, and the correspondence between these elements in different worlds is established by special functions called *runs*.

[4]In [23], written after the present paper was submitted, such formulas are called *monodic*.

Fix a variable x not occurring in φ and put

$$sub_x\varphi = \{\psi\{x/y\} : \psi(y) \in sub_1\varphi\}.$$

DEFINITION 3.1 (type). A *type* t for φ is any boolean-saturated subset of $sub_x\varphi$, i.e.,

- $\psi \wedge \chi \in t$ iff $\psi \in t$ and $\chi \in t$, for every $\psi \wedge \chi \in sub_x\varphi$;
- $\neg\psi \in t$ iff $\psi \notin t$, for every $\psi \in sub_x\varphi$.

We say that two types t and t' *agree on* $sub_0\varphi$ if $t \cap sub_0\varphi = t' \cap sub_0\varphi$. Given a type t for φ and a constant $c \in con\varphi$, the pair $\langle t, c\rangle$ will be called an *indexed type* for φ (indexed by c) and denoted by $t_c(x)$ or simply t_c.

For example, if

$$\varphi = \exists x_1\exists x_2\exists x_3(P(x_1,x_2) \wedge \Box\exists x_4 P(x_4,x_3))$$

then the set

$$\{\varphi, \exists x_2\exists x_3(P(x,x_2) \wedge \Box\exists x_4 P(x_4,x_3)),\ \Box\exists x_4(x_4,x),\ \neg\exists x_4 P(x_4,x)\}$$

is type for φ.

There are only finitely many types for φ, not more than

$$\flat(\varphi) = 2^{|sub_x\varphi|},$$

to be more precise. To a certain extent, every world in a model under a given assignment can be characterized (modulo φ, of course) by the set of types that are realized in this world and the set of types that hold on its constants. This motivates the following definition.

DEFINITION 3.2 (world candidate). Suppose T is a set of types for φ that agree on $sub_0\varphi$, and T^{con} is a set containing, for each $c \in con\varphi$, one indexed type t_c such that $t \in T$. Then the pair $\mathfrak{T} = \langle T, T^{con}\rangle$ is called a *world candidate* for φ.

Not all world candidates can represent worlds in models. To single out those that can, we require one more definition.

DEFINITION 3.3 (realizable world candidate). Consider an $\mathcal{L}$-structure

$$\mathfrak{D} = \left\langle D, P_0^{\mathfrak{D}}, \dots, c_0^{\mathfrak{D}}, \dots\right\rangle \tag{1}$$

and suppose that $a \in D$. The set

$$t^{\mathfrak{D}}(a) = \{\psi \in sub_x\varphi : \mathfrak{D} \models \overline{\psi}[a]\}$$

is clearly a type for φ. We say that $\mathfrak{D}$ *realizes* a world candidate $\langle T, T^{con}\rangle$ if the following conditions hold:

- $T = \{t^{\mathfrak{D}}(a) : a \in D\}$,
- $T^{con} = \{\left\langle t^{\mathfrak{D}}(c^{\mathfrak{D}}), c\right\rangle : c \in con\varphi\}$.

Denote by $\sharp(\varphi)$ the number of distinct realizable world candidates for φ. It should be clear that

$$\sharp(\varphi) \le 2^{\flat(\varphi)} \cdot \flat(\varphi)^{|con\varphi|}.$$

LEMMA 3.4. *A world candidate $\mathfrak{T} = \langle T, T^{con}\rangle$ for φ is realizable iff the $\mathcal{L}$-formula*

$$\alpha_{\mathfrak{T}} = \bigwedge_{t\in T} \exists x\, \bar{t}(x) \wedge \forall x \bigvee_{t\in T} \bar{t}(x) \wedge \bigwedge_{t_c\in T^{con}} \bar{t}(c)$$

is satisfiable in some $\mathcal{L}$-structure.

PROOF. Follows directly from the definitions. ⊣

LEMMA 3.5. *Let κ be a cardinal and let $\kappa \geq \aleph_0$. Then every realizable world candidate $\langle T, T^{con}\rangle$ is realized in an $\mathcal{L}$-structure $\mathfrak{D}$ of the form* (1) *such that, for every $t \in T$, the set*

$$\mathfrak{D}_t = \{a \in D : \mathfrak{D} \models \bar{t}[a]\}$$

is of cardinality κ.

PROOF. Follows from classical model theory, since the language $\mathcal{L}$ is countable and does not contain equality. ⊣

We are in a position now to define the central notion of this section, that of a quasimodel. Consider a frame $\mathfrak{F} = \langle W, R_1, \dots, R_l\rangle$ and a map f associating with each $w \in W$ a realizable world candidate $f(w) = \langle T_w, T_w^{con}\rangle$.

DEFINITION 3.6 (run). A *run* in $\langle \mathfrak{F}, f\rangle$ is a function r from W into the set of types for φ such that

- $r(w) \in T_w$, for all $w \in W$, and
- for every $\Box_i^\circ \psi \in sub_x\varphi$ and every $w \in W$, we have $\Box_i^\circ\psi \in r(w)$ iff $\psi \in r(v)$ for all $v \in W$ such that $wR_i^\circ v$.

DEFINITION 3.7 (quasimodel). The pair $\langle \mathfrak{F}, f\rangle$ is called a *quasimodel* for φ (based on $\mathfrak{F}$) if the following conditions hold:

- for every $c \in con\varphi$, the function r_c defined by $r_c(w) = t$, for $\langle t, c\rangle \in T_w^{con}$, $w \in W$, is a run in $\langle \mathfrak{F}, f\rangle$;
- for every pair $w \in W$ and $t \in T_w$, there exists a run r in $\langle \mathfrak{F}, f\rangle$ such that $r(w) = t$.

Say that φ is *satisfied* in the quasimodel $\langle \mathfrak{F}, f\rangle$ if there is $w \in W$ such that $\varphi \in t$, for some (or, equivalently, all) $t \in T_w$.

EXAMPLE 3.8. Consider the formula

$$\varphi = \exists y(\Box P(y) \wedge \neg\Box\forall z P(z)).$$

Let $\mathfrak{F} = \langle W, R\rangle$ contain two worlds, say u and v, such that uRv. Define realizable world candidates $\mathfrak{T}_u = \langle T_u, T_u^{con}\rangle$ and $\mathfrak{T}_v = \langle T_v, T_v^{con}\rangle$ by taking

$$T_u = \{t_1, t_2\},\ T_v = \{t_3, t_4\},\ T_u^{con} = T_v^{con} = \emptyset,$$
$$t_1 = \{\Box P(x) \wedge \neg\Box\forall z P(z),\ \Box P(x),\ \neg P(x)\} \cup \Phi_u,$$
$$t_2 = \{\neg(\Box P(x) \wedge \neg\Box\forall z P(z)),\ \neg\Box P(x),\ \neg P(x)\} \cup \Phi_u,$$
$$\Phi_u = \{\varphi,\ \neg\Box\forall z P(z),\ \neg\forall z P(z)\},$$
$$t_3 = \{\neg(\Box P(x) \wedge \neg\Box\forall z P(z)),\ \Box P(x),\ P(x)\} \cup \Phi_v,$$
$$t_4 = \{\neg(\Box P(x) \wedge \neg\Box\forall z P(z)),\ \Box P(x),\ \neg P(x)\} \cup \Phi_v,$$
$$\Phi_v = \{\neg\varphi,\ \Box\forall z P(z),\ \neg\forall z P(z)\},$$

and let $f(u) = \mathfrak{T}_u$, $f(v) = \mathfrak{T}_v$. The functions r_1 and r_2 defined by

$$r_1(u) = t_1,\ r_1(v) = t_2,\ r_2(u) = t_3,\ r_2(v) = t_4$$

are clearly runs in $\langle \mathfrak{F}, f\rangle$. So this pair is a quasimodel satisfying φ.

