Decidable fragments of first-order temporal logics

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Abstract

In this paper, we introduce a new fragment of the first-order temporal language, called the *monodic fragment,* in which all formulas beginning with a temporal operator (Since or Until) have at most one free variable. We show that the satisfiability problem for monodic formulas in various linear time structures can be reduced to the satisfiability problem for a certain fragment of classical first-order logic. This reduction is then used to single out a number of decidable fragments of first-order temporal logics and of two-sorted first-order logics in which one sort is intended for temporal reasoning. Besides standard first-order time structures, we consider also those that have only finite first-order domains, and extend the results mentioned above to temporal logics of finite domains. We prove decidability in three different ways: using decidability of monadic second-order logic over the intended flows of time, by an explicit analysis of structures with natural numbers time, and by a composition method that builds a model from pieces in finitely many steps.

1 Introduction

Temporal logic has found numerous applications in computer science, ranging from the traditional and well-developed fields of program specification and verification [34, 30, 31], temporal databases [12, 13, 3, 42, 17], and distributed and multi-agent systems [15], to more recent uses in knowledge representation and reasoning [6, 7, 8, 40, 46]. This is true of both propositional and first-order temporal logic. However, the mainstream of theoretical studies in the discipline has mostly been restricted to the propositional case—witness the surveys [14, 43], or the two-volume monograph [16, 17] where only one chapter is devoted to first-order temporal logics.

The reason for this seems clear. Though some axiomatizations of first-order temporal logics are known (e.g., [38] presents axiomatizations for first-order logics with Until and Since over the class of all linear flows and over the rationals), a series of incompleteness theorems [1, 4, 16, 19, 32, 44, 45], started by unpublished results of Scott and Lindström in the 1960s, show that many of the first-order temporal logics most useful in computer science are not even recursively enumerable. But in contrast to classical first-order logic, where the early undecidability results of Turing and Church stimulated research and led to a rich and profound theory concerned with classifying fragments of first-order logic according to their decidability (see, e.g., [9]), there were no serious attempts to convert the 'negative' results in first-order temporal logic into a classification problem. Apparently, the extremely weak expressive power of the temporal formulas required to prove undecidability left no hope that any useful decidable fragments located 'between' propositional and first-order temporal logics could ever be found. (See, e.g., Theorems 2 and 3 below.)

The main aim of this paper is to define and investigate a new kind of sub-language of the first-order temporal language which, on the one hand, is considerably more expressive than the propositional language, yet on the other hand gives rise to decidable fragments of first-order temporal logics. Roughly speaking, these fragments are obtained by:

(1) restricting the pure classical (non-temporal) part of the language to an arbitrary decidable fragment of first-order logic, and

(2) restricting the temporal part of the language to the *monodic* formulas whose subformulas beginning with a temporal operator have at most one free variable.

Condition (1) allows the use of classical decidability results to select a suitable first-order part of the language, while (2) leaves enough room for non-trivial interactions between quantifiers and temporal operators (as in the Barcan formula, $\exists x \Diamond \varphi(x) \leftrightarrow \Diamond \exists x \varphi(x)$). Thus, we can talk about objects in the intended domain using the full power of the selected fragment of first-order logic; however, temporal operators may be used to describe the development in time of only one object (two are enough to simulate the behaviour of Turing machines or tilings; see below).

The bulk of the paper is devoted to showing that these two conditions do result in decidable temporal fragments over various flows of time. As a consequence, we obtain for instance that the two-variable monodic fragment, and the temporal guarded monodic fragment, are decidable where the flow of time is arbitrary, finite, $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Z}, < \rangle$, $\langle \mathbb{Q}, < \rangle$, or (for finite domains at least) $\langle \mathbb{R}, < \rangle$. The obtained results and the developed techniques can be applied to prove the decidability of various propositional multi-dimensional modal logics, including some temporal epistemic logics close to those in [15] and used in multi-agent systems, and temporal description logics used in knowledge representation (cf. [46]). Thus, the results of the paper are of significance both for applications in CS and AI, and for theoretical studies in temporal logic. Moreover, we hope that the discovery of natural decidable fragments of first-order temporal logic will stimulate further research in this field.

In this paper, we confine ourselves to considering satisfiability of temporal formulas without equality or function symbols, interpreted in models with constant first-order domains and strictly linear flows of time: in particular, the aforementioned $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Z}, < \rangle$, and $\langle \mathbb{Q}, < \rangle$. We are interested both in models with arbitrary domains and in those with only finite domains. Actually, none of the decidable fragments to be constructed below has the *finite domain property*: the set of formulas (in these fragments) satisfiable in arbitrary temporal models properly contains the set of formulas satisfiable in models with finite domains. We show, however, that the decidability results mentioned above hold for the temporal logics (on $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Z}, < \rangle$, $\langle \mathbb{R}, < \rangle$, etc.) with finite domains.

Our results also apply to two-sorted first-order languages in which one sort is specially intended for talking about time. The predicate temporal language, '$\mathcal{TL}$', provides only 'implicit' access to time: quantification over points in time in the sense of first-order logic is not permitted, and the only means of expressing temporal properties is by the operators Since and Until. A common alternative is to reason about time explicitly, using first-order logic. Following this approach in the propositional case yields monadic first-order logic interpreted in strict linear orders, while in the predicate case it leads to a two-sorted first-order language, called '$\mathcal{TS}$' in what follows, one sort of which refers to points in time and the other to the first-order domain. The relation between $\mathcal{TL}$ and $\mathcal{TS}$ has been investigated intensively in the context of temporal databases (see, e.g., [2, 3, 12, 13]), while in the propositional case, both languages are known to have the same expressive power over most classes of flows of time—i.e., the temporal propositional language is *expressively complete*, see [26, 16]. This turns out not to be so in the first-order case: the formula

$$\exists t_1 \exists t_2 (t_1 < t_2 \land \forall x (P(t_1, x) \leftrightarrow P(t_2, x)))$$

is not expressible in $\mathcal{TL}$ over any interesting class of flows of time [2, 3, 12]. However, it remained open in the literature on temporal databases whether there is a natural characterization of the fragment of $\mathcal{TS}$ for which $\mathcal{TL}$ is expressively complete. We will show that a natural such fragment—called $\mathcal{TS}_{1t}$—consists of all formulas in which 'domain' quantifiers $\forall x$ are applied to formulas with at most one free temporal variable (observe that this condition, approximately dual to monodicity, is violated in the formula above). Moreover, the fragment $\mathcal{TL}_1$ of monodic $\mathcal{TL}$-formulas turns out to be expressively complete for the fragment $\mathcal{TS}_1$ of monodic $\mathcal{TS}_{1t}$-formulas. The translation from $\mathcal{TS}_1$ into $\mathcal{TL}_1$ is effective, so all our decidability results for fragments of $\mathcal{TL}_1$ carry over to the corresponding fragments of $\mathcal{TS}_1$.

We will give three different decidability proofs for monodic fragments. They all rely on representing a temporal model satisfying a given monodic formula φ in the form of a 'quasimodel', the most important feature of which is that the size of its domain is finitely bounded (in terms of φ). Our first algorithm expresses the existence of a quasimodel satisfying such a φ by a formula of monadic second-order logic. This fact, together with the Büchi and Rabin decidability theorems, makes it possible to reduce the satisfiability problem for monodic formulas in models based on $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Z}, < \rangle$, $\langle \mathbb{Q}, < \rangle$, and some other linear temporal structures to the satisfiability problem for a certain fragment of classical first-order logic. The complexity of the satisfiability-checking algorithm supplied by such a reduction is non-elementary. To construct an algorithm of better performance (at least for some flows of time) we investigate the structure of quasimodels on $\langle \mathbb{N}, < \rangle$ satisfying a given $\mathcal{TL}_1$-formula φ, and obtain a second, more explicit and elementary satisfiability-checking algorithm for $\langle \mathbb{N}, < \rangle$, provided of course that we have an 'elementary' oracle capable of deciding the satisfiability problem for the classical first-order formulas mentioned above. A modified algorithm checks satisfiability in models with finite domains. Our third algorithm covers the flow of time $\langle \mathbb{R}, < \rangle$ in the finite-domain case, and is an adaptation of the second proof of decidability of propositional temporal logic with Until and Since over $\langle \mathbb{R}, < \rangle$ given in [11].

The paper is organized in the following way. In Section 2 we define the syntax and semantics of the temporal logics under consideration and prove that their monadic two-variable fragments are undecidable. We then introduce the fragment $\mathcal{TL}_1$ of monodic formulas. In Section 3 we introduce quasimodels. In Section 4 we give our first decision procedure for monodic formulas, using monadic second-order logic. In Section 5 we give the second one, and in Section 6 its modified form for finite domains. In section 7 we describe the third algorithm, for $\langle \mathbb{R}, < \rangle$ in the finite-domain case. In Section 8, we prove the expressive completeness of $\mathcal{TL}_1$ for $\mathcal{TS}_1$, and use the obtained criteria to single out a number of decidable fragments of first-order temporal logics, including fragments of $\mathcal{TS}_1$, and the two-variable, monadic, and guarded monodic fragments of $\mathcal{TL}_1$. We show also some applications to temporal epistemic and description (propositional) logics. Finally, in Section 9, we list some open problems.

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2 First-order temporal logic

Denote by $\mathcal{TL}$ the *first-order temporal language* constructed in the standard way from the following alphabet:

- *predicate symbols* $P_0, P_1, \dots$, each of which is of some fixed arity,
- *individual variables* $x_0, x_1, \dots$,
- *individual constants* $c_0, c_1, \dots$,
- the *booleans* $\wedge$, $\neg$,
- the *universal quantifier* $\forall x$ for each individual variable x,
- the *temporal operators* $\mathcal{S}$ (Since) and $\mathcal{U}$ (Until).

The set of predicate symbols in $\mathcal{TL}$ is assumed to be non-empty. 0-ary predicates, i.e., *propositional variables*, are denoted by $p_0, p_1, \dots$. We will assume that there is a sufficient supply of those variables, unary predicate symbols, and an infinite set *var* of individual variables. $\mathcal{L}$ is the classical (non-temporal) first-order language that results from $\mathcal{TL}$ by omitting all formulas containing $\mathcal{S}$ or $\mathcal{U}$.

We will use the following standard abbreviations:

$$\begin{array}{rcl} \exists x\varphi & = & \neg\forall x\neg\varphi; \\ \Diamond\varphi & = & \top\mathcal{U}\varphi; \\ \Box\varphi & = & \neg\Diamond\neg\varphi; \\ \Box^+\varphi & = & \varphi\wedge\Box\varphi; \\ \Diamond^+\varphi & = & \varphi\vee\Diamond\varphi; \\ \bigcirc\varphi & = & \bot\mathcal{U}\varphi. \end{array}$$

$\mathcal{TL}$ is interpreted in *first-order temporal models* of the form $\mathfrak{M} = \langle\mathfrak{F}, D, I\rangle$, where $\mathfrak{F} = \langle W, <\rangle$ is a strict linear order[1] representing the flow of time, D is a non-empty set, the *domain* of $\mathfrak{M}$, and I is a function associating with every moment of time $w \in W$ a first-order $\mathcal{L}$-structure

$$I(w) = \left\langle D, P_0^{I(w)}, \dots, c_0^{I(w)}, \dots \right\rangle,$$

[1] I.e., $<$ is irreflexive, transitive and $\forall x, y \in W(x < y \vee y < x \vee x = y)$.

the *state* of $\mathfrak{M}$ at moment w, in which $P_i^{I(w)}$, for each i, is a predicate on D of the same arity as P_i (for a propositional variable p_i, the predicate $p_i^{I(w)}$ is simply one of the propositional constants $\top$, 'truth', or $\bot$, 'falsehood'), and $c_i^{I(w)}$ is an element of D. We require that $c_i^{I(w)} = c_i^{I(v)}$ for any $w, v \in W$ ('rigid constants'). To simplify notation, we will omit the superscript I and write P_i^w, p_i^w, c_i^w, etc., if I is clear from the context.

An *assignment in* D is a function $\mathfrak{a}$ from *var* to D. The *truth-relation* $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ (or simply $w \models^{\mathfrak{a}} \varphi$, if $\mathfrak{M}$ is understood) in the model $\mathfrak{M}$ under the assignment $\mathfrak{a}$ is defined inductively in the usual way:

- $w \models^{\mathfrak{a}} P_i(y_1, \dots, y_\ell)$ iff $P_i^w(\mathfrak{a}(y_1), \dots, \mathfrak{a}(y_\ell))$ is true in $I(w)$ (we write this also as $I(w) \models^{\mathfrak{a}} P_i(y_1, \dots, y_\ell)$, or $I(w) \models P_i[\mathfrak{a}(y_1), \dots, \mathfrak{a}(y_l)]$, or indeed as $(\mathfrak{a}(y_1), \dots, \mathfrak{a}(y_l)) \in P_i^{I(w)}$);
- $w \models^{\mathfrak{a}} \varphi \wedge \psi$ iff $w \models^{\mathfrak{a}} \varphi$ and $w \models^{\mathfrak{a}} \psi$;
- $w \models^{\mathfrak{a}} \neg\psi$ iff $w \not\models^{\mathfrak{a}} \psi$;
- $w \models^{\mathfrak{a}} \forall x \psi$ iff $w \models^{\mathfrak{b}} \psi$ for every assignment $\mathfrak{b}$ in D that may differ from $\mathfrak{a}$ only on x;
- $w \models^{\mathfrak{a}} \varphi \mathcal{S} \psi$ iff there is $v < w$ such that $v \models^{\mathfrak{a}} \psi$ and $u \models^{\mathfrak{a}} \varphi$ for every u in the interval $(v, w) = \{u \in W : v < u < w\}$;
- $w \models^{\mathfrak{a}} \varphi \mathcal{U} \psi$ iff there is $v > w$ such that $v \models^{\mathfrak{a}} \psi$ and $u \models^{\mathfrak{a}} \varphi$ for every $u \in (w, v)$.

It follows, in particular, that

- $w \models^{\mathfrak{a}} \Diamond\varphi$ iff there is $v > w$ such that $v \models^{\mathfrak{a}} \varphi$;
- $w \models^{\mathfrak{a}} \Diamond^+\varphi$ iff there is $v \geq w$ such that $v \models^{\mathfrak{a}} \varphi$;
- $w \models^{\mathfrak{a}} \Box\varphi$ iff $v \models^{\mathfrak{a}} \varphi$ for all $v > w$;
- $w \models^{\mathfrak{a}} \Box^+\varphi$ iff $v \models^{\mathfrak{a}} \varphi$ for all $v \geq w$;
- $w \models^{\mathfrak{a}} \bigcirc\varphi$ iff there exists an immediate successor v of w (i.e., $v > w$ and $(w, v) = \emptyset$) such that $v \models^{\mathfrak{a}} \varphi$.

For a class $\mathcal{F}$ of strict linear orders, we let $TL(\mathcal{F})$, 'the *temporal logic of* $\mathcal{F}$', denote the set of $\mathcal{TL}$-formulas that are valid in $\mathcal{F}$:

$$\begin{aligned} TL(\mathcal{F}) \quad = \quad \{\varphi \in \mathcal{TL} \ : \ & (\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi \text{ for all } \mathfrak{M} = \langle \mathfrak{F}, D, I \rangle \text{ with } \mathfrak{F} \in \mathcal{F}, \\ & \text{all } w \in D, \text{ and all assignments } \mathfrak{a} \text{ in } D\}. \end{aligned}$$

$TL_{fin}(\mathcal{F})$ stands for the set of those $\mathcal{TL}$-formulas that are valid in all models based on linear orders in $\mathcal{F}$ and having *finite domains*. Instead of $TL(\{\langle \mathbb{N}, < \rangle\})$, $TL_{fin}(\{\langle \mathbb{N}, < \rangle\})$ we write $TL(\mathbb{N})$ and $TL_{fin}(\mathbb{N})$, respectively; similar notation is used for $\langle \mathbb{Z}, < \rangle$, $\langle \mathbb{Q}, < \rangle$, and $\langle \mathbb{R}, < \rangle$.

Remark 1. In this paper we consider only models with *constant domains*. Satisfiability in models with expanding domains is known to be reducible to satisfiability in models with constant domains (see [47]).

2.1 Undecidable fragments of $\mathcal{TL}$

The following two theorems indicate some limits outside which one cannot hope to find decidable fragments of first-order temporal logics.

For $\ell < \omega$, let $\mathcal{TL}^\ell$ be the ℓ-variable fragment of $\mathcal{TL}$ (i.e., every formula in $\mathcal{TL}^\ell$ contains at most ℓ distinct individual variables). And by $\mathcal{TL}^{mo}$ we denote the *monadic* fragment of $\mathcal{TL}$ (i.e., the set of formulas which contain only unary predicates and propositional variables).

Theorem 2. *Let $\mathcal{F}$ be either $\{\langle \mathbb{N}, < \rangle\}$ or $\{\langle \mathbb{Z}, < \rangle\}$. Then the set $\mathcal{TL}^2 \cap \mathcal{TL}^{mo} \cap TL(\mathcal{F})$ is not recursively enumerable.*

Proof We show this by reducing the recurrent tiling problem for $\mathbb{N}\times\mathbb{N}$ (which is Σ_1^1-complete; see [24]) to the satisfiability problem for the monadic $\mathcal{TL}^2$-formulas in $\mathcal{F}$. Recall that a tile $\mathfrak{t}$ is a 1×1 square with fixed orientation and coloured edges $\mathit{right}(\mathfrak{t})$, $\mathit{left}(\mathfrak{t})$, $\mathit{up}(\mathfrak{t})$, and $\mathit{down}(\mathfrak{t})$. The $\mathbb{N}\times\mathbb{N}$ recurrent tiling problem is formulated as follows: given a finite set $\mathfrak{T}$ of tiles and a tile $\mathfrak{t}_0 \in \mathfrak{T}$, determine whether there is a tiling of $\mathbb{N}\times\mathbb{N}$ by $\mathfrak{T}$ such that $\mathfrak{t}_0$ occurs infinitely often in the first row. More precisely, the problem is to find out whether there exists a function f from $\mathbb{N}\times\mathbb{N}$ into $\mathfrak{T}$ such that, for all $m, n \in \mathbb{N}$,

- $\mathit{right}(f(n,m)) = \mathit{left}(f(n+1,m))$,
- $\mathit{up}(f(n,m)) = \mathit{down}(f(n,m+1))$,
- the set $\{n \in \mathbb{N} : f(n,0) = \mathfrak{t}_0\}$ is infinite.

With a given a set $\mathfrak{T} = \{\mathfrak{t}_0, \dots, \mathfrak{t}_n\}$ of tiles we associate unary predicates $P_0, \dots, P_n$. We also require two unary predicates, Q_1 and Q_2, which will be used in the formula

$$R(x,y) = \Diamond(Q_1(x) \wedge Q_2(y)).$$

Now define a first-order temporal formula $\varphi_{\mathfrak{T}}$ in $\mathcal{TL}^2 \cap \mathcal{TL}^{mo}$ as the conjunction of the following formulas:

$$\begin{array}{l}
\exists x \Box\Diamond(P_0(x) \wedge \Diamond\top), \\
\forall x \exists y R(x,y), \\
\forall x,y((R(x,y) \to \Box R(x,y)) \wedge (\neg R(x,y) \to \Box\neg R(x,y))), \\
\Box^+ \forall x (\bigvee\limits_{i=0}^{n} P_i(x) \wedge \bigwedge\limits_{i \neq j} (P_i(x) \to \neg P_j(x))), \\
\Box^+ \forall x,y (P_i(x) \wedge R(x,y) \to \bigvee\limits_{\mathit{up}(\mathfrak{t}_i) = \mathit{down}(\mathfrak{t}_j)} P_j(y)), \\
\Box^+ \forall x (P_i(x) \to \bigcirc \bigvee\limits_{\mathit{right}(\mathfrak{t}_i) = \mathit{left}(\mathfrak{t}_j)} P_j(x)).
\end{array}$$

Let us show that $\varphi_{\mathfrak{T}}$ is satisfiable in a model based on the frame in $\mathcal{F}$ iff there is a recurrent tiling of $\mathbb{N}\times\mathbb{N}$ by $\mathfrak{T}$.

