Semi-qualitative reasoning about distances: a preliminary report

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Abstract We introduce a family of languages intended for representing knowledge and reasoning about metric (and more general distance) spaces. While the simplest language can speak only about distances between individual objects and Boolean relations between sets, the more expressive ones are capable of capturing notions such as 'somewhere in (or somewhere out of) the sphere of a certain radius', 'everywhere in a certain ring', etc. The computational complexity of the satisfiability problem for formulas in our languages ranges from NP-completeness to undecidability and depends on the class of distance spaces in which they are interpreted. Besides the class of all metric spaces, we consider, for example, the spaces $\mathbb{R} \times \mathbb{R}$ and $\mathbb{N} \times \mathbb{N}$ with their natural metrics.

1 Introduction

The concept of 'distance between objects' is one of the most fundamental abstractions both in science and in everyday life. Imagine for instance (only imagine) that you are going to buy a house in London. You then inform your estate agent about your intention and provide her with a number of constraints:

(A) The house should not be too far from your college, say, not more than 10 miles.
(B) The house should be close to shops, restaurants, and a movie theatre; all this should be reachable, say, within 1 mile.
(C) There should be a 'green zone' around the house, at least within 2 miles in each direction.
(D) Factories and motorways must be far from the house, not closer than 5 miles.
(E) There must be a sports center around, and moreover, all sports centers of the district should be reachable on foot, i.e., they should be within, say, 3 miles.
(F) And of course there must be a tube station around, not too close, but not too far either—somewhere between 0.5 and 1 mile.

'Distances' can be induced by different measures. We may be interested in the physical distance between two cities a and b, i.e., in the length of the straight (or geodesic) line between a and b. More pragmatic would be to bother about the length of the railroad connecting a and b, or even better the time it takes to go from a to b by train (plane, ship, etc.). But we can also define the distance as the number of cities (stations, friends to visit, etc.) on the way from a to b, as the difference in altitude between a and b, and so forth.

The standard mathematical models capturing common features of various notions of distance are known as metric spaces (see e.g. [4]). We define a *metric space* as a pair $\mathfrak{D} = \langle W, d \rangle$, where W is a set (of points) and d a function from $W \times W$ into $\mathbb{R}$, the *metric* on W, satisfying the following conditions, for all $x, y, z \in W$:

$$d(x,y) = 0 \text{ iff } x = y, \tag{1}$$

$$d(x,z) \leq d(x,y) + d(y,z), \tag{2}$$

$$d(x,y) = d(y,x). \tag{3}$$

The value $d(x,y)$ is called the *distance* from the point x to the point y.[1]

It is to be noted, however, that although quite acceptable in many cases, the defined concept of metric space is not universally applicable to all interesting measures of distances between points, especially those used in everyday life. Here are some examples:

(i) Suppose that W consists of the villages in a certain district and $d(x,y)$ denotes the time it takes to go from x to y by train. Then the function d is not necessarily total, since there may be villages without stations.

(ii) If $d(x,y)$ is the flight-time from x to y then, as we know it too well, d is not necessarily symmetric, even approximately (just go from Malaga to Tokyo and back).

(iii) Often we do not measure distances by means of real numbers but rather using more fuzzy notions such as 'short', 'medium', 'long'. To represent these measures we can, of course, take functions d from $W \times W$ into the set $\{1, 2, 3\} \subseteq \mathbb{R}$ and define $short := 1$, $medium := 2$, and $long := 3$. So we can still regard these distances as real numbers. However, for measures of this type the triangle inequality (2) does not make sense (short plus short can still be short, but it can be also medium or long).

In this paper we assume first that distance functions are total and satisfy (1)–(3), i.e., we deal with standard metric spaces. But then, in Section 6, we discuss how far our results can be extended if we consider more general *distance spaces*.

Our main aim in the paper is to

> *design formal languages of metric (or more general distance) spaces that can be used to represent and reason about (a substantial part of) our*

[1] Usually axioms (2) and (3) are combined into one axiom $d(y,z) \leq d(x,y) + d(x,z)$ which implies the symmetry property (3); cf. [4]. In our case symmetry does not follow from the triangle inequality (2). We will use this fact in Section 6.

everyday knowledge of distances, and that are at the same time as computationally tractable as possible.

The next step will be to integrate the developed languages with formalisms intended for qualitative spatial reasoning (e.g. RCC-8), temporal reasoning, and maybe even combined spatio-temporal reasoning (e.g. [19]).

The requirement of computational effectiveness imposes rather severe limitations on possible languages of metric spaces. For instance, we can hardly use the full power of the common mathematical formalism which allows arithmetic operations and quantification over distances as in the usual definition of a continuous function f from $\mathfrak{D}$ to $\mathbb{R}$:

$$\forall x \in W \ \forall \epsilon > 0 \ \exists \delta > 0 \ \forall y \in W \ \ (d(x,y) < \epsilon \rightarrow |f(x) - f(y)| < \delta)\,.$$

On the other hand, in everyday life a great deal of assertions about distances can be (and are) made without such operations and quantification. Although we operate quantitative information about distances, as in examples (A)–(F) above, the reasoning is quite often rather qualitative, with numerical data being involved only in comparisons ('everywhere within 7 m distance', 'in more than 3 hours', etc.), which as we observed above can also encode such vague concepts as 'short', 'medium', 'long'. As travelling scientists, we don't care about the precise location of Malaga, being content with the (qualitative) information that it is in Spain, Spain is disconnected from Germany and the U.K., and the flight-time to any place in Spain from Germany or the U.K. is certainly less than 4 hours. That is why we call our formalisms *semi-qualitative*, following a suggestion of A. Cohn.

In the next section we propose a hierarchy of 'semi-qualitative' propositional languages intended for reasoning about distances. We illustrate their expressive power and formulate the results on the finite model property, decidability, and computational complexity we have managed to obtain so far. (The closest 'relatives' of our logics in the literature are the logics of place from [14, 18, 15, 11, 12] and metric temporal logics from [13]; see also [5].) Sections 3–5 show how some of these results can be proved. And in Section 6 we discuss briefly more general notions of 'distance spaces.'

The paper is a preliminary report on our ongoing research; that is why it contains more questions than answers (some of them will certainly be solved by the time of publication).

2 The logics of metric spaces

All the logics of metric spaces to be introduced in this section are based on the following *Boolean logic of space* $\mathcal{BS}$. The alphabet of $\mathcal{BS}$ contains

- an infinite list of *set* (or *region*) *variables* $X_1, X_2, \ldots$;
- an infinite list of *location variables* $x_1, x_2, \ldots$;
- the Boolean operators $\wedge$ and $\neg$.

Boolean combinations of set variables are called *set* (or *region*) *terms*. *Atomic formulas* in $\mathcal{BS}$ are of two types:

- $x \sqsubseteq t$, where x is a location variable and t a set term,
- $t_1 = t_2$, where t_1 and t_2 are set terms.

The intended meaning of these formulas should be clear from their syntax: $x \sqsubseteq t$ means that x belongs to t, and $t_1 = t_2$ says that t_1 and t_2 have the same extensions.

$\mathcal{BS}$-*formulas* are just arbitrary Boolean combinations of atoms.

The language $\mathcal{BS}$, as well as all other languages to be introduced below, is interpreted in metric spaces $\mathfrak{D} = \langle W, d \rangle$ by means of *assignments* $\mathfrak{a}$ associating with every set variable X a subset $\mathfrak{a}(X)$ of W and with every location variable x an element $\mathfrak{a}(x)$ of W. The *value* $t^{\mathfrak{a}}$ of a set term t in the model $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{a} \rangle$ is defined inductively:

$$\begin{aligned} &X_i^{\mathfrak{a}} = \mathfrak{a}(X_i), \ X_i \text{ a set variable,} \\ &(t_1 \wedge t_2)^{\mathfrak{a}} = t_1^{\mathfrak{a}} \cap t_2^{\mathfrak{a}}, \\ &(\neg t)^{\mathfrak{a}} = W - t^{\mathfrak{a}}. \end{aligned}$$

(If the space $\mathfrak{D}$ is not clear from the context, we write $t^{\mathfrak{M}}$ instead of $t^{\mathfrak{a}}$.)

The *truth-relation* for $\mathcal{BS}$-formulas reflects the intended meaning:

$$\begin{aligned} &\mathfrak{M} \models x \sqsubseteq t &&\text{iff } \ \mathfrak{a}(x) \in t^{\mathfrak{a}}, \\ &\mathfrak{M} \models t_1 = t_2 &&\text{iff } \ t_1^{\mathfrak{a}} = t_2^{\mathfrak{a}}, \end{aligned}$$

plus the standard clauses for the Booleans.

We write $\top$ instead of $\neg(X \wedge \neg X)$, $\emptyset$ instead of $X \wedge \neg X$, and $t_1 \sqsubseteq t_2$ instead of $\neg(t_1 \wedge \neg t_2) = \top$. It should be clear that $\mathfrak{M} \models t_1 \sqsubseteq t_2$ iff $t_1^{\mathfrak{a}} \subseteq t_2^{\mathfrak{a}}$.

