On the products of linear modal logics

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Abstract

We study two-dimensional Cartesian products of modal logics determined by infinite or arbitrarily long finite linear orders and prove a general theorem showing that in many cases these products are undecidable, in particular, such are the squares of standard linear logics like $\mathbf{K4.3}$, $\mathbf{S4.3}$, $\mathbf{GL.3}$, $\mathbf{Grz.3}$, or the logic determined by the Cartesian square of any infinite linear order. This theorem solves a number of open problems of Gabbay and Shehtman [7]. We also prove a sufficient condition for such products to be not recursively enumerable and give a simple axiomatisation for the square $\mathbf{K4.3} \times \mathbf{K4.3}$ of the minimal liner logic using non-structural Gabbay-type inference rules.

Keywords: Modal logic, Kripke frame, Cartesian product, decidability.

1 Introduction

The formation of Cartesian products of Kripke frames is probably the most natural way of introducing a concept of dimension in modal logic in order to reflect interactions between modal operators representing time, space, knowledge, actions, etc. Products of modal logics (i.e., sets of multi-modal formulas that are valid in the Cartesian products of Kripke frames for those logics) have been studied in both pure modal logic (see e.g. [23, 24, 18, 7, 15, 19, 31]) and applications in computer science and artificial intelligence (see e.g. [20, 3, 1, 21, 4, 27, 28, 29]) since the 1970s. It would not be an exaggeration to say that now multi-dimensional logics in general and Cartesian products in particular are becoming the subject of one of the most important and interesting research fields in pure and applied modal logic. (See e.g. applications of results and techniques developed in multi-dimensional logic to first-order classical, modal and temporal logics in [8, 30, 13].)

Unfortunately, as was observed by Gabbay and Shehtman [7], there is no general transfer theorem that could guarantee the preservation of such properties of logics as decidability or axiomatizability under the formation of products. If we consider only 2D products of "standard" modal systems, then the results obtained so far can roughly be described as follows (for more details consult [7] and [31]):

- logics of the form $\mathbf{K} \times L$ and $\mathbf{S5} \times L$ are usually decidable and finitely axiomatisable, in particular for $L \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}\}$ (see [19] for complexity results concerning some of these logics);
- the known undecidable (and not recursively axiomatizable) 2D products are the square of the logic of the frame $\langle \mathbb{N}, < \rangle$ (see [25]) and the "compass" logic of Venema [26] (see [17]).

As concerns 3D products, Maddux [16] proved (in the algebraic setting) that any logic between $[\mathbf{S5}, \mathbf{S5}, \mathbf{S5}]$ and $\mathbf{S5} \times \mathbf{S5} \times \mathbf{S5}$ is undecidable.[1] And quite recently Hirsch *et al.* [12] have shown that no logic between $\mathbf{K}^3$ and $\mathbf{S5}^3$ is decidable and finitely axiomatisable: see also [15]. (For more details consult [6].)

The main aim of this paper is to study the 2D products of modal logics determined by infinite or arbitrarily long finite linear frames. Namely, we show that the logic characterised by a class of product-frames containing the Cartesian product of two rooted linear orders, both having an infinite ascending chain (or both having an infinite descending chain), is undecidable. Moreover, if two classes of finite linear frames both contain arbitrarily long chains, then the product of these classes determines an undecidable logic as well. It follows that the Cartesian squares of all standard (non-tabular) "linear" modal logics, say **K4.3**, **S4.3**, **GL.3**, **Grz.3**, etc., are undecidable. We show also that some of these logics (say, $\mathbf{S4.3Dum} \times \mathbf{S4.3}$, $\mathbf{GL.3}^2$, $\mathbf{Grz.3}^2$, and the logics of $\langle \mathbb{N}, < \rangle \times \langle \mathbb{N}, < \rangle$, $\langle \mathbb{R}, < \rangle \times \langle \mathbb{R}, < \rangle$), are not even recursively enumerable (and so not recursively axiomatisable). These results give "negative" solutions to a number of open problems of [7], in particular to Problem 12 which asks whether there exists an infinite strict linear order $\mathfrak{F}$ such that the logic of $\mathfrak{F} \times \mathfrak{F}$ is decidable.

We prove our undecidability theorems using a tiling technique (save Theorem 2.2 (iii), where we simulate the behaviour of Turing machines). Reductions to the tiling problem are known to be rather straightforward when logics have both the "next-time" operator and the universal modality (see e.g. [19]). In our case the former is not available. To simulate it, we adopt the ideas of [17] and [25]. In particular, we enumerate the whole tiling plane along a defined diagonal and use various propositional variables as pointers to certain diagonal points.

In Section 7 we give a simple axiomatisation for the minimal linear logic $\mathbf{K4.3} \times \mathbf{K4.3}$ using non-structural Gabbay-type inference rules. Note that the standard techniques using Sahlqvist and negative definability (see [18]) are not appropriate here as the language has no difference operator, i.e., a modality which captures the concept of "anywhere but here." Unfortunately, we have not been able to eliminate the non-structural rules and so, as with many other multi-modal logics, it is not known whether there is an orthodox finite axiomatisation for $\mathbf{K4.3} \times \mathbf{K4.3}$.

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2 Preliminaries

In this paper we consider bimodal logics determined by the Cartesian products of Kripke frames $\mathfrak{F} = \langle W, < \rangle$ that are

- *transitive*, i.e., $\forall x, y, z \in W(x < y \wedge y < z \to x < z)$, and
- *weakly linear*, i.e., $\forall x, y \in W(x = y \vee x < y \vee y < x)$.

[1] $[\mathbf{S5}, \mathbf{S5}, \mathbf{S5}]$ is the fusion (or independent join) of three **S5**s extended with the commutativity and Church-Rosser axioms; see [7].

The *product* of two frames $\mathfrak{F}_1 = \langle W_1, <_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, <_2 \rangle$ is the birelational frame

$$\mathfrak{F}_1 \times \mathfrak{F}_2 = \langle W_1 \times W_2, <_h, <_v \rangle$$

in which, for all $u_1, u_2 \in W_1$, $v_1, v_2 \in W_2$,

$$\begin{aligned} (u_1, v_1) <_h (u_2, v_2) \quad &\text{iff} \quad u_1 <_1 u_2 \text{ and } v_1 = v_2, \\ (u_1, v_1) <_v (u_2, v_2) \quad &\text{iff} \quad u_1 = u_2 \text{ and } v_1 <_2 v_2. \end{aligned}$$

(The subscripts h and v appeal to the geometrical intuition of considering $<_h$ as the "horizontal" accessibility relation in $\mathfrak{F}_1 \times \mathfrak{F}_2$ and $<_v$ as the "vertical" one.)

Given two classes of frames $\mathcal{C}_1$ and $\mathcal{C}_2$, we define their *product* $\mathcal{C}_1 \times \mathcal{C}_2$ by taking

$$\mathcal{C}_1 \times \mathcal{C}_2 = \{\mathfrak{F}_1 \times \mathfrak{F}_2 : \mathfrak{F}_1 \in \mathcal{C}_1, \mathfrak{F}_2 \in \mathcal{C}_2\}.$$

Let $\mathcal{ML}_2$ be the *bimodal propositional language* with the boolean connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$) and two diamonds $\diamondminus$ and $\diamondbar$. We interpret $\mathcal{ML}_2$-formulas in models of the form $\mathfrak{M} = \langle \mathfrak{F}_1 \times \mathfrak{F}_2, \mathfrak{V} \rangle$, where $\mathfrak{V}$, a *valuation* in $\mathfrak{F}_1 \times \mathfrak{F}_2$, is a function associating with every propositional variable p a set of worlds $\mathfrak{V}(p) \subseteq W_1 \times W_2$ in which, according to $\mathfrak{M}$, p holds. The *truth-relation* $(\mathfrak{M}, x) \models \varphi$ (or simply $x \models \varphi$, if $\mathfrak{M}$ is understood), where $x \in W_1 \times W_2$ and φ is an $\mathcal{ML}_2$-formula, is defined in the standard way for the booleans, and

$$\begin{aligned} x \models \diamondminus\psi \text{ iff } \exists y >_h x \; y \models \psi, \\ x \models \diamondbar\psi \text{ iff } \exists y >_v x \; y \models \psi. \end{aligned}$$

The following modal operators are defined as abbreviations:

$$\begin{aligned} \diamondminus^+ \varphi &= \varphi \vee \diamondminus\varphi, \\ \diamondbar^+ \varphi &= \varphi \vee \diamondbar\varphi, \\ \boxminus\varphi &= \neg\diamondminus\neg\varphi, \\ \boxbar\varphi &= \neg\diamondbar\neg\varphi, \\ \Box^*\varphi &= \varphi \wedge \boxminus\varphi \wedge \boxbar\varphi \wedge \boxbar\boxminus\varphi. \end{aligned}$$

A formula φ is said to be *valid* in a frame $\mathfrak{F}$ if φ is true in all worlds in all models based on $\mathfrak{F}$; this fact is denoted by $\mathfrak{F} \models \varphi$.

The set of $\mathcal{ML}_2$-formulas that are valid in all frames in the class $\mathcal{C}_1 \times \mathcal{C}_2$ is called the *logic* of $\mathcal{C}_1 \times \mathcal{C}_2$ and denoted by $\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)$.

Given two Kripke complete unimodal logics L_1 and L_2, we define their *product* $L_1 \times L_2$ as $\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the classes of all Kripke frames for L_1 and L_2, respectively. As in the unimodal case, it is not hard to see that

$$L_1 \times L_2 = \mathsf{Log}\,(\mathcal{C}_1^r \times \mathcal{C}_2^r),$$

where $\mathcal{C}_i^r$, $i = 1, 2$, contains all *rooted* frames in $\mathcal{C}_i$ (in which $\exists x \forall y (x \neq y \rightarrow x < y)$). Say that $L_1 \times L_2$ has the *product finite model property* if $L_1 \times L_2 = \mathsf{Log}\,(\mathcal{C}_1^f \times \mathcal{C}_2^f)$, where $\mathcal{C}_i^f$ is the set of all finite frames in $\mathcal{C}_i$, $i = 1, 2$.

Here is a list of some standard unimodal logics (in the language with $\Diamond$ and $\Box$) determined by transitive weakly linear frames. It begins with the minimal logic, known as **K4.3** and characterised by the class of all such frames. The classes of (rooted transitive weakly linear) frames of the form $\langle W, <\rangle$ for other logics in the list are described in brackets. The operator $\oplus$ means: add the axiom and take the closure under modus ponens, substitution and necessitation $\varphi/\Box\varphi$.