Suppose first that $f : \mathbb{N}\times\mathbb{N} \to \mathfrak{T}$ defines a recurrent tiling. Put $D = \mathbb{N}$,

$$P_i^{I(n)} = \{m \in D : f(n,m) = \mathfrak{t}_i\},$$

for $n \in \mathbb{N}$, and select for every $i \in \mathbb{N}$ an infinite set $M_i \subseteq \mathbb{N}$ such that $M_i \cap M_{i'} = \emptyset$ whenever $i \neq i'$. Now put, for $i \in D$ and $n \in \mathbb{N}$, $i \in Q_1^{I(n)}$ and $i+1 \in Q_2^{I(n)}$ iff $n \in M_i$. Also specify that $0 \notin Q_2^{I(n)}$. It should be clear that $\varphi_{\mathfrak{T}}$ is satisfied in $\langle \langle \mathbb{N}, < \rangle, D, I \rangle$. It follows that $\varphi_{\mathfrak{T}}$ is satisfiable in $\mathcal{F}$.

Conversely, suppose $\varphi_{\mathfrak{T}}$ is satisfied in a model $\mathfrak{M} = \langle \mathfrak{F}, D, I \rangle$, for $\mathfrak{F} \in \mathcal{F}$. Then $\mathfrak{F} = \langle W, < \rangle$ contains an infinite ascending chain, say $0, 1, 2, \dots$ such that $0 \models \varphi_{\mathfrak{T}}$ and $i+1$ is the immediate successor of i. By the first conjunct of $\varphi_{\mathfrak{T}}$, we find an $a_0 \in D$ for which $0 \models P_0[a_0]$ and the set $\{n \in \mathbb{N} : n \models P_0[a_0]\}$ is infinite. Let $R^{I(n)} = \{\langle a, b \rangle \in D^2 : n \models \Diamond(Q_1 \wedge Q_2)[a, b]\}$. According to the second conjunct, we have an R-ascending chain $a_0 R^{I(0)} a_1 R^{I(0)} a_2 \dots$ of elements in D. By the third conjunct, for all $n, i, j \in \mathbb{N}$, we have $a_i R^{I(n)} a_j$ iff $a_i R^{I(0)} a_j$. Now define a function f by putting, for all $i, j \in \mathbb{N}$, $f(i, j) = \mathfrak{t}_k$ whenever $i \models P_k[a_j]$. It is straightforward to check that f is a recurrent tiling of $\mathbb{N} \times \mathbb{N}$. ❑

It follows, in particular, that if $\mathcal{F}$ is any one of the classes mentioned in the formulation of Theorem 2 then $TL(\mathcal{F})$ is not recursively axiomatizable (cf. [16]).

Theorem 3. *Let $\mathcal{F}$ be one of the following classes of temporal frames: $\{\langle \mathbb{N}, < \rangle\}$, $\{\langle \mathbb{Z}, < \rangle\}$, the class of all strict linear orders. Then $\mathcal{TL}^2 \cap \mathcal{TL}^{mo} \cap TL_{fin}(\mathcal{F})$ is not recursively enumerable.*

Proof We are going to reduce the following undecidable problem to the satisfaction problem for the monadic $\mathcal{TL}^2$-formulas in models with finite domains: given a Turing machine, determine whether it comes to a stop having started from the empty tape. Let $\mathfrak{A}$ be a single-tape right-infinite deterministic Turing machine with state space S, initial state s_0, halt state s_1, tape alphabet A ($b \in A$ stands for blank) and transition function δ. The configurations of $\mathfrak{A}$ will be represented by infinite words of the form $\pounds a_0 \dots a_i \dots a_n b^\omega$, where $\pounds$ marks the left side of the tape, all $a_0, \dots, a_n$ save one, say a_i, are in A, while a_i belongs to $S \times A$ and represents the active cell and the current state. The start configuration, for instance, is represented by $\pounds(s_0, b)b^\omega$. Let $A' = A \cup \{\pounds\} \cup (S \times A)$, and $A'' = A' \setminus \{\pounds\}$.

We want to construct a monadic $\mathcal{TL}^2$-formula $\varphi_{\mathfrak{A}}$ which is satisfiable in a model with a finite domain D (based on a frame in $\mathcal{F}$) iff $\mathfrak{A}$ comes to a stop (i.e., reaches the halt state) having started from $\pounds(s_0, b)b^\omega$. Roughly, the idea is to codify configurations of $\mathfrak{A}$ by elements $x \in D$ using the behaviour of x over time.

First, with every $\alpha \in A'$ we associate a unary predicate P_α. The sentence

$$\forall x\Big(P_\pounds(x) \wedge \Box \bigvee_{\alpha \in A''} (P_\alpha(x) \wedge \neg \bigvee_{\alpha \neq \beta \in A''} P_\beta(x))\Big) \tag{1}$$

means that 'now', all objects in D are in $P_\pounds$ while later each of them belongs to precisely one of the sets P_α, for $\alpha \in A''$. To mark the object representing the active cell of a given configuration and its immediate predecessor and successor, we use three unary predicates, S, L, and R, defined by the formulas:

$$\Box^+ \forall x (S(x) \leftrightarrow \bigvee_{(s,a) \in A'} P_{(s,a)}(x)), \tag{2}$$

$$\Box^+ \forall x \big((L(x) \leftrightarrow \bigcirc S(x)) \wedge (S(x) \leftrightarrow \bigcirc R(x))\big), \tag{3}$$

$$\Box^+ \forall x \neg (S(x) \wedge \Diamond S(x)). \tag{4}$$

The transition from one configuration to another is simulated by means of the formula:

$$\chi(x,y) = \bigwedge_{\delta(\alpha,\beta,\gamma)=(\alpha',\beta',\gamma')} \Diamond^+\Big(L(x) \wedge P_\alpha(x) \wedge \bigcirc(P_\beta(x) \wedge \bigcirc P_\gamma(x))\Big) \to \\ \Box^+\Big((L(x) \to P_{\alpha'}(y)) \wedge (S(x) \to P_{\beta'}(y)) \wedge (R(x) \to P_{\gamma'}(y)) \\ \wedge \bigwedge_{\alpha \in A'} (\neg L(x) \wedge \neg S(x) \wedge \neg R(x) \wedge P_\alpha(x) \to P_\alpha(y))\Big).$$

The following two formulas define a unary predicate C (clock); its intended meaning is to fix the moment of time the machine reaches this or that configuration.

$$\forall x(\Diamond^+ C(x) \wedge \Box^+\neg(C(x) \wedge \Diamond C(x))), \tag{5}$$

$$\forall x, y(\chi(x,y) \to \Diamond^+(C(x) \wedge \bigcirc C(y))). \tag{6}$$

It remains only to ensure that there exists a sequence representing a halt configuration:

$$\exists x \Diamond^+ \bigvee_{(s_1,a)\in A'} P_{(s_1,a)}(x), \tag{7}$$

and that each configuration save the start one on the empty tape has a predecessor:

$$\forall y\Big(\neg(P_{\pounds}(y) \wedge \bigcirc(P_{(s_0,b)}(y) \wedge \Box P_b(y)) \to \exists x\, \chi(x,y)\Big). \tag{8}$$

Let $\varphi_{\mathfrak{A}}$ be the conjunction of (1)–(8) and the formula $\Box^+ \bigcirc \top$, which ensures that every moment of time (starting from the one satisfying this formula) has an immediate successor. It is not hard to check that $\varphi_{\mathfrak{A}}$ is satisfied in a model with a finite domain (based on a frame in $\mathcal{F}$) iff $\mathfrak{A}$ comes to a stop having started from the empty tape. Indeed, the '$\Leftarrow$'-part of the proof should be clear. For the converse, suppose that $\varphi_{\mathfrak{A}}$ is satisfied in a world w of a model based on some linear order and having a finite domain, D. By (7), (1), and (2)–(4), there is $h \in D$ representing a halt configuration. Observe that, by (5) and (6), we cannot have objects $c_0, \dots, c_n \in D$ such that $c_0 = c_n$ and $w \models \chi[c_0, c_1] \wedge \dots \wedge \chi[c_{n-1}, c_n]$. Let $c_0, \dots, c_n$ be a maximal chain in D for which $c_n = h$ and $w \models \chi[c_i, c_{i+1}]$, $0 \le i < n$. Such a chain exists since D is finite. So there is no $c \in D$ with $w \models \chi[c, c_0]$. In view of (8), this can only mean that c_0 represents the start configuration on the empty tape. Thus, by definition of χ, $\mathfrak{A}$ reaches a halt configuration having started from the empty tape.

Thus, the set $\mathcal{TL}^2 \cap \mathcal{TL}^{mo} \cap TL_{fin}(\mathcal{F})$ is undecidable. On the other hand, its complement (in the set of monadic $\mathcal{TL}$-formulas) is recursively enumerable. For, it is not hard to see that satisfiability of monadic and indeed arbitrary $\mathcal{ML}$-formulas in models based in $\mathcal{F}$ and having domains of $\le n$ elements, for fixed n, can be reduced to satisfiability of propositional temporal formulas in $\mathcal{F}$, which is known to be decidable (see e.g. [16]). ❑

2.2 Monodic formulas

Note that both undecidability proofs above use temporal formulas of the form $\varphi\mathcal{U}\psi$ with *two free variables*. We now consider the 'monodic' fragment of $\mathcal{TL}$ without formulas of that sort.

Definition 4 (monodic formulas). Denote by $\mathcal{TL}_1$ the set of all $\mathcal{TL}$-formulas φ such that any subformula of φ of the form $\psi_1\mathcal{U}\psi_2$ or $\psi_1\mathcal{S}\psi_2$ has at most one free variable. Such formulas will be called *monodic.* In other words, monodic formulas allow quantification into temporal contexts only with one free variable. From now on we will be assuming that *all our formulas are monodic.*

For a set Γ of $\mathcal{TL}$-formulas, denote by $sub_n\Gamma$ the closure under negation of the set of all subformulas of formulas in Γ containing $\leq n$ free variables; $sub\varphi$ denotes the set of all subformulas in a formula φ, and $con\varphi$ the set of all constants in φ. Without loss of generality, we may identify ψ and $\neg\neg\psi$; so $sub_n\Gamma$ is finite whenever Γ is finite. In what follows we will not be distinguishing between a finite set Γ of formulas and the conjunction $\bigwedge\Gamma$ of formulas in it.

For every formula $\psi(x) = \varphi_1\mathcal{U}\varphi_2$ or $\psi(x) = \varphi_1\mathcal{S}\varphi_2$ with one free variable x, we reserve a unary predicate $P_\psi(x)$, and for every sentence $\psi = \varphi_1\mathcal{U}\varphi_2$ or $\psi = \varphi_1\mathcal{S}\varphi_2$ we fix a propositional variable p_ψ. $P_\psi(x)$ and p_ψ are called the *surrogates* of $\psi(x)$ and ψ, respectively.

Given a formula φ, we denote by $\overline{\varphi}$ the formula that results from φ by replacing all its subformulas of the form $\psi_1\mathcal{U}\psi_2$ and $\psi_1\mathcal{S}\psi_2$ which are not within the scope of another occurrence of $\mathcal{U}$ or $\mathcal{S}$ by their surrogates. Thus, $\overline{\varphi}$ contains no occurrences of temporal operators at all—i.e., it is an $\mathcal{L}$-formula; we will call $\overline{\varphi}$ the *$\mathcal{L}$-reduct* of φ. For a set Γ of $\mathcal{TL}_1$-formulas, we let $\overline{\Gamma} = \{\overline{\psi} : \psi \in \Gamma\}$.

3 Codifying models

Imagine that we need to find out whether a $\mathcal{TL}_1$-sentence φ is satisfiable. Following the motto 'divide and conquer', we separate the temporal and the pure first-order parts of $\mathcal{TL}_1$, focusing attention mainly on the former and pretending that we have a friend who knows how to deal with the latter. We assume that this friend can obtain for us an $\mathcal{L}$-structure realizing any given set of subsets of $\overline{sub\varphi}$, if such a structure exists at all. In this way, we build up a complete stock of such structures; one of them should satisfy the $\mathcal{L}$-reduct of φ. Our task is then to try to fit these structures together into a temporal model satisfying φ. When doing this, we need only take care of formulas of the form $\psi_1\mathcal{U}\psi_2$ and $\psi_1\mathcal{S}\psi_2$ in $sub\varphi$, relying upon our good friendship as far as other formulas are concerned.

The aim of this section is to show that modulo φ, every temporal model can be codified in a structure called a *quasimodel.* A quasimodel may be viewed as a model in which the states have pairwise disjoint domains, each domain has a bounded number of elements (depending on φ), and each domain element satisfies some specified set of subformulas of φ. The correspondence between elements in different states will be established by special functions called *runs.*

Let x be a variable not occurring in φ. Put

$$sub_x\varphi = \{\psi\{x/y\} : \psi(y) \in sub_1\varphi\}.$$

Definition 5 (type). By a *type* for φ we mean any boolean-saturated subset t of $sub_x\varphi$: that is,

- $\psi \wedge \chi \in t$ iff $\psi \in t$ and $\chi \in t$, for every $\psi \wedge \chi \in sub_x\varphi$;
- $\neg\psi \in t$ iff $\psi \notin t$, for every $\psi \in sub_x\varphi$.

We say that two types t and t' *agree on $sub_0\varphi$* if $t \cap sub_0\varphi = t' \cap sub_0\varphi$. Given a type t for φ and a constant $c \in con\varphi$, the pair $\langle t, c\rangle$ will be called an *indexed type* for φ (indexed by c) and denoted by $t_c(x)$ or simply t_c.

There are only finitely many types for φ—at most

$$\flat(\varphi) = 2^{|sub_x\varphi|},$$

to be more precise. To a certain extent, every state w in a model under a given assignment can be characterized (modulo φ, of course) by the set of types that are realized in this state and the set of types that hold on its constants. This motivates the following definition.

Definition 6 (state candidate). Suppose that T is a set of types for φ that agree on $sub_0\varphi$, and $T^{con} = \{\langle t, c\rangle : c \in con\varphi\}$ a set of indexed types such that $\{t : \langle t, c\rangle \in T^{con}\} \subseteq T$. Then the pair $\mathfrak{C} = \langle T, T^{con}\rangle$ is called a *state candidate* for φ.

Not all state candidates can represent states in temporal models. To single out those that can, we require one more definition.

Definition 7 (realizable state candidate). Consider a first-order $\mathcal{L}$-structure

$$\mathfrak{D} = \left\langle D, P_0^{\mathfrak{D}}, \ldots, c_0^{\mathfrak{D}}, \ldots \right\rangle \tag{9}$$

and suppose that $a \in D$. The set

$$t^{\mathfrak{D}}(a) = \{\psi \in sub_x\varphi : \mathfrak{D} \models \overline{\psi}[a]\}$$

is clearly a type for φ. Say that $\mathfrak{D}$ *realizes* a state candidate $\langle T, T^{con}\rangle$ if the following conditions hold:

- $T = \{t^{\mathfrak{D}}(a) : a \in D\}$,
- $T^{con} = \{\langle t^{\mathfrak{D}}(c^{\mathfrak{D}}), c\rangle : c \in con\varphi\}$.

A state candidate is said to be *finitely realizable* if there exists a finite $\mathcal{L}$-structure realizing it.

Denote by $\sharp(\varphi)$ the number of distinct realizable state candidates for φ. It should be clear that

$$\sharp(\varphi) \leq 2^{\flat(\varphi)} \cdot \flat(\varphi)^{|con\varphi|}.$$

Lemma 8. *A state candidate $\mathfrak{C} = \langle T, T^{con}\rangle$ for φ is (finitely) realizable iff the $\mathcal{L}$-formula*

$$\alpha_{\mathfrak{C}} \quad = \quad \bigwedge_{t\in T} \exists x\, \overline{t}(x) \quad \wedge \quad \forall x \bigvee_{t\in T} \overline{t}(x) \quad \wedge \quad \bigwedge_{\langle t,c\rangle\in T^{con}} \overline{t}(c)$$

is satisfied in some (respectively, finite) $\mathcal{L}$-structure.

Proof Follows immediately from the definitions. ❑

Lemma 9. *Let κ be a cardinal, $\kappa \geq \aleph_0$. Then every realizable state candidate $\langle T, T^{con}\rangle$ is realized in an $\mathcal{L}$-structure $\mathfrak{D}$ of the form (9) such that, for every $t \in T$, the set*

$$\mathfrak{D}_t = \{a \in D : \mathfrak{D} \models \overline{t}[a]\}$$

is of cardinality κ.

Proof Follows from classical model theory, since the language $\mathcal{L}$ is countable and does not contain equality. ❑

We are now in a position to define the central notion of this section, that of a quasimodel. Let $\mathfrak{F} = \langle W, < \rangle$ be a linear order.

Definition 10 (state function). A *state function* for φ over $\mathfrak{F}$ is a map f associating with each $w \in W$ a realizable state candidate $f(w) = \langle T_w, T_w^{con} \rangle$ for φ.

Definition 11 (run). Let f be a state function for φ over $\mathfrak{F} = \langle W, < \rangle$, with $f(w) = \langle T_w, T_w^{con} \rangle$ for $w \in W$. By a *run* in f we mean a function r from W into the set $\bigcup_{w \in W} T_w$ such that

- $r(w) \in T_w$, for all $w \in W$,
- for every $\psi_1 \mathcal{U} \psi_2 \in sub_x\varphi$ and every $w \in W$, we have $\psi_1 \mathcal{U} \psi_2 \in r(w)$ iff there is $v > w$ such that $\psi_2 \in r(v)$ and $\psi_1 \in r(u)$ for all $u \in (w, v)$,
- for every $\psi_1 \mathcal{S} \psi_2 \in sub_x\varphi$ and every $w \in W$, we have $\psi_1 \mathcal{S} \psi_2 \in r(w)$ iff there is $v < w$ such that $\psi_2 \in r(v)$ and $\psi_1 \in r(u)$ for all $u \in (v, w)$.

Definition 12 (quasimodel). Suppose f is a state function for φ over $\mathfrak{F}$ and $\mathcal{R}$ a set of runs in f. The pair $\mathfrak{m} = \langle f, \mathcal{R} \rangle$ is called a *quasimodel* for φ (over $\mathfrak{F}$) if the following conditions hold:

- for every $c \in con\varphi$, the function r_c defined by $r_c(w) = t$, for $\langle t, c \rangle \in T_w^{con}$, $w \in W$, is a run in $\mathcal{R}$,
- for every $w \in W$ and every $t \in T_w$, there exists a run $r \in \mathcal{R}$ such that $r(w) = t$.

In this case the state candidates $f(w)$ are called *quasistates* of $\mathfrak{m}$. Say that φ is *satisfied* in the quasimodel $\mathfrak{m}$ if there is $w \in W$ such that $\varphi \in t$, for some (or, equivalently, all) $t \in T_w$.