THEOREM 3.9. *An $\mathcal{ML}_1$-sentence φ is satisfied in a model based on a frame $\mathfrak{F}$ iff it is satisfied in a quasimodel for φ based on $\mathfrak{F}$.*

PROOF. Suppose φ is satisfied in a model $\mathfrak{M} = \langle \mathfrak{F}, D, I\rangle$, $\mathfrak{F} = \langle W, R_1, \dots, R_l\rangle$. For every $w \in W$, define $f(w) = \langle T_w, T_w^{con}\rangle$ by taking

$$\begin{array}{lcl} t_a^w & = & \{\psi \in sub_x\varphi : (\mathfrak{M}, w) \models^{\mathfrak{a}} \psi\},\ \text{where } a \in D \text{ and } \mathfrak{a}(x) = a,\\ T_w & = & \{t_a^w : a \in D\},\\ T_w^{con} & = & \{\langle t_{c^w}^w, c\rangle : c \in con\varphi\}. \end{array}$$

It is easy to see that $\langle \mathfrak{F}, f\rangle$ is a quasimodel satisfying φ (for every $a \in D$, the function $r(w) = t_a^w$, $w \in W$, is clearly a run in $\langle \mathfrak{F}, f\rangle$).

Conversely, suppose that φ is satisfied in a quasimodel $\langle \mathfrak{F}, f\rangle$ for φ. Take a cardinal $\kappa \geq \aleph_0$ exceeding the cardinality of the set Ω of all runs in $\langle \mathfrak{F}, f\rangle$ and put

$$D = \{\langle r, \xi\rangle : r \in \Omega, \xi < \kappa\}.$$

By Lemma 3.5, for every $w \in W$ there exists an $\mathcal{L}$-structure $I(w)$ with domain D realizing $f(w)$ and such that $c^w = \langle r_c, 0\rangle$, for every $c \in con\varphi$, and

$$r(w) = \{\psi \in sub_x\varphi : I(w) \models \overline{\psi}[\langle r, \xi\rangle]\},$$

for all $r \in \Omega$ and $\xi < \kappa$. Let $\mathfrak{M} = \langle \mathfrak{F}, D, I\rangle$ and let $\mathfrak{a}$ be an arbitrary assignment in D. We show by induction that, for all $\psi \in sub\varphi$ and $w \in W$,

$$I(w) \models^{\mathfrak{a}} \overline{\psi} \text{ iff } (\mathfrak{M}, w) \models^{\mathfrak{a}} \psi.$$

The basis of induction, i.e., the case when $\psi = P_i(x_1, \dots, x_n)$, is clear; for then $\psi = \overline{\psi}$. The induction step for $\psi = \psi_1 \wedge \psi_2$, $\psi = \neg\psi_1$, and $\psi = \forall x \psi_1$ follows by the induction hypothesis from the obvious equations:

$$\overline{\psi_1 \wedge \psi_2} = \overline{\psi_1} \wedge \overline{\psi_2},\ \overline{\neg\psi_1} = \neg\overline{\psi_1},\ \overline{\forall x \psi_1} = \forall x \overline{\psi_1}.$$

Let $\psi = \Box_i^{\circ}\chi(y)$ and $\mathfrak{a}(y) = \langle r, \xi\rangle$. We then have $\overline{\psi} = P_{\Box_i^{\circ}\chi}$, and so

$$\begin{array}{rll} I(w) \models^{\mathfrak{a}} P_{\Box_i^{\circ}\chi}(y) & \text{iff} & \Box_i^{\circ}\chi \in r(w)\\ & \text{iff} & \forall v \in W\ (wR_i^{\circ}v \to \chi \in r(v))\\ & \text{iff} & \forall v \in W\ (wR_i^{\circ}v \to I(v) \models^{\mathfrak{a}} \overline{\chi})\\ & \text{iff} & \forall v \in W\ (wR_i^{\circ}v \to (\mathfrak{M}, v) \models^{\mathfrak{a}} \chi(y))\\ & \text{iff} & (\mathfrak{M}, w) \models^{\mathfrak{a}} \Box_i^{\circ}\chi(y). \end{array}$$

Since $\varphi \in r(w)$ for some $w \in W$, it follows that $(\mathfrak{M}, w) \models \varphi$, as required. ⊣

§4. Satisfiability criterion. In this section we prove an effective satisfiability criterion for those sentences $\varphi \in \mathcal{ML}_1$ that have an oracle capable of deciding the realizability problem for the world candidates for φ. The idea is to represent quasimodels for φ as possibly infinite mosaics consisting of repeating finite patterns we call *blocks*.

Say that a frame $\mathfrak{F} = \langle W, R_1, \dots, R_l \rangle$ is an *intransitive tree* if $R_i \cap R_j = \emptyset$, for $i \neq j$, and the frame $\langle W, R \rangle$, with $R = \bigcup_{i=1}^{l} R_i$, is rooted and contains no distinct paths of the form $xRy_1R \dots Ry_nRy$ and $xRz_1R \dots Rz_mRy$ (in particular, $\langle W, R \rangle$ has no infinite descending chains). By a *tree quasimodel* for φ we mean a quasimodel for φ based on an intransitive tree. Using the standard unravelling technique of modal logic (see e.g. [9]) one can prove the following theorem.

THEOREM 4.1 (tree quasimodel completeness). *A formula φ is satisfiable iff it is satisfiable in a tree quasimodel for φ.*

Fix again a sentence $\varphi \in \mathcal{ML}_1$. To introduce the notion of a block for φ, we require two auxiliary definitions. Suppose that $\mathfrak{F}$ is a frame as above and f is a map from W into the set of realizable world candidates for φ, so that $f(w) = \langle T_w, T_w^{con} \rangle$. It will be convenient for us not to distinguish between w and $f(w)$, i.e., we will assume that the worlds w in $\mathfrak{F}$ *are* the realizable world candidates $\langle T_w, T_w^{con} \rangle$.