$\mathcal{BS}$ can only talk about relations between sets, about their members, but not about distances. For instance, we can construct the following knowledge base in $\mathcal{BS}$:

$$\begin{array}{ll} \mathit{Leipzig} \sqsubseteq \mathit{Germany}, & \mathit{Malaga} \sqsubseteq \mathit{Spain}, \\ \mathit{Germany} \sqsubseteq \mathit{Europe}, & \mathit{Spain} \sqsubseteq \mathit{Europe}, \\ \mathit{Spain} \wedge \mathit{Germany} = \emptyset. & \end{array}$$

The metric d in $\mathfrak{D}$ is irrelevant for $\mathcal{BS}$. 'Real' metric logics are defined by extending $\mathcal{BS}$ with a number of set term and formula constructs which involve distances. We define five such logics and call them $\mathcal{MS}_0, \ldots, \mathcal{MS}_4$.

$\boldsymbol{\mathcal{MS}_0}$. To begin with, let us introduce constructs which allow us to speak about distances between locations. Denote by $\mathcal{MS}_0$ the language extending $\mathcal{BS}$ with the possibility of constructing atomic formulas of the form

- $\boldsymbol{\delta}(x, y) = a$,
- $\boldsymbol{\delta}(x, y) < a$,

– $\boldsymbol{\delta}(x,y) = \boldsymbol{\delta}(x',y')$,
– $\boldsymbol{\delta}(x,y) < \boldsymbol{\delta}(x',y')$,

where x, y, x', y' are location variables and $a \in \mathbb{R}_+$ (i.e., a is a non-negative real number). The truth-conditions for such formulas are obvious:

$$\begin{array}{ll} \mathfrak{M} \models \boldsymbol{\delta}(x,y) = a & \text{iff } d(\mathfrak{a}(x), \mathfrak{a}(y)) = a, \\ \mathfrak{M} \models \boldsymbol{\delta}(x,y) < a & \text{iff } d(\mathfrak{a}(x), \mathfrak{a}(y)) < a, \\ \mathfrak{M} \models \boldsymbol{\delta}(x,y) = \boldsymbol{\delta}(x',y') & \text{iff } d(\mathfrak{a}(x), \mathfrak{a}(y)) = d(\mathfrak{a}(x'), \mathfrak{a}(y')), \\ \mathfrak{M} \models \boldsymbol{\delta}(x,y) < \boldsymbol{\delta}(x',y') & \text{iff } d(\mathfrak{a}(x), \mathfrak{a}(y)) < d(\mathfrak{a}(x'), \mathfrak{a}(y')). \end{array}$$

$\mathcal{MS}_0$ provides us with some primitive means for basic reasoning about regions and distances between locations. For example, constraint (A) from Section 1 can be represented as

$$(\boldsymbol{\delta}(house, college) < 10) \vee (\boldsymbol{\delta}(house, college) = 10). \tag{4}$$

The main reasoning problem we are interested in is satisfiability of finite sets of formulas in arbitrary metric spaces or in some special classes of metric spaces, say, finite ones, the Euclidean n-dimensional space $\langle \mathbb{R}^n, d_n \rangle$ with the standard metric

$$d_n(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

the subspace $\langle \mathbb{N}^n, d'_n \rangle$ of $\langle \mathbb{R}^n, d_n \rangle$ (with the induced metric), etc. The choice of metric spaces depends on applications. For instance, if we deal with time constraints then the intended space can be one-dimensional $\langle \mathbb{R}, d_1 \rangle$ or its subspaces based on $\mathbb{Q}$ or $\mathbb{N}$. If we consider a railway system, then the metric space is finite.

It is to be noted from the very beginning that the language $\mathcal{MS}_0$ as well as other languages $\mathcal{MS}_i$ are *uncountable* because all of them contain uncountably many formulas of the form $\boldsymbol{\delta}(x,y) = a$, for $a \in \mathbb{R}_+$. So in general it does not make sense to ask whether the satisfiability problem for such languages is decidable.

To make the satisfiability problem sensible we have to restrict the languages $\mathcal{MS}_i$ to at least recursive (under some coding) subsets of $\mathbb{R}_+$. Natural examples of such subsets are the non-negative rational numbers $\mathbb{Q}_+$ or the natural numbers $\mathbb{N}$.

Given a set $\mathbb{S} \subseteq \mathbb{R}_+$, we denote by $\mathcal{MS}_i[\mathbb{S}]$ the fragment of $\mathcal{MS}_i$ consisting of only those $\mathcal{MS}_i$-formulas all real numbers in which belong to $\mathbb{S}$.

For the logic $\mathcal{MS}_0$ we have the following:

Theorem 1. (i) *The satisfiability problem for $\mathcal{MS}_0[\mathbb{Q}]$-formulas in arbitrary metric spaces is decidable.*

(ii) *Every finite satisfiable set of $\mathcal{MS}_0$-formulas is satisfiable in a finite metric space, or in other words, $\mathcal{MS}_0$ has the finite model property.*

This theorem follows immediately from the proof of the finite model property of $\mathcal{MS}_2$ in Section 5. We don't know whether satisfiability of $\mathcal{MS}_0[\mathbb{Q}]$-formulas in $\mathbb{R}^n$ is decidable. We conjecture that it is and that the complexity of the satisfiability problem for both arbitrary metric spaces and $\mathbb{R}^n$ is in NP.

In $\mathcal{MS}_0$ we can talk about distances between *points* in metric spaces. Now we extend the language by providing constructs capable of saying that a point is within a certain distance from a set, which is required to represent constraint (B) from Section 1.

$\boldsymbol{\mathcal{MS}_1}$. Denote by $\mathcal{MS}_1$ the language that is obtained by extending $\mathcal{MS}_0$ with the following set term constructs:

– if t is a set term and $a \in \mathbb{R}_+$, then $\exists_{\leq a} t$ and $\forall_{\leq a} t$ are set terms as well.

The semantical meaning of the new set terms is defined by

$$(\exists_{\leq a} t)^{\mathfrak{a}} = \{x \in W : \exists y \in W \ (d(x,y) \leq a \wedge y \in t^{\mathfrak{a}})\},$$
$$(\forall_{\leq a} t)^{\mathfrak{a}} = \{x \in W : \forall y \in W \ (d(x,y) \leq a \rightarrow y \in t^{\mathfrak{a}})\}.$$

Thus $x \sqsubseteq \exists_{\leq a} t$ means that 'somewhere in or on the sphere with center x and radius a there is a point from t'; $x \sqsubseteq \forall_{\leq a} t$ says that 'the whole sphere with center x and radius a, including its surface, belongs to t.'

Constraints (B)–(D) are now expressible by the formulas:

$$house \sqsubseteq \exists_{\leq 1} shops \wedge \exists_{\leq 1} restaurants \wedge \exists_{\leq 1} cinemas, \tag{5}$$
$$house \sqsubseteq \forall_{\leq 2}\ green\_zone, \tag{6}$$
$$house \sqsubseteq \neg \exists_{\leq 5} (factories \vee motorways). \tag{7}$$

Here is what we know about this language:

Theorem 2. (i) *The satisfiability problem for $\mathcal{MS}_1[\mathbb{Q}]$-formulas in arbitrary metric spaces is decidable.*

(ii) *$\mathcal{MS}_1$ has the finite model property.*

(iii) *The satisfiability problem for $\mathcal{MS}_1[\{1\}]$-formulas in $\langle \mathbb{N}^2, d_2' \rangle$ is undecidable.*

Claims (i) and (ii) follow from the proof of the finite model property in Section 5. The proof of (iii) is omitted. It can be conducted similarly to the undecidability proof in Section 3. Note that at the moment we don't know whether the satisfiability in $\mathbb{R}^2$ is decidable and what is the complexity of satisfiability of $\mathcal{MS}_1[\mathbb{Q}]$-formulas.

$\boldsymbol{\mathcal{MS}_2}$. In the same manner we can enrich the language $\mathcal{MS}_1$ with the constructs for expressing 'somewhere outside the sphere with center x and radius a' and 'everywhere outside the sphere with center x and radius a'. To this end we add to $\mathcal{MS}_1$ two term-formation constructs:

– if t is a set term and $a \in \mathbb{R}_+$, then $\exists_{>a} t$ and $\forall_{>a} t$ are set terms.