Remark 13. Note that, for any two sets of runs $\mathcal{R}_1$ and $\mathcal{R}_2$ in f, if $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and $\langle f, \mathcal{R}_1 \rangle$ is a quasimodel for φ then $\langle f, \mathcal{R}_2 \rangle$ is a quasimodel for φ as well. Consequently, there exists a quasimodel for φ based on a state function f iff $\langle f, \Omega_f \rangle$ is a quasimodel for φ, where Ω_f is the set of all runs in f. If we are interested in satisfiability of temporal formulas in arbitrary models then it is enough to consider quasimodels of the form $\langle f, \Omega_f \rangle$; to simplify notation, we will denote such quasimodels by f. To deal with satisfiability in models with *finite* domains, we shall need quasimodels $\langle f, \mathcal{R} \rangle$ with *finite* $\mathcal{R}$.

Theorem 14. *A $\mathcal{TL}_1$-sentence φ is satisfiable in a model based on $\mathfrak{F} = \langle W, < \rangle$ iff it is satisfied in a quasimodel for φ over $\mathfrak{F}$.*

Proof Suppose φ is satisfied in a model $\mathfrak{M} = \langle \mathfrak{F}, D, I \rangle$. For every $w \in W$, define $f(w) = \langle T_w, T_w^{con} \rangle$ by taking

$$
\begin{array}{lcl}
t_a & = & \{\psi \in sub_x\varphi : (\mathfrak{M}, w) \models^{\mathfrak{a}} \psi\}, \text{ where } a \in D \text{ and } \mathfrak{a}(x) = a, \\
T_w & = & \{t_a : a \in D\}, \\
T_w^{con} & = & \{\langle t_{c^{I(w)}}, c \rangle : c \in con\varphi\}.
\end{array}
$$

It is easy to see that for every $a \in D$, the function $r(w) = t^{I(w)}(a)$, $w \in W$, is a run in f. Let $\mathcal{R}$ be the set of all such runs. Then $\langle f, \mathcal{R}\rangle$ is clearly a quasimodel satisfying φ. Note that $\mathcal{R}$ is finite whenever D is finite.

Conversely, suppose that φ is satisfied in a quasimodel f for φ over $\mathfrak{F}$. Take a cardinal $\kappa \geq \aleph_0$ exceeding the cardinality of the set Ω_f of all runs in f and put

$$D = \{\langle r, \xi\rangle : r \in \Omega_f, \xi < \kappa\}.$$

Then for any $w \in W$ and any type t,

$$|\{\langle r, \xi\rangle \in D : r(w) = t\}| \quad = \quad \begin{cases} \kappa, & \text{if } t \in T_w, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 9, for every $w \in W$ there exists an $\mathcal{L}$-structure $I(w)$ with domain D realizing $f(w)$ and such that $c^w = \langle r_c, 0\rangle$, for every $c \in con\varphi$, and

$$r(w) = \{\psi \in sub_x\varphi : I(w) \models \overline{\psi}[\langle r, \xi\rangle]\}, \tag{10}$$

for all $r \in \Omega_f$ and $\xi < \kappa$. Let $\mathfrak{M} = \langle \mathfrak{F}, D, I\rangle$. We show by induction on ψ that for all $\psi \in sub\varphi$ and $w \in W$, and any assignment $\mathfrak{a}$ in D,

$$I(w) \models^{\mathfrak{a}} \overline{\psi} \text{ iff } (\mathfrak{M}, w) \models^{\mathfrak{a}} \psi.$$

The basis of induction—i.e., the case when $\psi = P_i(x_1, \dots, x_\ell)$—is clear; for then, $\psi = \overline{\psi}$. The induction step for $\psi = \psi_1 \wedge \psi_2$, $\psi = \neg\psi_1$, and $\psi = \forall x\psi_1$ follows by the induction hypothesis from the obvious equations:

$$\overline{\psi_1 \wedge \psi_2} = \overline{\psi_1} \wedge \overline{\psi_2},\ \overline{\neg\psi_1} = \neg\overline{\psi_1},\ \overline{\forall x\psi_1} = \forall x\overline{\psi_1}.$$

Let $\psi(y) = \chi_1 \mathcal{U} \chi_2$ and $\mathfrak{a}(y) = \langle r, \xi\rangle$. We then have $\overline{\psi} = P_\psi(y)$, so by (10) and the induction hypothesis,

$$\begin{array}{lll} I(w) \models^{\mathfrak{a}} P_\psi(y) & \text{iff} & \chi_1\mathcal{U}\chi_2 \in r(w) \\ & \text{iff} & \exists v > w(\chi_2 \in r(v)\ \&\ \forall u \in (w,v)\chi_1 \in r(u)) \\ & \text{iff} & \exists v > w(I(v) \models^{\mathfrak{a}} \overline{\chi_2}\ \&\ \forall u \in (w,v) I(u) \models^{\mathfrak{a}} \overline{\chi_1}) \\ & \text{iff} & \exists v > w((\mathfrak{M}, v) \models^{\mathfrak{a}} \chi_2\ \&\ \forall u \in (w,v)(\mathfrak{M}, u) \models^{\mathfrak{a}} \chi_1) \\ & \text{iff} & (\mathfrak{M}, w) \models^{\mathfrak{a}} \chi_1\mathcal{U}\chi_2. \end{array}$$

The formula $\psi(y) = \chi_1 \mathcal{S} \chi_2$ is considered analogously.

Since $\varphi \in r(w)$ for some $w \in W$, we must have also $(\mathfrak{M}, w) \models \varphi$, as required. ❑

4 Embedding into second-order monadic theories

We can now quickly deduce decidability results by translating into monadic second-order logic the statement that a quasimodel satisfying φ exists.

We will use some auxiliary formulas. Introduce a unary predicate variable R_ψ for each $\psi \in sub_x\varphi$. If t is any type for φ, let

$$\chi_t(x) \quad = \quad \bigwedge_{\psi \in t} R_\psi(x) \ \ \wedge \bigwedge_{\psi \in (sub_x\varphi)\setminus t} \neg R_\psi(x),$$

saying that the $R_\psi(x)$ define the type t at x. Also, ρ denotes the conjunction of the two formulas

$$\forall x \bigwedge_{\psi_1 \mathcal{U} \psi_2 \in sub_x \varphi} \big(R_{\psi_1 \mathcal{U} \psi_2}(x) \leftrightarrow \exists y (x < y \wedge R_{\psi_2}(y) \wedge \forall z (x < z < y \rightarrow R_{\psi_1}(z)))\big),$$
$$\forall x \bigwedge_{\psi_1 \mathcal{S} \psi_2 \in sub_x \varphi} \big(R_{\psi_1 \mathcal{S} \psi_2}(x) \leftrightarrow \exists y (y < x \wedge R_{\psi_2}(y) \wedge \forall z (y < z < x \rightarrow R_{\psi_1}(z)))\big)$$

—this says that the $R_\psi(x)$ define a run.

Let Σ be the set of all realizable state candidates for φ, and P_s ($s \in \Sigma$) a unary predicate variable. We now define the monadic second-order sentence σ_φ as follows:

$$\begin{aligned} \mathop{\Large\exists}_{s \in \Sigma} P_s \Bigg(& \forall x \Big[\bigvee_{s \in \Sigma} P_s(x) \wedge \bigwedge_{\substack{s,s' \in \Sigma \\ s \neq s'}} \neg (P_s(x) \wedge P_{s'}(x)) \Big] \;\wedge \bigvee_{\substack{\langle T, T^{con} \rangle \in \Sigma \\ \varphi \in \bigcup T}} \exists x \, P_{\langle T, T^{con} \rangle}(x) \\ & \wedge \bigwedge_{c \in con \varphi} \mathop{\Large\exists}_{\psi \in sub_x \varphi} R_\psi \Big[\rho \wedge \forall x \bigwedge_{\substack{\langle T, T^{con} \rangle \in \Sigma \\ \langle t, c \rangle \in T^{con}}} (P_{\langle T, T^{con} \rangle}(x) \rightarrow \chi_t(x)) \Big] \\ & \wedge \; \forall x \bigwedge_{\substack{\langle T, T^{con} \rangle \in \Sigma \\ t \in T}} \Big[P_{\langle T, T^{con} \rangle}(x) \rightarrow \mathop{\Large\exists}_{\psi \in sub_x \varphi} R_\psi (\rho \wedge \chi_t(x)) \Big] \Bigg). \end{aligned}$$

If $\mathfrak{F} = \langle W, < \rangle$ is a linear order, then $\mathfrak{F} \models \sigma_\varphi$ iff there exist (possibly empty) subsets $P_s \subseteq W$ ($s \in \Sigma$) which partition W in such a way that the state function $f : W \rightarrow \Sigma$ defined by $w \in P_{f(w)}$, for all $w \in W$, is a quasimodel for φ in the sense of Remark 13: the second line states that each constant of $con\varphi$ defines a run coded by the R_ψ, and the third line expresses the second condition of Definition 12. The last conjunct on the first line says that φ is satisfied in this quasimodel. Hence, $\mathfrak{F} \models \sigma_\varphi$ iff φ is satisfied in a quasimodel for φ over $\mathfrak{F}$.

Note that if Σ can be constructed from φ by an algorithm, then so can σ_φ.

We can now apply known facts on decidability of monadic second-order logic to obtain decidability results for monodic fragments.

Theorem 15. *Let $\mathcal{TL}' \subseteq \mathcal{TL}_1$ and suppose that there is an algorithm which is capable of deciding, for any $\mathcal{TL}'$-sentence φ, whether an arbitrarily-given state candidate for φ is realizable. Let $\mathcal{F}$ be one of the following classes of flows of time:*

1. $\{\langle \mathbb{N}, < \rangle\}$,
2. $\{\langle \mathbb{Z}, < \rangle\}$,
3. $\{\langle \mathbb{Q}, < \rangle\}$,
4. *the class of all finite strict linear orders,*
5. *any first-order-definable class of strict linear orders—for example, the class of all linear orders.*

Then the satisfiability problem for the $\mathcal{TL}'$-sentences in $\mathcal{F}$, and so the decision problem for the fragment $TL(\mathcal{F}) \cap \mathcal{TL}'$, are decidable.

Proof By assumption, the sentence σ_φ is constructible effectively from φ.

1. By Theorem 14, φ is satisfiable in a model based on $\langle \mathbb{N}, < \rangle$ iff φ is satisfied in a quasimodel for φ over $\langle \mathbb{N}, < \rangle$, iff (by the foregoing) $\langle \mathbb{N}, < \rangle \models \sigma_\varphi$. This last statement is decidable, by a result of Büchi [10].

2. The case of $\langle \mathbb{Z}, < \rangle$ is similar.

3. The case of $\langle \mathbb{Q}, < \rangle$ is again similar, except that the decidability of the problem '$\langle \mathbb{Q}, < \rangle \models \sigma_\varphi$' follows from Rabin's theorem on the decidability of $S2S$ [35].

4. As before, we see that φ is satisfiable in a model based on a finite linear order iff σ_φ is true in some finite linear order. As is well-known, it follows from Büchi's theorem [10] that this last statement is decidable.

5. By considering the standard translation of φ into two-sorted first-order logic (see Section 8.1) and applying the downward Löwenheim–Skolem–Tarski theorem, it can be seen that φ has a model with flow of time in $\mathcal{F}$ iff it has a model with countable flow of time in $\mathcal{F}$. By Theorem 14, this holds iff φ is satisfied in a quasimodel for φ over a countable order in $\mathcal{F}$.

   Let ψ be a formula of monadic second-order logic, and let P be a monadic predicate variable not occurring in ψ. Define the relativization ψ^P of ψ to P, by $\psi^P = \psi$ for atomic ψ, $(\neg\psi)^P = \neg\psi^P$, $(\psi_1 \wedge \psi_2)^P = \psi_1^P \wedge \psi_2^P$, $(\forall x \psi)^P = \forall x (P(x) \rightarrow \psi^P)$, and $(\forall Q \psi)^P = \forall Q \psi^P$. Evidently, for any sentence ψ and any linear order $\mathfrak{F}$, we have $\mathfrak{F} \models \exists P(\exists x P(x) \wedge \psi^P)$ iff $\mathfrak{F}' \models \psi$ for some (non-empty) suborder $\mathfrak{F}'$ of $\mathfrak{F}$—the intended interpretation of P is the domain of $\mathfrak{F}'$.

   Now any countable strict linear order is a sub-order of $\langle \mathbb{Q}, < \rangle$. Let λ be a sentence of linear order defining $\mathcal{F}$. Then σ_φ (assumed not to involve P) is satisfiable in some countable $\mathfrak{F} \in \mathcal{F}$ iff

   $$\langle \mathbb{Q}, < \rangle \models \exists P(\exists x P(x) \wedge (\lambda \wedge \sigma_\varphi)^P).$$

   By Rabin's theorem, this last statement is decidable.

This completes the proof of the theorem. ❑

Remark 16. A similar result for scattered orders (those not embedding $\langle \mathbb{Q}, < \rangle$) can be obtained by combining these methods. A similar encoding will establish decidability of fragments $TL_{fin}(\mathbb{N}) \cap \mathcal{TL}'$, for $\mathcal{TL}'$ as in Theorem 15, using Theorem 29 below in place of Theorem 14. This proves Theorem 26 below.

Various applications of Theorem 15 can be found in Section 8.

5 Satisfiability in $\langle \mathbb{N}, < \rangle$: arbitrary models

The translation into monadic second-order logic given in the preceding section reduces the satisfiability problem for monodic sentences to decidable problems of high computational complexity—for example, the complexity of monadic second-order logic over $\langle \mathbb{N}, < \rangle$ (that is, 'S1S') is itself non-elementary [36]. In this section we demonstrate another way of proving decidability of fragments of linear temporal logics, which is more direct, makes plain the structure of these models, and does yield an elementary decision procedure, provided of course that determining the realizability of state candidates is elementary. For simplicity we will be considering here the logic $TL(\mathbb{N})$ in the language with only one temporal operator Until; it is easy to add Since if required. The idea is to show that every quasimodel satisfying a given $\mathcal{TL}_1$-formula φ can be converted into another quasimodel which also satisfies φ and is based on a periodical state function, with the period being of some bounded length. In the next section we will use this idea to obtain a satisfiability criterion for $\mathcal{TL}_1$-formulas in models (on $\langle \mathbb{N}, < \rangle$) with finite domains.

Fix a $\mathcal{TL}_1$-sentence φ.

We will use the following notation regarding certain sequences of elements, in particular, state functions $f = f(0), f(1), \dots$ and runs $r = r(0), r(1), \dots$. Given a sequence $s = s(0), s(1), \dots$ and $i \geq 0$, we denote by $s^{\leq i}$ and $s^{>i}$ the head $s(0), \dots, s(i)$ and the tail $s(i+1), s(i+2), \dots$ of s, respectively; $s_1 * s_2$ denotes the concatenation of sequences s_1 and s_2; $|s|$ denotes the length of s, and

$$s^* = s * s * s * \dots$$

An infinite subsequence $g = f(n_0), f(n_1), \dots$ of a state function f for φ will also be understood as a state function for φ defined by $g(i) = f(n_i)$, $i \in \mathbb{N}$.

Lemma 17. *Let $\langle f, \mathcal{R} \rangle$ be a quasimodel for φ such that $f(n) = f(m)$ for some $n < m$. Then $\left\langle f^{\leq n} * f^{>m}, \mathcal{R}^{\leq n} * \mathcal{R}^{>m} \right\rangle$ is also a quasimodel for φ, where*

$$\mathcal{R}^{\leq n} * \mathcal{R}^{>m} = \{ r_1^{\leq n} * r_2^{>m} : r_1, r_2 \in \mathcal{R}, r_1(n) = r_2(m) \}.$$

Proof It suffices to observe that if r_1 and r_2 are runs in f and $r_1(n) = r_2(m)$, then $r_1^{\leq n} * r_2^{>m}$ is a run in $f^{\leq n} * f^{>m}$. Let us check, for instance, the '$\Rightarrow$'-condition for $\psi_1 \mathcal{U} \psi_2 \in sub_x \varphi$. Suppose that $\psi_1 \mathcal{U} \psi_2 \in r_1(k)$ for some $k \leq n$. Then, since r_1 is a run, there is $l > k$ such that $\psi_2 \in r_1(l)$ and $\psi_1 \in r_1(l')$ for all $l' \in (k, l)$. If $l \leq n$ then we are done. Otherwise, when $l > n$, we have $\psi_1 \mathcal{U} \psi_2 \in r_1(n) = r_2(m)$, and so are done again, since r_2 is a run.

Now, because $\langle f, \mathcal{R} \rangle$ is a quasimodel, for every $r_1 \in \mathcal{R}$ there is $r_2 \in \mathcal{R}$ such that $r_1(n) = r_2(m)$, and vice versa (swapping n, m). It now follows that $\left\langle f^{\leq n} * f^{>m}, \mathcal{R}^{\leq n} * \mathcal{R}^{>m} \right\rangle$ is a quasimodel for φ. ❑

Definition 18. If g is a subsequence of f, and both $\langle f, \mathcal{R} \rangle$ and $\langle g, \mathcal{Q} \rangle$ are quasimodels for φ, then we call $\langle g, \mathcal{Q} \rangle$ a *subquasimodel* of $\langle f, \mathcal{R} \rangle$.

For instance, $\left\langle f^{\leq n} * f^{>m}, \mathcal{R}^{\leq n} * \mathcal{R}^{>m} \right\rangle$ in the formulation of Lemma 17 is a subquasimodel of $\langle f, \mathcal{R} \rangle$.

Lemma 19. *Every quasimodel f for φ contains a subquasimodel $f' = f_1 * f_2$ such that $|f_1| \leq \sharp(\varphi)$ and each quasistate in f_2 occurs in this sequence infinitely many times.*

Proof If each $f(n)$, for $n \in \mathbb{N}$, occurs infinitely often in f then let $f' = f = f_2$ (f_1 is empty). Otherwise, we take n to be the maximal number such that $f(n) \neq f(m)$, for all $m > n$, and apply Lemma 17 to the quasimodel f deleting from its head $f^{\leq n}$ all repeating quasistates, which yields us a subquasimodel $f' = f_1 * f^{>n}$ satisfying the required properties. ❑

Definition 20. Suppose that $f = f(0), f(1), \dots$ is a sequence of realizable state candidates for φ of the form $f(i) = \langle T_i, T_i^{con} \rangle$, r is a sequence of elements from T_i, $i \in \mathbb{N}$, such that $r(i) \in T_i$, and $n \in \mathbb{N}$. Suppose also that a formula $\psi_1 \mathcal{U} \psi_2 \in sub_x(\varphi)$ occurs in $r(n)$. Then we say that r *realizes* $\psi_1 \mathcal{U} \psi_2$ *in* m *steps* (starting from n), if there is $l \in (0, m)$ such that $\psi_2 \in r(n+l)$ and $\psi_1 \in r(n+k)$ for all $k \in (0, l)$.