DEFINITION 4.2 (weak run). A *weak run* in $\langle \mathfrak{F}, f \rangle$ is a function r from W into $\bigcup_{w \in W} T_w$ such that

- $r(w) \in T_w$, for all $w \in W$, and
- for every $\Box_i^{\circ}\psi \in sub_x\varphi$ and every $w \in W$, if $\Box_i^{\circ}\psi \in r(w)$ then $\psi \in r(v)$ for all v such that $wR_i^{\circ}v$.

The difference between runs and weak runs is only that the latter do not necessarily contain 'witnesses' for formulas of the form $\neg\Box_i^{\circ}\psi \in r(w)$ (in a run we must have a 'witness' v such that $wR_i^{\circ}v$ and $\neg\psi \in r(v)$). The difference between quasimodels and weak quasimodels to be defined below is of the same sort.

DEFINITION 4.3 (weak quasimodel). The pair $\langle \mathfrak{F}, f \rangle$ is called a *weak quasimodel* for φ if the following holds:

- for every $c \in con\varphi$, the function r_c is a weak run in $\langle \mathfrak{F}, f \rangle$;
- for all $w \in W$ and $t \in T_w$, there exists a weak run r in $\langle \mathfrak{F}, f \rangle$ such that $r(w) = t$.

Say that an intransitive tree $\mathfrak{F} = \langle W, R_1, \dots, R_l \rangle$ is a *bouquet* if no point in W different from its root has more than one R-successor, where $R = \bigcup_{i=1}^{l} R_i$.

DEFINITION 4.4 (block). Let $\mathfrak{B} = \langle \mathfrak{F}, f \rangle$ be a weak quasimodel based on a finite bouquet $\mathfrak{F}$ with root w_0. A weak run r in $\mathfrak{B}$ is called *root-saturated* if, for every $\Box_i^{\circ}\psi(x) \in sub_x\varphi$, whenever $\Box_i^{\circ}\psi(x) \notin r(w_0)$ then there is $w \in W$ such that $w_0R_i^{\circ}w$ and $\psi(x) \notin r(w)$.

Now, we say $\mathfrak{B}$ is a *block* for φ if it satisfies conditions (a) and (b) below:

(a) for every $w \in W$ and every $t \in T_w$ there is a root-saturated weak run r in $\mathfrak{B}$ such that $r(w) = t$;
(b) every weak run r_c in $\mathfrak{B}$, for $c \in con\varphi$, is root-saturated.

The following notion will be used for organizing blocks in a mosaic-like structure which is intended to represent a quasimodel satisfying φ.

DEFINITION 4.5 (satisfying set). A set $\mathcal{S}$ of blocks for φ is called a *satisfying set* for φ if (i) it contains a block with root w_0 such that $w_0 \models \varphi$ (i.e, $\varphi \in t$ for all $t \in T_{w_0}$) and (ii) for every realizable world candidate $\mathfrak{T}$ in every block in $\mathcal{S}$, there exists a block in $\mathcal{S}$ having $\mathfrak{T}$ as its root.

We are going to show now that φ is satisfiable iff there is a satisfying set for φ whose blocks contain at most N worlds, where $N < \omega$ is effectively given by φ.

THEOREM 4.6 (satisfiability criterion). *An $\mathcal{ML}_1$-sentence φ is satisfiable iff there is a satisfying set for φ, each block in which contains at most*

$$N = \sharp(\varphi) \cdot 2|sub_x\varphi|^2 \cdot 2^{|sub_x\varphi|}$$

worlds.

PROOF. To simplify notation, we will confine ourself to considering the language with only one necessity operator $\Box$ and its 'star' version $\Box^*$. The general case is treated in the same way.

($\Rightarrow$) Suppose φ is satisfiable. Then, by Theorem 4.1, there is a tree quasimodel $\mathfrak{m} = \langle \mathfrak{F}, f \rangle$ for φ satisfying φ at its root. Moreover, without loss of generality we may also assume that if wRu in $\mathfrak{F} = \langle W, R \rangle$ then there are sufficiently many worlds $v \in W$, called *twins of* u, such that wRv, $f(u) = f(v)$ and the subquasimodels of $\mathfrak{m}$ generated by u and v are isomorphic. In any case, we can achieve this by duplicating (as many times as we need) the subtree of $\mathfrak{m}$ generated by u and connecting it with w via R. The resulting structure will clearly be again a tree quasimodel satisfying φ (for it can be p-morphically mapped onto the original quasimodel).

We begin our construction of a satisfying set for φ by associating with each world $w \in W$ a block $\mathfrak{B}_w = \langle \mathfrak{F}_w, f_w \rangle$ with $\mathfrak{F}_w = \langle W_w, R_w \rangle$.

For every type $t \in T_w$ we fix a run r in $\mathfrak{m}$ coming through t; if $t_c \in T_w^{con}$, $c \in con\varphi$, then we take $r = r_c$. Then, for every $\Box^\circ\psi(x) \in sub_x\varphi$ such that $\Box^\circ\psi(x) \notin r(w)$, we select a world $w' \in W$, for which $wR^\circ w'$, $\psi(x) \notin r(w')$, and put it into an auxiliary set $Sel(w)$ *together with one of its twins* w''. The number of the selected worlds does not exceed

$$2 \cdot |sub_x\varphi| \cdot 2^{|sub_x\varphi|}.$$

Without loss of generality we may assume that all of them are pairwise distinct. For each selected w' there is a unique path from w to w', namely the set $(w, w') = \{w_1 : wR^*w_1R^*w'\}$. Again, without loss of generality we assume that distinct paths (w, w') and (w, w'') (for $w' \neq w''$) have no common worlds save w (otherwise the duplication technique will do the job).

Finally, we define W_w to be the set of all worlds in the paths (w, w'), for $w' \in Sel(w)$, R_w to be the restriction of R to W_w, and f_w the restriction of f to W_w (taking into account all the duplications, of course).

The constructed structure $\mathfrak{B}_w$ is a block. Indeed, it is clearly a weak quasimodel for φ based on a finite bouquet, having root w (for it can be regarded as a subquasimodel of $\mathfrak{m}$) and satisfying condition (b) by the construction. To show that it satisfies (a) as well, suppose that $u \in W_w$, $t \in T_u$, and let r be a weak run in $\mathfrak{B}_w$ coming through t, i.e., $r(u) = t$. Consider the type $r(w)$ and the set $\mathcal{B}$ of formulas $\chi = \Box^\circ\psi \in sub_x\varphi$ for which $\chi \notin r(w)$. For each of them there is a weak run r_χ such that $r_\chi(w) = r(w)$, $\psi \notin r_\chi(w_\chi)$, for some $w_\chi \in Sel(w)$, and $u \notin (w, w_\chi)$. Using these weak runs and r we can define a function r' by taking, for every $w' \in W_w$,

$$r'(w') = \begin{cases} r(w') & \text{if } w' \notin (w, w_\chi), \text{ for any } \chi \in \mathcal{B}, \\ r_\chi(w') & \text{otherwise.} \end{cases}$$

Clearly, r' is a root-saturated weak run in $\mathfrak{B}_w$ coming through t, which establishes (a).