- $\mathbf{K4.3} = \mathbf{K4} \oplus \Box(p \wedge \Box p \to q) \vee \Box(q \wedge \Box q \to p)$;
- $\mathbf{KD4.3} = \mathbf{K4.3} \oplus \Diamond\top$ (serial, i.e., $\forall x \exists y x < y$);
- $\mathbf{K4.3Den} = \mathbf{K4.3} \oplus \Box\Box p \to \Box p$ (dense, i.e., $\forall x, z(x < z \to \exists y\ x < y < z)$);
- $\mathbf{K4.1.3} = \mathbf{K4.3} \oplus \Box\Diamond p \to \Diamond\Box p$ ($\forall x \exists y(x < y \wedge \forall z(y < z \to y = z))$; this first-order formula is known as the McKinsey condition);
- $\mathbf{KD4.3Z} = \mathbf{KD4.3} \oplus \Box(\Box p \to p) \to (\Diamond\Box p \to \Box p)$ (serial and without an infinite ascending chain $x_0 < x_1 < \dots$ (of not necessarily distinct points) having an upper bound);
- $\mathbf{GL.3} = \mathbf{K4.3} \oplus \Box(\Box p \to p) \to \Box p$ (irreflexive and noetherian, i.e., containing no infinite ascending chain of distinct points);
- $\mathbf{S4.3} = \mathbf{K4.3} \oplus \Box p \to p$ (reflexive);
- $\mathbf{S4.1.3} = \mathbf{S4.3} \oplus \Box\Diamond p \to \Diamond\Box p$ (reflexive and satisfying the McKinsey condition);
- $\mathbf{Grz.3} = \mathbf{S4.3} \oplus \Box(\Box(p \to \Box p) \to p) \to p$ (reflexive, noetherian and containing no proper cluster, i.e., distinct x, y such that $x < y$ and $y < x$);
- $\mathbf{S4.3Dum} = \mathbf{S4.3} \oplus \Box(\Box(p \to \Box p) \to p) \to (\Diamond\Box p \to p)$ (reflexive and without an infinite ascending chain $x_0 < x_1 < \dots$ which has an upper bound and contains at least two distinct points).

Transitive weakly linearly ordered frames that are *antisymmetric* in the sense that

$$\forall x, y(x < y \wedge y < x \to x = y)$$

will be called in this paper *linear orders*. A linear order all points in which are irreflexive is known also as a *strict linear order*.

Using the standard bulldozing technique of modal logic (see e.g. [2]) and the obvious fact that $\mathfrak{F}_1 \times \mathfrak{F}_2$ is a p-morphic image of $\mathfrak{G}_1 \times \mathfrak{G}_2$ whenever $\mathfrak{F}_i$ is a p-morphic image of $\mathfrak{G}_i$, for $i = 1, 2$, one can prove the following:

PROPOSITION 2.1
Suppose L_1 and L_2 are logics from the list: **K4.3**, **KD4.3**, **K4.3Den**, **KD4.3Z**, **K4.1.3**, **S4.3**, **S4.1.3**, **S4.3Dum**. Suppose also that $\mathcal{C}_1$ and $\mathcal{C}_2$ are the classes of all rooted linear orders for L_1 and L_2, respectively. Then $L_1 \times L_2 = \mathsf{Log}(\mathcal{C}_1 \times \mathcal{C}_2)$.

All the unimodal logics in the list above are known to have the finite model property and to be decidable (consult e.g. [2]). Our main aim in this paper is to show that this is not the case for their products. Namely, we are going to prove the following general theorem:

THEOREM 2.2
(i) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of linear orders such that at least one frame in each of these classes contains an infinite ascending chain of distinct points. Then $\mathsf{Log}(\mathcal{C}_1 \times \mathcal{C}_2)$ is undecidable and does not have the product finite model property.

(ii) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of rooted noetherian linear orders such that at least one frame in each of these classes contains an infinite descending chain. Then $\mathsf{Log}(\mathcal{C}_1 \times \mathcal{C}_2)$ is undecidable and does not have the product finite model property.

(iii) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of finite linear orders both containing arbitrarily long (but finite) chains. Then $\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)$ is undecidable.

As a consequence we shall have, in particular, the following results.

THEOREM 2.3
If $\mathfrak{F}$ is an infinite linear order then $\mathsf{Log}\,(\mathfrak{F} \times \mathfrak{F})$ is undecidable.

PROOF. Let $\mathcal{C}_i = \{\mathfrak{F}\}$, $i = 1, 2$. If the $\mathcal{C}_i$ satisfy neither (i) nor (ii), then $\mathfrak{F}$ is an infinite noetherian linear order without root. Let $\mathcal{C}_i'$ be the class of all point-generated (finite) subframes of $\mathfrak{F}$. It should be clear that $\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2) = \mathsf{Log}\,(\mathcal{C}_1' \times \mathcal{C}_2')$, and so we can apply Theorem 2.2 (iii). ∎

THEOREM 2.4
(i) The product of any two unimodal logics from the intervals

$$\mathbf{K4.3} \subseteq L \subseteq \mathsf{Log}\,\langle \omega + 1, \leq \rangle \quad \text{or} \quad \mathbf{K4.3} \subseteq L \subseteq \mathbf{S4.3Dum} = \mathsf{Log}\,\langle \mathbb{N}, \leq \rangle$$

is undecidable and does not have the product finite model property.

(ii) The following logics are undecidable and do not have the product finite model property: $\mathsf{Log}\,\langle \mathbb{N}, < \rangle^2$, $\mathsf{Log}\,\langle \mathbb{N}, \leq \rangle^2$, $\mathsf{Log}\,\langle \mathbb{Q}, < \rangle^2$, $\mathsf{Log}\,\langle \mathbb{Q}, \leq \rangle^2$, $\mathsf{Log}\,\langle \mathbb{R}, < \rangle^2$, $\mathsf{Log}\,\langle \mathbb{R}, \leq \rangle^2$.

(iii) $\mathbf{GL.3}^2$, $\mathbf{Grz.3}^2$, and $\mathbf{GL.3} \times \mathbf{Grz.3}$ do not have the product finite model property and are undecidable.

(iv) The logics $\mathsf{Log}\,\langle \mathbb{N}, > \rangle^2$ and $\mathsf{Log}\,\langle \mathbb{N}, \geq \rangle^2$ are undecidable (and so not recursively enumerable).[2]

REMARK 2.5
(1) Note that the transitivity of frames is essential in the proof of Theorem 2.2. For instance, the logics $\mathbf{Alt}_1$ and $\mathbf{Alt}_1 \oplus \Diamond\top$ can be considered as the logics of some *intransitive* linear frames containing infinite ascending chains; yet their squares have the product finite model property and are decidable, as was shown in [7].

(2) Although some of the logics mentioned in theorems above do not have the product finite model property, they may still enjoy the finite model property in the class of all frames, not only the products.

(3) It may be worth noting that the following logics are decidable:

- $\mathbf{Lin} \times \mathbf{S5}$, where **Lin** is the temporal logic with the diamonds in the future and the past on arbitrary strict linear orders (cf. [21]).
- Actually, the products of many other temporal logics with **S5** turn out to be decidable, in particular, $\mathbf{S5} \times \mathsf{Log}\,\langle \mathbb{N}, < \rangle$, $\mathbf{S5} \times \mathsf{Log}\,\langle \mathbb{Q}, < \rangle$; see [13] for more details.
- $\mathbf{Lin} \times \mathbf{K}$, $\mathbf{K} \times \mathsf{Log}\,\langle \mathbb{N}, < \rangle$, $\mathbf{K} \times \mathsf{Log}\,\langle \mathbb{Q}, < \rangle$ (cf. [31]).
- $\mathbf{S4} \times \mathbf{K}$, $\mathbf{K4} \times \mathbf{K}$ and even $\mathbf{CPDL} \times \mathbf{K}$ (cf. [31]).

On the other hand, it is not known whether such natural logics as $\mathbf{K4} \times \mathbf{K4}$, $\mathbf{S4} \times \mathbf{S4}$, $\mathbf{K4} \times \mathsf{Log}\,\langle \mathbb{N}, < \rangle$, $\mathbf{S4} \times \mathsf{Log}\,\langle \mathbb{N}, < \rangle$ and similar ones are decidable.

[2] $\mathsf{Log}\,\langle \mathbb{N}, > \rangle^2$ and $\mathsf{Log}\,\langle \mathbb{N}, \geq \rangle^2$ are different from $\mathbf{GL.3}^2$ and $\mathbf{Grz.3}^2$, although $\mathsf{Log}\,\langle \mathbb{N}, > \rangle = \mathbf{GL.3}$ and $\mathsf{Log}\,\langle \mathbb{N}, \geq \rangle = \mathbf{Grz.3}$. Indeed, both $\mathsf{Log}\,\langle \mathbb{N}, > \rangle^2$ and $\mathsf{Log}\,\langle \mathbb{N}, \geq \rangle^2$ have clearly the product finite model property, while $\mathbf{GL.3}^2$ and $\mathbf{Grz.3}^2$ do not enjoy it.

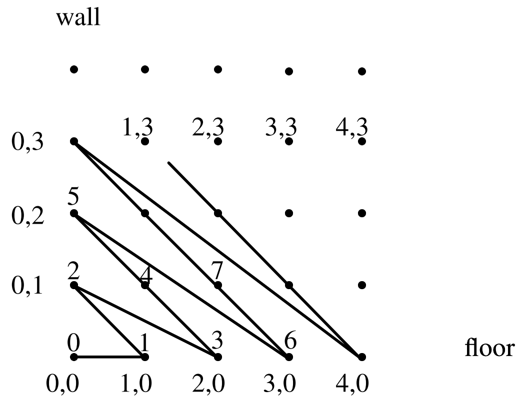

FIGURE 1. The enumeration $pair$.

3 Linear orders with infinite ascending chains

In this section we prove part (i) of Theorem 2.2 by reducing the tiling problem for $\mathbb{N} \times \mathbb{N}$, which is known to be undecidable [22], to the satisfiability problem in $\mathcal{C}_1 \times \mathcal{C}_2$.

Let us recall first that *tiles* are 4-tuples of colours

$$t = \langle left(t), right(t), up(t), down(t) \rangle \, .$$

A finite set T of tiles is said to *tile* $\mathbb{N} \times \mathbb{N}$ if there is a map $\tau : \mathbb{N} \times \mathbb{N} \mapsto T$ such that for each $i, j \in \mathbb{N}$,

- $up(\tau(i,j)) = down(\tau(i, j+1))$ and
- $right(\tau(i,j)) = left(\tau(i+1, j))$.