Lemma 21. *Let $f = f_1 * f_2$ be a quasimodel for φ (with quasistates of the form $\langle T_i, T_i^{con} \rangle$ for $i \in \mathbb{N}$) such that $n = |f_1| \leq \sharp(\varphi)$ and each quasistate in f_2 occurs in it infinitely often. Then f contains a subquasimodel of the form $f_1 * f_0 * f_2^{>l}$, for some $l \geq 0$, such that*

(i) $|f_0| \leq |sub_x\varphi| \cdot \sharp(\varphi) \cdot \flat^2(\varphi) + \sharp(\varphi)$;

(ii) *for every $t \in T_n$ there is a run r in $f_1 * f_0 * f_2^{>l}$ coming through t and realizing all formulas of the form $\psi_1 \mathcal{U} \psi_2 \in r(n)$ in $|f_0|$ steps (for $t_c \in T_n^{con}$ the run r_c realizes all formulas of the form $\psi_1 \mathcal{U} \psi_2 \in r_c(n)$ in $|f_0|$ steps);*

(iii) $f_0(0) = f_2^{>l}(0)$.

Proof Suppose $t \in T_n$, $\psi_1 \mathcal{U} \psi_2 \in t$ and r is a run in f through t, i.e., $r(n) = t$. Take the minimal $m > 0$ such that $\psi_2 \in r(n+m)$ and $\psi_1 \in r(n+k)$ for all $k \in (0, m)$. Assume now that $0 < i < j < m$, $r(n+i) = r(n+j)$ and $f(n+i) = f(n+j)$. In view of Lemma 17, $f_1 * f_2^{\leq i} * f_2^{>j}$ is a subquasimodel of f and $r^{\leq n+i} * r^{>n+j}$ is a run in it coming through $t \in T_n$. It follows that we can construct a subquasimodel $f_1 * f_2^{\leq 0} * f_3$ of f and a run r_1 in it which comes through $t \in T_n$ and realizes $\psi_1 \mathcal{U} \psi_2$ in $m_1 \leq \flat(\varphi) \cdot \sharp(\varphi)$ steps.

Then we consider another formula of the form $\psi_1' \mathcal{U} \psi_2' \in t$ and assume that it is realized in $m_2 > m_1$ steps in r_1. Using Lemma 17 once again (and deleting repeating quasistates in the interval $f_3(m_1), \dots, f_3(m_2)$) we select a subquasimodel $f_1 * f_2^{\leq 0} * f_3^{\leq m_1} * f_4$ of f and a run r_2 through $t \in T_n$ which realizes both $\psi_1 \mathcal{U} \psi_2$ and $\psi_1' \mathcal{U} \psi_2'$ in $2 \cdot \flat(\varphi) \cdot \sharp(\varphi)$ steps.

Having analyzed all distinct formulas of the form $\psi_1 \mathcal{U} \psi_2$ in $t \in T_n$ we obtain a subquasimodel $f_1 * f_2^{\leq 0} * f' * f^{>k}$ of f and a run r' through t which realizes all such formulas in $m' \leq |sub_x\varphi| \cdot \flat(\varphi) \cdot \sharp(\varphi)$ steps.

After that we consider in the same manner another type $t' \in T_n$. However this time we can delete quasistates only after $f'(m')$, and so to realize in some run through t' a formula $\psi_1 \mathcal{U} \psi_2 \in t'$, we need again $\leq \flat(\varphi) \cdot \sharp(\varphi)$ new steps. Since $|T_n| \leq \flat(\varphi)$, at most $|sub_x\varphi| \cdot \flat^2(\varphi) \cdot \sharp(\varphi)$ quasistates are required to satisfy (ii).

Finally, not more than $\sharp(\varphi)$ quasistates may be needed to comply with (iii). ❑

Definition 22 (suitable pair). A pair t, t' of types for φ is called *suitable* if for every $\psi_1 \mathcal{U} \psi_2 \in sub_x\varphi$,

$$\psi_1 \mathcal{U} \psi_2 \in t \text{ iff either } \psi_2 \in t' \text{ or } \psi_1 \in t' \text{ and } \psi_1 \mathcal{U} \psi_2 \in t'.$$

Lemma 23. *Suppose that f_1 and f_2 are finite sequences of realizable state candidates for φ of length l_1 and l_2, respectively, and let*

$$f = f_1 * f_2^*$$

with $f(n) = \langle T_n, T_n^{con} \rangle$, $n \in \mathbb{N}$. Then f is a quasimodel for φ whenever the following conditions hold:

1. *for every i, $0 \le i \le l_1 + l_2$, and every $t_i \in T_i$, there are $t_{i-1} \in T_{i-1}$ (only if $i > 0$) and $t_{i+1} \in T_{i+1}$ (only if $i < l_1 + l_2$)[2] such that the pairs t_{i-1}, t_i and t_i, t_{i+1} are suitable;*

2. *for every $i \le l_1$ and every $t_i \in T_i$, all formulas of the form $\psi_1 \mathcal{U} \psi_2 \in t_i$ are realized in $l_1 + l_2 - i$ steps in some sequence $t_i, t_{i+1}, \dots, t_{l_1+l_2}$ in which $t_{i+j} \in T_{i+j}$ and every pair of adjacent elements is suitable;*

3. *every pair of adjacent elements in $t_c^0, \dots, t_c^{l_1+l_2}$, where $t_c^i \in T_i^{con}$, is suitable and, for every $i \le l_1$, all formulas of the form $\psi_1 \mathcal{U} \psi_2 \in t_i$ are realized in this sequence in l_1+l_2-i steps.*

Proof We have to show that there is a run coming through an arbitrarily given $t_n \in T_n$, for every $n \in \mathbb{N}$. If $n \le l_1$, then we first use condition 1 to construct a sequence $t_0, \dots, t_n$ such that $t_i \in T_i$ and every pair of adjacent elements in it is suitable. After that, in accordance with condition 2, we continue this sequence to $t_0, \dots, t_n, \dots, t_{l_1+l_2}$ in order to realize all formulas of the form $\psi_1 \mathcal{U} \psi_2 \in t_n$. Then we again use 2 to continue it to $t_0, \dots, t_{l_1+l_2}, \dots, t_{2(l_1+l_2)}$, realizing all $\mathcal{U}$-formulas in $t_{l_1+l_2}$. And so forth. The resulting sequence is clearly a run in f.

If $n > l_1$ then, using 1, we construct a sequence $t_0, \dots, t_n, \dots, t_m$ such that $t_i \in T_i$, every pair of adjacent elements in it is suitable and $m = k(l_1 + l_2)$, for some $k \ge 1$. After that, by 2, we run on this sequence to

$$t_0, \dots, t_n, \dots, t_m, \dots, t_{(k+1)(t_1+t_2)}$$

realizing all the $\mathcal{U}$-formulas in t_m, and so on, thus obtaining a run through t_n.

Finally, we observe that the sequence

$$t_c^0, \dots, t_c^{l_1-1} * (t_c^{l_1}, \dots, t_c^{l_1+l_2-1})^*$$

is a run in f, for every $c \in con\varphi$. ❑

As a consequence of the two preceding lemmas we immediately obtain the following:

Theorem 24. *A $\mathcal{TL}_1$-sentence φ is satisfiable iff there are two sequences f_1 and f_2 of realizable state candidates for φ such that $f_1 * f_2^*$ satisfies conditions 1–3 of Lemma 23, all state candidates in f_1 are distinct (and so $|f_1| \le \sharp(\varphi)$),*

$$|f_2| \le |sub_x\varphi| \cdot \flat^2(\varphi) \cdot \sharp(\varphi) + \sharp(\varphi),$$

and $\varphi \in t$ for all $t \in T_0$.

Proof By Theorem 14 and Lemmas 19, 21, φ is satisfiable iff φ is true in the first quasistate of a quasimodel of the form $f_1 * f_0 * f_2^{>l}$ described in Lemma 21. It remains to observe that $f_1 * f_0^*$ satisfies the conditions of Lemma 23. ❑

Given two finite sequences f_1 and f_2 of state candidates for φ, we can effectively check whether they satisfy conditions 1–3 of Lemma 23. The only missing thing to make the criterion of Theorem 24 effective is therefore an algorithm for detecting whether a given state candidate for φ is realizable. Modulo such an (elementary) algorithm, we obtain an (elementary) algorithm for deciding φ.

Now we extend the developed technique to obtain a similar satisfiability criterion in models with finite domains.

[2] Note that $f(l_1 + l_2) = f(l_1) = f_2(0)$.

6 Satisfiability in $\langle \mathbb{N}, < \rangle$: finite domains

To begin with, let us observe that formulas in $\mathcal{TL}_1$ behave differently in models with arbitrary and finite domains.

For $\ell \geq 1$, put $\mathcal{TL}_1^\ell = \mathcal{TL}_1 \cap \mathcal{TL}^\ell$.

Theorem 25. *For every $\mathcal{TL}' \supseteq \mathcal{TL}_1^1 \cap \mathcal{TL}^{mo}$,*

$$TL_{fin}(\mathbb{N}) \cap \mathcal{TL}' \supsetneqq TL(\mathbb{N}) \cap \mathcal{TL}'.$$

Proof Let

$$\varphi = \Box \exists x \big(P(x) \wedge \neg(\top \mathcal{S} P(x))\big).$$

In English: 'at every moment, someone starts to get old'—or perhaps, 'every day has its dog'. Then $\varphi \in \mathcal{TL}_1^1 \cap \mathcal{TL}^{mo}$, and it is readily checked that φ is satisfied in the model $\mathfrak{M} = \langle \langle \mathbb{N}, < \rangle, \mathbb{N}, I \rangle$ with

$$I(n) = \langle \mathbb{N}, P^n = \{0, \ldots, n\} \rangle,$$

but is false in all models with finite domains. Indeed, if we interpret $\Box$ as 'at all times', then in any model of φ with linear flow of time W and domain D we have $|D| \geq |W|$. Thus, $\neg\varphi \in TL_{fin}(\mathbb{N}) \cap \mathcal{TL}'$ and $\neg\varphi \notin TL(\mathbb{N}) \cap \mathcal{TL}'$. ❑

Our aim in this section is to prove the following analogue of Theorem 15 (1):

Theorem 26. *Let $\mathcal{TL}' \subseteq \mathcal{TL}_1$ and suppose that there is an algorithm which is capable of deciding, for a $\mathcal{TL}'$-sentence φ, whether an arbitrarily-given state candidate for φ is finitely realizable. Then the satisfiability problem for $\mathcal{TL}'$-formulas in models with finite domains, and so the decision problem for the fragment $TL_{fin}(\mathbb{N}) \cap \mathcal{TL}'$, are decidable.*

To this end we will modify Theorem 24 to show that a $\mathcal{TL}_1$-sentence φ is satisfied in a model with a finite domain iff there is a quasimodel based on a state function $f = f_1 * f_2^*$ as in Theorem 24, $f(n)$ being a finitely realizable state candidate for all $n \in \mathbb{N}$, and the quasimodel having a *finite* set of runs $\mathcal{R}$ in it. The idea is to strengthen conditions 1 and 2 of Lemma 23 in such a way that sequences $t_{l_1}, \ldots, t_{l_1+l_2}$, realizing formulas of the form $\psi_1 \mathcal{U} \psi_2$, could be short-circuited, i.e., $t_{l_1+l_2} = t_{l_1}$. Then we will be able to compose infinite runs of the form

$$t_0, \ldots, t_{l_1-1}, \langle t_{l_1}, \ldots, t_{l_1+l_2-1} \rangle^*,$$

the number of which is clearly finite. Yet, there remains one more technical problem: to ensure that we have enough runs, i.e., that every type in every quasistate lies on some run. To solve it, we will need two kinds of sequences of types in quasistates: one,

$$s_2 = \langle t_{l_1}, \ldots t_{l_1+l_2-1} \rangle,$$

to realize $\mathcal{U}$-formulas, and another one,

$$s_3 = \left\langle t'_{l_1}, \ldots t'_{l_1+l_2-1} \right\rangle,$$

to make sure that we have enough runs. The resulting runs will then have the forms

$$t_0, \ldots, t_{l_1-1}, (s_2 * s_3)^* \text{ and } t_0, \ldots, t_{l_1-1}, (s_3 * s_2)^*.$$

Let us fix a $\mathcal{TL}_1$-sentence φ and an enumeration $\langle t_1, \ldots, t_{n_\varphi} \rangle$ of all types for φ, $n_\varphi \leq \flat(\varphi)$.

The following claim is a 'finite version' of Lemma 9:

Lemma 27. *There is $m < \omega$ such that, for every finitely realizable state candidate $\mathfrak{C} = \langle T, T^{con}\rangle$ and every sequence $\langle n_i : 0 < i \leq n_\varphi\rangle$, in which $n_i = 0$ whenever $t_i \notin T$ and $n_i > m$ otherwise, $\mathfrak{C}$ is realized in an $\mathcal{L}$-structure $\mathfrak{D}$ such that $|\mathfrak{D}_{t_i}| = n_i$, for every $i \leq n_\varphi$.*

Proof Suppose that $\mathfrak{C}_1, \dots, \mathfrak{C}_k$ are all distinct finitely realizable state candidates for φ (so that $k \leq \sharp(\varphi)$) and that $\mathfrak{C}_j$ is finitely realized in $\mathfrak{D}^j$. Then it is enough to take $m = \max\{\left|\mathfrak{D}^j_{t_i}\right| : 0 < i \leq n_\varphi, 0 < j \leq k\}$. ❑

Definition 28. A quasimodel $\langle f, \mathcal{R}\rangle$ for φ over a linear order $\mathfrak{F} = \langle W, <\rangle$ is said to be *finitary* if $f(w)$ is a finitely realizable state candidate for all $w \in W$, and $\mathcal{R}$ is finite.

Now we can prove a finite analogue of Theorem 14; it holds for any linear flow of time.

Theorem 29. *A $\mathcal{TL}_1$-sentence φ is satisfied in a model with a finite domain iff it is satisfied in a finitary quasimodel $\langle f, \mathcal{R}\rangle$ for φ.*

Proof The implication ($\Rightarrow$) was established in the proof of Theorem 14.

($\Leftarrow$) Suppose φ is satisfied in a finitary quasimodel $\langle f, \mathcal{R}\rangle$ for φ, and let m be the number supplied by Lemma 27. Define the domain of the model to be constructed by taking

$$D = \{\langle r, \xi\rangle : r \in \mathcal{R}, \xi < m\}.$$

By Lemma 27, for every $n \in \mathbb{N}$ there exists an $\mathcal{L}$-structure $I(n)$ with domain D realizing $f(n)$ and such that $c^n = \langle r_c, 0\rangle$, for every $c \in con\varphi$, and

$$r(n) = \{\psi \in sub_x\varphi : I(n) \models \overline{\psi}[\langle r, \xi\rangle]\},$$

for all $r \in \mathcal{R}$ and $\xi < m$. Let $\mathfrak{M} = \langle D, I\rangle$. In precisely the same way as in the proof of Theorem 14 one can show that φ is satisfied in $\mathfrak{M}$. ❑

Let $\langle f, \mathcal{R}\rangle$ be a quasimodel for φ. Define an equivalence relation $\sim_\mathcal{R}$ on $\mathbb{N}$ by taking

$$i \sim_\mathcal{R} j \text{ iff } f(i) = f(j) \text{ and } \forall r \in \mathcal{R}\ r(i) = r(j),$$

and denote by $[n]_\mathcal{R}$ the $\sim_\mathcal{R}$-equivalence class generated by n.

Besides, for each $n \in \mathbb{N}$, we define one more equivalence relation $\sim^n_\mathcal{R}$ on $\mathbb{N}$ by taking $i \sim^n_\mathcal{R} j$ iff $f(i) = f(j)$ and

- for every $r \in \mathcal{R}$ there is $r' \in \mathcal{R}$ such that $r(n) = r'(n)$ and $r(i) = r'(j)$,
- for every $r \in \mathcal{R}$ there is $r' \in \mathcal{R}$ such that $r(n) = r'(n)$ and $r(j) = r'(i)$.

Lemma 30. *For every $n \in \mathbb{N}$, the number of pairwise distinct $\sim^n_\mathcal{R}$-equivalence classes does not exceed*

$$\natural(\varphi) = \sharp(\varphi) \cdot 2^{2 \cdot 2^{|sub_x\varphi|}}.$$

Proof Fix some $n \in \mathbb{N}$ and define a function $\sigma_i(k, l)$, for $i \in \mathbb{N}$, $k, l \leq n_\varphi$, by taking

$$\sigma_i(k,l) = \begin{cases} 1 & \text{if } \exists r \in \mathcal{R}\ r(n) = t_k \ \&\ r(i) = t_l, \\ 0 & \text{otherwise.} \end{cases}$$

We then have $i \sim^n_\mathcal{R} j$ iff $f(i) = f(j)$ and $\sigma_i(k,l) = \sigma_j(k,l)$, for all $k, l \leq n_\varphi$. It remains to observe that the number of functions from $\{1, \dots, n_\varphi\}^2$ into $\{0, 1\}$ is $2^{n_\varphi^2}$. ❑

The finite analogue of Lemma 19 we need is proved similarly to that lemma with the help of Lemma 17.

Lemma 31. *Every quasimodel $\langle f, \mathcal{R}\rangle$ for φ with finite $\mathcal{R}$ contains a subquasimodel $\langle f_1 * f_2, \mathcal{Q}\rangle$ with finite $\mathcal{Q}$ such that $|f_1| \leq \sharp(\varphi)$ and $[n]_{\mathcal{Q}}$ is infinite, for every $n \geq |f_1|$.*

However, to prove a finite version of Lemma 21, a somewhat subtler deleting technique is required:

Lemma 32. *Let $\langle f, \mathcal{R}\rangle$ be a quasimodel for φ, $n < i < j$, and $i \sim^n_{\mathcal{R}} j$.*
Then $\left\langle f^{\leq i} * f^{>j}, \mathcal{Q} = \mathcal{R}^{\leq i} *_n \mathcal{R}^{>j}\right\rangle$ is also a quasimodel for φ, where

$$\mathcal{R}^{\leq i} *_n \mathcal{R}^{>j} = \{r_1^{\leq i} * r_2^{>j} : r_1, r_2 \in \mathcal{R}, r_1(i) = r_2(j), r_1(n) = r_2(n)\}.$$

Moreover, for all $n' > j$, if $n \sim_{\mathcal{R}} n'$ then $n \sim_{\mathcal{Q}} n' - (j - i)$.