The resulting language is denoted by $\mathcal{MS}_2$. The intended semantical meaning of the new constructs is as follows:

$$(\exists_{>a}t)^{\mathfrak{a}} = \{x \in W : \exists y \in W \ (d(x,y) > a \wedge y \in t^{\mathfrak{a}})\},$$
$$(\forall_{>a}t)^{\mathfrak{a}} = \{x \in W : \forall y \in W \ (d(x,y) > a \rightarrow y \in t^{\mathfrak{a}})\}.$$

Constraint (E) can be represented now as the formula

$$\mathit{house} \sqsubseteq \exists_{\leq 3}\mathit{district\_sports\_center} \wedge \forall_{>3}\neg\ \mathit{district\_sports\_center}. \qquad (8)$$

The language $\mathcal{MS}_2$ is quite expressive. First, it contains an analogue of the difference operator from modal logic (see [6]), because using $\forall_{>0}$ we can say 'everywhere but here':

$$\mathfrak{M} \models x \sqsubseteq \forall_{>0}t \text{ iff } \mathfrak{M} \models y \sqsubseteq t \text{ for all } y \neq x.$$

We also have the universal modalities of [9]: the operators $\forall$ and $\exists$ can be defined by taking

$$\forall t = t \wedge \forall_{>0}t, \qquad \text{i.e., } \forall t \text{ is } \emptyset \text{ if } t \neq \top \text{ and } \top \text{ otherwise,}$$
$$\exists t = t \vee \exists_{>0}t, \qquad \text{i.e., } \forall t \text{ is } \top \text{ if } t \neq \emptyset \text{ and } \emptyset \text{ otherwise.}$$

Second, we can simulate the nominals of [1]. Denote by $\mathcal{MS}_2'$ the language that results from $\mathcal{MS}_2$ by allowing set terms of the form $\{x\}$, for every location variable x, with the obvious interpretation:

– $\mathfrak{a}(\{x\}) = \{\mathfrak{a}(x)\}$.

In $\mathcal{MS}_2'$ we can say, for example, that

$$(\exists_{\leq 1100}\{\mathit{Leipzig}\} \wedge \exists_{\leq 1100}\{\mathit{Malaga}\}) \sqsubseteq \mathit{France},$$

i.e., 'if you are not more than 1100 km away from Leipzig and not more than 1100 km away from Malaga, then you are in France'.

As far as the satisfiability problem is concerned, $\mathcal{MS}_2'$ is not more expressive than $\mathcal{MS}_2$. To see this, consider a finite set of $\mathcal{MS}_2'$-formulas Γ and suppose that $x_1, \ldots, x_n$ are all location variables which occur in Γ as set terms $\{x_i\}$. Take fresh set variables $X_1, \ldots, X_n$ and let Γ' be the result of replacing all $\{x_i\}$ in Γ with X_i. It is readily checked that Γ is satisfiable in a model based on a metric space $\mathfrak{D}$ iff the set of $\mathcal{MS}_2$-formulas

$$\Gamma' \cup \{(X_i \wedge \neg\exists_{>0}X_i) \neq \emptyset : i \leq n\}$$

is satisfiable in $\mathfrak{D}$.

It is worth noting that, as will become obvious in the next section, the relation between the operators $\forall_{\leq a}$ and $\forall_{>a}$ corresponds to the relation between modal operators $\Box$ and $\Box^-$ interpreted in Kripke frames by an accessibility relation R and its complement $\overline{R}$, respectively; see [8] for a study of modal logics with such boxes.

Theorem 3. (i) *The satisfiability problem for $\mathcal{MS}_2[\mathbb{Q}]$-formulas in arbitrary metric spaces is decidable.*

(ii) *$\mathcal{MS}_2$ has the finite model property.*

This result will be proved in Section 5. We don't know, however, what is the complexity of the satisfiability problem from (i).

***$\mathcal{MS}_3$**.* To be able to express the last constraint (F) from Section 1, we need two more constructs:

– if t is a set term and $a < b$, then $\exists^{>a}_{\leq b} t$ and $\forall^{>a}_{\leq b} t$ are set terms.

The extended language will be denoted by $\mathcal{MS}_3$. The truth-conditions for these operators are as follows:

$$(\exists^{>a}_{\leq b} t)^{\mathfrak{a}} = \{x \in W : \exists y \in W \ (a < d(x,y) \leq b \wedge y \in t^{\mathfrak{a}})\},$$
$$(\forall^{>a}_{\leq b} t)^{\mathfrak{a}} = \{x \in W : \forall y \in W \ (a < d(x,y) \leq b \rightarrow y \in t^{\mathfrak{a}})\}.$$

In other words, $x \sqsubseteq \exists^{>a}_{\leq b} t$ iff 'somewhere in the ring with center x, the inner radius a and the outer radius b, including the outer circle, there is a point from t'.

Constraint (F) is represented then by the formula:

$$house \sqsubseteq \exists^{>0.5}_{\leq 1} tube\_station. \tag{9}$$

(By the way, the end of the imaginary story about buying a house in London was not satisfactory. Having checked her knowledge base, the estate agent said: "Unfortunately, your constraints (4)–(9) are not satisfiable in London, where we have

$$tube\_station \sqsubseteq \exists_{\leq 3.5}(factory \vee motorway).$$

In view of the triangle inequality, this contradicts constraints (7) and (9).")

Unfortunately, the language $\mathcal{MS}_3$ is too expressive for many important classes of metric spaces.

Theorem 4. *Let $\mathcal{K}$ be a class of metric spaces containing $\mathbb{R}^2$. Then the satisfiability problem for $\mathcal{MS}_3[\{0, \ldots, 100\}]$-formulas in $\mathcal{K}$ is undecidable.*

This result will be proved in the next section (even for a small fragment of $\mathcal{MS}_3$).

***$\mathcal{MS}_4$**.* The most expressive language $\mathcal{MS}_4$ we have in mind is an extension of $\mathcal{MS}_3$ with the operators $\exists_{<a} t$, $\forall_{<a} t$, $\exists_{\geq a} t$, $\forall_{\geq a} t$, $\exists^{\geq a}_{<b} t$, $\forall^{\geq a}_{<b} t$.

Here is what we know about these operators: the satisfiability problem for the full language in the class of all metric spaces is of course undecidable—it contains $\mathcal{MS}_3$. Moreover, the operators $\forall^{\geq a}_{<b}$ alone determine an undecidable language for the class of arbitrary metric spaces (this can be proved similarly to the undecidability proof in Section 3). Also, a similar proof shows that the language with the operators $\forall_{<a}$ only is undecidable both in $\langle \mathbb{R}^2, d_2 \rangle$ and in $\langle \mathbb{N}^2, d_2' \rangle$. Still, various questions are open, however: for example, whether the language with the operators $\forall_{<a}$ only is decidable in arbitrary metric spaces or whether there are interesting classes of metric spaces in which $\mathcal{MS}_4$ is decidable.

3 Undecidability

In this section we prove a rather general undecidability result. In particular, Theorem 4 is its immediate consequence.

Theorem 5. *Let $\mathcal{K}$ be a class of metric spaces containing $\mathbb{R}^2$. Then the satisfiability problem for $\mathcal{MS}_3[\{0,9,10,20,80\}]$-formulas (even for those with the operators $\forall^{>0}_{\leq a}$ and $\exists_{\leq a}$ only) in $\mathcal{K}$ is undecidable.*

Proof. To prove this result, we reduce the undecidable $\mathbb{N}\times\mathbb{N}$-tiling problem (see [17, 2] and references therein) to the satisfiability problem in $\mathcal{K}$. We remind the reader that the tiling problem for $\mathbb{N}\times\mathbb{N}$ is formulated as follows: given a finite set $\mathcal{T} = \{T_1,\ldots,T_l\}$ of tiles (i.e., squares T_i with colors $left(T_i)$, $right(T_i)$, $up(T_i)$, and $down(T_i)$ on their edges), determine whether tiles in $\mathcal{T}$ can cover the grid $\mathbb{N}\times\mathbb{N}$ in such a way that the colors of adjacent edges on adjacent tiles match, or more precisely, whether there exists a function $\tau : \mathbb{N}\times\mathbb{N} \to \mathcal{T}$ such that for all $n, m \in \mathbb{N}$:

(a) $right(\tau(n,m)) = left(\tau(n+1,m))$,
(b) $up(\tau(n,m)) = down(\tau(n,m+1))$.

So, suppose a set of tiles $\mathcal{T} = \{T_1,\ldots,T_l\}$ is given. Our aim is to construct a finite set of $\mathcal{MS}_3[\{0,9,10,20,80\}]$-formulas which is satisfiable in $\mathcal{K}$ iff $\mathcal{T}$ can tile $\mathbb{N}\times\mathbb{N}$.