If we think of a tile as a physical 1×1-square with colours along its four edges, then a tiling τ of $\mathbb{N} \times \mathbb{N}$ is just a way of placing an infinite number of tiles, each of a type from T, together to cover the first quarter of the infinite plane, with no rotation of the tiles allowed and the colours on adjacent edges of adjacent tiles matching.

The *tiling problem* for $\mathbb{N} \times \mathbb{N}$ is formulated as follows: "given a finite set T of tiles, does T tile $\mathbb{N} \times \mathbb{N}$?"

The reduction we are looking for would be pretty simple if our frames were intransitive or the language contained the "next-time" operators (see e.g. [10] or [19]); for then we would be able to refer to the tiles on the right and above directly. As this is not the case, we will use the idea of [17] to enumerate the pairs of natural numbers and refer to the right and above pairs indirectly via special pointers. And a little trick of [25] will make it possible to deal with both strict and non-strict linear orders.

Let $pair : \mathbb{N} \mapsto \mathbb{N} \times \mathbb{N}$ be the enumeration of the pairs of natural numbers defined recursively by:

- $pair(0) = (0, 0)$,
- if $pair(n) = (0, j)$ then $pair(n+1) = (j+1, 0)$,
- otherwise, if $pair(n) = (i+1, j)$ then $pair(n+1) = (i, j+1)$;

see Fig. 1. The pairs of the form $(n, 0)$ are said to be on the *floor*, while those of the form $(0, n)$ are on the *wall*. Let $right(n)$ denote the number of the pair to the right of $pair(n)$ and $above(n)$ the number of the pair above $pair(n)$. For instance, $right(3) = 6$, $above(3) = 7$. An important property of the enumeration is that

$$right(n+1) = \begin{cases} above(n), & \text{if } pair(n) \text{ is not on the wall;} \\ above(n) + 1, & \text{if } pair(n) \text{ is on the wall.} \end{cases} \tag{3.1}$$

To explain the general idea of the reduction, let us imagine first that we need to show that $\mathsf{Log}\,(\langle \mathbb{N}, < \rangle \times \langle \mathbb{N}, < \rangle)$ is undecidable. Then, given a set of tiles T, we will construct a formula φ_T with the propositional variables

- t, for every tile $t \in T$,
- tile $(= \bigvee\{t : t \in T\})$,
- next (a pointer to the next tile according to the enumeration),
- right (a pointer to the tile on the right),
- above (a pointer to the tile above),
- wall (a tile is on the wall),
- floor (a tile is on the floor)

in such a way that $\tau : \mathbb{N} \times \mathbb{N} \to T$ is a tiling iff φ_T holds at $(0, 0)$ under the valuation in $\langle \mathbb{N}, < \rangle^2$ defined by:

$$(n, m) \models t \text{ iff } m = n \text{ and } \tau(pair(n)) = t; \tag{3.2}$$
$$(n, m) \models \mathsf{tile} \text{ iff } n = m; \tag{3.3}$$
$$(n, m) \models \mathsf{next} \text{ iff } m = n + 1; \tag{3.4}$$
$$(n, m) \models \mathsf{right} \text{ iff } m = right(n); \tag{3.5}$$
$$(n, m) \models \mathsf{above} \text{ iff } m = above(n); \tag{3.6}$$
$$(n, m) \models \mathsf{wall} \text{ iff } m = n \text{ and } pair(n) \text{ is on the wall;} \tag{3.7}$$
$$(n, m) \models \mathsf{floor} \text{ iff } m = n \text{ and } pair(n) \text{ is on the floor.} \tag{3.8}$$

(This situation is depicted in Fig. 2.) It will follow that $\mathsf{Log}\,\langle \mathbb{N}, < \rangle^2$ is undecidable.

Suppose now that we have the product $\mathfrak{F} = \langle W_1 \times W_2, <_h, <_v \rangle$ of two arbitrary rooted linear orders $\mathfrak{F}_1 = \langle W_1, <_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, <_2 \rangle$ each of which contains an infinite ascending chain of distinct points. One important difference from $\langle \mathbb{N}, < \rangle \times \langle \mathbb{N}, < \rangle$ is that the relations $<_h$ and $<_v$ are not in general irreflexive (the diamond operators can't say "here but not later"), and our first task is to simulate strict orders in both directions of $\mathfrak{F}$.

To this end, we partition $W_1 \times W_2$ into "black" and "white" squares using the following formulas containing propositional variables h (horizontal) and v (vertical):

$$\boxminus((h \to \diamondminus\neg h) \wedge (\neg h \to \diamondminus h)),$$
$$\boxbar((v \to \diamondbar\neg v) \wedge (\neg v \to \diamondbar v)),$$
$$\Box^*(h \to \boxbar h), \;\; \Box^*(\neg h \to \boxbar\neg h),$$
$$\Box^*(v \to \boxminus v), \;\; \Box^*(\neg v \to \boxminus\neg v).$$

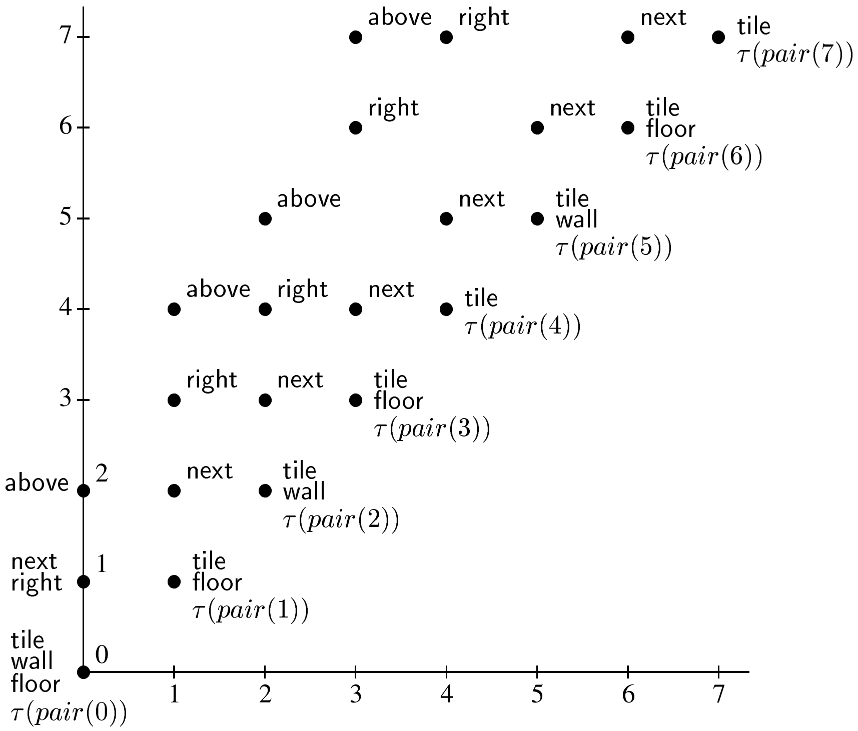

FIGURE 2. The case of infinite ascending chains.

The conjunction of these formulas will be denoted by Chessboard. Let

$$\mathsf{w} = (h \wedge v) \vee (\neg h \wedge \neg v), \quad \mathsf{b} = (\neg h \wedge v) \vee (h \wedge \neg v).$$

Say that a point x in $\mathfrak{F}$ (under some valuation) is *white* (*black*) if $x \models \mathsf{w}$ (respectively, $x \models \mathsf{b}$). Let $square(x)$ denote the largest set S of points in $\mathfrak{F}$ such that $x \in S$, all points in S have the same colour, and S is continuous in the sense that if $y, z \in S$ then $y <_h u <_h z$ or $y <_v u <_v z$ imply $u \in S$. Thus, if Chessboard is true at the root r of $\mathfrak{F}$ (under some valuation) then $\mathfrak{F}$ can be viewed as an infinite chessboard: it is divided into infinite columns and rows of black and white squares in such a way that every square has a horizontal and a vertical successors of different colour.

Now we can define new possibility operators ⬘ and ⬗ by taking, for any formula ψ,

$$⬘\psi = (h \to \Diamond_h(\neg h \wedge \Diamond_h^{+}\psi)) \wedge (\neg h \to \Diamond_h(h \wedge \Diamond_h^{+}\psi)),$$
$$⬗\psi = (v \to \Diamond_v(\neg v \wedge \Diamond_v^{+}\psi)) \wedge (\neg v \to \Diamond_v(v \wedge \Diamond_v^{+}\psi)).$$

It should be clear that for every x on the chessboard we have:

$$x \models ⬘\psi \text{ iff } \exists y >_h x \; (y \models \psi \;\&\; square(x) \neq square(y)),$$
$$x \models ⬗\psi \text{ iff } \exists y >_v x \; (y \models \psi \;\&\; square(x) \neq square(y)).$$

A square S on the chessboard is called a *p-square*, p a propositional variable, if the following two conditions hold:

- $\forall x \in S\ x \models p$;
- $\forall x \in S \forall y \notin S(x <_h y \vee x <_v y \to y \not\models p)$.

Let p-square be the conjunction of the following formulas (in which q' and q'' are auxiliary variables different from p):

$$\Box^*(p \to (\neg \text{⬗} p \wedge \neg \text{⬘} p)),$$
$$\Box^*((\Diamond_{\leftarrow} p \wedge \neg \text{⬘} p) \to p),$$
$$\Box^*((\Diamond_{\uparrow} p \wedge \neg \text{⬗} p) \to p),$$
$$\Box^*(p \to (\text{⬘} q' \wedge \text{⬗} q'' \wedge \neg \text{⬘}\text{⬘} q' \wedge \neg \text{⬗}\text{⬗} q'')),$$
$$\Box^*(q' \to \neg \text{⬘} q'),$$
$$\Box^*(q'' \to \neg \text{⬗} q''),$$
$$\Box^*(p \to \boxminus(\neg q' \wedge \text{⬘} q' \to p)),$$
$$\Box^*(p \to \Box_{\uparrow}(\neg q'' \wedge \text{⬗} q'' \to p)).$$

The reader can readily check that if p-square and Chessboard hold at the root of $\mathfrak{F}$ and $x \models p$ then $square(x)$ is a p-square: the first conjunct of p-square ensures that the second p-square condition holds; the second and third conjuncts ensure that where p holds in a square it also holds to the left and downwards within that square; the last five formulas say that the auxiliary variables hold in the immediate right and upwards neighbours of a p-square and use this to ensure that where p holds in a square it also holds to the right and upwards in that square.