Proof Follows immediately from the definition of $i \sim^n_{\mathcal{R}} j$. ❑

Lemma 33. *Let $\langle f = f_1 * f_2, \mathcal{R}\rangle$ be a quasimodel for φ (with quasistates having the form $\langle T_i, T_i^{con}\rangle$) such that $n = |f_1| \leq \sharp(\varphi)$, $\mathcal{R}$ is finite, and $[m]_{\mathcal{R}}$ is infinite for all $m > n$. Then $\langle f, \mathcal{R}\rangle$ contains a subquasimodel of the form $\left\langle f_1 * f_0 * f_2^{>l}, \mathcal{Q}\right\rangle$, for some $l \geq 0$, such that $\mathcal{Q}$ is finite and*

(i) *$|f_0| \leq |sub_x\varphi| \cdot \natural(\varphi) \cdot 2^{2\cdot 2^{|sub_x\varphi|}} + \natural(\varphi)$;*

(ii) *for every $t \in T_n$ there is a run $r \in \mathcal{Q}$ through t realizing all formulas of the form $\psi_1 \mathcal{U} \psi_2 \in r(n)$ in $|f_0|$ steps (for $t_c \in T_n^{con}$ the run r_c realizes all formulas of the form $\psi_1\mathcal{U}\psi_2 \in r_c(n)$ in $|f_0|$ steps);*

(iii) *$n \sim_{\mathcal{Q}} |f_1 * f_0|$.*[3]

Proof Suppose $t \in T_n$, $\psi_1\mathcal{U}\psi_2 \in t$ and r is a run in $\mathcal{R}$ through t. Then there exists $m > 0$ such that $\psi_2 \in r(n+m)$ and $\psi_1 \in r(n+k)$ for all $k \in (0, m)$. Assume now that $0 < i < j < m$, $r(n+i) = r(n+j)$ and $n + i \sim^{\mathcal{R}}_n n + j$. In view of Lemma 32, $\left\langle f_1 * f_2^{\leq i} * f_2^{>j}, \mathcal{Q}_0 = \mathcal{R}^{\leq n+i} *_n \mathcal{R}^{>n+j}\right\rangle$ is a subquasimodel of $\langle f, \mathcal{R}\rangle$, $r^{\leq n+i} * r^{>n+j}$ is a run through t, and for all $n' > n + j$ we have $n \sim_{\mathcal{Q}_0} n' - (j - i)$ whenever $n \sim_{\mathcal{R}} n'$. Thus we obtain a subquasimodel

$$\left\langle f_1 * f_2^{\leq 0} * f_3, \mathcal{Q}_0\right\rangle$$

of $\langle f, \mathcal{R}\rangle$ such that $\mathcal{Q}_0$ is finite, there is a run $r_1 \in \mathcal{Q}_0$ through t, realizing $\psi_1\mathcal{U}\psi_2$ in $m_1 \leq 2^{|sub_x\varphi|} \cdot \natural(\varphi)$ steps, and such that, for all $n' > n + m_1$ we have $n \sim_{\mathcal{Q}_0} n' - (j - i)$ whenever $n \sim_{\mathcal{R}} n'$. In particular, $[n]_{\mathcal{Q}_0}$ is infinite.

After that we consider another formula $\psi_1'\mathcal{U}\psi_2' \in t$ and assume that it is realized in $m_2 > m_1$ steps in r_1. Using Lemma 32 once again (and deleting quasistates in the interval $f_3(m_1), \ldots, f_3(m_2)$) we construct a subquasimodel

$$\left\langle f_1 * f_2^{\leq 0} * f_3^{\leq m_1} * f_4, \mathcal{Q}_1\right\rangle$$

of $\langle f, \mathcal{R}\rangle$ and a run r_2 through t realizing both $\psi_1\mathcal{U}\psi_2$ and $\psi_1'\mathcal{U}\psi_2'$ in $2 \cdot 2^{|sub_x\varphi|} \cdot \natural(\varphi)$ steps, with $[n]_{\mathcal{Q}_1}$ being infinite.

[3]Note that $f(n) = f_0(0) = f_2^{>l}(0) = f(|f_1 * f_0|)$.

Having analyzed all distinct formulas of the form $\psi_1 \mathcal{U} \psi_2$ in t we obtain a subquasimodel

$$\left\langle f_1 * f_2^{\leq 1} * f', \mathcal{Q}' \right\rangle$$

of $\langle f, \mathcal{R} \rangle$ with finite $\mathcal{Q}'$ and a run $r' \in \mathcal{Q}'$ through t realizing all $\mathcal{U}$-formulas in $m' \leq |sub_x \varphi| \cdot 2^{|sub_x \varphi|} \cdot \natural(\varphi)$ steps. The class $[n]_{\mathcal{Q}'}$ is infinite.

Then we consider in the same manner another type $t' \in T_n$. However, this time we can delete quasistates only after $f'(m')$. And so forth. Thus we arrive at a subquasimodel

$$\left\langle f_1 * f_2^{\leq 0} * f'', \mathcal{Q}'' \right\rangle$$

of $\langle f, \mathcal{R} \rangle$ with finite $\mathcal{Q}''$, infinite $[n]_{\mathcal{Q}''}$ and such that all formulas of the form $\psi_1 \mathcal{U} \psi_2$ in all $t \in T_n$ are realized by some $r \in \mathcal{Q}'$ in $\leq |sub_x \varphi| \cdot \natural(\varphi) \cdot 2^{2 \cdot 2^{|sub_x \varphi|}}$ steps.

Finally, we need at most $\natural(\varphi)$ new quasistates to comply with (iii). ❑

We are in a position now to prove the finite analogue of Lemma 23.

Lemma 34. *Suppose f_1 and f_2 are finite sequences of realizable state candidates for φ of length l_1 and l_2, respectively, and let*

$$f = f_1 * f_2^*$$

with $f(n) = \langle T_n, T_n^{con} \rangle$, for $n \in \mathbb{N}$. Suppose also that the following conditions hold:

1. *for every $i < l_1 + l_2$ and every $t_i \in T_i$, there is a sequence $t_0, \dots, t_{l_1+l_2-1}$ of types for φ such that*
   - 1.1. *$t_j \in T_j$, for every $j < l_1 + l_2$,*
   - 1.2. *the pair t_i, t_{i+1} is suitable, for every $i < l_1 + l_2 - 1$,*
   - 1.3. *the pair $t_{l_1+l_2-1}, t_{l_1}$ is suitable;*
2. *for every $i \leq l_1$ and every $t_i \in T_i$, there is a sequence $t_0, \dots, t_{l_1+l_2-1}$ such that*
   - 2.1. *all formulas of the form $\psi_1 \mathcal{U} \psi_2$ are realized in $l_1 + l_2 - i$ steps in $t_0, \dots, t_{l_1+l_2-1}$,*
   - 2.2. *$t_j \in T_j$, for $j < l_1 + l_2$,*
   - 2.3. *every pair of adjacent types in the sequence is suitable,*
   - 2.4. *the pair $t_{l_1+l_2-1}, t_{l_1}$ is suitable.*
3. *all pairs of adjacent elements in $t_c^0, \dots, t_c^{l_1+l_2-1}$, where $t_c^i \in T_i^{con}$, as well as the pair $t_c^{l_1+l_2-1}, t_c^{l_1}$, are suitable, and, for every $i \leq l_1$, all formulas of the form $\psi_1 \mathcal{U} \psi_2 \in t_i$ are realized in $t_c^0, \dots, t_c^{l_1+l_2-1}$ in $l_1 + l_2 - i$ steps.*

Then there is a finite set $\mathcal{R}$ of runs in f such that $\langle f, \mathcal{R} \rangle$ is a quasimodel for φ.

Proof We have to define a finite set of runs $\mathcal{R}$ in f. Say that a sequence $t_0, \dots, t_{l_1+l_2-1}$ is of *type 1* (*type 2*) if it satisfies condition 1 (respectively, condition 2 for $i = l_1$) in the formulation of the lemma. Clearly, there are finitely many sequences of type 1, and every sequence of type 2 is also a sequence of type 1.

Let $\mathcal{R}$ consist of all infinite words of the form

$$s_1 * (s_2^{\geq l_1} * s_3^{\geq l_1})^* \text{ and } s_1 * (s_3^{\geq l_1} * s_2^{\geq l_1})^*,$$

where s_1, s_3 are sequences of type 1 and s_2 is a sequence of type 2 such that

- the pair $s_1(l_1+l_2-1)$, $s_2(l_1)$ is suitable and
- $s_2(l_1) = s_3(l_1)$.

It is readily checked that every such word is a run in f and that $\langle f, \mathcal{R}\rangle$ is a quasimodel for φ. Needless to say that $\mathcal{R}$ is finite. ❑

Putting together the two preceding lemmas we obtain:

Theorem 35. *A $\mathcal{TL}_1$-sentence φ is satisfied in a model on $\langle \mathbb{N}, <\rangle$ with a finite domain iff there are two sequences f_1 and f_2 of finitely realizable state candidates for φ such that $f_1 * f_2^*$ satisfies conditions 1–3 of Lemma 34, all state candidates in f_1 are distinct (and so $|f_1| \leq \sharp(\varphi)$),*

$$|f_2| \leq |sub_x\varphi| \cdot \natural(\varphi) \cdot 2^{2 \cdot 2^{|sub_x \varphi|}} + \natural(\varphi),$$

and $\varphi \in t$, for all $t \in T_0$.

The criterion of Theorem 26, reducing the satisfiability problem for $\mathcal{TL}'$-formulas in models with finite domains to the finite satisfiability problem for $\mathcal{L}$-formulas of the form $\alpha_{\mathfrak{C}}$, follows immediately.

7 Satisfiability in $\langle \mathbb{R}, <\rangle$: finite domains

Now we will present a third method of reducing decidability of monodic fragments to classical decidability problems. We will consider only finite domains, with flow of time the real numbers, $\langle \mathbb{R}, <\rangle$. (The decidability problems for finite domains over an arbitrary first-order definable class of flows and over $\langle \mathbb{N}, <\rangle$, $\langle \mathbb{Z}, <\rangle$, and $\langle \mathbb{Q}, <\rangle$ reduce to this case; see Corollary 37. The case of $\langle \mathbb{R}, <\rangle$ with arbitrary domains remains open.)

We will prove the following theorem.

Theorem 36. *Let $\mathcal{TL}' \subseteq \mathcal{TL}_1$ and suppose that there is an algorithm which is capable of deciding, for any $\mathcal{TL}'$-sentence φ, whether an arbitrarily-given state candidate for φ is finitely realizable. Then it is decidable whether such a sentence φ is satisfied in a model with flow of time $\langle \mathbb{R}, <\rangle$ and finite domain: that is, $TL_{fin}(\mathbb{R}) \cap \mathcal{TL}'$ is decidable.*

The proof will occupy most of this section. The method is model-theoretic, based on that of [11, 23, 28]; see also [16, chapter 6.9]. Very roughly, the idea of the proof is as follows. By Theorem 29, we need only decide whether there is a finitary quasimodel of a given sentence $\varphi \in \mathcal{TL}'$ with flow of time $\langle \mathbb{R}, <\rangle$. Such a quasimodel consists of a finite set of runs over $\langle \mathbb{R}, <\rangle$, a 'snapshot' of the runs at any moment of time giving a finitely realizable state candidate. Thus, the finitely realizable state candidate gives an instantaneous description of the runs in the quasimodel. We will show how to describe the runs over longer intervals of $\mathbb{R}$, ranging from one-point intervals as above, to the whole of $\mathbb{R}$. We may decide whether each possible description of the runs is satisfiable: for one-point intervals, using the algorithm deciding $\mathcal{TL}'$, and for more complex ones by decomposing them into simpler parts for which we can already decide satisfiability (cf. Lemma 46(2,4) below). We will then show that a description of the runs on the whole of $\mathbb{R}$ can always be built up in finitely many steps from instantaneous descriptions (finitely realizable state candidates)—cf. Lemma 46(3). Combining these ideas serves to prove Theorem 36; formally, the theorem follows from Lemma 46.

Decidability over various other flows of time or classes of flows of time reduce to Theorem 36.

Corollary 37. *Let $\mathcal{TL}' \subseteq \mathcal{TL}_1$ and suppose that there is an algorithm deciding, for any $\mathcal{TL}'$-sentence φ, whether an arbitrarily-given state candidate for φ is finitely realizable. Then it is decidable whether a $\mathcal{TL}'$-sentence is satisfied in a model with finite domain and with any of the following (classes of) flows of time:*

1. $\langle \mathbb{N}, < \rangle$,
2. $\langle \mathbb{Z}, < \rangle$,
3. $\langle \mathbb{Q}, < \rangle$,
4. *the class of all finite linear orders,*
5. *any first-order-definable class of linear orders.*

Proof We prove part 1. Given φ, introduce a new propositional variable p, and define the $\mathcal{TL}_1$-sentence

$$\nu = \blacklozenge\neg(\top\mathcal{S}p) \wedge \blacksquare(\top\mathcal{U}p \wedge \neg p\mathcal{S}\top \wedge \neg p\mathcal{U}\top),$$

where $\blacklozenge\psi$ abbreviates $\psi \vee \top\mathcal{U}\psi \vee \top\mathcal{S}\psi$ and $\blacksquare\psi$ abbreviates $\neg\blacklozenge\neg\psi$. So ν states that p is bounded below, unbounded above, and that there is no accumulation point of p. Clearly, the models of ν with flow of time $\langle \mathbb{R}, < \rangle$ are precisely those in which the interpretation of p is isomorphic to $\langle \mathbb{N}, < \rangle$. Now define the relativization φ^p of the temporal connectives in φ to p, by induction in the usual way: $\alpha^p = \alpha$ for atomic α, $(\neg\psi)^p = \neg\psi^p$, $(\psi_1 \wedge \psi_2)^p = \psi_1^p \wedge \psi_2^p$, $(\forall x\psi)^p = \forall x\psi^p$, and $(\psi_1\mathcal{U}\psi_2)^p = (p \to \psi_1^p)\mathcal{U}(p \wedge \psi_2^p)$, plus a similar clause for $\mathcal{S}$. Then it is easily seen that φ has a model with flow of time $\langle \mathbb{N}, < \rangle$ and finite domain iff $\nu \wedge \varphi^p$ has a model with flow of time $\langle \mathbb{R}, < \rangle$ and finite domain. This proves part 1.

Parts 2 and 4 of the corollary are proved by similar reductions. For part 5, we also use the downward Löwenheim–Skolem–Tarski theorem (as in Theorem 15) and the expressive completeness of $\mathcal{U}$ and $\mathcal{S}$ over $\langle \mathbb{R}, < \rangle$ [26, 16]. Part 3 follows, because φ has a model with dense flow of time without endpoints (a first-order definable property) iff it has a model over $\langle \mathbb{Q}, < \rangle$. The details are left to the interested reader. ❑

This gives an alternative proof of Theorem 26.

7.1 3-theories

We begin our proof of Theorem 36 with the definitions needed to describe runs over intervals of $\mathbb{R}$. To simplify notation, we will frequently identify (notationally) a structure with its domain: hence, we write W rather than $\mathfrak{F} = \langle W, < \rangle$ for a linear order.

Let $\mathcal{L}_\varphi$ denote the first-order language (with equality, say, though it is immaterial for our purposes) in the signature $\{<, R_\psi : \psi \in sub_x\varphi\}$, where the R_ψ are unary predicates. An *$\mathcal{L}_\varphi$-order* is an $\mathcal{L}_\varphi$-structure $M = \langle W, R_\psi^M : \psi \in sub_x\varphi \rangle$ where W is a linear order and the R_ψ^M are subsets of W.

Definition 38 (3-theory). A *3-theory* (in $\mathcal{L}_\varphi$) is a set σ of first-order $\mathcal{L}_\varphi$-sentences of the form $\mathit{3\text{-}th}(M) = \{\theta : \theta \text{ an } \mathcal{L}_\varphi\text{-sentence of quantifier depth at most 3}, M \models \theta\}$, for some $\mathcal{L}_\varphi$-order M.

Up to logical equivalence, there are finitely many 3-theories. Note that by definition, any 3-theory has a model. Let $\mathcal{T}$ be the set of types for φ; recall that $\mathcal{T}$ is finite, with $|\mathcal{T}| \leq \flat(\varphi)$. If W is a linear order and $r : W \to \mathcal{T}$, define the $\mathcal{L}_\varphi$-order M_r to be

$$\langle W, \{w \in W : \psi \in r(w)\} : \psi \in sub_x\varphi\rangle.$$

That is, M_r has underlying order W, and $M_r \models R_\psi(w)$ iff $\psi \in r(w)$, for $w \in W$ and $\psi \in sub_x\varphi$. We let *3-th*(r) denote *3-th*(M_r).

Definition 39 (endpoints, degenerate). Let σ be a 3-theory. We say that σ *has a left endpoint* if $\sigma \vdash \exists x \forall y \neg(y < x)$, that σ *has a right endpoint* if $\sigma \vdash \exists x \forall y \neg(x < y)$, and we say that σ *is degenerate* if $\sigma \vdash \forall xy \neg(x < y)$.

Let I be a linear order and $M_i = \langle W_i, R^i_\psi : \psi \in sub_x\varphi\rangle$ $(i \in I)$ be $\mathcal{L}_\varphi$-orders. We write $\sum_{i\in I} M_i$ for the $\mathcal{L}_\varphi$-order M with underlying order $W = \bigcup_{i\in I} W_i \times \{i\}$, ordered lexicographically by $\langle w, i\rangle < \langle w', j\rangle$ iff either $i < j$, or $i = j$ and $w < w'$ in W_i, and with $M \models R_\psi(\langle w, i\rangle)$ iff $M_i \models R_\psi(w)$, for $\langle w, i\rangle \in W$ and $\psi \in sub_x\varphi$. We write the underlying order of M as $\sum_{i\in I} W_i$. When $I = \{0, 1\}$ with $0 < 1$, we write simply $M_0 + M_1$ and $W_0 + W_1$.

A well-known Feferman–Vaught argument (see, e.g., [25, Theorem A.6.2]) shows that if M_i, N_i $(i \in I)$ are $\mathcal{L}_\varphi$-orders and *3-th*(M_i) = *3-th*(N_i) for all i, then *3-th*$(\sum_{i\in I} M_i)$ = *3-th*$(\sum_{i\in I} N_i)$. Hence, we may use the following notation. Let I be a linear order and for each $i \in I$ let σ_i be a 3-theory. We write $\sum_{i\in I} \sigma_i$ for the unique 3-theory σ such that σ = *3-th*$(\sum_{i\in I} M_i)$ for any $\mathcal{L}_\varphi$-orders $M_i \models \sigma_i$ $(i \in I)$.

As with $\mathcal{L}_\varphi$-orders, we write $\sigma_0 + \sigma_1$ when $I = \{0, 1\}$ with $0 < 1$.

7.2 Characters

We will describe runs over intervals of $\mathbb{R}$ by *characters.*

Definition 40 (character).

1. A *character* is a pair $\langle S, S^{con}\rangle$, where S is a set of 3-theories and $S^{con} : con\varphi \to S$ is a function. There are only finitely many characters.

2. A character $\langle S, S^{con}\rangle$ is said to have a *left (right) endpoint* if every $\sigma \in S$ has a left (right) endpoint.

3. A character $\langle S, S^{con}\rangle$ is said to be *degenerate* if

   - each $\sigma \in S$ is degenerate,
   - for each $\sigma \in S$, the set $t_\sigma = \{\psi \in sub_x\varphi : \sigma \vdash \exists x R_\psi(x)\}$ is a type for φ,
   - $\big\langle \{t_\sigma : \sigma \in S\}, \{\langle t_{S^{con}(c)}, c\rangle : c \in con\varphi\}\big\rangle$ is a finitely realizable state candidate for φ.

We may combine simpler characters into more complex ones as follows. We will need the notion of a *condensation of* $\mathbb{R}$: namely, a linear order $\langle I, <\rangle$ where I is the set of equivalence classes of some equivalence relation on $\mathbb{R}$ whose equivalence classes are convex, the ordering $<$ on I being induced from the ordering on $\mathbb{R}$ in the obvious way. For more information on condensations see, e.g., [39].

Definition 41 ($\approx$, $\equiv$). Let I be a linear order, and $\chi = \langle S, S^{con}\rangle$ and $\chi_i = \langle S_i, S_i^{con}\rangle$ $(i \in I)$ be characters. We write $\chi \approx \sum_{i\in I} \chi_i$ if

(i) for each $c \in con\varphi$, $S^{con}(c) = \sum_{i\in I} S_i^{con}(c)$,

(ii) for each $\sigma \in S$ there are $\sigma_i \in S_i$ $(i \in I)$ such that $\sigma = \sum_{i\in I} \sigma_i$,

(iii) for all $i \in I$ and $\sigma_i \in S_i$, there are $\sigma_j \in S_j$ $(j \in I \setminus \{i\})$ such that $\sum_{j\in I} \sigma_j \in S$.