Take set variables $Z_1,\ldots,Z_l, X_0,\ldots,X_4, Y_0,\ldots,Y_4$. Let $\chi_{ij} = \forall_{\leq 9}(X_i \wedge Y_j)$, for $i,j \leq 4$, and let Γ be the set of the following formulas, where $i,j \leq 4$ and $k \leq l$:

$$X_i \wedge Y_j \sqsubseteq \exists_{\leq 9}\chi_{ij},\;\; \chi_{ij} \sqsubseteq \forall^{>0}_{\leq 80}\neg\chi_{ij},\;\; \chi_{ij} \sqsubseteq \neg\chi_{mn}\ ((i,j)\neq(m,n)), \tag{10}$$

$$\chi_{ij} \sqsubseteq \bigvee_{k\leq l} \forall_{\leq 9} Z_k,\;\; Z_m \sqsubseteq \neg Z_n\ (n \neq m), \tag{11}$$

$$\chi_{ij} \wedge Z_k \sqsubseteq \exists_{\leq 20}(\chi_{i+_5 1 j} \wedge \bigvee_{right(T_k)=left(T_m)} Z_m), \tag{12}$$

$$\chi_{ij} \wedge Z_k \sqsubseteq \exists_{\leq 20}(\chi_{ij+_5 1} \wedge \bigvee_{up(T_k)=down(T_m)} Z_m), \tag{13}$$

where $+_5$ denotes addition modulo 5.

The first formula in (10) is satisfied in a model $\mathfrak{M} = \langle W, d, \mathfrak{a}\rangle$ iff $\mathfrak{a}(X_i \wedge Y_j)$ is the union of a set of spheres of radius 9. The second one is satisfied in $\mathfrak{M}$ iff the distance between any two distinct centers of spheres, all points in which belong to $\mathfrak{a}(X_i \wedge Y_j)$, is more than 80.

We are going to show that the set $\{x \sqsubseteq \chi_{00}\} \cup \Gamma$ is satisfiable in $\mathcal{K}$ iff $\mathcal{T}$ can tile $\mathbb{N}\times\mathbb{N}$.

Lemma 1. *If $\mathcal{T}$ can tile $\mathbb{N}\times\mathbb{N}$, then $\{x \sqsubseteq \chi_{00}\} \cup \Gamma$ is satisfiable in $\mathbb{R}^2$.*

Proof. Suppose $\tau : \mathbb{N} \times \mathbb{N} \to \mathcal{T}$ is a tiling. For $r \in \mathbb{R}^2$, put

$$S(r) = \{y \in \mathbb{R}^2 : d_2(r, y) \leq 9\}.$$

Define an assignment $\mathfrak{a}$ into $\mathbb{R}^2$ by taking, for $i, j \leq 4$ and $k \leq l$:

- $\mathfrak{a}(X_i) = \bigcup\{S(50m + 10i, 20n) : m, n \in \mathbb{N}\}$,
- $\mathfrak{a}(Y_j) = \bigcup\{S(20n, 50m + 10j) : m, n \in \mathbb{N}\}$,
- $\mathfrak{a}(Z_k) = \bigcup\{S(n, m) : \tau(n, m) = T_k\}$.

It is not difficult to see that $\langle \mathbb{R}^2, \mathfrak{a} \rangle$ satisfies $\{x \sqsubseteq \chi_{00}\} \cup \Gamma$.

Lemma 2. *Suppose a model $\mathfrak{M} = \langle W, d, \mathfrak{a} \rangle$ satisfies $\{x \sqsubseteq \chi_{00}\} \cup \Gamma$. Then there exists a function $f : \mathbb{N} \times \mathbb{N} \to W$ such that, for all $i, j \leq 4$ and $k_1, k_2 \in \mathbb{N}$,*

- $f(5k_1 + i, 5k_2 + j) \in \chi_{ij}^{\mathfrak{a}}$,
- $d(f(k_1, k_2), f(k_1 + 1, k_2)) \leq 20$,
- $d(f(k_1, k_2), f(k_1, k_2 + 1)) \leq 20$.

The map $\tau : \mathbb{N} \times \mathbb{N} \to \mathcal{T}$ defined by taking $\tau(n, m) = T_k$ iff $f(n, m) \in Z_k^{\mathfrak{a}}$, for all $k \leq l$ and all $n, m \in \mathbb{N}$, is a tiling.

Proof. We define f inductively. Put $f(0, 0) = \mathfrak{a}(x)$. By (12), we find a sequence $w_n \in W$, $n \in \mathbb{N}$, such that

- $w_0 = f(0, 0)$,
- $w_{5k+i} \in \chi_{i0}^{\mathfrak{a}}$, for all $i \leq 4$ and $k \in \mathbb{N}$,
- $d(w_n, w_{n+1}) \leq 20$.

We put $f(n, 0) = w_n$ for all $n \in \mathbb{N}$. Similarly, by (13) we find a sequence v_n, $n \in \mathbb{N}$, such that

- $v_0 = f(0, 0)$,
- $v_{5k+j} \in \chi_{0j}^{\mathfrak{a}}$, for all $j \leq 4$ and $k \in \mathbb{N}$,
- $d(v_n, v_{n+1}) \leq 20$.

Put $f(0, m) = v_m$ for all $m \in \mathbb{N}$. Suppose now that f satisfies the conditions listed in the formulation of the lemma (on its defined domain), that it has been defined for all (m', n') with $m' + n' < m + n$, but not for (m, n). Without loss of generality we can assume that $n = 5k_1$, $m = 5k_2 + 1$, for some $k_1, k_2 \in \mathbb{N}$. Then $f(n, m - 1) \in \chi_{00}^{\mathfrak{a}}$, and so $f(n, m - 1) \in (\exists_{\leq 20}\chi_{01})^{\mathfrak{a}}$. So we can find a $w' \in W$ with $d(f(n, m - 1), w') \leq 20$ such that $w' \in \chi_{01}^{\mathfrak{a}}$. We then put $f(n, m) = w'$. It remains to prove that f still has the required properties. To this end it suffices to show that $d(f(n - 1, m), w') \leq 20$. We have $f(n - 1, m) \in \chi_{41}^{\mathfrak{a}}$, and so there exists a w'' such that $w'' \in \chi_{01}^{\mathfrak{a}}$ and $d(f(n - 1, m), w'') \leq 20$. So it is enough to show that $w' = w''$. Suppose otherwise. Then

- $d(w'', f(n - 1, m)) \leq 20$,
- $d(f(n - 1), m), f(n - 1, m - 1)) \leq 20$,
- $d(f(n - 1, m - 1), f(n, m - 1)) \leq 20$,
- $d(f(n, m - 1), w') \leq 20$.

By the triangle inequality, we then have $d(w'', w') \leq 80$, contrary to the second formula in (10).

The reader can readily check that τ is a tiling.

4 Relational semantics

To prove the finite model property of $\mathcal{MS}_2$, we require a relational representation of metric space models defined in Section 2. Let $M \subseteq \mathbb{R}_+$.

A *relational metric M-model* is a quadruple of the form

$$\mathfrak{S} = \langle W, (R_a)_{a \in M}, (R_{\overline{a}})_{a \in M}, \mathfrak{a} \rangle ,$$

where W is a non-empty set, $(R_a)_{a \in M}$ and $(R_{\overline{a}})_{a \in M}$ are families of binary relations on W, and $\mathfrak{a}$ is an assignment in W. The *value* $t^{\mathfrak{S}}$ of a set term t in $\mathfrak{S}$ is defined inductively. The basis of induction and the case of Booleans are the same as in metric space models. And for set terms of the form $\forall_{\leq a} t$ and $\forall_{> a} t$ we put

- $(\forall_{\leq a} t)^{\mathfrak{S}} = \{w \in W : \forall v \in W \ (w R_a v \to v \in t^{\mathfrak{S}})\}$,
- $(\forall_{> a} t)^{\mathfrak{S}} = \{w \in W : \forall v \in W \ (w R_{\overline{a}} v \to v \in t^{\mathfrak{S}})\}$.

The values of $\exists_{\leq a} t$ and $\exists_{> a} t$ are defined dually.

Say that the model $\mathfrak{S}$ is *M-standard* if the following conditions are satisfied for all $a, b \in M$ and $w, u, v \in W$:

(i) $R_a \cup R_{\overline{a}} = W \times W$,
(ii) $R_a \cap R_{\overline{a}} = \emptyset$,
(iii) if $u R_a v$ and $a \leq b$, then $u R_b v$,
(iv) if $u R_{\overline{a}} v$ and $a \geq b$, then $u R_{\overline{b}} v$,
(v) $u R_0 v$ iff $u = v$,
(vi) if $u R_a v$ and $v R_b w$, then $u R_{a+b} w$ whenever $a + b \in M$,
(vii) $u R_a v$ iff $v R_a u$.

Note that as a consequence of (i), (ii) and (vi) we have:

(viii) if $u R_a v$ and $u R_{\overline{a+b}} w$ then $v R_{\overline{b}} w$.