To simulate the tiling problem we use the formula Tiling which is the conjunction of the four formulas:

$$\Box^*(\mathsf{tile} \leftrightarrow \bigvee_{t \in T} t),$$
$$\Box^* \bigwedge_{t \neq t'} \neg(t \wedge t'),$$
$$\Box^* \bigwedge_{up(t) \neq down(t')} \neg(t \wedge \text{⬗}(\mathsf{above} \wedge \text{⬘} t')),$$
$$\Box^* \bigwedge_{right(t) \neq left(t')} \neg(t \wedge \text{⬗}(\mathsf{right} \wedge \text{⬘} t')).$$

The intended meaning of these formulas should be clear.

We are in a position now to write down a formula φ_T which is satisfiable in $\mathfrak{F}$ iff T tiles $\mathbb{N} \times \mathbb{N}$. Define φ_T to be the conjunction of Chessboard, Tiling, t-square, for all $t \in T$, tile-square, next-square, right-square, above-square, wall-square, floor-square,[3] and the fol-

[3]We always take distinct pairs of auxiliary variables q' and q'' in the square-formulas.

lowing formulas as well:

$$\Box^*(\mathsf{tile} \to ⬗\mathsf{next}), \tag{3.9}$$

$$\Box^*(\mathsf{next} \to ⬘\mathsf{tile}), \tag{3.10}$$

$$\Box^*\neg(⬗\mathsf{next} \wedge ⬘\mathsf{tile}), \tag{3.11}$$

$$\Box^*(\mathsf{tile} \to ⬗\mathsf{right}), \tag{3.12}$$

$$\Box^*(\mathsf{right} \to ⬘\mathsf{tile}), \tag{3.13}$$

$$\Box^*(\mathsf{next} \to \neg⬘(⬘\mathsf{tile} \wedge ⬗\mathsf{right})), \tag{3.14}$$

$$\Box^*\neg(⬗\mathsf{right} \wedge ⬘\mathsf{right}), \tag{3.15}$$

$$\Box^*(\mathsf{tile} \to ⬗\mathsf{above}), \tag{3.16}$$

$$\Box^*(\mathsf{above} \to ⬘\mathsf{tile}), \tag{3.17}$$

$$\neg⬗(\mathsf{next} \wedge \neg\mathsf{right}), \tag{3.18}$$

$$\Box^*(\mathsf{right} \to ⬗\mathsf{above}), \tag{3.19}$$

$$\Box^*(⬗\mathsf{above} \wedge ⬘\mathsf{tile} \to \mathsf{right} \vee ⬗\mathsf{right}), \tag{3.20}$$

$$\mathsf{floor} \wedge \mathsf{wall} \wedge \neg⬘⬗(\mathsf{floor} \wedge \mathsf{wall}), \tag{3.21}$$

$$\Box^*(\mathsf{floor} \vee \mathsf{wall} \to \mathsf{tile}), \tag{3.22}$$

$$\Box^*(\mathsf{wall} \to \neg⬗(\mathsf{next} \wedge ⬘(\mathsf{tile} \wedge \neg\mathsf{floor}))), \tag{3.23}$$

$$\Box^*(\mathsf{wall} \to \neg⬗(\mathsf{above} \wedge ⬘(\mathsf{tile} \wedge \neg\mathsf{wall}))), \tag{3.24}$$

$$\Box^*(\mathsf{tile} \wedge \neg\mathsf{wall} \to \neg⬗(\mathsf{above} \wedge ⬘\mathsf{wall})), \tag{3.25}$$

$$\neg⬗(⬘(\mathsf{tile} \wedge \neg\mathsf{wall}) \wedge \neg(⬘\mathsf{right} \vee \mathsf{right})), \tag{3.26}$$

$$\Box^*\neg(⬘\mathsf{tile} \wedge ⬗\mathsf{tile}), \tag{3.27}$$

$$\neg⬗(⬘\mathsf{wall} \wedge ⬘\mathsf{right}). \tag{3.28}$$

Figure 2 will give the reader some general intuitions about these formulas. In particular, (3.14) says that as we go rightwards from a next-square to a tile-square we do not pass below a right-square. (3.18) says that the only next-square above the origin is also a right-square. (3.26) says that every tile-square which is not a wall-square is on the same horizontal level as some right-square. We will see how all of these formulas are used in the proof of the next lemma.

LEMMA 3.1
T tiles $\mathbb{N} \times \mathbb{N}$ iff φ_T is satisfiable in a frame $\mathfrak{F} \in \mathcal{C}_1 \times \mathcal{C}_2$.

PROOF. ($\Rightarrow$) Suppose $\tau : \mathbb{N} \times \mathbb{N} \to T$ is a tiling and $\mathfrak{F} \in \mathcal{C}_1 \times \mathcal{C}_2$ is the product of two frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$ with infinite ascending chains. Then the formula Chessboard is satisfied in $\mathfrak{F}$ under some valuation $\mathfrak{V}$ of h and v. Let us select two infinite sequences of squares on the chessboard: one in the first horizontal row and another one in the first vertical row so that the root of $\mathfrak{F}$ belongs to the first square in both sequences. We enumerate them by natural numbers; (n, m) is then the square with the horizontal coordinate n and the vertical coordinate m according to this enumeration. Now we extend $\mathfrak{V}$ to the other variables in φ_T

by taking (3.2)–(3.8) as the definition in which $(n,m) \models p$ means that p holds at every point in the square (n,m). It is not hard to check that under this valuation we have $(0,0) \models \varphi_T$.

($\Leftarrow$) Suppose φ_T is satisfied at the root x_0 of some $\mathfrak{F} \in \mathcal{C}_1 \times \mathcal{C}_2$ under some valuation. Then x_0 belongs to a tile-square; let us denote this square by $(0,0)$. By (3.9) and (3.10), we have an infinite sequence of squares

$$(0,0) <_v (0,1) <_h (1,1) <_v (1,2) <_h \cdots <_h (i,i) <_v (i,i+1) <_h \dots,$$

in which every (i,i), $i \in \mathbb{N}$, is a tile-square and every $(i,i+1)$ is a next-square ($(0,0) <_v (0,1)$ means that for every $x \in (0,0)$ there is $y \in (0,1)$ such that $x <_v y$). Thus $(n,n) \models \mathsf{tile}$ and $(n,n+1) \models \mathsf{next}$, for every $n \in \mathbb{N}$. Denote by (m,n) the square located in the same column with (m,m) and the same row with (n,n).

Observe that by (3.21), $(0,0) \models \mathsf{floor} \wedge \mathsf{wall}$. By (3.23) we then have $(1,1) \models \mathsf{floor}$ and by (3.21) $(1,1) \models \neg\mathsf{wall}$.

Consider an arbitrary square (n,n). By (3.12) and (3.13) there is a right-square r_n such that $(n,n) <_v r_n <_h t$, for some tile-square t. We will say that r_n *points to* t. Notice that by (3.18) $r_0 = (0,1)$, and so r_0 points to $(1,1)$.

By (3.16) and (3.17) there is an above-square a_n such that $(n,n) <_v a_n <_h t$, for some tile-square t. We will say that a_n *points to* t. Because of the second square condition, r_n and a_n are the unique right-square and above-square above (n,n), respectively. Notice that by (3.19) and (3.20) $a_0 = (0,2)$ points to $(2,2)$: if a_0 was further above $(0,2)$ then (3.20) would imply that $r_0 = (0,1)$ is not a unique right-square above $(0,0)$. By (3.24), we have $(2,2) \models \mathsf{wall}$.

Now we prove by induction on $n \geq 0$ that

(i) if $pair(right(n))$ is on the floor then $(right(n), right(n)) \models \mathsf{floor}$;

(ii) $(right(n), right(n)) \models \neg\mathsf{wall}$;

(iii) $r_n = (n, right(n))$ (i.e., r_n points to $(right(n), right(n))$);

(iv) $a_n = (n, above(n))$ (i.e., a_n points to $(above(n), above(n))$;

(v) if $pair(above(n))$ is on the wall then $(above(n), above(n)) \models \mathsf{wall}$.

The basis of induction was established above. Consider the induction step for $n > 0$.

(i) If $pair(right(n))$ is on the floor then $pair(n-1)$ is on the wall. By (3.1), we have $r(n) - 1 = a(n-1)$ and so

$$pair(right(n) - 1) = pair(above(n-1))$$

is on the wall too, whence by IH (v), $(right(n) - 1, right(n) - 1) \models \mathsf{wall}$. It follows directly by (3.23) that we must have $(right(n), right(n)) \models \mathsf{floor}$.

(ii) If $pair(right(n))$ is on the floor then $(right(n), right(n)) \models \neg\mathsf{wall}$ by (3.21) and (i). Otherwise $right(n) = above(right(k))$ for some k, $0 \leq k < n$. By IH (ii), we have $(right(k), right(k)) \models \neg\mathsf{wall}$ and $(n-1, n-1) \models \neg\mathsf{wall}$, because $n - 1 = right(k)$. Since IH (iv) tells us that a_{n-1} points to $(above(n-1), above(n-1))$, by (3.25) we obtain $(above(n-1), above(n-1)) \models \neg\mathsf{wall}$ which is $(right(n), right(n)) \models \neg\mathsf{wall}$ as required.

(iii) It follows from (ii) and (3.26) that there is a right-square r in the same horizontal line as $(right(n), right(n))$. According to (3.12) and (3.15), this is possible only when we have $r <_h (right(n), right(n))$. We will show that $r = (n, right(n))$.

By (3.14) there is some i such that $0 \leq i < right(n)$ with $(i,i) <_v r$. It follows that $r = (i, right(i))$. If $i < n$ then by IH $r_i = (i, right(i))$, which is impossible. Suppose that

$i > n$. Clearly, r_n cannot be in the same row with $(right(n), right(n))$. If r_n is below that row then, by (3.11) and (3.27) $r_n = (n, j)$ for some $0 < j < right(n)$. If $j = right(k)$, for some $k < n$, then by IH $r_k = (k, j)$, which is a contradiction. And if $j \neq right(k)$ for any $k < n$ then $pair(j)$ is on the wall, i.e., $j = above(k)$ for some $k < n$, from which by IH $(j, j) \models \mathsf{wall}$, contrary to (3.28). Finally, the case when r_n is above the row with $(right(n), right(n))$ is impossible by (3.15). Thus, $i = n$.