We write $\chi \equiv \sum_{i\in I} \chi_i$ if one of the following holds:

(iv) I is a 2-element order, say $\{0, 1\}$ with $0 < 1$, either χ_0 has a right endpoint or χ_1 a left endpoint (not both), and $\chi \approx \chi_0 + \chi_1$,

(v) $I = \langle \mathbb{N}, <\rangle$, $\chi_i = \chi_0$ for all $i \in \mathbb{N}$, χ_0 has either a left or a right endpoint (not both), condition (i) above holds, and $S = \{\sum_{i\in I} \sigma_i : \sigma_i \in S_0,\ \sigma_i = \sigma_0 \text{ for all } i \in I\}$.

(vi) As for (v) but with $I = \langle \mathbb{N}, >\rangle$.

(vii) I is a dense condensation of $\langle \mathbb{R}, <\rangle$ without endpoints, conditions (i) and (ii) above hold, and for all $i \in I$ (so that i is a convex subset of $\mathbb{R}$):

- i and χ_i have a left and a right endpoint,
- i is a singleton subset of $\mathbb{R}$ iff $\sigma \vdash \forall xy\neg(x < y)$ for all $\sigma \in S_i$,
- for each $\sigma \in S_i$ there are $\sigma_j \in S_j$ $(j \in I)$ with $\sum_{j\in I} \sigma_j \in S$, $\langle \chi_j, \sigma_j\rangle = \langle \chi_i, \sigma\rangle$ for some $j \in I$, and for each $j \in I$, the set $\{k \in I : \langle \chi_k, \sigma_k\rangle = \langle \chi_j, \sigma_j\rangle\}$ is dense in I.

We will see later that the conditions for $\chi \equiv \sum_{i\in I} \chi_i$ are decidable.

7.3 Legal and perfect characters

We now define those characters that are reachable from degenerate ones by finitely many applications of Definition 41.

Definition 42 (legal character). Let Λ denote the smallest set of characters containing all degenerate characters and such that if I is a linear order, $\chi_i \in \Lambda$ for $i \in I$, and $\chi \equiv \sum_{i\in I} \chi_i$, then $\chi \in \Lambda$. A character χ is said to be *legal* if $\chi \in \Lambda$.

We also define those characters that may be descriptions of runs over the whole of the quasimodel.

Definition 43 (perfect character). A character $\chi = \langle S, S^{con}\rangle$ is said to be *perfect* if for every $\sigma \in S$,

- $\sigma \vdash \forall x(R_{\psi_1 \mathcal{U} \psi_2}(x) \leftrightarrow \exists y(x < y \wedge R_{\psi_2}(y) \wedge \forall z(x < z < y \rightarrow R_{\psi_1}(z))))$ for every $\psi_1 \mathcal{U} \psi_2 \in sub_x\varphi$,
- $\sigma \vdash \forall x(R_{\psi_1 \mathcal{S} \psi_2}(x) \leftrightarrow \exists y(y < x \wedge R_{\psi_2}(y) \wedge \forall z(y < z < x \rightarrow R_{\psi_1}(z))))$ for every $\psi_1 \mathcal{S} \psi_2 \in sub_x\varphi$,
- $\sigma \vdash \forall x \exists y, z(y < x < z)$,

and for some $\sigma \in S$ we have $\sigma \vdash \exists x R_\varphi(x)$.

By an *interval of* $\mathbb{R}$ we mean a linear order whose domain is a non-empty convex subset of $\mathbb{R}$, the ordering on it being induced from $\langle \mathbb{R}, < \rangle$. We will often abuse notation by identifying the subset of $\mathbb{R}$ with the linear order. Note that up to isomorphism there are just five intervals of $\mathbb{R}$, represented by $[0,1]$, $[0,1)$, $(0,1]$, $(0,1)$, and $\{0\}$. Here and below, we use standard notation for intervals: $[x,y) = \{z \in \mathbb{R} : x \leq z < y\}$ if $x \leq y$, etc.

Characters describe runs over some interval of a potential finitary quasimodel. We now make this precise.

Definition 44 (pre-quasimodel). A *pre-quasimodel* is a triple $\mathfrak{p} = \langle W, f, \mathcal{R} \rangle$, where W is a linear order isomorphic to an interval of $\mathbb{R}$, f is a state function for φ over W, $f(w) = \langle T_w, T_w^{con} \rangle$ is a finitely realizable state candidate for φ for each $w \in W$, and $\mathcal{R}$ is a finite set of functions $r : W \to \mathcal{T}$, satisfying the conditions:

- for each $c \in con\varphi$, the map $r_c^f : W \to \mathcal{T}$ defined by $r_c^f(w) = t$, where $\langle t, c \rangle \in T_w^{con}$, is in $\mathcal{R}$,
- for each $w \in W$ and $t \in T_w$ there is $r \in \mathcal{R}$ with $r(w) = t$.

Definition 45 (model of a character). Let $\mathfrak{p} = \langle W, f, \mathcal{R} \rangle$ be a pre-quasimodel for φ, and let $\chi = \langle S, S^{con} \rangle$ be a character. We write $\mathfrak{p} \models \chi$ if

- $\textit{3-th}(r_c^f) = S^{con}(c)$ for each $c \in con\varphi$,
- $\{\textit{3-th}(r) : r \in \mathcal{R}\} = S$.

7.4 The main lemma

We will prove:

Lemma 46.

1. *If χ is a perfect character, $\mathfrak{p} = \langle W, f, \mathcal{R} \rangle$ is a pre-quasimodel, and $\mathfrak{p} \models \chi$, then $\langle f, \mathcal{R} \rangle$ is a finitary quasimodel for φ over W in which φ is satisfied, and $W \cong \langle \mathbb{R}, < \rangle$.*
2. *If χ is a legal character, then there exists a pre-quasimodel $\mathfrak{p}$ with $\mathfrak{p} \models \chi$.*
3. *If $\mathfrak{m}$ is a finitary quasimodel for φ in which φ is satisfied, then there is a legal perfect character χ with $\mathfrak{m} \models \chi$.*
4. *Given an oracle that determines whether a given state candidate for φ is finitely realizable, it is decidable whether there exists a legal perfect character. The algorithm is uniform in φ.*

Theorem 29 and parts 1–3 of the lemma show that φ has a model with flow of time $\langle \mathbb{R}, < \rangle$ and finite domain iff there exists a perfect legal character. By part 4 of the lemma, given $\mathcal{TL}' \subseteq \mathcal{TL}_1$ and an algorithm that decides for any sentence $\varphi \in \mathcal{TL}'$ whether a given state candidate for φ is finitely realizable, it is decidable whether such a character exists. Hence, Theorem 36 follows from Lemma 46.

7.5 Proof of Lemma 46(1)

This is straightforward. Let $\chi = \langle S, S^{con} \rangle$ be a perfect character, $\mathfrak{p} = \langle W, f, \mathcal{R} \rangle$ a pre-quasimodel, and let $\mathfrak{p} \models \chi$. Then by the definitions, $\mathcal{R}$ is finite, and if $r \in \mathcal{R}$ we have *3-th*$(r) \in S$ and so r is a run in f. So $\mathfrak{m} = \langle f, \mathcal{R} \rangle$ is a finitary quasimodel for φ over W. Let $\sigma \in S$ be such that $\sigma \vdash \exists x R_\varphi(x)$, and let $r \in \mathcal{R}$ satisfy *3-th*$(r) = \sigma$. Then clearly, $\varphi \in r(w)$ for some $w \in W$, so that φ is satisfied in $\mathfrak{m}$. Since $\sigma \vdash \forall x \exists yz(y < x < z)$, W is isomorphic to an interval of $\mathbb{R}$ and has no endpoints, so we must have $W \cong \langle \mathbb{R}, < \rangle$.

7.6 Proof of Lemma 46(2)

Here, we prove the 'soundness' part of Lemma 46. (Some may wonder if it should be called 'completeness'.) We will show that if χ is a legal character then there is a pre-quasimodel $\mathfrak{p} \models \chi$. By definition of Λ, it suffices to prove that this holds for any degenerate χ, and that if I is a linear order, χ_i $(i \in I)$ are characters having pre-quasimodels, and $\chi \equiv \sum_{i \in I} \chi_i$, then $\mathfrak{p} \models \chi$ for some pre-quasimodel $\mathfrak{p}$.

Let $\chi = \langle S, S^{con} \rangle$ be a degenerate character. As in Definition 40, for $\sigma \in S$ let $t_\sigma = \{\psi \in sub_x\varphi : \sigma \vdash \exists x R_\psi(x)\}$, a type for φ. Let W be a one-point ordering with domain $\{w\}$, define $f(w)$ to be the finitely realizable state candidate $\big\langle \{t_\sigma : \sigma \in S\}, \{\langle t_{S^{con}(c)}, c \rangle : c \in con\varphi\} \big\rangle$ for φ, and for $\sigma \in S$ define $r_\sigma : W \to \mathcal{T}$ by $r_\sigma(w) = t_\sigma$.

Observe that *3-th*$(r_\sigma) = \sigma$. For, by definition of M_{r_σ}, for every $\psi \in sub_x\varphi$ we have $M_{r_\sigma} \models R_\psi(w)$ iff $\psi \in r_\sigma(w)$ iff $\sigma \vdash \exists x R_\psi(x)$. As $\sigma \vdash \forall xy \neg(x < y)$, we see that if $N \models \sigma$ then $N \cong M_{r_\sigma}$. Since such an N exists, we have $M_{r_\sigma} \models \sigma$. Hence, *3-th*$(r_\sigma) =$ *3-th*$(M_{r_\sigma}) = \sigma$.

As W is isomorphic to a (one-point) interval of $\mathbb{R}$, $\mathfrak{p} = \langle W, f, \{r_\sigma : \sigma \in S\} \rangle$ is evidently a pre-quasimodel, and $\mathfrak{p} \models \chi$.

For the inductive step, let I be a linear order and $\chi = \langle S, S^{con} \rangle$, $\chi_i = \langle S_i, S_i^{con} \rangle$, $\mathfrak{p}_i = \langle W_i, f_i, \mathcal{R}_i \rangle$ characters and pre-quasimodels with $\mathfrak{p}_i \models \chi_i$ (for all $i \in I$), and suppose that $\chi \equiv \sum_{i \in I} \chi_i$. We will define a pre-quasimodel $\langle W, f, \mathcal{R} \rangle$ and show that $\langle W, f, \mathcal{R} \rangle \models \chi$.

Let $W = \sum_{i \in I} W_i$. We show first that $(*)$ W is isomorphic to an interval of $\mathbb{R}$. If I is the order $0 < 1$, then our assumptions show that either W_0 has a right endpoint or W_1 a left endpoint, and not both, so that $(*)$ is clear. (For example, if $W_0 \cong [0,1]$ and $W_1 \cong (1,2)$ then $W \cong [0,2)$, an interval of $\mathbb{R}$.) If $I = \langle \mathbb{N}, < \rangle$, then each W_i has a left (say) endpoint, so again, $\sum_{i \in I} W_i$ is isomorphic to an interval of $\mathbb{R}$; the case $I = \langle \mathbb{N}, > \rangle$ is similar. Finally, suppose that I is a dense condensation of $\mathbb{R}$ without endpoints whose elements have left and right endpoints. Then by definition of $\equiv$, $W_i \cong i$ for each $i \in I$, so $(*)$ follows. All cases in the definition of $\equiv$ are now covered, and we are done.

For any functions g_i defined on W_i $(i \in I)$, we write $\sum_{i \in I} g_i$ for the function g on W defined by $g(\langle w, i \rangle) = g_i(w)$.

Lemma 47. *If $r_i : W_i \to \mathcal{T}$ $(i \in I)$, then 3-th*$(\sum_{i \in I} r_i) = \sum_{i \in I}$ *3-th*(r_i).

Proof Write r for $\sum_{i \in I} r_i$. By definition, *3-th*$(r) =$ *3-th*(M_r) and *3-th*$(r_i) =$ *3-th*(M_{r_i}) for each $i \in I$. Clearly, $M_r = \sum_{i \in I} M_{r_i}$. So $\sum_{i \in I}$ *3-th*(r_i) is by definition *3-th*(r). ❑

Define a state function $f = \sum_{i \in I} f_i$ on W, and write $f(w) = \langle T_w, T_w^{con} \rangle$ for $w \in W$. The definition of $\mathcal{R}$ will divide into cases according to the parts of the definition of $\equiv$, but in all cases we will arrange that $r_c^f \in \mathcal{R}$ for each $c \in con\varphi$. Given this much, we can already check that $\mathit{3\text{-}th}(r_c^f) = S^{con}(c)$. As $\mathfrak{p}_i \models \chi_i$ for each i, we have $\mathit{3\text{-}th}(r_c^{f_i}) = S_i^{con}(c)$. By the definitions and Lemma 47, we obtain

$$\left.\begin{array}{l} r_c^f = \sum_{i\in I} r_c^{f_i}, \\ \mathit{3\text{-}th}(r_c^f) = \mathit{3\text{-}th}\Big(\sum_{i\in I} r_c^{f_i}\Big) = \sum_{i\in I} \mathit{3\text{-}th}(r_c^{f_i}) = \sum_{i\in I} S_i^{con}(c) = S^{con}(c) \in S. \end{array}\right\} \qquad (11)$$

Now we go through the cases of Definition 41, defining $\mathcal{R}$ and checking that $\mathfrak{p} = \langle W, f, \mathcal{R}\rangle$ is a pre-quasimodel and $\mathfrak{p} \models \chi$.

4. ($I = \{0,1\}$) We define $\mathcal{R} = \{r_0 + r_1 : r_0 \in \mathcal{R}_0\ r_1 \in \mathcal{R}_1,\ \mathit{3\text{-}th}(r_0 + r_1) \in S\}$. This is clearly finite, since $\mathcal{R}_0$, $\mathcal{R}_1$ are finite.
   - By (11), $r_c^f \in \mathcal{R}$.
   - Let $w \in W$ and $t \in T_w$; we seek $r \in \mathcal{R}$ with $r(w) = t$. Let $w = \langle w', i\rangle$ for $w' \in W_i$, $i \in I$. As $\mathfrak{p}_i$ is a pre-quasimodel, there is $r_i \in \mathcal{R}_i$ with $r_i(w') = t$. As $\mathfrak{p}_i \models \chi_i$, $\mathit{3\text{-}th}(r_i) = \sigma_i \in S_i$. As $\chi \approx \chi_0 + \chi_1$, there is $\sigma_{1-i} \in S_{1-i}$ with $\sigma_0 + \sigma_1 \in S$, and as $\mathfrak{p}_{1-i} \models \chi_{1-i}$, there is $r_{1-i} \in \mathcal{R}_{1-i}$ with $\mathit{3\text{-}th}(r_{1-i}) = \sigma_{1-i}$. Then by Lemma 47, $r = r_0 + r_1$ satisfies
   $$\mathit{3\text{-}th}(r) = \mathit{3\text{-}th}(r_0 + r_1) = \mathit{3\text{-}th}(r_0) + \mathit{3\text{-}th}(r_1) = \sigma_0 + \sigma_1 \in S,$$
   so clearly, $r \in \mathcal{R}$ and $r(w) = r_i(w') = t$.
   - To prove $S = \{\mathit{3\text{-}th}(r) : r \in \mathcal{R}\}$, we only need check '$\subseteq$'—that if $\sigma \in S$ then there is $r \in \mathcal{R}$ with $\sigma = \mathit{3\text{-}th}(r)$. By condition (ii) of Definition 41, there are $\sigma_i \in S_i$ ($i = 0, 1$) with $\sigma = \sigma_0 + \sigma_1$, and since $\mathfrak{p}_i \models \chi_i$, there are $r_i \in \mathcal{R}_i$ with $\mathit{3\text{-}th}(r_i) = \sigma_i$, for each i. We may take $r = r_0 + r_1$.

5. ($I = \langle \mathbb{N}, <\rangle$) We may assume that $\mathfrak{p}_i = \mathfrak{p}_0$ for all $i \in I$, since $\chi_i = \chi_0$. We define
   $$\mathcal{R} = \{r : r = \sum_{i\in I} r_i \text{ for some } r_i \in \mathcal{R}_i\ (i \in I),\ r_i = r_0 \text{ for all } i,\ \mathit{3\text{-}th}(r) \in S\}.$$
   Clearly, $|\mathcal{R}| = |\mathcal{R}_0|$, so $\mathcal{R}$ is finite.
   - If $c \in con\varphi$ then $r_c^{f_i} = r_c^{f_0}$ for all $i \in I$, since $\mathfrak{p}_i = \mathfrak{p}_0$. It now follows from (11) that $r_c^f \in \mathcal{R}$.
   - We let $w \in W$ and $t \in T_w$ and find $r \in \mathcal{R}$ with $r(w) = t$. Suppose that $w = \langle w', n\rangle$, for $w' \in W_n$, $n \in \mathbb{N}$. As $\mathfrak{p}_n$ is a pre-quasimodel, we may pick $r_n \in \mathcal{R}_n$ with $r_n(w') = t$. As $\mathfrak{p}_n \models \chi_n$, $\mathit{3\text{-}th}(r_n) \in S_n$. Define $r_i = r_n$ for all $i \in I$. Then $\mathit{3\text{-}th}(\sum_{i\in I} r_i) = \sum_{i\in I} \mathit{3\text{-}th}(r_i) \in S$, so $r = \sum_{i\in I} r_i \in \mathcal{R}$ and $r(w) = r_n(w') = t$.
   - Each $\sigma \in S$ has the form $\sum_{i\in I} \sigma_i$ for $\sigma_i \in S_i$ ($i \in I$) with all σ_i equal to σ_0. By $\mathfrak{p}_0 \models \chi_0$, there is $r_0 \in \mathcal{R}_0$ with $\mathit{3\text{-}th}(r_0) = \sigma_0$. Let $r_i = r_0$, for each i, and $r = \sum_{i\in I} r_i$. Then $\mathit{3\text{-}th}(r) = \sigma$, so $r \in \mathcal{R}$. Hence, $S \subseteq \{\mathit{3\text{-}th}(r) : r \in \mathcal{R}\}$, and the converse inclusion is clear by definition of $\mathcal{R}$.

6. ($I = \langle \mathbb{N}, >\rangle$) This is similar to the preceding case.

7. (I is a dense condensation of $\mathbb{R}$) This is the most involved case. Again, we may as well suppose that if $\chi_i = \chi_j$ then $\mathfrak{p}_i = \mathfrak{p}_j$, for $i, j \in I$. The definition of $\mathcal{R}$ has two parts. First, observe that by Definition 41(ii), for each $\sigma \in S$ there are $\sigma_i \in S_i$ $(i \in I)$ such that $\sigma = \sum_{i \in I} \sigma_i$. For each i, pick $r_i \in \mathcal{R}_i$ with $\mathit{3\text{-}th}(r_i) = \sigma_i$, and let $r_\sigma = \sum_{i \in I} r_i$. Next, noting that it follows from the definition of $\equiv$ that for each character χ, the set $I_\chi = \{i \in I : \chi_i = \chi\}$ is either empty or dense in I, choose an equivalence relation $\sim$ on I with the properties:

   (∗) $\forall i, j \in I\ (i \sim j \Rightarrow \chi_i = \chi_j)$,
   for all $i \in I$, I_{χ_i} is partitioned by $\sim$ into $|S_i|$ equivalence classes, each dense in I.