With every metric space model $\mathfrak{M} = \langle W, d, \mathfrak{a} \rangle$ we can associate the relational metric M-model

$$\mathfrak{S}(\mathfrak{M}) = \langle W, (R_a)_{a \in M}, (R_{\overline{a}})_{a \in M}, \mathfrak{a} \rangle ,$$

in which the relations R_a and $R_{\overline{a}}$ are defined as follows:

$$\forall w, v \in W \ (w R_a v \leftrightarrow d(w, v) \leq a),$$
$$\forall w, v \in W \ (w R_{\overline{a}} v \leftrightarrow d(w, v) > a).$$

It is easy to see that $\mathfrak{S}(\mathfrak{M})$ is M-standard. Note that (v), (vi) and (vii) reflect axioms (1)–(3) of metric spaces.

The model $\mathfrak{S}(\mathfrak{M})$ can be regarded as a relational representation of $\mathfrak{M}$. For we clearly have the following:

Lemma 3. *For every metric space model $\mathfrak{M}$ and every set term $t \in \mathcal{MS}_2[M]$, the value of t in $\mathfrak{M}$ coincides with the value of t in $\mathfrak{S}(\mathfrak{M})$.*

5 The finite model property of $\mathcal{MS}_2$

In this section we prove that $\mathcal{MS}_2$ has the finite model property. The idea of the proof is as follows.

Let φ be an $\mathcal{MS}_2$-formula and let $\mathfrak{M} \models \varphi$ for some metric space model $\mathfrak{M} = \langle W, d, \mathfrak{a} \rangle$. Depending on $\mathfrak{M}$, we transform φ into a set Φ, containing only formulas of the form $x \sqsubseteq t$, $s = t$, $s \neq t$, and $\boldsymbol{\delta}(x, y) = a$, in such a way that φ is satisfiable in a finite model whenever Φ is finitely satisfiable. Starting from Φ, we compute a finite set $M[\Phi]$ of real numbers containing, in particular, all the numbers occurring in Φ. Then we replace the metric d by a new metric d' with (finite) range $M[\Phi]$. The new model $\mathfrak{M}_1$ still satisfies Φ. The next step is to filtrate (as in modal logic; see e.g. [3]) the relational metric model $\mathfrak{S} = \mathfrak{S}(\mathfrak{M}_1)$ through some suitable set of terms $cl(\Phi)$. To define $cl(\Phi)$, we first transform Φ into a set Φ' which, roughly speaking, is obtained from Φ by replacing every formula of the form $\boldsymbol{\delta}(y, z) = a$ with two formulas $z \sqsubseteq X^z$ and $y \sqsubseteq \exists_{\leq a} X^z$, where the X^z are fresh set variables. $cl(\Phi)$ will be the closure of the terms in Φ' under syntactical rules that are similar to the rules of the Fischer–Ladner closure for **PDL**-formulas (cf. [10]). (Note, however, that in contrast to the Fischer–Ladner closure the closure considered here results in an exponential blow up.)

As a result of the filtration we get a *finite* relational metric model $\mathfrak{S}^f$. But unlike $\mathfrak{S}$, in general $\mathfrak{S}^f$ is not $M[\Phi]$-standard, which means that we cannot directly transform it into a finite metric space model. However, $\mathfrak{S}^f$ still has all the properties of $M[\Phi]$-standard models save (ii): there may exist $v \in W^f$ such that $wR_a v$ and $wR_{\overline{a}} v$, for some $w \in W^f$, and $a \in M[\Phi]$. To 'cure' these defects, we make copies of such 'bad' points v and modify the relations R_a and $R_{\overline{a}}$ in $\mathfrak{S}^f$ obtaining a finite standard relational metric model $\mathfrak{S}^*$. (The 'copying-method' was developed by the Bulgarian school of modal logic; see [7, 16]. Our technique follows [8]). The final step is to transform $\mathfrak{S}^*$ into a metric space model $\mathfrak{M}^*$.

Let us now turn to details. Denote by $term(\varphi)$ the set of all set terms occurring in φ; $sub(\varphi)$ stands for the set of all subformulas of φ. Define a set $\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3$ by taking:

$$
\begin{aligned}
\Phi_1 &= \{x \sqsubseteq t : (x \sqsubseteq t) \in sub(\varphi),\ \mathfrak{M} \models x \sqsubseteq t\} \cup \\
&\quad\ \{x \sqsubseteq \neg t : (x \sqsubseteq t) \in sub(\varphi),\ \mathfrak{M} \not\models x \sqsubseteq t\}, \\
\Phi_2 &= \{s = t : (s = t) \in sub(\varphi),\ \mathfrak{M} \models s = t\} \cup \\
&\quad\ \{s \neq t : (s = t) \in sub(\varphi),\ \mathfrak{M} \models s \neq t\}, \\
\Phi_3 &= \{\boldsymbol{\delta}(y, z) = a : \boldsymbol{\delta}(y, z) \in term(\varphi),\ a = d(\mathfrak{a}(y),\ \mathfrak{a}(z))\}.
\end{aligned}
$$

It should be clear from the definition that we have

Lemma 4. (1) $\mathfrak{M} \models \Phi$.

(2) *For every metric space model $\mathfrak{M}'$, if $\mathfrak{M}' \models \Phi$ then $\mathfrak{M}' \models \varphi$.*

Next we construct $M[\Phi]$ and Φ'. Let

$$M(\Phi) = \{a \in \mathbb{R} : a \text{ occurs in } \Phi\}.$$

Denote by γ the smallest natural number that is greater than all numbers in $M(\Phi) \cup \{0\}$ and define $M[\Phi]$ as

$$M[\Phi] = \{a_1 + \cdots + a_n < \gamma : a_1, \ldots, a_n \in M(\Phi),\ n < \omega\} \cup \{\gamma\} \cup \{0\}.$$

Let $\mu = \min\{M(\Phi) - \{0\}\}$ and let χ be the least natural number such that $\chi \geq \gamma/\mu$. An easy (but tedious) computation yields:

Lemma 5. *$|M[\Phi]| \leq |M(\Phi)|^{\chi}$, whenever $|M(\Phi)| \geq 2$.*

For each location variable x occurring in Φ_3 we pick a new set variable X^x and define Φ_3', Φ', and $t(\Phi)$ by taking

$$\begin{aligned}
\Phi_3' = {} & \{y \sqsubseteq \exists_{\leq a} X^z : \boldsymbol{\delta}(y,z) = a \in \Phi_3\} \cup \\
& \{z \sqsubseteq X^z : \boldsymbol{\delta}(y,z) = a \in \Phi_3\} \cup \\
& \{y \sqsubseteq \forall_{\leq b} \neg X^z : \boldsymbol{\delta}(y,z) = a \in \Phi_3,\ b < a,\ b \in M[\Phi]\}, \\
\Phi' = {} & \Phi_1 \cup \Phi_2 \cup \Phi_3', \\
t(\Phi) = {} & \{\, t : t \in term(\Phi')\}.
\end{aligned}$$

The *closure* $cl(\Phi)$ of $t(\Phi)$ is the smallest set of terms T such that $t(\Phi) \subseteq T$ and

1. T is closed under subterms;
2. if $t \in T$, then $\forall_{\leq 0} t \in T$ whenever t is not of the form $\forall_{\leq 0} s$;
3. if $\forall_{\leq a} t \in T$ and $a \geq a_1 + \cdots + a_n$, for $a_i \in M[\Phi] - \{0\}$, then $\forall_{\leq a_1} \ldots \forall_{\leq a_n} t \in T$;
4. if $\forall_{>a} t \in T$ and $b \in M[\Phi]$, then $\neg\forall_{\leq b} \neg\forall_{>a} t \in T$;
5. if $\forall_{>a} t \in T$ and $b > a$, for $b \in M[\Phi]$, then $\forall_{>b} t \in T$ and $\neg\forall_{>b} \neg\forall_{>a} t \in T$.

By an easy but tedious computation the reader can check that we have:

Lemma 6. *If $|M(\Phi)| \geq 4$ and $\chi \geq 3$, then*

$$|cl(\Phi)| \leq S(\Phi) = |t(\Phi)| \cdot |M[\Phi]|^{(\chi+1)\cdot(|M(\Phi)|+1)}.$$

We are in a position now to prove the following:

Theorem 6. *Φ is satisfied in a metric space model $\mathfrak{M}^* = \langle W^*, d^*, \mathfrak{b}^* \rangle$ such that $|W^*| \leq 2 \cdot 2^{S(\Phi)}$ and the range of d^* is a subset of $M[\Phi]$.*

Proof. We first show that Φ is satisfied in a metric space model $\langle W, d', \mathfrak{a} \rangle$ with the range of d' being a subset of $M = M[\Phi]$. Indeed, define d' by taking

$$d'(w,v) = \min\{\gamma, a \in M : d(w,v) \leq a\},$$

for all $w, v \in W$, and let $\mathfrak{M}_1 = \langle W, d', \mathfrak{a} \rangle$. Clearly, the range of d' is a subset of M. We check that d' is a metric. It satisfies (1) because $0 \in M$. That d' is symmetric follows from the symmetry of d. To show (2), suppose $d'(w,v) + d'(v,u) \leq a$, for $a \in M$. By the definition of d', we then have $d(w,v) + d(v,u) \leq a$, and so $d(w,u) \leq a$. Hence $d'(w,u) \leq a$. Thus we have shown that

$$\{a \in M : d'(w,v) + d'(v,u) \leq a\} \subseteq \{a \in M : d'(w,u) \leq a\},$$

from which one easily concludes that $d'(w,u) \leq d'(w,v) + d'(v,u)$.