The problem now is that the branches in $\mathfrak{B}_w$ may be too long. So our next step is to extract from $\mathfrak{B}_w$ a substructure $\mathfrak{A}_w$ which is still a block for φ and whose branches are of length $\leq \sharp(\varphi) \cdot |sub_x\varphi|$. We will do this by cutting out certain fragments of branches in $\mathfrak{B}_w$.

Consider a branch (w, w') and suppose that w' was selected to 'saturate' $\Box\psi(x) \in sub_x\varphi$ in w by means of a weak run r. Since wRw', the branch $(w, w') = \{w, w'\}$ does not need 'truncation': we simply leave it as it is.

Suppose now that w' in the branch (w, w') was selected to saturate the formula $\Box^*\psi(x) \in sub_x\varphi$ by means of a weak run r. The truncation of (w, w') can be done in the following way. Let $w_1, \dots, w_n$ be all the worlds in (w, w') such that $w = w_1 R_w w_2 R_w \dots R_w w_n = w'$. If $n \leq \sharp(\varphi) \cdot |sub_x\varphi|$ then we do nothing with this branch. Otherwise let r be the weak run in $\mathfrak{B}_w$ such that $r(w) = t$ and $\psi(x) \notin r(w')$; since $n > \sharp(\varphi) \cdot |sub_x\varphi|$, there must be two worlds w_i and w_j, $1 \leq i < j \leq n$, such that $f(w_i) = f(w_j)$ and $r(w_i) = r(w_j)$. Then we cut out from (w, w') all the worlds in the interval (w_i, w_j) save w_i and connect w_i directly to the R_w-successor of w_j, i.e., put $w_i R_w w^\dagger$ in the truncated bouquet iff $w_j R_w w^\dagger$ in the original bouquet, for all $w^\dagger \in W_w$. It should be clear that the resulting structure is still a block for φ, and so by deleting repeating worlds in the branches of $\mathfrak{B}_w$ we can construct a block $\mathfrak{A}_w$ for φ whose branches are of length $\leq \sharp(\varphi) \cdot |sub_x(\varphi)|$.

The satisfying set for φ we are looking for can be constructed now by taking the blocks $\mathfrak{A}_w$ for all $w \in W$.

($\Leftarrow$) Let $\mathcal{S}$ be a satisfying set for φ. We are going to construct a quasimodel $\mathfrak{m}$ satisfying φ as the limit of a sequence of weak quasimodels

$$\mathfrak{m}_n = \langle \mathfrak{F}_n, f_n \rangle, \ \mathfrak{F}_n = \langle W_n, R_n \rangle, \ n = 1, 2, \dots,$$

the first of which, $\mathfrak{m}_1$, is a block in $\mathcal{S}$ satisfying φ at its root. Suppose we have already constructed a weak quasimodel $\mathfrak{m}_n$. For every world $w \in W_n - W_{n-1}$ (W_0 consists of the root in $\mathfrak{F}_1$) select a block $\mathfrak{B}_{w'} \in \mathcal{S}$ with root w' such that $f_n(w) = f_{w'}(w')$. Without loss of generality we may assume all the selected blocks and the weak quasimodel $\mathfrak{m}_n$ to be disjoint. The weak quasimodel $\mathfrak{m}_{n+1}$ is then the result of hooking the selected blocks $\mathfrak{B}_{w'}$ to $\mathfrak{m}_n$ by identifying their roots w' with $w \in W_n - W_{n-1}$.

Define the limit $\mathfrak{m} = \langle \mathfrak{F}, f \rangle$, $\mathfrak{F} = \langle W, R \rangle$, of the constructed sequence by taking

$$W = \bigcup_{n \geq 1} W_n, \quad R = \bigcup_{n \geq 1} R_n, \quad f = \bigcup_{n \geq 1} f_n,$$

and show that $\mathfrak{m}$ is a quasimodel for φ indeed. To this end it suffices to observe that any r constructed in the following manner is a run in $\mathfrak{m}$. Let r^1 be an arbitrary weak root-saturated run in $\mathfrak{m}_1$, and suppose inductively that we have already defined r^m. For any $w \in W_m - W_{m-1}$ take a weak root-saturated run r_w in $\mathfrak{B}_w$ such that $r^m(w) = r_w(w)$ and put for $w' \in W_{m+1}$

$$r^{m+1}(w') = \begin{cases} r^m(w') & \text{if } w' \in W_m; \\ r_w(w') & \text{if } w' \in W_w - W_m. \end{cases}$$

Finally, let

$$r = \bigcup_{m>0} r^m.$$

It should be clear that r is a run in $\mathfrak{m}$. The runs of this form come through all types in all $w \in W$. Moreover, the function r_c, for $c \in con\varphi$, can also be constructed in this way. Thus, $\mathfrak{m}$ is a quasimodel satisfying φ. ⊣

As an immediate consequence we obtain the following satisfiability criterion.

THEOREM 4.7. *Let $\mathcal{ML}' \subseteq \mathcal{ML}_1$ and suppose that there is an algorithm that is capable of deciding, given $\varphi \in \mathcal{ML}'$ and a world candidate $\mathfrak{T} = \langle T, T^{con}\rangle$ for φ, whether $\mathfrak{T}$ is realizable, or equivalently, whether the formula $\alpha_{\mathfrak{T}}$ is satisfiable. Then the fragment* $\mathbf{QK}^* \cap \mathcal{ML}'$ *is decidable.*

In other words, this theorem reduces the satisfiability problem for formulas in $\mathcal{ML}'$ to the satisfiability problem for $\mathcal{L}$-formulas of the form $\alpha_{\mathfrak{T}}$, $\mathfrak{T} = \langle T, T^{con}\rangle$, where T and T^{con} are just (arbitrary) subsets of $sub_x\varphi$. A few examples of the use of this reduction will be given in the next section.

Remark 4.8. In principle it is not hard to extend Theorem 4.7 to some other standard modal logics, e.g. **D**, **KD45**, **S5**, **GL**, **S4.3** (see [46]). Using the technique developed in [49, 50] one can prove similar criteria for first-order temporal logics (see [23]), first-order dynamic logic, and first-order epistemic logics with the common knowledge operator (see [41]).

Remark 4.9. According to Theorem 2.3, the logic $\mathbf{QK}^*$ and even its monadic two-variable fragment are not recursively axiomatizable. It turns out that the monodic fragment of $\mathbf{QK}^*$, i.e., $\mathbf{QK}^* \cap \mathcal{ML}_1$, does have a finite Hilbert-style axiomatization (proofs for the temporal and epistemic cases can be found in [41, 48]). However, the addition of equality or functional symbols makes the fragment $\mathbf{QK}^* \cap \mathcal{ML}_1$ not recursively enumerable [41, 48].