(iv) By (iii) $r_n \models \mathsf{right}$, and so by (3.19) there is an above-square $a >_v r_n$. It follows by (3.11) and (3.20) that $a = (n, above(n))$.

(v) If $pair(above(n))$ is on the wall then $pair(n)$ is on the wall as well. Since $n > 0$, $pair(n) = pair(above(k))$ for some $k < n$, and so by IH $(n, n) \models \mathsf{wall}$. By (iv) we have $a_n = (n, above(n))$ and by (3.24), $(above(n), above(n)) \models \mathsf{wall}$.

A tiling τ of $\mathbb{N} \times \mathbb{N}$ is defined now as follows: $\tau(m, n)$ is the unique $t \in T$ for which $t_{pair^{-1}(m,n)} \models t$. Using the formula Tiling it is readily seen that τ is a tiling indeed. ∎

Suppose now that $\mathcal{C}_1$ and $\mathcal{C}_2$ are classes of linear frames each of which contains a frame with an infinite ascending chain of distinct points. To complete the proof of Theorem 2.2 (i), it remains only to observe that, for any set of tiles T, we clearly have: $\neg\varphi_T \in \mathsf{Log}(\mathcal{C}_1 \times \mathcal{C}_2)$ iff φ_T is not satisfiable in $\mathcal{C}_1 \times \mathcal{C}_2$ iff, by Lemma 3.1, T does not tile $\mathbb{N} \times \mathbb{N}$. It follows that $\mathsf{Log}(\mathcal{C}_1 \times \mathcal{C}_2)$ is undecidable.

4 Linear orders with infinite descending chains

Now let us prove the second claim of Theorem 2.2. Suppose $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are rooted noetherian linear orders containing infinite descending chains, i.e., infinite sequences of the form

$$\cdots <_i x_n <_i \cdots <_i x_1 <_i x_0,$$

for $i = 1, 2$. As in Section 3, we partition $\mathfrak{F} = \langle W_1 \times W_2, <_h, <_v \rangle$ into "black" and "white" squares. This can be done using the formulas:

$$\Box^*((h \vee \Diamond_h h \to \Box_h h) \wedge (\neg h \vee \Diamond_h \neg h \to \Box_h \neg h)), \tag{4.1}$$

$$\Box^*((v \vee \Diamond_v v \to \Box_v v) \wedge (\neg v \vee \Diamond_v \neg v \to \Box_v \neg v)), \tag{4.2}$$

$$\neg h \wedge \neg v \wedge \Diamond_h \Diamond_v \Box^*(h \wedge v), \tag{4.3}$$

$$\Box^*(\text{◨}\bot \wedge \text{⬒}\bot \to p), \tag{4.4}$$

$$\Box_v \text{⬗}(p \wedge \text{◨}\neg p), \tag{4.5}$$

$$\Box_h \text{⬘}(\neg p \wedge \text{⬒} p), \tag{4.6}$$

$$\Box^*(p \to \text{⬒}(p \wedge \text{⬗} p)), \tag{4.7}$$

$$\Box^*(\neg p \to \text{◨}(\neg p \wedge \text{⬘}\neg p)), \tag{4.8}$$

where $\text{⬘}\varphi$ and $\text{⬗}\varphi$ are defined as in Section 3; ⬒ and ◨ are the duals of ⬘ and ⬗, respectively.

Denote the conjunction of (4.1)–(4.8) by $\mathsf{Diagonal}$. If it holds somewhere in $\mathfrak{F}$ (under some valuation) then, in view of (4.3)–(4.4), there is a top-right square in $\mathfrak{F}$ all points in which validate h, v, and p. We denote this square by $(0, 0)$. Let $(1, 0)$ be its left neighbour (its points validate $\neg h$ and v), $(0, 1)$ the neighbour below (there h and $\neg v$ hold), if they exist, etc. The square (m, n) is located in the column with $(m, 0)$ and in the row with $(0, n)$. We write $x \lneqq_h (m,n)$ to say that $x \notin (m,n)$ and there is $y \in (m,n)$ such that $x <_h y$; $(m,n) \models \psi$ means that $x \models \psi$ for every $x \in (m,n)$,

LEMMA 4.1
If $\mathsf{Diagonal}$ is true at some point x in $\mathfrak{F}$ under some valuation then, for all $n, m \in \mathbb{N}$,

(a) $(m,n) \models p$ if $m \leq n$,

(b) $(m,n) \models \neg p$ if $m > n$, and

(c) $\exists y \; x \lneqq_h y \lneqq_v (n,n)$.

PROOF. Suppose $x \models \mathsf{Diagonal}$ for some x in $\mathfrak{F}$. By induction on n we are going to show that (i) there is a point y in $\mathfrak{F}$ such that $x \lneqq_h y \lneqq_v (n,n)$, and (ii) $(m,n) \models p$ if $m \leq n$ and $(n,m) \models \neg p$ whenever $m < n$. The basis of induction follows from (4.1)–(4.4).

Assume now that $n > 0$ and our claim holds for all $k < n$. So we have y such that $x \lneqq_h y \lneqq_v (n-1, n-1)$, p is true everywhere in the triangle $(0,0)$, $(n-1, n-1)$, $(0, n-1)$ and is false everywhere above this triangle. By (4.6), there are points z and u such that $x \lneqq_v z \lneqq_h u \lneqq_h (n-1, n-1)$ and $u \models \neg p$. So there is a square $(n, n-1)$, that is the left neighbour of $(n-1, n-1)$. By (4.7), $(n,k) \not\models p$, for all $k \leq n-1$. In view of (4.5), we have some v such that $x \lneqq_h v \lneqq_v (n,n)$. And by (4.8), $(k,n) \models p$ for all $k \leq n$. ∎

LEMMA 4.2
$\mathsf{Diagonal}$ is satisfied in $\mathfrak{F}$.

PROOF. Since $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are noetherian, we can define a valuation of h and v in $\mathfrak{F}$ so that $(0,0)$ consists of the point in the top-right corner, $(1,0)$ of its immediate $<_h$-predecessor, $(0,1)$ of the immediate $<_v$-predecessor of $(0,0)$, and so forth. Besides, we put $z \models p$ iff $z = (m,n)$ for some $m \leq n$. Let x be the point such that condition (c) of Lemma 4.1 holds for all $n \in \mathbb{N}$ and whenever $y \gneqq_h x$ or $y \gneqq_v x$ then $y = (m,n)$ for some $m, n \in \mathbb{N}$ (it exists, since $\mathfrak{F}_i$ are rooted and contain no infinite ascending chains). The reader can easily check that under this valuation $x \models \mathsf{Diagonal}$. ∎

Note by the way that as a consequence of these two lemmas we obtain

COROLLARY 4.3
Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be as in the formulation of Theorem 2.2 (ii). Then $\mathsf{Log}\,(\mathcal{C}_1 \times \mathcal{C}_2)$ does not have the product finite model property.

Given a finite set T of tiles, we construct now a formula φ_T which is satisfiable in $\mathfrak{F}$ iff T tiles $\mathbb{N} \times \mathbb{N}$. As in the previous section, we will use the enumeration $pair : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ and represent the tiles on $\mathbb{N} \times \mathbb{N}$ by the squares (n,n) on the diagonal of $\mathfrak{F}$ using the variables tile and t, for $t \in T$. Let $left(n)$ denote the number of the pair to the left of $pair(n)$ and $below(n)$ the number of the pair below $pair(n)$. As before, wall and floor will indicate that a certain pair is on the wall or on the floor. But instead of the pointers next, right and above we will use now the variables prev, left and below which are supposed to point to the previous pair in the enumeration, to that on the left and below, respectively (see Fig. 3).

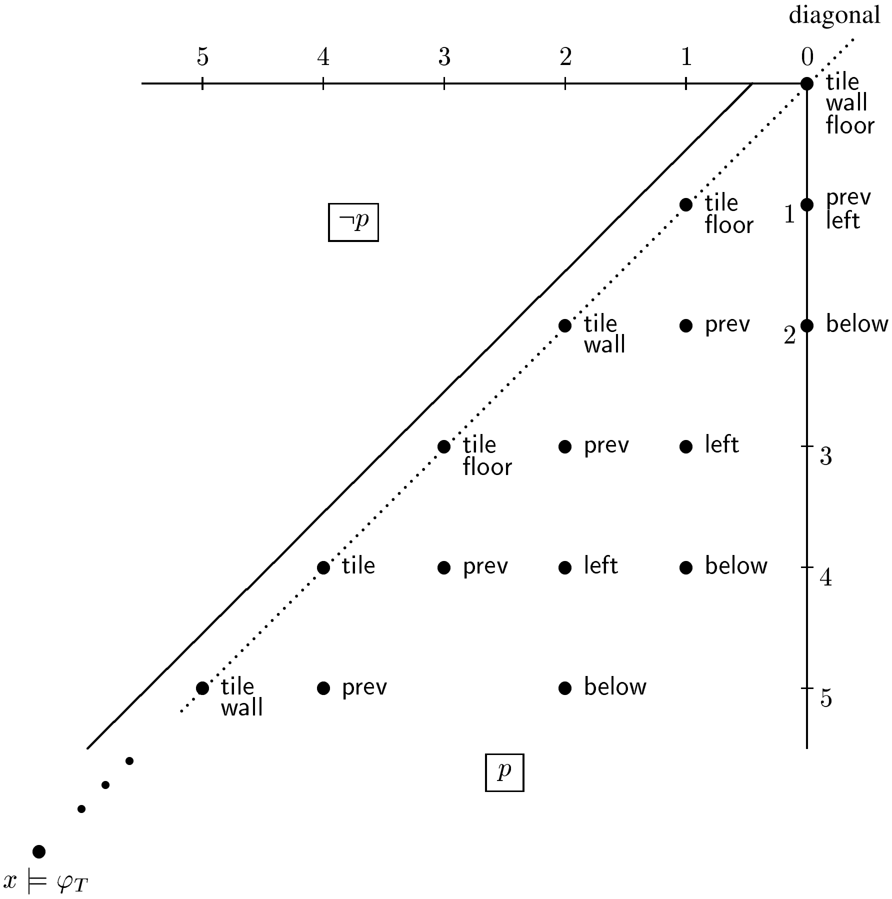

FIGURE 3. The noetherian case.