Lemma 7. *The set Φ is satisfied in $\mathfrak{M}_1$.*

Proof. Clearly, for each $(\boldsymbol{\delta}(y,z)=a) \in \Phi_3$, $d(\mathfrak{a}(y),\mathfrak{a}(z)) = d'(\mathfrak{a}(y),\mathfrak{a}(z)) = a$. So $\mathfrak{M}_1 \models \Phi_3$. To show $\mathfrak{M}_1 \models \Phi_1 \cup \Phi_2$ it suffices to prove that

$$\forall w \in W \forall t \in t(\Phi)\ (w \in t^{\mathfrak{M}} \leftrightarrow w \in t^{\mathfrak{M}_1}).$$

This can be done by a straightforward induction on the construction of t. The basis of induction and the case of Booleans are trivial. So suppose t is $\forall_{\leq a} s$ (then $a \in M$). Then we have:

$$\begin{aligned} w \in t^{\mathfrak{M}} &\Leftrightarrow_1 \forall v \in W\ (d(w,v) \leq a \to v \in s^{\mathfrak{M}}) \\ &\Leftrightarrow_2 \forall v \in W\ (d'(w,v) \leq a \to v \in s^{\mathfrak{M}_1}) \\ &\Leftrightarrow_3 w \in t^{\mathfrak{M}_1}. \end{aligned}$$

The equivalences $\Leftrightarrow_1$ and $\Leftrightarrow_3$ are obvious. $\Leftrightarrow_2$ holds by the induction hypothesis and the fact that, for all $w, v \in W$ and every $a \in M$, $d(x,y) \leq a$ iff $d'(x,y) \leq a$. The case $\forall_{>a} s$ is considered in a similar way.

Before filtrating $\mathfrak{M}_1$ through $\Theta = cl(\Phi)$, we slightly change its assignment. Recall that Θ contains the new set variables X^z which function as nominals and which will help to fix the distances between the points occurring in Φ_3. Define $\mathfrak{b}$ to be the assignment that acts as $\mathfrak{a}$ on all variables save the X^z, where

- $\mathfrak{b}(X^z) = \{\mathfrak{a}(z)\}$.

Let $\mathfrak{M}_2 = \langle W, d', \mathfrak{b}\rangle$. It should be clear from the definition and Lemma 7 that we have:

(a) $t^{\mathfrak{M}_1} = t^{\mathfrak{M}_2}$, for all set terms $t \in t(\Phi)$;
(b) $\mathfrak{M}_1 \models \psi$ iff $\mathfrak{M}_2 \models \psi$, for all formulas $\psi \in \mathcal{MS}_2(\Phi)$;
(c) $\mathfrak{M}_2 \models \Phi$;
(d) $\mathfrak{M}_2 \models \Phi'$.

Consider the relational counterpart of $\mathfrak{M}_2$, i.e., the model

$$\mathfrak{S}(\mathfrak{M}_2) = \langle W, (R_a)_{a \in M}, (R_{\overline{a}})_{a \in M}, \mathfrak{b}\rangle$$

which, for brevity, will be denoted by $\mathfrak{S}$. Define an equivalence relation $\equiv$ on W by taking $u \equiv v$ when $u \in t^{\mathfrak{S}}$ iff $v \in t^{\mathfrak{S}}$ for all $t \in \Theta$. Let $[u] = \{v \in W : u \equiv v\}$. Note that if $(z \sqsubseteq X^z) \in \Phi'_3$ then $[\mathfrak{b}(z)] = \{\mathfrak{b}(z)\}$, since $X^z \in \Theta$.

Construct a filtration $\mathfrak{S}^f = \left\langle W^f, (R^f_a)_{a \in M}, (R^f_{\overline{a}})_{a \in M}, \mathfrak{b}^f \right\rangle$ of $\mathfrak{S}$ through Θ by taking

- $W^f = \{[u] : u \in W\}$;
- $\mathfrak{b}^f(x) = [\mathfrak{b}(x)]$;
- $\mathfrak{b}^f(X) = \{[u] : u \in \mathfrak{b}(X)\}$;
- $[u] R^f_a [v]$ iff for all terms $\forall_{\leq a} t \in \Theta$,
  - $u \in (\forall_{\leq a} t)^{\mathfrak{S}}$ implies $v \in t^{\mathfrak{S}}$ and

- $v \in (\forall_{\leq a} t)^{\mathfrak{S}}$ implies $u \in t^{\mathfrak{S}}$;

– $[u] R^f_{\overline{a}} [v]$ iff for all terms $\forall_{>a} t \in \Theta$,
  - $u \in (\forall_{>a} t)^{\mathfrak{S}}$ implies $v \in t^{\mathfrak{S}}$ and
  - $v \in (\forall_{>a} t)^{\mathfrak{S}}$ implies $u \in t^{\mathfrak{S}}$.

Since Θ is finite, W^f is finite as well. Note also that $\mathfrak{b}^f(X^z) = \{\mathfrak{b}^f(z)\}$ whenever $(z \sqsubseteq X^z) \in \Phi'_3$.

Lemma 8. (1) *For every* $t \in \Theta$ *and every* $u \in W$, $u \in t^{\mathfrak{S}}$ *iff* $[u] \in t^{\mathfrak{S}^f}$.

(2) *For all* $(\boldsymbol{\delta}(y, z) = a) \in \Phi_3$, $a = \min\{b \in M : \mathfrak{b}^f(y) R^f_b \mathfrak{b}^f(z)\}$.

(3) $\mathfrak{S}^f$ *satisfies* (i), (iii)–(vii) *in Section 4.*

Proof. (1) is proved by an easy induction on the construction of t. To prove (2), take $(\boldsymbol{\delta}(y, z) = a) \in \Phi_3$. We must show that $\mathfrak{b}^f(y) R^f_a \mathfrak{b}^f(z)$ and $\neg \mathfrak{b}^f(y) R^f_b \mathfrak{b}^f(z)$, for all $a > b \in M$. Notice first that $u R_a v$ implies $[u] R^f_a [v]$ and $u R_{\overline{a}} v$ implies $[u] R^f_{\overline{a}} [v]$. Since $\mathfrak{M}_2 \models \Phi$, we have $\mathfrak{M}_2 \models \boldsymbol{\delta}(y, z) = a$, and so $d'(\mathfrak{b}(y), \mathfrak{b}(z)) = a$. Hence $\mathfrak{b}(y) R_a \mathfrak{b}(z)$ and $\mathfrak{b}^f(y) R^f_a \mathfrak{b}^f(z)$. Suppose now that $b < a$ and consider $\forall_{\leq b} \neg X^z$. By definition, $\mathfrak{b}(X^z) = \{\mathfrak{b}(z)\}$. Hence $\mathfrak{b}(z) \notin (\neg X^z)^{\mathfrak{S}}$. On the other hand, we have $b < d'(\mathfrak{b}(y), \mathfrak{b}(z))$, from which $\mathfrak{b}(y) \in (\forall_{\leq b} \neg X^z)^{\mathfrak{S}}$. Since $(\forall_{\leq b} \neg X^z) \in \Theta$, we then obtain $\neg \mathfrak{b}^f(y) R^f_b \mathfrak{b}^f(z)$.

Now let us prove (3). Condition (vii), i.e., $[w] R^f_a [u]$ iff $[u] R^f_a [w]$, holds by definition.

(i), i.e, $R^f_a \cup R^f_{\overline{a}} = W^f \times W^f$. If $\neg [u] R^f_a [v]$ then $\neg u R_a v$, and so $u R_{\overline{a}} v$, since $\mathfrak{S}$ satisfies (i). Thus $[u] R^f_{\overline{a}} [v]$.

(iii), i.e., if $[u] R^f_a [v]$ and $a \leq b$ then $[u] R^f_b [v]$. Let $[u] R^f_a [v]$ and $a < b$, for $b \in M$. Suppose $u \in (\forall_{\leq b} t)^{\mathfrak{S}}$. By the definition of $\Theta = cl(\Phi)$, $\forall_{\leq a} t \in \Theta$, and so $u \in (\forall_{\leq a} t)^{\mathfrak{S}}$. Hence $v \in t^{\mathfrak{S}}$. The other direction is considered in the same way.

(iv), i.e., if $[u] R^f_{\overline{a}} [v]$ and $a \geq b$ then $[u] R^f_{\overline{b}} [v]$. Let $[u] R^f_{\overline{a}} [v]$ and $a > b$, and suppose that $u \in (\forall_{>b} t)^{\mathfrak{S}}$. Then $\forall_{>a} t \in \Theta$, $u \in (\forall_{>a} t)^{\mathfrak{S}}$, and so $v \in t^{\mathfrak{S}}$. Again, the other direction is treated analogously.