   If $r_i \in \mathcal{R}_i$ for $i \in I$, the sequence $(r_i)_{i \in I}$ is said to be *simple* if $i \sim j$ implies $r_i = r_j$, for all $i, j \in I$. Note that there are only finitely many simple sequences. We let

   $$\mathcal{R} \;=\; \{r_\sigma : \sigma \in S\} \cup \Big\{ \sum_{i \in I} r_i : (r_i)_{i \in I} \text{ a simple sequence, } \mathit{3\text{-}th}(\sum_{i \in I} r_i) \in S \Big\}.$$

   - Observe that if $c \in \mathit{con}\varphi$ then by (∗), $(r_c^{f_i})_{i \in I}$ is simple, so by (11) as before, $r_c^f \in \mathcal{R}$.
   - Since by Lemma 47, $\mathit{3\text{-}th}(r_\sigma) = \sigma \in S$, we have $S = \{\mathit{3\text{-}th}(r) : r \in \mathcal{R}\}$.
   - Let $\langle w, j \rangle \in W$, and $t \in T_w$. We require $r \in \mathcal{R}$ with $r(\langle w, j \rangle) = t$. As $\mathfrak{p}_j$ is a pre-quasimodel, we may pick $r_j \in \mathcal{R}_j$ with $r_j(w) = t$. By Definition 41(vii), there are $\sigma_i \in S_i$ for $i \in I$ such that $\sum_{i \in I} \sigma_i \in S$, $\langle \chi_i, \sigma_i \rangle = \langle \chi_j, \mathit{3\text{-}th}(r_j) \rangle$ for some $i \in I$, and $\{k \in I : \langle \chi_k, \sigma_k \rangle = \langle \chi_i, \sigma_i \rangle\}$ is dense in I for each $i \in I$. We may therefore choose a new equivalence relation $\sim'$ on I satisfying the conditions (∗), such that if $i \sim' i'$ then $\sigma_i = \sigma_{i'}$. So, writing $i/{\sim'}$ for the $\sim'$-class of i (and similarly for $\sim$), we may define $\sigma_{i/\sim'}$ to be σ_i, for $i \in I$. Let $I_\chi/{\sim}$ denote the set of $\sim$-classes contained in I_χ, and define $I_\chi/{\sim'}$ similarly. By (∗), $|I_\chi/{\sim}| = |I_\chi/{\sim'}|$ for every χ, and we know that $\mathit{3\text{-}th}(r_j) = \sigma_e$ for some $e \in I_{\chi_j}/{\sim'}$. Since $j \in I_{\chi_j}$, we may pick a bijection $\theta : I/{\sim} \to I/{\sim'}$ such that

     (a) $\theta(I_\chi/{\sim}) = I_\chi/{\sim'}$, for all characters χ,
     (b) $\theta(j/{\sim}) = e$, so that $\sigma_{\theta(j/\sim)} = \mathit{3\text{-}th}(r_j)$.

     Now pick $r_i \in \mathcal{R}_i$ for each $i \in I \setminus \{j\}$ in such a way that $\forall i \in I(\mathit{3\text{-}th}(r_i) = \sigma_{\theta(i/\sim)} \in S_i)$ and $\forall i, k \in I\,(i \sim k \Rightarrow r_i = r_k)$. Thus, the sequence $(r_i)_{i \in I}$ is simple. For every $i \in I$, the set $\{k \in I : \langle \chi_k, \mathit{3\text{-}th}(r_k) \rangle = \langle \chi_i, \mathit{3\text{-}th}(r_i) \rangle\}$ contains $i/{\sim}$, so by (∗) it is dense in I. We saw that an analogous property holds for $(\sigma_i)_{i \in I}$. A Feferman–Vaught argument (cf. [25, Theorem A.6.2]) now shows that $\sum_{i \in I} \mathit{3\text{-}th}(r_i) = \sum_{i \in I} \sigma_i \in S$. Hence, $r = \sum_{i \in I} r_i \in \mathcal{R}$, and $r(\langle w, j \rangle) = r_j(w) = t$.

     Remark 48. This part of the argument seems to fail in the arbitrary-domain case—there is no obvious analogue for the last, density condition of Definition 41(vii) in that case. This does not necessarily mean that the finite-domain case is 'easier', as opposed to 'different'. We conjecture that the argument of the first half of [11] may apply in arbitrary domains.

7.7 Proof of Lemma 46(3)

The argument is very similar to one in [11]. Let $\mathfrak{m} = \langle f, \mathcal{R} \rangle$ be a finitary quasimodel for φ over $\langle \mathbb{R}, < \rangle$ in which φ is satisfied. For any interval E of $\mathbb{R}$, we write $\mathfrak{m}{\restriction}E$ for

$\langle E, f\restriction E, \{r\restriction E : r \in \mathcal{R}\}\rangle$; note that $\mathfrak{m}\restriction E$ is a pre-quasimodel. We write χ_E for the character

$$\chi_E = \Big\langle \{\textit{3-th}(r\restriction E) : r \in \mathcal{R}\},\ (c \mapsto \textit{3-th}(r_c^f\restriction E))_{c \in con\varphi} \Big\rangle .$$

It is clear that $\mathfrak{m}\restriction E \models \chi_E$ for all E, and that $\chi_{\mathbb{R}}$ is perfect. We are going to show that $\chi_{\mathbb{R}}$ is legal.

Lemma 49. *Let I be a linear order and let E_i be an interval of $\mathbb{R}$ for each $i \in I$, such that $E = \bigcup_{i \in I} E_i$ is also an interval of $\mathbb{R}$, and $x < y$ whenever $i < j$ in I, $x \in E_i$, and $y \in E_j$. Then*

1. *$\textit{3-th}(r\restriction E) = \sum_{i \in I} \textit{3-th}(r\restriction E_i)$ for each $r \in \mathcal{R}$,*
2. *$\chi_E \approx \sum_{i \in I} \chi_{E_i}$.*

Proof Let $r \in \mathcal{R}$. Then by definition, $\textit{3-th}(r\restriction E) = \textit{3-th}(M_{r\restriction E})$ and $\textit{3-th}(r\restriction E_i) = \textit{3-th}(M_{r\restriction E_i})$ for each i. Clearly, $M_{r\restriction E} = \sum_{i \in I} M_{r\restriction E_i}$. So $\sum_{i \in I} \textit{3-th}(r\restriction E_i)$ is by definition $\textit{3-th}(r\restriction E)$.

We now check that $\chi_E \approx \sum_{i \in I} \chi_{E_i}$. Let $\chi_E = \langle S, S^{con}\rangle$, and $\chi_{E_i} = \langle S_i, S_i^{con}\rangle$ for $i \in I$. If $c \in con\varphi$, then by definition, $S^{con}(c) = \textit{3-th}(r_c^f\restriction E)$ and $S_i^{con}(c) = 3th(r_c^f\restriction E_i)$ for $i \in I$. Of course, $r_c^f \in \mathcal{R}$. By part 1, we conclude that $S^{con}(c) = \sum_{i \in I} S_i^{con}(c)$.

Parts (ii) and (iii) of Definition 41 follow easily from the fact that

$$S = \{\textit{3-th}(r\restriction E) : r \in \mathcal{R}\} = \Big\{ \sum_{i \in I} \textit{3-th}(r\restriction E_i) : r \in \mathcal{R} \Big\}. \tag{12}$$

❑

Definition 50 (good interval). We say that an interval E of $\mathbb{R}$ is *good* if χ_E is legal.

Lemma 51. *Any one-point interval of $\mathbb{R}$ is good.*

Proof Let E be such, with domain $\{e\}$; we claim that $\chi_E = \langle S, S^{con}\rangle$, say, is degenerate. Each $\sigma \in S$ has the form $\textit{3-th}(r\restriction E)$ for some $r \in \mathcal{R}$. Then $M_{r\restriction E} \models \forall xy\neg(x < y)$, so $\sigma = \textit{3-th}(M_{r\restriction E})$ is degenerate. Further,

$$t_\sigma = \{\psi \in sub_x\varphi : \sigma \vdash \exists x R_\psi(x)\} = \{\psi \in sub_x\varphi : M_{r\restriction E} \models R_\psi(e)\} = r(e),$$

a type for φ. As $\mathfrak{m}$ is a finitary quasimodel for φ,

$$\big\langle \{t_\sigma : \sigma \in S\}, \{\langle t_{S^{con}(c)}, c\rangle : c \in con\varphi\} \big\rangle = \Big\langle \{r(e) : r \in \mathcal{R}\},\ \{\langle r_c^f(e), c\rangle : c \in con\varphi\} \Big\rangle = f(e),$$

a finitely realizable state candidate for φ. ❑

Lemma 52. *Assume the conditions of Lemma 49, that $I = \{0,1\}$ with $0 < 1$, and that E_0 and E_1 are good. Then E is good too.*

Proof It suffices to prove that $\chi_E \equiv \chi_{E_0} + \chi_{E_1}$.

As E is an interval of $\mathbb{R}$, either E_0 has a right endpoint or E_1 a left endpoint. Assume the former; the other case is similar. If $r \in \mathcal{R}$ then by definition, $\textit{3-th}(r\restriction E_0) = \textit{3-th}(M_{r\restriction E_0})$, so as $M_{r\restriction E_0} \models \exists x \forall y \neg(x < y)$, $\textit{3-th}(r\restriction E_0) \vdash \exists x \forall y \neg(x < y)$. Hence, χ_{E_0} has a left endpoint.

By Lemma 49(2), $\chi_E \approx \chi_{E_0} + \chi_{E_1}$, and we conclude that $\chi_E \equiv \chi_{E_0} + \chi_{E_1}$. ❑

Lemma 53. *Again assume the conditions of Lemma 49, that $I \in \{\langle \mathbb{N}, < \rangle, \langle \mathbb{N}, > \rangle, \langle \mathbb{Z}, < \rangle\}$, and that every E_i ($i \in I$) is good. Then E is good.*

Proof We only consider the case $I = \langle \mathbb{N}, < \rangle$; the case $\langle \mathbb{N}, > \rangle$ is similar, and $\langle \mathbb{Z}, < \rangle$ is handled using $\langle \mathbb{N}, > \rangle$, $\langle \mathbb{N}, < \rangle$, and Lemma 52. For $i < j$ in $\mathbb{N}$, write E_{ij} for the interval $\bigcup_{i \leq k < j} E_k$ of $\mathbb{R}$. By Lemma 52 and induction on $j - i$, E_{ij} is good. There are only finitely many characters, so by Ramsey's theorem [37], there is infinite $X \subseteq \mathbb{N}$ such that $\chi_{E_{ij}}$ is constant for all $i < j$ in X. Let $x \in X$ be minimal. As $E_{0,x}$ is good, by Lemma 52 it suffices to prove that $\bigcup_{i \geq x} E_i$ is good. Therefore, by renaming, we may assume that $\chi_{E_{ij}}$ is constant for all $i < j$ in $\mathbb{N}$. As $\mathcal{R}$ is finite, we may further assume (by Ramsey's theorem) that for each $r \in \mathcal{R}$, *3-th*$(r \restriction E_{ij})$ is the same for all $i < j$ in $\mathbb{N}$.

We will show that $\chi_E \equiv \sum_{i \in I} \chi_{E_i}$.

We know that $\chi_{E_i} = \chi_{E_0}$ for all $i \in I$. Since E_0, E_1 are disjoint convex subsets of $\mathbb{R}$ whose union is convex, either E_0 has a right endpoint or E_1 a left endpoint—and not both. It follows as in Lemma 49 that χ_{E_0} has either a left or right endpoint.

Let $\chi_E = \langle S, S^{con} \rangle$ and $\chi_{E_i} = \langle S_i, S_i^{con} \rangle$ for $i \in I$, as usual.

- By Lemma 49, $S^{con}(c) = \textit{3-th}(r_c^f \restriction E) = \sum_{i \in I} \textit{3-th}(r_c^f \restriction E_i) = \sum_{i \in I} S_i^{con}(c)$ for each $c \in con\varphi$.
- $S = \{\sum_{i \in I} \sigma_i : \sigma_i \in S_0,\ \sigma_i = \sigma_0 \text{ for all } i \in I\}$ is true because (12) holds, and by the above, $r \restriction E_i = r \restriction E_0$ for each $r \in \mathcal{R}$, $i \in I$.

Now Definition 41(v) gives $\chi_E \equiv \sum_{i \in I} \chi_{E_i}$. Since the χ_{E_i} are assumed legal, so is χ_E, and we conclude that E is good. ❑

Definition 54. We define a binary relation $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $x = y$, or $x < y$ and every convex subset contained in $[x, y]$ is good, or $y < x$ and every convex subset contained in $[y, x]$ is good.

Lemma 55. *$\sim$ is an equivalence relation on W, and any $\sim$-class is itself an interval of $\mathbb{R}$.*

Proof Only transitivity needs a proof. Assume that $x \sim y \sim z$ in $\mathbb{R}$; we check that $x \sim z$. There are various cases, depending on the order-type of x, y, z. If $x < z < y$, it is clear. Assume that $x < y < z$, let E be a convex subset of $[x, z]$, $E_0 = E \cap [x, y)$, and $E_1 = E \cap [y, z]$. If either E_0 or E_1 is empty, then certainly E is good. Otherwise, we are in the situation of Lemma 52, so again E is good. The other cases are similar. Hence, $x \sim z$, as required.

It is clear by definition that any $\sim$-class is convex. ❑

Lemma 56. *Any subinterval E of any $\sim$-class is good.*

Proof There are four cases, depending on the endpoints of E. If $E = [x, y]$ for some $x < y$ in $\mathbb{R}$, then $x \sim y$ and the result is trivial. Assume that E has a left-hand endpoint x_0 but no right-hand endpoint. Choose an increasing sequence $x_0 < x_1 < \cdots$ in E, of order type $\langle \mathbb{N}, < \rangle$ and unbounded in E, and let $E_i = [x_i, x_{i+1})$. Since $x_i \sim x_{i+1}$, E_i is good. Now we are in the situation of Lemma 53, and we conclude that E is good. The other two cases, when E has no left-hand endpoint, can be covered using the cases $\langle \mathbb{N}, > \rangle$ and $\langle \mathbb{Z}, < \rangle$ of Lemma 53. ❑

Lemma 57. *Each $\sim$-class is a closed interval of $\mathbb{R}$.*

Proof Let E be a $\sim$-class, and suppose that E has a least upper bound $b \in \mathbb{R}$. We show that $b \in E$. Take $e \in E$, and any interval D of $\mathbb{R}$ with $D \subseteq [e, b]$. Lemma 56 shows that $D \cap E$ is good. If $D \subseteq E$, we are done. Otherwise, $D = (D \cap E) \cup \{b\}$, and Lemmas 51, 52, and 56 show that D is good. So $b \sim e$ and $b \in E$.

Similarly, E contains any greatest lower bound for it. So it is closed. ❑

We aim to show that $\mathbb{R}$ is a single $\sim$-class. To this end, assume not: so the condensation $C = \mathbb{R}/\sim$ given by $\sim$ has at least two elements. Because $\mathbb{R}$ is dense, Lemma 57 now shows that C is a dense ordering. Enumerate $\mathcal{R}$ as $\langle r^n : n < N \rangle$, and choose an open interval I of C such that the finite set

$$\{\langle \chi_E, \textit{3-th}(r^n \restriction E) : n < N \rangle : E \in I\}$$

has least possible cardinality. It follows that for each open interval $J \subseteq I$ and each sequence $\xi = \langle \chi, \sigma_n : n < N \rangle$ of a character and N 3-theories, $\{E \in J : \langle \chi_E, \textit{3-th}(r^n \restriction E) : n < N \rangle = \xi\}$ is empty or dense in J.

It can now be seen that $\chi_{\bigcup J} \equiv \sum_{E \in J} \chi_E$ by dint of Definition 41(vii). Certainly, J is isomorphic to a dense condensation of $\langle \mathbb{R}, < \rangle$ without endpoints. By Lemma 49(2), conditions (i) and (ii) hold. By Lemma 57, each $E \in J$ has a right and a left endpoint, and since if $r \in \mathcal{R}$ and $E \in J$ then $M_{r \restriction E} \models \textit{3-th}(r \restriction E)$ and the underlying order of $M_{r \restriction E}$ is E, χ_E has left and right endpoints too. Similarly, $|E| = 1$ iff $\textit{3-th}(r \restriction E) \vdash \forall xy \neg (x < y)$ for all $r \in \mathcal{R}$. The last part of Definition 41(vii) holds because for any $r \in \mathcal{R}$ and $E \in J$, $\{E' \in J : \langle \chi_{E'}, \textit{3-th}(r \restriction E') \rangle = \langle \chi_E, \textit{3-th}(r \restriction E) \rangle$ is dense in J.

So $\bigcup J$ is good. By Lemma 56, each $E \in J$ is good, and Lemma 52 now shows that if J is any subinterval of I then $\bigcup J$ is good.

Take $x < y$ in $\bigcup I$ with $x \not\sim y$. So there is an interval $X \subseteq [x, y]$ that is no good. Let $\overline{X} = \{E \in I : E \subseteq X\}$. Then $\overline{X}$ is a subinterval of I, so $\bigcup \overline{X}$ is good. Let $X_< = \{z \in X : z < v$ for all $v \in \bigcup \overline{X}\}$, and define $X_>$ similarly. By Lemma 56, $X_<$ and $X_>$ are good. We have $X = X_< + \bigcup \overline{X} + X_>$, so by Lemma 52, X is itself good, a contradiction.

Hence indeed, $\mathbb{R}$ is a single $\sim$-class, so is good—$\chi_{\mathbb{R}}$ is legal. This completes the proof.

7.8 Proof of Lemma 46(4)

Assume that we have an oracle telling whether a given state candidate for φ is finitely realizable. We show how to use it to decide whether there exists a legal perfect character. The decision procedure is uniform in φ. Our method is to reduce the problem to the satisfiability of certain existential monadic second-order sentences in $\langle \mathbb{R}, < \rangle$. By [11, Theorem 2.9(d)], such problems are decidable. This reduction is quite quick to present, avoiding several semantic subtleties, but since [11] uses much the same methods as here, it is a very convoluted way of obtaining decidability. It is easy but tedious to give a more direct algorithm.

Recall that up to logical equivalence there are finitely many 3-theories. Indeed, we may easily construct from φ a finite set T of $\mathcal{L}_\varphi$-sentences of quantifier depth at most 3, closed under single negations and containing every such sentence up to logical equivalence, and in particular containing the sentences $\exists x \forall y \neg (y < x)$, $\exists x \forall y \neg (x < y)$, $\forall xy \neg (x < y)$, and $\exists x R_\psi(x)$ for $\psi \in sub_x \varphi$, and their negations. Any 3-theory can be taken to be a certain subset of T, and a character a pair $\langle S, S^{con} \rangle$ where $S \subseteq \wp T$ ($\wp$ denotes the power set) and $S^{con} : con\varphi \to S$.

Note that not every such object is a 3-theory (or character). Nonetheless, we have:

Lemma 58. *Given $\sigma \subseteq T$ and $\chi = \langle S, S^{con} \rangle$ where $S \subseteq \wp T$ and $S^{con} : con\varphi \to S$, it is decidable whether σ is a 3-theory and χ is a character.*

Proof $\sigma \subseteq T$ is a 3-theory iff it contains every sentence in T or its negation, and the sentence $\exists_{\psi \in sub_x\varphi} R_\psi \bigwedge \sigma$ is true in some linear order. Hence, by the decidability of the universal monadic second-order theory of linear order [22, 11], it is decidable whether σ is a 3-theory or not. Therefore, whether χ is a character is also decidable. ❑

By this result, it suffices to show that it is decidable (using the oracle) whether a given character is legal or perfect. We can decide by inspection whether a character is perfect. For legality, there are two parts.