Let Tiling be the conjunction of the following formulas:

$$\Box^*(\mathsf{tile} \leftrightarrow \bigvee_{t \in T} t),$$

$$\Box^*(\bigwedge_{t \neq t'} \neg(t \wedge t')),$$

$$\bigwedge_{down(t) \neq up(t')} \Box^* \neg(t \wedge ⬘(\mathsf{below} \wedge ⬗ t')),$$

$$\bigwedge_{left(t) \neq right(t')} \Box^* \neg(t \wedge ⬘(\mathsf{left} \wedge ⬗ t')).$$

Define φ_T to be the conjunction of Tiling, Diagonal, t-square, for all $t \in T$, prev-square, left-square, below-square, wall-square, floor-square,[4] and the formulas:

$$\Box^*(\mathsf{tile} \leftrightarrow p \wedge \neg ⬖ p), \tag{4.9}$$

$$\Box^*(\mathsf{prev} \leftrightarrow ⬖\mathsf{tile} \wedge \neg ⬖⬖\mathsf{tile}), \tag{4.10}$$

$$\Box^*(\mathsf{tile} \wedge \neg\mathsf{wall} \to ⬘\mathsf{left}), \tag{4.11}$$

$$\Box^*(\mathsf{tile} \wedge \mathsf{wall} \to \neg ⬘\mathsf{left}), \tag{4.12}$$

$$\Box^*(\mathsf{left} \to ⬖\mathsf{tile} \wedge \neg ⬘\mathsf{left}), \tag{4.13}$$

$$\Box^*(\mathsf{tile} \wedge \neg\mathsf{floor} \to ⬘\mathsf{below}), \tag{4.14}$$

$$\Box^*(\mathsf{tile} \wedge \mathsf{floor} \to \neg ⬘\mathsf{below}), \tag{4.15}$$

$$\Box^*(\mathsf{below} \to ⬖\mathsf{tile} \wedge \neg ⬘\mathsf{below}), \tag{4.16}$$

$$\Box^*(\mathsf{tile} \wedge \neg\mathsf{floor} \wedge \neg\mathsf{wall} \to ⬘(\mathsf{left} \wedge ⬘\mathsf{below} \wedge \neg ⬘⬘\mathsf{below})), \tag{4.17}$$

$$\Box^*(⬒\bot \wedge ◨\bot \leftrightarrow \mathsf{floor} \wedge \mathsf{wall}), \tag{4.18}$$

$$\Box^*((⬒\bot \wedge ◨\bot) \vee (\mathsf{tile} \wedge ⬘(\mathsf{prev} \wedge ⬖\mathsf{wall})) \leftrightarrow \mathsf{floor}), \tag{4.19}$$

$$\Box^*(⬖\top \to (\mathsf{tile} \wedge ⬘(\mathsf{below} \wedge ⬖\mathsf{wall}) \leftrightarrow \mathsf{wall})), \tag{4.20}$$

$$\Box^*(\mathsf{tile} \wedge \neg\mathsf{floor} \to ⬒(\mathsf{below} \to ⬖\mathsf{left} \wedge \neg ⬖⬖\mathsf{left})), \tag{4.21}$$

$$\Box^*(\mathsf{tile} \wedge \mathsf{floor} \to \neg ⬘(⬖\mathsf{floor} \wedge ⬘\mathsf{left})), \tag{4.22}$$

$$\Box^*(\mathsf{floor} \to ⬒(\mathsf{left} \to ⬖\mathsf{floor})). \tag{4.23}$$

LEMMA 4.4
T tiles $\mathbb{N} \times \mathbb{N}$ iff φ_T is satisfiable in $\mathfrak{F}$.

PROOF. ($\Rightarrow$) Suppose $\tau : \mathbb{N} \times \mathbb{N} \to T$ is a tiling, and let x be the "limit" of the diagonal in $\mathfrak{F}$ as in the proof of Lemma 4.2. Define a valuation in $\mathfrak{F}$ by taking, for any $y \in W_1 \times W_2$,

$y \models h$ iff $y = (m, n)$ and m is even;
$y \models v$ iff $y = (m, n)$ and n is even;
$y \models p$ iff $y = (m, n)$, for some $m \leq n$;
$y \models t$ iff $y = (n, n)$ and $\tau(pair(n)) = t$;
$y \models$ tile iff $y = (n, n)$ for some $n \in \mathbb{N}$;
$y \models$ prev iff $y = (n-1, n)$;
$y \models$ left iff $y = (m, n)$ and $m = left(n)$;
$y \models$ below iff $y = (m, n)$ and $m = below(n)$;
$y \models$ wall iff $y = (n, n)$ and $pair(n)$ is on the wall;
$y \models$ floor iff $y = (n, n)$ and $pair(n)$ is on the floor.

It is a matter of routine to check that under the defined valuation φ_T holds at x.

($\Leftarrow$) Suppose φ_T is true at x in $\mathfrak{F}$ under some valuation. Since $x \models$ Diagonal and in view of Lemma 4.1, p is true in all squares (m, n), for $m \leq n$, and false in all (m, n) with $m > n$, $m, n \in \mathbb{N}$. Moreover, all squares on the diagonal are accessible from x via $<_h$ and $<_v$.

[4]These formulas were constructed in Section 3.

By (4.9), tile holds in (n, n) on the diagonal and is false in (n, m) whenever $n \neq m$. And by (4.10), prev holds in (m, n) iff $n = m + 1$, i.e., it holds only in the squares located immediately below the diagonal ones. (Note that p, tile, prev may be true at some other points that do not belong to the depicted grid, but they are of no concern to us.)

By (4.18), $(0, 0) \models \mathsf{floor} \wedge \mathsf{wall}$. By (4.19), $(1, 1) \models \mathsf{floor}$ and in view of (4.18), we have $(1, 1) \models \neg\mathsf{wall}$. By (4.11), $(0, 1) \models \mathsf{left}$. And by (4.15), $(0, 1) \models \neg\mathsf{below}$.

Now by induction on $n \geq 0$ we show that

(i) $pair(n)$ is on the floor iff $(n, n) \models \mathsf{floor}$;

(ii) $(m, n) \models \mathsf{below}$ iff $m = below(n)$;

(iii) $pair(n)$ is on the wall iff $(n, n) \models \mathsf{wall}$;

(iv) $(m, n) \models \mathsf{left}$ iff $m = left(n)$.

The case $n \leq 1$ has been already considered. So let $n > 1$. Observe that by (4.13) and (4.16), if $(m, n) \models \mathsf{left}$ or $(m, n) \models \mathsf{below}$ then $n > m$, i.e., (m, n) is below the diagonal.

Suppose that $pair(n)$ is on the floor. Then $pair(n - 1)$ is on the wall, and so by IH $(n - 1, n - 1) \models \mathsf{wall}$. It follows that $(n, n) \models \mathsf{tile} \wedge \text{⬘}(\mathsf{prev} \wedge \text{⬗}\mathsf{wall})$. By (4.19) we then have $(n, n) \models \mathsf{floor}$. The converse implication also follows by IH from (4.19). This proves (i).

Assume now that $(m, n) \models \mathsf{below}$. By (4.15), $(n, n) \models \neg\mathsf{floor}$. And by (4.21) we have $(m, n - 1) \models \mathsf{left}$, which by IH means that $m = left(n - 1)$, and so $pair(m)$ is below $pair(n)$ indeed. Conversely, suppose $m = below(n)$. Then n is not on the floor, and so $(n, n) \models \neg\mathsf{floor}$. By (4.14), $(k, n) \models \mathsf{below}$ for some $k < n$. But then, as we know, $k = m$ (uniquely by the definition of $\mathsf{below\text{-}square}$. This yields (ii).

Suppose $pair(n)$ is on the wall and $pair(m)$ is located immediately below $pair(n)$. Then by (i) $(n, n) \models \neg\mathsf{floor}$, and so by (4.14) $(k, n) \models \mathsf{below}$ for some $k < n$. By (ii) $k = m$. So by IH $(n, m) \models \mathsf{wall}$ and we have $(n, n) \models \mathsf{tile} \wedge \text{⬘}(\mathsf{below} \wedge \text{⬗}\mathsf{wall})$. And since $(n, n) \models \text{⬗}\top$, we obtain by (4.20) that $(n, n) \models \mathsf{wall}$. The converse implication follows from IH and (4.20). Thus we have (iii).

Finally, to prove (iv), suppose that $(m, n) \models \mathsf{left}$. By (4.12), we have $(n, n) \models \neg\mathsf{wall}$. If $(n, n) \models \neg\mathsf{floor}$ then by (4.17) and (ii) $pair(m - 1)$ is below $pair(n)$, whence $m = left(n)$. And if $(n, n) \models \mathsf{floor}$ (and so $pair(n)$ is on the floor) then by (4.22) $(k, k) \models \mathsf{floor}$ for no k such that $m < k < n$, which means by IH and (4.23) that m is the greatest integer $< n$ with $pair(m)$ on the floor. Thus $pair(m)$ is on the floor immediately to the left of $pair(n)$. Thus again $m = left(n)$. Conversely, suppose $m = left(n)$. Then $(n, n) \models \neg\mathsf{wall}$. By (4.11), $(k, n) \models \mathsf{left}$ for some $k < n$, from which $k = m$, as required.

It follows from (ii), (iv) and $x \models \mathsf{Tiling}$ that the map τ defined by taking $\tau(n, n) = t$ iff $(n, n) \models t$, for $t \in T$, is a tiling of $\mathbb{N} \times \mathbb{N}$. ∎

Theorem 2.2 (ii) follows immediately from Lemma 4.4.

5 Finite linear orders

In this section we prove the last claim of Theorem 2.2, namely that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are classes of finite linear orders (no matter reflexive or irreflexive) both containing arbitrarily long (but finite) chains, then the logic $\mathsf{Log}(\mathcal{C}_1 \times \mathcal{C}_2)$ is undecidable.

For simplicity we will assume here that both $\mathcal{C}_1$ and $\mathcal{C}_2$ contain only *strict* linear orders. If this is not the case, one can use variables h and v satisfying (4.1) and (4.2) to partition frames into black and white squares as in the two previous sections.

We are going to reduce the following undecidable problem to the satisfiability problem in $\mathcal{C}_1 \times \mathcal{C}_2$: given a Turing machine, to determine whether it comes to a stop having started from the empty tape. The basic idea of the proof is to represent a run of the Turing machine as a sequence of columns, each of which represents a configuration.