(v), i.e., $[u] R^f_0 [v]$ iff $[u] = [v]$. The implication ($\Leftarrow$) is obvious. So suppose $[u] R^f_0 [v]$. Take some $t \in \Theta$ with $u \in t^{\mathfrak{S}}$. Without loss of generality we may assume that t is not of the form $\forall_{\leq 0} s$. Then, by the definition of Θ, $u \in (\forall_{\leq 0} t)^{\mathfrak{S}}$ and $\forall_{\leq 0} t \in \Theta$. Hence $v \in t^{\mathfrak{S}}$. In precisely the same way one can show that for all $t \in \Theta$, $v \in t^{\mathfrak{S}}$ implies $u \in t^{\mathfrak{S}}$. Therefore, $[u] = [v]$.

(vi), i.e., if $[u] R^f_a [v]$ and $[v] R^f_b [w]$, then $[u] R^f_{a+b} [w]$, for $(a + b) \in M$. Suppose $u \in (\forall_{\leq a+b} t)^{\mathfrak{S}}$. Then $\forall_{\leq a} \forall_{\leq b} t \in \Theta$ and $u \in (\forall_{\leq a} \forall_{\leq b} t)^{\mathfrak{S}}$. Hence $w \in t^{\mathfrak{S}}$. For the other direction, assume $w \in (\forall_{\leq a+b} t)^{\mathfrak{S}}$. Again, we have $\forall_{\leq a} \forall_{\leq b} t \in \Theta$ and $w \in (\forall_{\leq a} \forall_{\leq b} t)^{\mathfrak{S}}$. In view of (vii) we then obtain $u \in t^{\mathfrak{S}}$.

(viii), i.e., if $[u] R^f_a [v]$ and $[u] R^f_{\overline{a+b}} [w]$ then $[v] R^f_{\overline{b}} [w]$, for $(a + b) \in M$. Suppose $v \in (\forall_{>b} t)^{\mathfrak{S}}$. Then $\neg \forall_{\leq a} \neg \forall_{>b} t \in \Theta$ and $u \in (\neg \forall_{\leq a} \neg \forall_{>b} t)^{\mathfrak{S}}$. Hence $u \in (\forall_{>(a+b)} t)^{\mathfrak{S}}$ and so $w \in t^{\mathfrak{S}}$. For the other direction, suppose $w \in (\forall_{>b} t)^{\mathfrak{S}}$. Then $u \in (\neg \forall_{>(a+b)} \neg \forall_{>b} t)^{\mathfrak{S}}$ and $\neg \forall_{>(a+b)} \neg \forall_{>b} t \in \Theta$. Hence $u \in (\forall_{\leq a} t)^{\mathfrak{S}}$ and so $v \in t^{\mathfrak{S}}$.

Unfortunately, $\mathfrak{S}^f$ does not necessarily satisfy (ii) which is required to construct the model $\mathfrak{M}^*$ we need: it may happen that for some points $[u]$, $[v]$ in W^f and $a \in M$, we have both $[u]R_a^f[v]$ and $[u]R_{\overline{a}}^f[v]$. To 'cure' these defects, we have to perform some surgery. The defects form the set

$$D(W^f) = \{d \in W^f : \exists a \in M \exists x \in W^f \ (xR_a^f d \ \& \ xR_{\overline{a}}^f d)\}.$$

Let

$$W^* = \{\langle d, i\rangle : d \in D(W^f), i \in \{0,1\}\} \cup \{\langle c, 0\rangle : c \in W^f - D(W^f)\}.$$

So for each $d \in D(W^f)$ we have now two copies $\langle d, 0\rangle$ and $\langle d, 1\rangle$. Define an assignment $\mathfrak{b}^*$ in W^* by taking

- $\mathfrak{b}^*(x) = \left\langle \mathfrak{b}^f(x), 0\right\rangle$ and
- $\mathfrak{b}^*(X) = \{\langle c, i\rangle \in W^* : c \in \mathfrak{b}^f(X)\}$.

Finally, we define accessibility relations R_a^* and $R_{\overline{a}}^*$ as follows:

- if $a > 0$ then $\langle c, i\rangle \, R_a^* \, \langle d, j\rangle$ iff either
  - $cR_a^f d$ and $\neg cR_{\overline{a}}^f d$, or
  - $cR_a^f d$ and $i = j$;
- if $a = 0$ then $\langle c, i\rangle \, R_a^* \, \langle d, j\rangle$ iff $\langle c, i\rangle = \langle d, j\rangle$;
- $R_{\overline{a}}^*$ is defined as the complement of R_a^*, i.e., $\langle c, i\rangle \, R_{\overline{a}}^* \, \langle d, j\rangle$ iff $\neg \langle c, i\rangle \, R_a^* \, \langle d, j\rangle$.

Lemma 9. *$\mathfrak{S}^* = \langle W^*, (R_a^*)_{a\in M}, (R_{\overline{a}}^*)_{a\in M}, \mathfrak{b}^*\rangle$ is an M-standard relational metric model.*

Proof. That $\mathfrak{S}^*$ satisfies (i), (ii), and (v) follows immediately from the definition. Let us check the remaining conditions.

(iii) Suppose $\langle c, i\rangle \, R_a^* \, \langle d, j\rangle$ and $a < b \in M$. If $i = j$ then clearly $\langle c, i\rangle \, R_b^* \, \langle d, j\rangle$. So assume $i \neq j$. Then, by definition, $cR_a^f d$ and $\neg cR_{\overline{a}}^f d$. Since $\mathfrak{S}^f$ satisfies (iii) and (iv), we obtain $cR_b^f d$ and $\neg cR_{\overline{b}}^f d$. Thus $\langle c, i\rangle \, R_b^* \, \langle d, j\rangle$.

(iv) Suppose that $\langle c, i\rangle \, R_{\overline{a}}^* \, \langle d, j\rangle$ and $a > b \in M$, but $\neg \langle c, i\rangle \, R_{\overline{b}}^* \, \langle d, j\rangle$. By (i), $\langle c, i\rangle \, R_b^* \, \langle d, j\rangle$. And by (iii), $\langle c, i\rangle \, R_a^* \, \langle d, j\rangle$. Finally, (ii) yields $\neg \langle c, i\rangle \, R_{\overline{a}}^* \, \langle d, j\rangle$, which is a contradiction.

(vi) Suppose $\langle c, i\rangle \, R_a^* \, \langle d, j\rangle$, $\langle d, j\rangle \, R_b^* \, \langle e, k\rangle$ and $a + b \in M$. Then $cR_a^f d$ and $dR_b^f e$. As $\mathfrak{S}^f$ satisfies (vii), we have $cR_{a+b}^f e$. If $i = k$ then clearly $\langle c, i\rangle \, R_{a+b}^* \, \langle e, k\rangle$. So assume $i \neq k$. If $i = j \neq k$ then $\neg cR_{\overline{a+b}}^f e$, since $cR_a^f d$ and $\neg dR_{\overline{b}}^f e$. The case $i \neq j = k$ is considered analogously using the fact that the relations in $\mathfrak{S}^f$ are symmetric.

(vii) follows from the symmetry of R_a^f and $R_{\overline{a}}^f$.

Lemma 10. *For all $\langle d, i\rangle \in W^*$ and $t \in \Theta$, we have $\langle d, i\rangle \in t^{\mathfrak{S}^*}$ iff $d \in t^{\mathfrak{S}^f}$.*

Proof. The proof is by induction on t. The basis of induction and the case of Booleans are trivial. The cases $t = (\forall_{\leq a} s)$ and $t = (\forall_{>a} s)$ are consequences of the following claims:

Claim 1: if $cR_a^f d$ and $i \in \{0,1\}$, then there exists j such that $\langle c,i\rangle\, R_a^* \,\langle d,j\rangle$. Indeed, this is clear for $i=0$. Suppose $i=1$. If d was duplicated, then $\langle d,1\rangle$ is as required. If d was not duplicated, then $\neg cR_{\overline{a}}^f d$, and so $\langle d,0\rangle$ is as required.

Claim 2: if $\langle c,i\rangle\, R_a^* \,\langle d,j\rangle$ then $cR_a^f d$. This is obvious.

Claim 3: if $cR_{\overline{a}}^f d$ and $i \in \{0,1\}$ then there exists j such that $\neg \langle c,i\rangle\, R_a^* \,\langle d,j\rangle$. Suppose $i=0$. If d was not duplicated, then $\neg cR_a^f d$. Hence $\neg \langle c,0\rangle\, R_a^* \,\langle d,0\rangle$. If d was duplicated, then $\neg \langle c,0\rangle\, R_a^* \,\langle d,1\rangle$. In the case $i=1$ we have $\neg \langle c,1\rangle\, R_a^* \,\langle d,0\rangle$.