§5. Decidable fragments. Now we apply the criterion of Theorem 4.7 to single out a number of decidable fragments of $\mathbf{QK}^*$, **QK**, **QT**, **QK4**, and **QS4**. But before that let us observe that **QK** is just a fragment of $\mathbf{QK}^*$, **QS4** is embedded into $\mathbf{QK}^*$ by the translation replacing every $\Box_i$ with $\Box_i^*$, **QT** is embedded into $\mathbf{QK}^*$ by the translation replacing every $\Box_i\varphi$ with $\varphi \wedge \Box_i\varphi$, and **QK4** by the translation replacing every $\Box_i$ with $\Box_i\Box_i^*$. (Using a somewhat more sophisticated translation one can reduce the satisfiability problem in **S5**-frames to the satisfiability problem in **K4**- or **S4**-frames.)

Denote by $\mathcal{ML}^2$ the set of all $\mathcal{ML}$-formulas containing at most two individual variables, and let $\mathcal{ML}_1^2 = \mathcal{ML}_1 \cap \mathcal{ML}^2$. That is to say, $\mathcal{ML}_1^2$ consists of all $\mathcal{ML}$-formulas with at most two variables, the modal operators in which are applied to subformulas with at most one free variable.

THEOREM 5.1. *The fragment* $L \cap \mathcal{ML}_1^2$, *for* $L \in \{\mathbf{QK}^*, \mathbf{QK}, \mathbf{QT}, \mathbf{QK4}, \mathbf{QS4}\}$, *is decidable.*

PROOF. The $\mathcal{L}$-formula $\alpha_{\mathfrak{T}}$, corresponding to a world candidate $\mathfrak{T}$ for a formula $\varphi \in \mathcal{ML}_1^2$, contains not more than two individual variables. As was mentioned in the introduction, the satisfiability problem for such formulas is decidable. It remains to use the criterion of Theorem 4.7. ⊣

Since $\mathcal{ML}_1^2$ contains the set $\mathcal{ML}^1$ of $\mathcal{ML}$-formulas with at most one variable, we also have:

THEOREM 5.2. *The fragment* $L \cap \mathcal{ML}^1$, *for* $L \in \{\mathbf{QK}^*, \mathbf{QK}, \mathbf{QT}, \mathbf{QK4}, \mathbf{QS4}\}$, *is decidable.*

We have obtained this result using the mosaic technique, which in general constructs only an *infinite* model satisfying a given formula. The more familiar way of establishing decidability in (propositional) modal logic—proving the finite model property, that is the existence of finite models for satisfiable formulas—does not work for the fragments mentioned in Theorem 5.2. To show this, let us note first that in the modal predicate case one should distinguish between two types of finite model property.

Say that a fragment $\mathcal{ML}' \subseteq \mathcal{ML}$ has the *finite frame property* with respect to a first-order modal logic L if every formula in $\mathcal{ML}'$ satisfiable in some model for L can be satisfied in a model for L based on a finite Kripke frame. $\mathcal{ML}'$ has the *finite domain property* with respect to L if every formula in $\mathcal{ML}'$ satisfiable in a model for L can be satisfied in a model for L with a finite domain.

THEOREM 5.3. *The one-variable fragment of* $\mathcal{ML}$ *with a single unary predicate has neither the finite frame nor the finite domain property with respect to* $\mathbf{QK}^*$, $\mathbf{QK4}$, *and* $\mathbf{QS4}$.

PROOF. Consider the conjunction φ of the following formulas:

$$\exists x P(x),\ \forall x \Box^+(P(x) \to \Box P(x)),\ \Box^+ \exists x \neg P(x),\ \Box^+ \forall x(\neg P(x) \to \Diamond P(x)),$$

where $\Box^+\psi = \psi \wedge \Box\psi$. One can readily check that φ is satisfied in the model $\mathfrak{M} = \langle \mathfrak{F}, \mathbb{N}, I \rangle$ based on $\mathfrak{F} = \langle \mathbb{N}, < \rangle$ or $\mathfrak{F} = \langle \mathbb{N}, \leq \rangle$, where for each $n \in \mathbb{N}$,

$$I(n) = \left\langle \mathbb{N}, P^{I(n)} \right\rangle, \quad P^{I(n)} = \{0, \ldots, n\}.$$

It is not hard to see, however, that every transitive model satisfying φ has both an infinite underlying Kripke frame and an infinite domain. ⊣

Remark 5.4. The same result holds, of course, in the case of expanding domains (the finite domain property means then that the universe D is finite).

The transitivity of the underlying frames was essential in the proof above. For we have the following:

THEOREM 5.5. *The monodic fragment* $\mathcal{ML}_1$ *does have the finite frame property with respect to* $\mathbf{QK}$ *and* $\mathbf{QT}$.

PROOF. The proof of Theorem 4.6 shows that every satisfiable $\mathcal{ML}_1$-formula φ is satisfied at the root of a tree quasimodel, the set of R_i-successors of every world w in which is finite. By transforming this quasimodel (in accordance with Theorem 3.9) into a real model and then taking the submodel of the latter consisting of the worlds accessible from the root in $\leq md(\varphi)$ steps, we construct a model based on a finite Kripke frame and still satisfying φ at its root. $\dashv$

$\mathcal{ML}_1$ contains the full classical first-order language. So it cannot have the finite domain property. The following observation shows how one can extract fragments with both the finite frame and the finite domain properties with respect to **QK** and **QT**.

THEOREM 5.6. *Let $\mathcal{ML}' \subseteq \mathcal{ML}_1$ and suppose that for every $\varphi \in \mathcal{ML}'$ and every realizable world candidate $\mathfrak{T}$ for φ, the formula $\alpha_{\mathfrak{T}}$ is satisfiable in a finite first-order structure. Then $\mathcal{ML}'$ has both the finite frame and the finite domain properties with respect to* **QK** *and* **QT**.

PROOF. The result follows immediately from Theorem 5.5 and the fact that if $\alpha_{\mathfrak{T}}$, $\mathfrak{T} = \langle T, T^{con}\rangle$, is satisfiable in a finite first-order structure then there exists $m \in \omega$ such that, for every $n \geq m$, $\mathfrak{T}$ is realized in an $\mathcal{L}$-structure $\mathfrak{D}$ of the form (1) in which $|\mathfrak{D}_t| = n$, for every $t \in T$. $\dashv$

COROLLARY 5.7. *$\mathcal{ML}_1^2$ and $\mathcal{ML}^1$ have both the finite frame and the finite domain properties with respect to* **QK** *and* **QT**.