Let $\mathfrak{A}$ be a single-tape right-infinite deterministic Turing machine with state space S, initial state s_0, halt state s_1, tape alphabet A ($b \in A$ stands for blank) and transition function δ. We assume that $\mathfrak{A}$ comes to a stop only if it reaches s_1; if the current state is different from s_1 then the machine makes another step. The configurations of $\mathfrak{A}$ will be represented by finite words of the form $\pounds a_0 \dots a_i \dots a_n b^m$, where $\pounds$ marks the left side of the tape, $m > 0$, all $a_0, \dots, a_n$ save one, say a_i, are in A, while a_i belongs to $S \times A$ and represents the active cell and the current state. The start configuration, for instance, is represented by any word of the form: $\pounds(s_0, b)b^m$, where $m > 0$. Let $A' = A \cup \{\pounds\} \cup (S \times A)$. With each $\alpha \in A'$ we associate a propositional variable p_α.

We show how to construct a formula $\varphi_{\mathfrak{A}}$ which is satisfiable in a frame from $\mathcal{C}_1 \times \mathcal{C}_2$ iff $\mathfrak{A}$ comes to a stop having started from $\pounds(s_0, b)b^m$. Define $\varphi_{\mathfrak{A}}$ to be the conjunction of the following formulas, where p ranges over all the variables occurring in $\varphi_{\mathfrak{A}}$:

$$d, \tag{5.1}$$

$$\Box^*\neg(p \wedge (\Diamond_v p \vee \Diamond_h p)), \tag{5.2}$$

$$\Box^*(d \wedge \Diamond_v \top \to \Diamond_v \mathsf{next} \wedge \neg\Diamond_v\Diamond_v \mathsf{next}), \tag{5.3}$$

$$\Box^*(\mathsf{next} \wedge \Diamond_h \top \to \Diamond_h d \wedge \neg\Diamond_h\Diamond_h d), \tag{5.4}$$

$$\mathsf{floor} \wedge \mathsf{wall} \wedge \neg\Diamond_v\Diamond_h(\mathsf{floor} \wedge \mathsf{wall}), \tag{5.5}$$

$$\Box_v(\mathsf{next} \to \mathsf{right}), \tag{5.6}$$

$$\Box^*(\mathsf{right} \wedge \Diamond_v \top \to \Diamond_v \mathsf{above} \wedge \neg\Diamond_v\Diamond_v \mathsf{above} \wedge \neg\Diamond_h \mathsf{above}), \tag{5.7}$$

$$\Box^*(\neg\mathsf{wall} \wedge d \to \Box_v(\mathsf{above} \wedge \Diamond_h \top \to \Diamond_h \mathsf{right} \wedge \neg\Diamond_h\Diamond_h \mathsf{right})), \tag{5.8}$$

$$\Box^*(\mathsf{wall} \wedge d \to \Box_v(\mathsf{above} \wedge \Diamond_h \top \to \Diamond_h q \wedge \neg\Diamond_h\Diamond_h q \wedge \neg\Diamond_h \mathsf{right})), \tag{5.9}$$

$$\Box^*(q \wedge \Diamond_v \top \to \Diamond_v \mathsf{right} \wedge \neg\Diamond_v\Diamond_v \mathsf{right}), \tag{5.10}$$

$$\Box^*(\mathsf{wall} \to \Box_v(\mathsf{next} \to \Box_h(d \to \mathsf{floor}))), \tag{5.11}$$

$$\Box^*(\mathsf{wall} \to \Box_v(\mathsf{above} \to \Box_h(d \to \mathsf{wall}))), \tag{5.12}$$

$$\Box^*(\mathsf{above} \to \neg\Diamond_h \mathsf{floor}), \tag{5.13}$$

$$\Box^*(\mathsf{right} \to \neg\Diamond_h \mathsf{wall}), \tag{5.14}$$

$$\Box^*(d \leftrightarrow \bigvee_{\alpha \in A'} p_\alpha), \tag{5.15}$$

$$\Box^* \bigwedge_{\alpha \in A'} (p_\alpha \to \neg \bigvee_{\alpha \neq \beta \in A'} p_\beta), \tag{5.16}$$

$$\Box^*(l \leftrightarrow d \wedge \Diamond_v(\mathsf{above} \wedge \Diamond_h s)), \tag{5.17}$$

$$\Box^*(s \leftrightarrow \bigvee_{(s,a) \in A'} p_{(s,a)}), \tag{5.18}$$

$$\Box^*(s \leftrightarrow \Diamond_v(\mathsf{above} \wedge \Diamond_h(d \wedge r))), \tag{5.19}$$

$$\Box^* \bigwedge_{\delta(\alpha,\beta,\gamma)=(\alpha',\beta',\gamma')} \big[p_\alpha \wedge l \wedge \Diamond(\mathsf{above} \wedge \Diamond(p_\beta \wedge \Diamond(\mathsf{above} \wedge \Diamond p_\gamma))) \to$$

$$\Diamond(\mathsf{right} \wedge \Diamond(p_{\alpha'} \wedge \Diamond(\mathsf{above} \wedge \Diamond(p_{\beta'} \wedge \Diamond(\mathsf{above} \wedge \Diamond p_{\gamma'})))))\big]\ , \quad (5.20)$$

$$\Box^* \bigwedge_{\alpha \in A'} (p_\alpha \wedge \neg l \wedge \neg s \wedge \neg r \to \boxminus(\mathsf{right} \to \boxminus(d \to p_\alpha))), \quad (5.21)$$

$$p_{\pounds} \wedge \Diamond(\mathsf{above} \wedge \Diamond(p_{(s_0,b)} \wedge \Diamond(\mathsf{above} \wedge \Diamond p_b))). \quad (5.22)$$

The intended meaning of this formula is as follows. Suppose $\varphi_{\mathfrak{A}}$ is true at the root of a frame $\mathfrak{F} \in \mathcal{C}_1 \times \mathcal{C}_2$ under some valuation. Then the variable d marks the diagonal of $\mathfrak{F}$ starting from the root. Let $0, \ldots, \ell$ be the consecutive points on this diagonal. We will think of them as representing the first $\ell + 1$ cells in the $\mathbb{N} \times \mathbb{N}$ grid under the enumeration defined in Section 3. For any $i, j \leq \ell$, denote by x_{ij} the point located in the column containing i and in the row containing j. Then we have $x_{ij} \models \mathsf{right}$ and $x_{ik} \models \mathsf{above}$, for $i, j, k \leq \ell$, iff $right(i) = j$ and $above(i) = k$. The variables wall and floor mark the wall and the floor, respectively. The formulas (5.1) to (5.14) are used to ensure this in a variation on the argument in Lemma 3.1.

Every point on the diagonal is marked also by a variable p_α, for some $\alpha \in A'$. Configurations are represented by tuples $\langle i_0, \ldots, i_k \rangle$ such that $i_0 \models \mathsf{floor}$ and $x_{i_j i_{j+1}} \models \mathsf{above}$. We have $i_j \models s$ iff $i_j \models p_{(s,a)}$ for some $(s, a) \in A'$, $i_{j-1} \models l$ and $i_{j+1} \models r$. The formulas (5.20) and (5.21) describe the transition function: we move to the column on the right and change the active cell with its left and right neighbours, leaving other cells intact. Finally, (5.22) says that $\mathfrak{A}$ starts working on the empty tape.

With this explanation, it is not hard to show that $\varphi_{\mathfrak{A}}$ is as required. We leave this to the reader.

6 Dedekind complete linear orders

Harel [11] proved that the following problem is Σ^1_1-complete:

- Given a finite set T of tiles and a tile $t \in T$, can T tile $\mathbb{N} \times \mathbb{N}$ such that t appears infinitely often in the first column?

We will use this result to show that logics of certain classes of products of linear frames are not recursively enumerable, and so not recursively axiomatisable.

Say that a linear order $\mathfrak{F} = \langle W, < \rangle$ is *Dedekind complete* if every bounded (with respect to $<$) infinite ascending chain of distinct points in W has a least upper bound in $\mathfrak{F}$.

THEOREM 6.1
Let $\mathcal{C}$ be a class of Cartesian products of linear orders satisfying the following conditions:

min there is a frame $\langle W_1, <_1 \rangle \times \langle W_2, <_2 \rangle \in \mathcal{C}$ with each $\langle W_i, <_i \rangle$, for $i = 1, 2$, containing an infinite ascending chain;

max if $\langle W_1, <_1 \rangle \times \langle W_2, <_2 \rangle \in \mathcal{C}$, then $\langle W_1, <_1 \rangle$ is Dedekind complete.

Then the satisfiability problem in $\mathcal{C}$ is Σ^1_1-hard, whence $\mathsf{Log}\,\mathcal{C}$ is not recursively enumerable.

PROOF. Given a set T of tiles containing t, let ψ_T be the conjunction of φ_T defined in Section 3 and the following three formulas:

$$\Box^*(\mathsf{tile} \to \text{⬗}\text{⬘}(t \wedge \mathsf{wall})), \tag{6.1}$$

$$\text{⬒}\Big((\neg\text{⬗}\mathsf{tile} \wedge \neg\text{⬗}\text{⬘}\mathsf{tile}) \vee \text{⬗}\mathsf{tile} \vee \text{⬗}(\text{⬘}\mathsf{tile} \wedge \neg\mathsf{next} \wedge \neg\text{⬘}\mathsf{next})\Big), \tag{6.2}$$

$$\text{◨}(\text{⬘}\mathsf{tile} \to \mathsf{next} \vee \text{⬘}\mathsf{next}). \tag{6.3}$$

We will show that ψ_T is satisfiable in $\mathcal{C}$ iff T tiles $\mathbb{N} \times \mathbb{N}$ with t appearing infinitely often up the wall.

If we have a tiling with t appearing infinitely often along the wall then it is clear that ψ_T has a model based on a frame in $\mathcal{C}$: just use the construction given in Section 3 above on one of the frames having ascending chains in both dimensions acting as the natural numbers. The three new conjuncts can easily seen to be satisfied.

Conversely, if ψ_T has a model based on a frame $(W_1, <_1) \times (W_2, <_2)$ from $\mathcal{C}$ then this is also a model of φ_T. So in the same way as in Section 3 we can construct a tiling of $\mathbb{N} \times \mathbb{N}$. (Recall that we use a sequence of squares $(0, 0), (1, 1), \ldots$ at which tile holds.)

We will show that the only points at which wall holds are those in squares (i, i) for which $pair(i)$ is on the wall. (6.1) then tells us that t appears infinitely often as a tile on the wall in the tiling, which is precisely what we need.

In fact it is sufficient to show that there are no tile-squares apart from (n, n), $n \in \mathbb{N}$. This is because by (3.22) wall is only true in tile-squares and in Section 3 we had shown that (n, n) is a wall-square iff $pair(n)$ is on the wall (i.e. not right of any other $pair(k)$).