Claim 4: if $\neg \langle c,i\rangle\, R_a^* \,\langle d,j\rangle$ then $cR_{\overline{a}}^f d$. Indeed, if $i=j$ then $\neg cR_a^f d$ and so $cR_{\overline{a}}^f d$. And if $i \neq j$ then $cR_{\overline{a}}^f d$.

To complete the proof of Theorem 6, we transform $\mathfrak{S}^*$ into a finite metric space model and show that this model satisfies Φ. Put $\mathfrak{M}^* = \langle W^*, d^*, \mathfrak{b}^*\rangle$, where for all $w, v \in W^*$,

$$d^*(w,v) = min\{\gamma, a \in M : wR_a^* v\}.$$

As M is finite, d^* is well-defined. Using (v)–(vii), it is easy to see that d^* is a metric. So $\mathfrak{M}^*$ is a finite metric space model. It remains to show that $\mathfrak{M}^*$ satisfies Φ. Note first that

(†) for all $w \in W^*$ and $t \in t(\Phi)$, we have $w \in t^{\mathfrak{S}^*}$ iff $w \in t^{\mathfrak{M}^*}$.

This claim is proved by induction on t. The basis and the Boolean cases are clear. So let $t = (\forall_{\leq a} s)$ for some $a \in M$. Then

$$\begin{aligned} w \in (\forall_{\leq a} s)^{\mathfrak{S}^*} &\Leftrightarrow_1 \forall v\ (wR_a^* v \to v \in s^{\mathfrak{S}^*}) \\ &\Leftrightarrow_2 \forall v\ (wR_a^* v \to v \in s^{\mathfrak{M}^*}) \\ &\Leftrightarrow_3 \forall v\ (d^*(w,v) \leq a \to v \in s^{\mathfrak{M}^*}) \\ &\Leftrightarrow_4 w \in (\forall_{\leq a} s)^{\mathfrak{M}^*}. \end{aligned}$$

Equivalences $\Leftrightarrow_1$ and $\Leftrightarrow_4$ are obvious; $\Leftrightarrow_2$ holds by the induction hypothesis; $\Leftarrow_3$ is an immediate consequence of the definition of d^*, and $\Rightarrow_3$ follows from (iii). The case $t = (\forall_{>a} s)$ is proved analogously.

We can now show that $\mathfrak{M}^* \models \Phi$. Let $(x \sqsubseteq t) \in \Phi_1$. Then we have:

$$\mathfrak{M}^* \models x \sqsubseteq t \Leftrightarrow_1 \mathfrak{b}^*(x) \in t^{\mathfrak{M}^*} \Leftrightarrow_2 \mathfrak{b}^*(x) \in t^{\mathfrak{S}^*} \Leftrightarrow_3 \left\langle \mathfrak{b}^f(x), 0\right\rangle \in t^{\mathfrak{S}^*} \Leftrightarrow_4$$
$$\mathfrak{b}^f(x) \in t^{\mathfrak{S}^f} \Leftrightarrow_5 [\mathfrak{b}(x)] \in t^{\mathfrak{S}^f} \Leftrightarrow_6 \mathfrak{b}(x) \in t^{\mathfrak{S}} \Leftrightarrow_7 \mathfrak{b}(x) \in t^{\mathfrak{M}_2} \Leftrightarrow_8 \mathfrak{M}_2 \models x \sqsubseteq t.$$

Equivalences $\Leftrightarrow_1$ and $\Leftrightarrow_8$ are obvious; $\Leftrightarrow_2$ follows from (†); $\Leftrightarrow_3$ and $\Leftrightarrow_5$ hold by definition; $\Leftrightarrow_4$ follows from Lemma 10, $\Leftrightarrow_6$ from Lemma 8, and $\Leftrightarrow_7$ from Lemma 3.

Since $\mathfrak{M}_2 \models \Phi$, we have $\mathfrak{M}^* \models \Phi_1$. That $\mathfrak{M}^* \models \Phi_2$ is proved analogously using (†).

It remains to show that $\mathfrak{M}^* \models \Phi_3$. Take any $\boldsymbol{\delta}(y,z) = a$ from Φ_3. We must show that $d^*(\mathfrak{b}^*(y), \mathfrak{b}^*(z)) = a$. By Lemma 8 (2),

$$a = min\{b \in M : \mathfrak{b}^f(y) R_b^f \mathfrak{b}^f(z)\}.$$

So $a = min\{b \in M : \langle \mathfrak{b}^f(y), 0 \rangle \, R_b^* \, \langle \mathfrak{b}^f(z), 0 \rangle\}$. By the definition of $\mathfrak{b}^*$ we have $a = min\{b \in M : \mathfrak{b}^*(y) R_b^* \mathfrak{b}^*(z)\}$, which means that $d^*(\mathfrak{b}^*(y), \mathfrak{b}^*(z)) = a$.

This completes the proof of Theorem 6.

Thus, by Theorem 6 and Lemma 4 (2), φ is satisfied in the finite model $\mathfrak{M}^*$.

Yet this is not enough to prove the decidability of $\mathcal{MS}_2[\mathbb{Q}]$: we still do not know an effectively computable upper bound for the size of a finite model satisfying φ. Indeed, the set $M(\Phi)$ depends not only on φ, but also on the initial model $\mathfrak{M}$ satisfying φ. Note, however, that by Lemmas 5 and 6 the size of $\mathfrak{M}^*$ can be computed from the maximum of $M(\Phi)$, the minimum of $M(\Phi) - \{0\}$, and φ. Hence, to obtain an effective upper bound we need, it suffices to start the construction with a model satisfying φ for which both the maximum of $M(\Phi)$ and the minimum of $M(\Phi) - \{0\}$ are known. The next lemma shows how to obtain such a model.

Lemma 11. *Suppose a formula $\varphi \in \mathcal{MS}_2[\mathbb{Q}]$ is satisfied in a metric space model $\langle W, d, \mathfrak{a} \rangle$. Denote by $\mathcal{D}$ the set of all $\boldsymbol{\delta}(x,y)$ occurring in φ, and let a and b be the minimal positive number and the maximal number occurring in φ, respectively (if no such number exists, then put $a = b = 1$). Then there is a metric d' on W such that φ is satisfied in $\langle W, d', \mathfrak{a} \rangle$ and*

$$\min\{d'(\mathfrak{a}(x), \mathfrak{a}(y)) > 0 : \boldsymbol{\delta}(x,y) \in \mathcal{D}\} \geq a/2,$$
$$\max\{d'(\mathfrak{a}(x), \mathfrak{a}(y)) : \boldsymbol{\delta}(x,y) \in \mathcal{D}\} \leq 2b.$$

Proof. Let

$$a' = \min\{d(\mathfrak{a}(x), \mathfrak{a}(y)) > 0 : \boldsymbol{\delta}(x,y) \in \mathcal{D}\},$$
$$b' = \max\{d(\mathfrak{a}(x), \mathfrak{a}(y)) : \boldsymbol{\delta}(x,y) \in \mathcal{D}\}.$$

We consider here the case when $a' < a/2$ and $2b < b'$. The case when this is not so is easy; we leave it to the reader. Define d' by taking

$$d'(x,y) = \begin{cases} d(x,y) & \text{if } a \leq d(x,y) \leq b \text{ or } d(x,y) = 0, \\ b + (b/(b'-b)) \cdot (d(x,y) - b) & \text{if } d(x,y) > b, \\ a + (a/2(a-a')) \cdot (d(x,y) - a) & \text{if } 0 < d(x,y) < a. \end{cases}$$

One can readily show now that d' is a metric and $\langle W, d', \mathfrak{a} \rangle$ satisfies φ.

6 Weaker distance spaces

As was mentioned in Section 1, our everyday life experience gives interesting measures of distances which lack some of the features characteristic to metric spaces. Not trying to cover all possible cases, we list here some possible ways of defining such alternative measures by modifying the axioms of standard metric spaces:

– we can omit either the symmetry axiom or the triangular inequality;
– we can omit both of them;
– we can allow d to be a partial function satisfying the following conditions for all $w, v, u \in W$, where $dom(d)$ is the domain of d:
  - $\langle w, w\rangle \in dom(d)$ and $d(w, w) = 0$,
  - if $\langle w, v\rangle \in dom(d)$ and $d(w, v) = 0$, then $w = v$,
  - if $\langle w, v\rangle \in dom(d)$ and $\langle v, u\rangle \in dom(d)$, then $\langle w, u\rangle \in dom(d)$ and $d(w, u) \leq d(w, v) + d(v, u)$,
  - if $\langle w, v\rangle \in dom(d)$, then $\langle v, w\rangle \in dom(d)$ and $d(w, v) = d(v, w)$.

Using almost the same techniques as above one can generalize the obtained results on the decidability and finite model property of $\mathcal{MS}_2$ to these weaker metric spaces as well.

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