PROOF. It is well known that the two-variable fragment of first-order logic has the finite model property (see [6]). $\dashv$

One more interesting fragment of $\mathcal{ML}$ is the set of *monadic formulas*, all predicate symbols in which are at most unary. Denote this fragment by $\mathcal{ML}^{mon}$, and let $\mathcal{ML}_1^{mon} = \mathcal{ML}_1 \cap \mathcal{ML}^{mon}$. In this case the formula $\alpha_{\mathfrak{T}}$, corresponding to a world candidate $\mathfrak{T}$ for $\varphi \in \mathcal{ML}_1^m$, is a monadic $\mathcal{L}$-formula. As is well-known, the monadic fragment of first-order logic is decidable, which yields us:

THEOREM 5.8. *Suppose that $L \in \{\mathbf{QK}^*, \mathbf{QK}, \mathbf{QT}, \mathbf{QK4}, \mathbf{QS4}\}$. Then the fragment $L \cap \mathcal{ML}_1^{mon}$ is decidable.*

As the monadic fragment of classical first-order logic has the finite model property (see [6]), we also have:

COROLLARY 5.9. *$\mathcal{ML}_1^{mon}$ has both the finite frame and the finite domain properties with respect to* **QK** *and* **QT**.

J. van Benthem (private communication) suggested the following natural generalization of the guarded formulas of [1] to the modal case.

DEFINITION 5.10 (guarded fragment). Denote by $\mathcal{MGF}$ the smallest set of $\mathcal{ML}$-formulas such that

- every atomic formula is in $\mathcal{MGF}$;
- if φ and ψ are in $\mathcal{MGF}$, then so are $\varphi \wedge \psi$, $\neg\varphi$, $\Box_i\varphi$ and $\Box_i^*\varphi$;
- if $\overline{x}, \overline{y}$ are tuples of variables, $G(\overline{x}, \overline{y})$ is atomic, $\varphi(\overline{x}, \overline{y}) \in \mathcal{MGF}$, and every free variable occurring in $\varphi(\overline{x}, \overline{y})$ occurs in $G(\overline{x}, \overline{y})$ as well, then the formula $\exists\overline{y}(G(\overline{x}, \overline{y}) \wedge \varphi(\overline{x}, \overline{y}))$ is in $\mathcal{MGF}$.

The set $\mathcal{MGF}$ is called the *guarded fragment* of first-order modal logic.

THEOREM 5.11. *The two-variable fragment of* $\mathbf{QK}^* \cap \mathcal{MGF}$ *is not recursively enumerable.*

PROOF. The proof is similar to the proof of Theorem 2.3. Now with every set $\mathcal{T} = \{\mathfrak{t}_0, \dots, \mathfrak{t}_n\}$ of tiles we associate the conjunction $\varphi_{\mathcal{T}}$ of the following formulas:

$$\begin{array}{l}
\exists x(Q(x) \wedge P_0(x) \wedge \Box^* \Diamond^* (P_0(x) \wedge \Diamond \top)),\\
\Box^* \forall x(Q(x) \to \exists y(R(x,y) \wedge Q(y))),\\
\Box^* \forall x,y(R(x,y) \to \Box^* R(x,y)),\\
\Box^* \forall x(Q(x) \to \Box Q(x)),\\
\Box^* \forall x(Q(x) \to (\bigvee_{i=0}^{n} P_i(x) \wedge \bigwedge_{i \neq j} (P_i(x) \to \neg P_j(x)))),\\
\Box^* \forall x(P_i(x) \to \forall y(R(x,y) \to \bigvee_{right(\mathfrak{t}_i)=left(\mathfrak{t}_j)} P_j(y))),\\
\Box^* \forall x(P_i(x) \to \Box \bigvee_{up(\mathfrak{t}_i)=down(\mathfrak{t}_j)} P_j(x)).
\end{array}$$

The reader can easily show that $\varphi_{\mathcal{T}}$ is satisfiable iff $\mathcal{T}$ recurrently tiles $\mathbb{N} \times \mathbb{N}$. ⊣

Let $\mathcal{MGF}_1 = \mathcal{MGF} \cap \mathcal{ML}_1$.

LEMMA 5.12. (i) *There is an algorithm that is capable of deciding, given a formula* $\varphi \in \mathcal{MGF}_1$ *and a world-candidate* $\mathfrak{T} = \langle T, T^{con} \rangle$ *for* φ, *whether* $\mathfrak{T}$ *is realizable.*

(ii) *Let* $\mathfrak{T} = \langle T, T^{con} \rangle$ *be a realizable world-candidate for* $\varphi \in \mathcal{MGF}$. *Then* $\mathfrak{T}$ *is realizable in a finite model.*

PROOF. As was shown in [1] and [19], the guarded fragment $\mathcal{GF}$ of classical first-order logic is decidable and has the finite model property. But we cannot apply these results directly to the formula

$$\alpha_{\mathfrak{T}} = \bigwedge_{t \in T} \exists x\, \bar{t}(x) \wedge \forall x \bigvee_{t \in T} \bar{t}(x) \wedge \bigwedge_{t_c \in T^{con}} \bar{t}(c)$$

(although both $\bar{t}(x)$ and $\bar{t}(c)$ are in $\mathcal{GF}$, the two quantifiers in $\alpha_{\mathfrak{T}}$ are not guarded). However, we can transform $\alpha_{\mathfrak{T}}$ into a guarded sentence in the following way.

Let P be a new unary predicate. Note that if $\psi \in \mathcal{GF}$ then the relativization ψ^P of ψ to P is logically equivalent to a $\mathcal{GF}$-formula. The proof is by induction on the construction of ψ^P. For atomic ψ, $\psi^P = \psi \in \mathcal{GF}$; the boolean cases are trivial; and for guarded $\psi(\overline{x}, \overline{y})$ and atomic $G(\overline{x}, \overline{y})$, $\left((\exists y_1 \dots \exists y_n (G(\overline{x}, \overline{y}) \wedge \psi)\right)^P$ is by definition $\exists y_1 \dots \exists y_n (\bigwedge_{1 \le i \le n} P(y_i) \wedge G(\overline{x}, \overline{y}) \wedge \psi^P)$, which is equivalent to $\exists y_1 \dots \exists y_n (G(\overline{x}, \overline{y}) \wedge (\bigwedge_{1 \le i \le n} P(y_i) \wedge \psi^P))$ and hence is (inductively) equivalent to a guarded formula. Now,

$$(\alpha_{\mathfrak{T}})^P = \bigwedge_{t \in T} \exists x(P(x) \wedge \bar{t}^P(x)) \wedge \forall x(P(x) \to \bigvee_{t \in T} \bar{t}^P(x)) \wedge \bigwedge_{t_c \in T^{con}} \bar{t}^P(c),$$

and we see that, up to logical equivalence, $(\alpha_{\mathfrak{T}})^P \in \mathcal{GF}$.