For each $i < \omega$, choose $x_i \in W_1$ and $y_i \in W_2$ such that $(x_i, y_i) \in (i, i)$. If $x_0 <_1 x_1 <_1 \ldots$ is unbounded in $\langle W_1, <_1\rangle$ then we are done as (3.9) and (3.11) can easily be used to show that there are no other tile-squares.

So consider the situation in which $x_0 <_1 x_1 <_1 \ldots$ is bounded in $\langle W_1, <_1\rangle$. As $\langle W_1, <_1\rangle$ is Dedekind complete, there is some least upper bound z to the sequence. We will show that "from z on" there are no more tile-squares, i.e.,

$$\forall u \in W_2 \forall z' \in W_1 (z' \geq_1 z \to (z', u) \not\models \mathsf{tile}).$$

By (6.2) there are three possibilities for the pair (z, y_0):

case 1: We may have $\neg\text{⬗}\mathsf{tile} \wedge \neg\text{⬗}\text{⬘}\mathsf{tile}$ true at (z, y_0). But then indeed tile is false from z on. Let us show that the other cases cannot occur.

case 2: (z, y_0) makes $\text{⬗}\mathsf{tile}$ true. Then tile is true at (z, u) for some $u \geq_2 y_0$. By (6.3), next is true at (z', u) for some $z' \geq_1 x_0$. By (3.10) and the definition of a tile-square, $z' <_1 z$. As z is the least upper bound of the sequence of x_is, we know there is some x_i such that $z' <_1 x_i$. There are three cases depending on the ordering of u and y_i. We can not have $u <_2 y_i$ by (3.27). We cannot have $u >_2 y_i$ by (3.11). And finally, we cannot have $u = y_i$ by the definition of a tile-square.

case 3: $\text{⬗}(\text{⬘}\mathsf{tile} \wedge \neg\mathsf{next} \wedge \neg\text{⬘}\mathsf{next})$ is true at (z, y_0). This case is similar to case 2 and cannot happen.

So all the tile-squares lie before the least upper bound z and we are done by (3.9) and (3.11). ∎

COROLLARY 6.2

The following logics are not recursively enumerable: $\mathsf{Log}\langle\mathbb{N}, <\rangle^2$, $\mathsf{Log}\ \langle\mathbb{N}, \leq\rangle^2$, $\mathsf{Log}\ \langle\mathbb{R}, <\rangle^2$, $\mathsf{Log}\ \langle\mathbb{R}, \leq\rangle^2$, $\mathsf{Log}\ \langle\mathbb{Z}, <\rangle^2$, and $\mathsf{Log}\ \langle\mathbb{Z}, \leq\rangle^2$.

COROLLARY 6.3
Let $L_1 \in \{\mathbf{KD4.3Z}, \mathbf{S4.3Dum}\}$ and let $L_2 = \mathsf{Log}\,\mathcal{C}$, where $\mathcal{C}$ is a class of linear orders at least one of which contains an infinite ascending chain of distinct points (e.g. L_2 is any logic mentioned in Theorem 2.4 (i)). Then $L_1 \times L_2$ is not recursively enumerable.

By using an equally devious set of extra formulas we can also produce a similar theorem for classes of frames with infinite dscending chains.

THEOREM 6.4
Let $\mathcal{C}$ be a class of Cartesian products of linear orders satisfying the following conditions:

min there is a frame $\langle W_1, <_1\rangle \times \langle W_2, <_2\rangle \in \mathcal{C}$ with each $\langle W_i, <_i\rangle$, for $i = 1, 2$, being rooted and containing an infinite descending chain;

max if $\langle W_1, <_1\rangle \times \langle W_2, <_2\rangle \in \mathcal{C}$, then both $\langle W_1, >_1\rangle$ and $\langle W_2, >_2\rangle$ are Dedekind complete.

Then the satisfiability problem in $\mathcal{C}$ is Σ^1_1-hard, whence $\mathsf{Log}\,\mathcal{C}$ is not recursively enumerable.

PROOF. As before, suppose that we have a finite set T of tiles containing t. We extend φ_T of Section 4 to ψ_T by adding the following conjuncts to φ_T:

$$\neg \text{⬗}\text{⬘}\varphi_T, \tag{6.4}$$

$$\Box^*(\neg p \to \neg r), \tag{6.5}$$

$$\Box^*(t \wedge \mathsf{wall} \to r \wedge \boxminus r), \tag{6.6}$$

$$\text{⍊}(\text{⟐} r \to \text{⟐}(t \wedge \mathsf{wall})), \tag{6.7}$$

$$\text{⬒}\text{⬗} r. \tag{6.8}$$

These are designed to ensure that:

- ψ_T can be true only at the Dedekind limit of the diagonal squares (n, n), $n \in \mathbb{N}$, (6.4) (for φ_T holds at this limit),
- r is false above the diagonal squares (6.5),
- r is true in every t-square and to the right it whenever this square validates wall (6.6),
- and only there (6.7), and
- if one moves even a bit to the right from the square where ψ_T holds, one sees r above (6.8).

Thus, if ψ_T is satisfied in some square then r (and so t on the wall) must be infinitely close to this square.

Condition **min** guarantees that if T tiles $\mathbb{N} \times \mathbb{N}$ so that t appears infinitely often on the wall, then ψ_T is satisfiable in $\mathcal{C}$. ■

COROLLARY 6.5
$\mathbf{GL.3}^2$, $\mathbf{Grz.3}^2$, and $\mathbf{GL.3} \times \mathbf{Grz.3}$ are not recursively enumerable.

7 An axiomatisation of $\mathbf{K4.3} \times \mathbf{K4.3}$

The results of the previous section showed that it is often impossible to axiomatize products of linear modal logics. Even when it is possible it is still hard. Here we give an example for $\mathbf{K4.3}^2$. It is difficult to see how to find a complete axiom system which only uses the traditional structural rules: there is a problem with marshalling maximal consistent sets into a grid-like pattern. As we will see, we solve the problem by using Gabbay-style irreflexivity rules to allow us to introduce new atoms into the construction which indicate lattitudes and longitudes of the sets of formulas. Because the connectives do not have duals in the language we rely on a series of irreflexivity rules (rather than just one) for each dimension. The proof then follows standard techniques for creating canonical structures using such rules (see Section 6.11 in [9]) and standard techniques for modifying the canonical structures as seen in [26] (for an interval logic), [9] (for Vlach and Åqvist's logic), [5] (for a two-dimensional until-since logic) and [18] (for the compass logic). Thus we do not present the proof.

So here is a complete axiom system—call it $K4.3^2$—for $\mathbf{K4.3} \times \mathbf{K4.3}$. The inference rules of $K4.3^2$ are modus ponens and the rules of necessitation for both $\Box_h$ and $\Box_v$:

$$\frac{\varphi,\ \varphi \to \psi}{\psi}, \qquad \frac{\varphi}{\Box_v \varphi}, \qquad \frac{\varphi}{\Box_h \varphi}$$

along with Gabbay-style irreflexivity rules (cf. [9]) for each $n < \omega$:

$$IRR_n: \qquad \frac{\neg(\phi_n \wedge \Diamond_n(\phi_{n-1} \wedge \Diamond_{n-1}(\cdots \wedge \Diamond_1(\phi_0 \wedge \psi(\overline{q}))) \cdots))}{\neg(\phi_n \wedge \Diamond_n(\phi_{n-1} \wedge \Diamond_{n-1}(\cdots \wedge \Diamond_1(\phi_0)) \cdots))}$$

where $\psi, \phi_0, \phi_1, \ldots, \phi_n$ are any formulas, $\psi(\overline{q})$ is a substitution instance of ψ by any atoms $\overline{q}$ not occurring in the ϕ_i, and $\Diamond_1, \ldots, \Diamond_n$ are each either $\Diamond_v$ or $\Diamond_h$.

The axiom schemas of $K4.3^2$ include the usual ones for **K4.3**:

X_0	the schemas of classical logic,
X_{1v}	$\Box_v(\varphi \to \psi) \to (\Box_v\varphi \to \Box_v\psi)$
X_{2v}	$\Box_v\varphi \to \Box_v\Box_v\varphi$
X_{3v}	$\Diamond_v\varphi \wedge \Diamond_v\psi \to \Diamond_v(\varphi \wedge \psi) \vee \Diamond_v(\varphi \wedge \Diamond_v\psi) \vee \Diamond_v(\psi \wedge \Diamond_v\varphi)$

with horizontal versions (X_{1h}, X_{2h}, X_{3h}) of the latter three and two schemas to describe the interaction of the two dimensions:

X_4	$\Diamond_v\Diamond_h\varphi \leftrightarrow \Diamond_h\Diamond_v\varphi$	"commutativity "
X_5	$\Diamond_v\Box_h\varphi \to \Box_h\Diamond_v\varphi$	"Church-Rosser."

THEOREM 7.1
The axiom system $K4.3^2$ is sound and complete for $\mathbf{K4.3}^2$.

8 Questions

As we have seen, products of linear logics above **K4** are usually undecidable and sometimes even not recursively enumerable. We did not consider, however, products of frames one of which contains an infinite ascending chain and the other an infinite ascending chain. So the following problem remains open:

(1) Are $\mathbf{K4.3} \times \mathbf{GL.3}$ and similar logics decidable (recursively enumerable)?

Nothing is known about the behaviour of products of linear and non-linear logics above **K4**. For instance, it would be interesting to answer the following questions:

(2) Are logics of the form $\mathbf{K4.3} \times \mathbf{K4}$, $\mathbf{S4.3} \times \mathbf{S4}$, etc. decidable?

(3) Is the product of a logic with the difference operator and a Priorean temporal logic decidable?

And of course products of "transitive" non-linear logics remain a challenge:

(4) Are logics of the form $\mathbf{K4} \times \mathbf{K4}$, $\mathbf{S4} \times \mathbf{S4}$, $\mathbf{GL} \times \mathbf{GL}$ etc. decidable?

Turning to axiomatizations, we have seen, once again, the benefits of using IRR-style rules. However, as with many other two-dimensional temporal logics, we have the usual open problem:

(5) Is there a finite axiomatization for $\mathbf{K4.3}^2$ (or for some other product of linear logics) without using IRR-style rules?

Even making use of IRR-style rules we have only presented a complete axiom system for $\mathbf{K4.3}^2$. Other products may be straightforward modifications of this system but some seem to be more difficult:

(6) Give complete axiomatisations for products such as $\mathbf{K4.1.3}^2$ and $\mathbf{K4.3} \times \mathbf{GL.3}$.

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