By classical model theory, $\alpha_{\mathfrak{T}}$ has a (finite) model iff $(\alpha_{\mathfrak{T}})^P$ has a (respectively, finite) model. ⊣

THEOREM 5.13. (i) *Let* $L \in \{\mathbf{QK}^*, \mathbf{QK}, \mathbf{QT}, \mathbf{QK4}, \mathbf{QS4}\}$. *Then the fragment* $L \cap \mathcal{MGF}_1$ *is decidable.*

(ii) $\mathcal{MGF}_1$ *has both the finite frame and the finite domain property with respect to* $\mathbf{QK}$ *and* $\mathbf{QT}$.

PROOF. Follows from the previous lemma and Theorems 4.7, 5.6. ⊣

Similar decidability results can be obtained for some other natural generalizations of decidable fragments of classical first-order logic to the modal predicate case, for example, the loosely guarded fragment of [44] (cf. [22]) or the fluted fragment (see [32] and references therein). We conjecture that more sophisticated classical decidable fragments, such as the Maslov class (see [51] and references therein), can also give rise to decidable fragments of $\mathcal{ML}_1$.

§6. Other applications. Another kind of decidable fragments of first-order modal logics can be associated with the so-called *modal description logics*, that is extensions of concept description logics with various modal operators (see e.g. [3, 17, 26, 49, 46, 50, 47]).

From the technical point of view pure (non-modal) description logics can be characterized as variable free fragments of first-order predicate logics (sometimes extended with fixed-point operators). They originate from practical knowledge representation systems which, in turn, can be traced back to semantic networks and Minsky frames; see e.g. [7]. An application domain is represented in the language of a description logic in terms of concepts (unary predicates), roles (binary predicates), and object names (constants). The expressive power of such a language depends on the available concept and role constructs, for instance, conjunction and negation of concepts, and composition, union, and the reflexive transitive closure of roles. Modal operators are added to description logics in order to capture various dynamic features of application domains.

Let us consider first a modal extension of the basic description logic called $\mathcal{ALC}$ (see [35]).

DEFINITION 6.1. The *modal description language* $\mathcal{ALC}_{\mathcal{M}}$ is based on a list of

- *concept names* $C_0, C_1, \dots$,
- *role names* $R_0, R_1, \dots$,
- *object names* $a_0, a_1, \dots$.

Starting from these, we define complex *concepts* and *formulas* using the following constructs. Atomic concepts are simply concept names, and

- if C and D are concepts, then so are $\top$, $C \wedge D$, $\neg C$,
- if C is a concept and R a role name, then $\exists R.C$ is a concept,
- if C is a concept then so are $\Box_i C$ and $\Box_i^* C$, for $i = 1, \dots, l$.

Atomic formulas are expressions of the form $\top$, $C = D$, $a : C$, aRb, where C and D are concepts, R is a role name and a, b are object names. Every atomic formula is a formula, and if φ and ψ are formulas then so are $\varphi \wedge \psi$, $\neg\varphi$, $\Box_i\varphi$, and $\Box_i^*\varphi$.

EXAMPLE 6.2. The following 'car salesman knowledge base' illustrates the expressive capacities of this language:

$$
\begin{aligned}
&\mathit{John} : \exists\mathsf{has}.\mathsf{Car}\\
&\mathit{John}\ \mathsf{likes}\ \mathit{Golf}\\
&\mathit{Golf} : \mathsf{VW}\\
&\mathsf{VW} \to \mathsf{Car} = \top\\
&\mathsf{Male\_customer} = \mathsf{Male} \wedge \mathsf{Customer}\\
&\mathsf{Modern\_car} = \mathsf{Car} \wedge \exists\mathsf{has}.\mathsf{Computer}\\
&\mathsf{Customer} = \mathsf{Homo\_sapiens} \wedge \langle\text{sometime in the past}\rangle\, \exists\mathsf{buys}.\mathsf{Car}\\
&\mathsf{Potential\_customer} = \langle\text{eventually}\rangle\, \mathsf{Customer}\\
&\langle\text{John believes}\rangle\, \langle\text{next year}\rangle\, (\mathsf{Male\_customer} \to \exists\mathsf{buys}.\mathsf{Modern\_car} = \top)
\end{aligned}
$$

Here *John* and *Golf* are object names, Car, Computer, VW, Male, Customer, Male_customer, Potential_customer, Homo_sapiens, Modern_car are concept names, has, likes, buys role names, and ⟨sometime in the past⟩, ⟨eventually⟩, ⟨next year⟩, and ⟨John believes⟩ are modal operators. So the last formula, for instance, says that according to John's belief next year every male customer will buy a modern car.

The intended meaning of concepts and formulas will become clear from the following translation T of $\mathcal{ALC}_{\mathcal{M}}$ into the language of first-order modal logic with constants $a_0, a_1, \dots$.

Fix two distinct variables x, y. The translation C^T of a concept C is the $\mathcal{ALC}_{\mathcal{M}}$-formula with one free variable defined inductively by taking

$$
\begin{aligned}
C_i^T &= P_i(x)\\
\top^T &= \top\\
(C \wedge D)^T &= C^T \wedge D^T\\
(\neg C)^T &= \neg C^T\\
(\Box_i^\circ C)^T &= \Box_i^\circ C^T\\
(\exists R.C)^T &= \exists x'(R(y', x') \wedge C^T),
\end{aligned}
$$

where x' is the free variable in C^T and $\{y'\} = \{x, y\} - \{x'\}$.

The translation φ^T of an $\mathcal{ALC}_{\mathcal{M}}$-formula φ is an $\mathcal{ML}$-sentence defined in the following way (without loss of generality we assume that x is the free variable in C^T and D^T):

$$
\begin{aligned}
(C = D)^T &= \forall x(C^T \leftrightarrow D^T)\\
(a : C)^T &= C^T\{a/x\}\\
(aR_ib)^T &= R_i(a, b)\\
(\varphi \wedge \psi)^T &= \varphi^T \wedge \psi^T\\
(\neg\varphi)^T &= \neg\varphi^T\\
(\Box_i^\circ \varphi)^T &= \Box_i^\circ \varphi^T
\end{aligned}
$$

φ is said to be *satisfiable* if its translation φ^T is satisfiable in a first-order Kripke model. It follows from the definition that $\varphi^T \in \mathcal{ML}_1^2$, for every $\mathcal{ALC}_{\mathcal{M}}$-formula φ. Hence we obtain:

THEOREM 6.3. *The satisfiability problem for $\mathcal{ALC}_{\mathcal{M}}$-formulas is decidable.*

This theorem covers various decidability results for modal description logics, e.g. those of [3, 17, 26] and some of [49].

Remark 6.4. In [47] we extend the language $\mathcal{ALC}_{\mathcal{M}}$ by allowing applications of the modal operators $\Box_i$ and $\Diamond_i$ not only to concepts and formulas, but to roles as well, i.e., if R is a role then so are $\Box_i R$ and $\Diamond_i R$. The intended meaning of the modalized roles is described by the clauses:

$$\begin{aligned}(\Box_i R)^T &= \Box_i R(x,y)\\ (\Diamond_i R)^T &= \Diamond_i R(x,y).\end{aligned}$$

For example, the formula

$$\mathsf{Faithful\_customer} = \mathsf{Customer} \wedge \exists[\text{always}]\mathsf{buys.Car}$$

defines 'a faithful customer' as a customer who always buys a car of the same type, say Golf.

It is shown in [47] that the satisfiability problem for this language is decidable in the classes of arbitrary frames, **S5**-frames and **KD45**-frames. However, it becomes undecidable in arbitrary frames if we add the operators $\Box_i^*$.

The reader must have already noticed that $\mathcal{ALC}$—the non-modal part of $\mathcal{ALC}_{\mathcal{M}}$—is in principle nothing else but the polymodal propositional logic **K** (this fact was first observed in [33]). There are, however, much more expressive (and yet decidable) concept description languages; see e.g. [8]. Here is only one example, the logic $\mathcal{CI}$ introduced in [15, 16]. Its syntax allows the formation of the union $R \vee S$ of roles R and S, their composition $R \circ S$, the reflexive and transitive closure R^*, and the inversion R^- of R; moreover, every concept C gives rise to the role $C?$ (the set of all pairs $\langle x, x\rangle$ such that $x \in C$). This language may be regarded as a terminological variant of propositional dynamic logic **PDL**.

The first-order modal language we deal with in this paper does not have enough expressive power to represent all these constructs, say that in each world of a model R^* should be the transitive reflexive closure of R. But we can interpret $\mathcal{CI}$ in models whose worlds are suitable first-order $\mathcal{L}$-structures, and use the following generalization of Theorem 4.7, which is proved in precisely the same way.

THEOREM 6.5. *Let $\mathcal{FO}$ be a class of first-order structures and $\mathcal{ML}' \subseteq \mathcal{ML}_1$. Suppose also that*

(i) *there is an algorithm that is capable of deciding, given a world candidate for $\varphi \in \mathcal{ML}'$, whether it is realizable in a structure from $\mathcal{FO}$, and*
(ii) *there is a cardinal $\kappa \geq \aleph_0$ such that for any $\kappa' \geq \kappa$, every world candidate $\langle T, T^{con}\rangle$ realizable in a structure from $\mathcal{FO}$ is realizable in a structure $\mathfrak{D}$ from $\mathcal{FO}$ in which $|\mathfrak{D}_t| = \kappa'$ for every $t \in T$.*

Then the satisfiability problem for $\mathcal{ML}'$-formulas in first-order modal models, the worlds of which are structures from $\mathcal{FO}$, is decidable.

COROLLARY 6.6. *The satisfiability problem for the modal description logic $\mathcal{CI}_{\mathcal{M}}$ (the modal operators $\Box_i^\circ$ are applied to both concepts and formulas) is decidable.*

PROOF. That (i) holds was actually shown in [15], and (ii) follows from the fact that the class of first-order structures for $\mathcal{CI}$ is closed under the disjoint unions. ⊣

For more decidability results of this sort see [49, 46, 50, 47].

As was observed in [12], the one-variable fragment of a first-order modal logic can be regarded as the Cartesian product of its propositional fragment with **S5**. Indeed, let L be a Kripke complete propositional modal logic in the language with $\Box_1$ and let **S5** be formulated in the language with $\Box_2$. The *product* $L \times \mathbf{S5}$ is defined as the set of all bimodal formulas in the combined language with $\Box_1$ and $\Box_2$ that are valid in all *product frames* $\mathfrak{F} \times \mathfrak{G}$, where $\mathfrak{F} = \langle W, R \rangle$ is a Kripke frame for L and $\mathfrak{G} = \langle V, S \rangle$ a Kripke frame for **S5**. More precisely, $\mathfrak{F} \times \mathfrak{G} = \langle W \times V, \overline{R}, \overline{S} \rangle$ where, for all $\langle w, v \rangle, \langle w', v' \rangle \in W \times V$, $\langle w, v \rangle \overline{R} \langle w', v' \rangle$ iff wRw' and $v = v'$, and $\langle w, v \rangle \overline{S} \langle w', v' \rangle$ iff $w = w'$ and vSv'; $\Box_1$ is interpreted by the relation $\overline{R}$ and $\Box_2$ by $\overline{S}$. It is easy to see that without loss of generality we may assume S to be the universal relation on V. Thus, a propositional model $\mathfrak{M}$ based on $\mathfrak{F} \times \mathfrak{G}$ can be viewed as a first-order Kripke model for the one-variable fragment of $\mathcal{ML}$: it is based on $\mathfrak{F}$, has domain V and interprets $P_i(x)$ in a world $w \in W$ as the set $P_i^w = \{v \in V : (\mathfrak{M}, \langle w, v \rangle) \models p_i\}$. Conversely, every first-order Kripke model based on $\mathfrak{F}$ and having domain V gives rise to a propositional model based on the product of $\mathfrak{F}$ and $\langle V, V \times V \rangle$. On the syntactical level, this observation corresponds to the translation * from the language of $L \times \mathbf{S5}$ onto $\mathcal{ML}^1$ defined by taking

$$
\begin{aligned}
p_i^\star &= P_i(x), \\
(\varphi \wedge \psi)^\star &= \varphi^\star \wedge \psi^\star, \\
(\neg \varphi)^\star &= \neg \varphi^\star, \\
(\Box_1 \varphi)^\star &= \Box \varphi^\star, \\
(\Box_2 \varphi)^\star &= \forall x \varphi^\star.
\end{aligned}
$$

It is easy to see that a propositional bimodal formula φ belongs to $L \times \mathbf{S5}$ iff $\varphi^\star$ is valid in all first-order modal models based on the Kripke frames for L. So, as a consequence of Theorem 5.2, we obtain that the products of **K**, **T**, **K4**, **S4** and some other logics with **S5** are decidable; cf. [12]. (By the way, 3-dimensional products $L \times \mathbf{S5} \times \mathbf{S5}$ can be embedded into the two-variable fragment of the first-order counterpart of L. As was shown in [21], all 3D product logics between $\mathbf{K}^3$ and $\mathbf{S5}^3$ are undecidable, and so the two-variable fragments of the corresponding first-order modal logics are undecidable as well; cf. [11].)

§7. Open problems. We conclude the paper with some open problems.

(1) Can our results be extended to first-order modal logics **QK**, **QK4**, etc. with equality?
(2) Are the modal guarded fragments of **QK**, **QK4**, etc. decidable?

(3) Is there a decidable fragment of classical first-order logic the monodic extension of which is undecidable (for some natural modal predicate logic)?
(4) What is the computational complexity of the decidable fragments listed in Section 5? How is it connected with the complexity of the underlying fragment of classical logic?

One more challenging problem is to develop 'implementable' decision algorithms for the obtained decidable fragments, say tableau- or resolution-type, preferably by combining available procedures for the modal and first-order components in a modular way. That such kind of tableau algorithms can be constructed for description logics with modal and temporal operators has been shown in [40, 28].

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