Lemma 59. *Given $S \subseteq \wp T$ and $S^{con} : con\varphi \to S$, it is decidable (using the oracle) whether $\chi = \langle S, S^{con} \rangle$ is a degenerate character.*

Proof We simply check that χ is a character and that each $\sigma \in S$ contains $\forall xy\neg(x < y)$. Then we check by inspection (cf. Definition 5) that for each $\sigma \in S$, the set $t_\sigma = \{\psi \in sub_x\varphi : \exists x R_\psi(x) \in \sigma\}$ is a type for φ. Finally, we check with the oracle that $\big\langle \{t_\sigma : \sigma \in S\}, \{\langle t_{S^{con}(c)}, c\rangle : c \in con\varphi\}\big\rangle$ is a finitely realizable state candidate for φ. χ is a degenerate character iff all these checks succeed. ❑

Lemma 60. *Let $\mathcal{S}$ be a set of characters and χ be a character. It is decidable whether there exist a linear order I and characters $\chi_i \in \mathcal{S}$ $(i \in I)$ such that $\chi \equiv \sum_{i \in I} \chi_i$.*

Proof We refer to Definition 41. We can certainly decide whether a character has a left or right endpoint. For the remainder, we need some notation. If $\alpha(x)$ is a first-order formula with x and perhaps other variables free, and θ is a first-order formula, we define the relativization $\theta^{\alpha(x)}$ of θ to $\alpha(x)$ in the usual way, by first renaming variables of θ so that they do not occur in α, and then setting $\theta^\alpha = \theta$ for atomic θ, $(\theta \wedge \theta')^\alpha = \theta^\alpha \wedge \theta'^\alpha$, $(\neg\theta)^\alpha = \neg\theta^\alpha$, and $(\exists y\theta)^\alpha = \exists y(\alpha(y/x) \wedge \theta^\alpha)$. We will always use the variable x for relativisation, and θ will always be a sentence, so that it is harmless to rename its variables. We note that any 3-theory σ is satisfiable in a countable $\mathcal{L}_\varphi$-order, and that any countable linear order embeds in $\langle \mathbb{R}, < \rangle$. Hence, if P is a new unary predicate, $\sigma^{P(x)}$ is true in some expansion of $\langle \mathbb{R}, < \rangle$ interpreting the symbols of $\mathcal{L}_\varphi \cup \{P\}$.

Now we go through the cases in Definition 41 once more.

4. ($I = \{0, 1\}$) Introduce new unary predicates P_0, P_1. For 3-theories $\sigma, \sigma_0, \sigma_1$, we have $\sigma = \sigma_0 + \sigma_1$ iff the conjunction of the following sentences is true in some expansion of $\langle \mathbb{R}, < \rangle$: $(\bigwedge \sigma_0)^{P_0(x)}$, $(\bigwedge \sigma_1)^{P_1(x)}$, $(\bigwedge \sigma)^{P_0 \vee P_1(x)}$, $\bigwedge_{i<2} \exists x P_i(x)$, and $\forall xy(P_0(x) \wedge P_1(y) \to x < y)$. By the result of [11] already mentioned, this is decidable. The definition of $\chi \equiv \chi_0 + \chi_1$ is a boolean combination of such conditions, and is therefore decidable. So we can decide whether $\chi \equiv \chi_0 + \chi_1$ for some $\chi_0, \chi_1 \in \mathcal{S}$, by considering all of the finitely many possibilities for χ_0, χ_1.

5. ($I = \langle \mathbb{N}, < \rangle$) Let P, Q be new unary predicates and let ν be the conjunction of the sentences $\forall x\neg(P(x) \wedge Q(x))$, $\exists x(Q(x) \wedge \forall y < x(\neg P(y) \wedge \neg Q(y)))$, $\forall x \exists y > x Q(y)$, $\forall x \exists y < x \forall z \in (y, x)\neg Q(z)$, $\forall x \exists y > x \forall z \in (x, y)\neg Q(z)$, and $\forall x \exists y > x P(x)$. An expansion of $\langle \mathbb{R}, < \rangle$ is a model of ν iff (the interpretations of) P, Q are disjoint and

unbounded above in $\mathbb{R}$, Q has order type $\langle \mathbb{N}, < \rangle$, and there is no P before the first Q. Let $\alpha(x, y)$ be the formula $P(x) \wedge \forall z((x \leq z \leq y \vee y \leq z \leq x) \to \neg Q(z))$.

Let σ, σ_i $(i \in I)$ be 3-theories with $\sigma_i = \sigma_0$ for all i. Then $\sigma = \sum_{i \in I} \sigma_i$ iff the conjunction of the following sentences is true in some expansion of $\langle \mathbb{R}, < \rangle$: ν, $\sigma^{P(x)}$, and $\forall y(P(y) \to (\sigma_0)^{\alpha(x,y)})$ (relativizing on x as said before). This statement is decidable, so given characters χ, $\chi_0 = \chi_1 = \cdots$, we can check effectively whether $S^{con}(c) = \sum_{i \in I} S_i^{con}(c)$ for all $c \in con\varphi$ and whether $S = \{\sum_{i \in I} \sigma_i : \sigma_i \in S_i,\ \sigma_i = \sigma_0 \text{ for all } i\}$. Thus, whether $\chi \equiv \sum_{i \in I} \chi_i$ for some $\chi_0 = \chi_1 = \cdots$ in $\mathcal{S}$ is decidable.

6. $(I = \langle \mathbb{N}, > \rangle)$ This is no different.

7. (I is a dense condensation of $\mathbb{R}$) We will need to make copies $\mathcal{L}_s$ of the signature $\mathcal{L}_\varphi = \{<, R_\psi : \psi \in sub_x\varphi\}$, for various objects s, by renaming the symbols R_ψ. We assume that if $s \neq t$ then $\mathcal{L}_s \cap \mathcal{L}_t$ consists of just the symbol $<$ for the order. If $\mathcal{L}_s$ is such a copy, and θ is an $\mathcal{L}_\varphi$-sentence, we write $\theta_{\mathcal{L}_s}$ for the result of replacing the relation symbols of $\mathcal{L}_\varphi$ in θ by the corresponding ones in $\mathcal{L}_s$.

   For a unary predicate P, we let $\alpha(x, y, P)$ be $\forall z((x \leq z \leq y \vee y \leq z \leq x) \to P(z))$.

   Let $\{\chi_0, \ldots, \chi_{n-1}\}$ be a set of characters, with $n \geq 2$, and let $\chi = \langle S, S^{con} \rangle$ be another character. Write $\chi_i = \langle S_i, S_i^{con} \rangle$, as usual. Introduce new unary predicates X_i $(i < n)$, and consider the following sentences:

   - $\forall x \bigvee_{i<n} (X_i(x) \wedge \bigwedge_{j \neq i} \neg X_j(x))$,
   - $\bigwedge_{i<n} \forall x \exists yz (y < x < z \wedge X_i(y) \wedge X_i(z))$,
   - $\forall xy \bigwedge_{i \neq j} (x < y \wedge X_i(x) \wedge X_j(y) \to \bigwedge_{k<n} \exists z \in (x, y) X_k(z))$.
     These three say that the condensation given by '$x \sim y$ iff $\bigvee_{i<n} \alpha(x, y, X_i)$' is dense without endpoints, and indeed that the classes included in any X_i occur densely.
   - For each $c \in con\varphi$, take a copy $\mathcal{L}_c$ of $\mathcal{L}_\varphi$ and add the sentences $(\bigwedge S^{con}(c))_{\mathcal{L}_c}$ and $\forall y(X_i(y) \to (\bigwedge S_i^{con}(c))_{\mathcal{L}_c}^{\alpha(x,y,X_i)})$ for each $i < n$.
   - For each $\sigma \in S$, take a copy $\mathcal{L}_\sigma$ of $\mathcal{L}_\varphi$, and add the sentences $(\bigwedge \sigma)_{\mathcal{L}_\sigma}$ and $\forall y \bigwedge_{i<n} (X_i(y) \to \bigvee_{\sigma_i \in S_i} (\bigwedge \sigma_i)_{\mathcal{L}_\sigma}^{\alpha(x,y,X_i)})$.
   - Finally, for each $\pi = \langle j, \sigma \rangle$ where $j < n$ and $\sigma \in S_j$, introduce new unary predicates $Q_{\pi,i,\sigma'}$ for $i < n$ and $\sigma' \in S_i$, and add the sentences:
     - 'The $Q_{\pi,i,\sigma'}$ are pairwise disjoint',
     - $\bigwedge_{i<n} \forall x (X_i(x) \leftrightarrow \bigvee_{\sigma' \in S_i} Q_{\pi,i,\sigma'}(x))$,
     - $(\exists x Q_\eta(x)) \to \forall xy(x < y \wedge Q_{\eta'}(x) \wedge Q_{\eta''}(y) \to \exists z \in (x, y) Q_\eta(z))$, for any three triples η, η', η'' of the form $\langle \pi, i, \sigma' \rangle$ for fixed π as above and with $\eta' \neq \eta''$,
     - $\forall y(Q_{\pi,i,\sigma'}(y) \to (\bigwedge \sigma')_{\mathcal{L}_\pi}^{\alpha(x,y,Q_{\pi,i,\sigma'})})$, for each i, σ',
     - $\bigvee_{\sigma \in S} (\bigwedge \sigma)_{\mathcal{L}_\pi}$.

It is not so hard to check that the conjunction of these sentences is true in some expansion of $\langle \mathbb{R}, < \rangle$ iff $\chi \equiv \sum_{i \in I} \chi_i$, where I is a condensation of $\langle \mathbb{R}, < \rangle$, $\{\chi_i : i \in I\} = \{\chi_0, \ldots, \chi_{n-1}\}$, and the provisions of Definition 41(vii) are met. Hence, as before, it is decidable whether $\chi \equiv \sum_{i \in I} \chi_i$ via Definition 41(vii) for some $\chi_i \in \mathcal{S}$.

❑

Now we decide whether a character λ is legal as follows. Build the set Λ_0 of all degenerate characters, using Lemmas 58 and 59. Given Λ_n, check for each character $\chi \notin \Lambda_n$ whether $\chi \equiv \sum_{i \in I} \chi_i$ for some linear order I and some $\chi_i \in \Lambda_n$, using Lemma 60. If so, put χ in Λ_{n+1}. Increment n, and repeat. Terminate when $\Lambda_{n+1} = \Lambda_n$, and check whether $\lambda \in \Lambda_n$. This determines whether λ is legal, and completes the proof of Lemma 46 and Theorem 36.

8 Applications

In this section, we apply the conditional decidability criteria obtained above in order to single out a number of decidable fragments of various temporal logics. We begin by discussing a major alternative approach to temporal reasoning, via two-sorted first-order logic (see, e.g., [2, 3, 12, 13]).

8.1 Two-sorted temporal logic

Consider a first-order logic with two sorts: domain and time. The language $\mathcal{TS}$ of the logic is based on the following alphabet:

- an infinite set of individual variables $x_0, x_1, \dots$ and a set of constants $c_0, c_1, \dots$ of domain sort,
- an infinite set of individual variables $t_0, t_1, \dots$ of temporal sort,
- the binary predicate symbol $<$ of sort 'temporal $\times$ temporal',
- predicate symbols $P_0, P_1, \dots$ of sort 'temporal $\times$ domain^n', $n < \omega$.

Formulas of $\mathcal{TS}$ are defined inductively:

- $t_i < t_j$ is an (atomic) formula, for temporal variables t_i, t_j,
- $P(t, x_1, \dots, x_n)$ is an (atomic) formula, for a predicate symbol P of sort temporal $\times$ domain^n, t a temporal variable, and $x_1, \dots, x_n$ domain variables,
- if φ and ψ are formulas, t a temporal variable, and x a domain variable, then $\neg\varphi$, $\varphi \wedge \psi$, $\forall t \varphi$, and $\forall x \varphi$ are formulas.

$\mathcal{TS}$ is interpreted in *first-order temporal models* of the usual form $\mathfrak{M} = \langle \mathfrak{F}, D, I \rangle$, where $\mathfrak{F} = \langle W, < \rangle$ is a flow of time (i.e., a strict linear order), D is a non-empty set, the *domain* of $\mathfrak{M}$, and I is a function associating with every moment of time $w \in W$ a first-order $\mathcal{L}$-structure

$$I(w) = \left\langle D, P_0^{I(w)}, \dots, c_0^{I(w)}, \dots \right\rangle,$$

in which $P_i^{I(w)}$, for each i, is a predicate on D of arity n whenever P_i is of arity $n+1$, and $c_i^{I(w)} \in D$.

An *assignment* in $\mathfrak{M}$ is a function $\mathfrak{a} = \mathfrak{a}_1 \cup \mathfrak{a}_2$ such that $\mathfrak{a}_1$ associates with every temporal variable t a moment of time $\mathfrak{a}_1(t) \in W$ and $\mathfrak{a}_2$ associates with every domain variable x an element $\mathfrak{a}_2(x)$ of D.

The truth relation $\mathfrak{M} \models^{\mathfrak{a}} \varphi$ is defined inductively as follows:

- $\mathfrak{M} \models^{\mathfrak{a}} t_i < t_j$ iff $\mathfrak{F} \models \mathfrak{a}_1(t_i) < \mathfrak{a}_1(t_j)$,
- $\mathfrak{M} \models^{\mathfrak{a}} P(t, x_1, \dots, x_n)$ iff $\langle \mathfrak{a}_2(x_1), \dots, \mathfrak{a}_2(x_n)\rangle \in P^{I(\mathfrak{a}_1(t))}$,
- $\mathfrak{M} \models^{\mathfrak{a}} \forall t\varphi$ iff $\mathfrak{M} \models^{\mathfrak{b}} \varphi$ for every assignment $\mathfrak{b}$ that may differ from $\mathfrak{a}$ only on t,
- $\mathfrak{M} \models^{\mathfrak{a}} \forall x\varphi$ iff $\mathfrak{M} \models^{\mathfrak{b}} \varphi$ for every assignment $\mathfrak{b}$ that may differ from $\mathfrak{a}$ only on x,

and the standard clauses for the booleans.

It should be clear that the temporal operators $\mathcal{U}$ and $\mathcal{S}$ of $\mathcal{TL}$ are expressible in $\mathcal{TS}$. On the other hand, there are $\mathcal{TS}$-formulas that are not expressible in $\mathcal{TL}$ over any interesting class of flows of time (see below). It turns out, however, that $\mathcal{TL}$ and $\mathcal{TL}_1$ are expressively complete for some natural fragments of $\mathcal{TS}$.

Definition 61. Let $\mathcal{TS}_{1t}$ (respectively, $\mathcal{TS}_{1x}$) consist of all $\mathcal{TS}$-formulas φ without subformulas of the form $\forall x\psi$ ($\forall t\psi$) such that ψ contains more than one free temporal (respectively, domain) variable. Let $\mathcal{TS}_1 = \mathcal{TS}_{1t} \cap \mathcal{TS}_{1x}$.

Suppose that each n-ary predicate symbol Q_i of $\mathcal{TL}$ is associated with the $(n+1)$-ary predicate symbol P_i of $\mathcal{TS}$. Define a translation $\dagger$ from $\mathcal{TL}$ into $\mathcal{TS}$ by taking, for some fixed temporal variable t,

$$\begin{aligned}
Q_i(x_1, \dots, x_n)^\dagger &= P_i(t, x_1, \dots, x_n),\\
(\varphi \wedge \psi)^\dagger &= \varphi^\dagger \wedge \psi^\dagger,\\
(\neg\varphi)^\dagger &= \neg(\varphi^\dagger),\\
(\forall x\varphi)^\dagger &= \forall x(\varphi^\dagger),\\
(\psi\mathcal{U}\varphi)^\dagger &= \exists t'(t < t' \wedge \varphi^\dagger\{t'/t\} \wedge \forall t''(t < t'' < t' \to \psi^\dagger\{t''/t\})),\\
(\psi\mathcal{S}\varphi)^\dagger &= \exists t'(t' < t \wedge \varphi^\dagger\{t'/t\} \wedge \forall t''(t' < t'' < t \to \psi^\dagger\{t''/t\})),
\end{aligned}$$

where t' and t'' are new temporal variables.

Note that for every $\mathcal{TL}$-formula φ, we have $\varphi^\dagger \in \mathcal{TS}_{1t}$, and for every $\varphi \in \mathcal{TL}_1$ we have $\varphi^\dagger \in \mathcal{TS}_1$.

The meaning of the translation $\dagger$ is explained by:

Definition 62. Let $\mathfrak{M} = \langle \mathfrak{F}, D, I\rangle$ be a $\mathcal{TS}$-model and $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2)$ an assignment in $\mathfrak{M}$. Let $\mathfrak{N} = \langle \mathfrak{F}, D, J\rangle$ be a $\mathcal{TL}$-model, $\mathfrak{b}$ an assignment in $\mathfrak{N}$. We say that $\langle \mathfrak{M}, \mathfrak{a}\rangle$ and $\langle \mathfrak{N}, \mathfrak{b}\rangle$ are *equivalent,* and write $\langle \mathfrak{M}, \mathfrak{a}\rangle \sim \langle \mathfrak{N}, \mathfrak{b}\rangle$, if $P_i^{I(w)} = Q_i^{J(w)}$ for all w, i, and $\mathfrak{a}_2 = \mathfrak{b}$.

Lemma 63. *Suppose $\langle \mathfrak{M}, \mathfrak{a}\rangle \sim \langle \mathfrak{N}, \mathfrak{b}\rangle$. Then for every $\mathcal{TL}$-formula φ and every moment of time w, if $\mathfrak{a}(t) = w$ then*

$$\langle \mathfrak{N}, w\rangle \models^{\mathfrak{b}} \varphi \text{ iff } \mathfrak{M} \models^{\mathfrak{a}} \varphi^\dagger.$$

Proof An easy induction on φ. ❑

Definition 64. Let $\mathcal{F}$ be a class of flows of time, $\mathcal{L}' \subseteq \mathcal{TL}$, and $\mathcal{L}'' \subseteq \mathcal{TS}$. We say that $\mathcal{L}'$ is *expressively complete* for $\mathcal{L}''$ on $\mathcal{F}$ if for every $\varphi \in \mathcal{L}''$ with at most one free temporal variable, there exists a formula $\widehat{\varphi} \in \mathcal{L}'$ such that $(\widehat{\varphi})^\dagger$ and φ are equivalent in all models based on flows of time in $\mathcal{F}$.

Theorem 65. *Let $\mathcal{F}$ be any class of dedekind-complete flows of time (for example, the class $\{\langle \mathbb{N}, < \rangle, \langle \mathbb{Z}, < \rangle, \langle \mathbb{R}, < \rangle\} \cup \{\mathfrak{F} : \mathfrak{F}$ a finite linear order$\}$). Then*

1. *$\mathcal{TL}$ is expressively complete for $\mathcal{TS}_{1t}$ on $\mathcal{F}$.*
2. *$\mathcal{TL}_1$ is expressively complete for $\mathcal{TS}_1$ on $\mathcal{F}$.*

Proof By Kamp's theorem ([26]; see also [16, chapters 9–12]), the propositional temporal logic with $\mathcal{S}$ and $\mathcal{U}$ is expressively complete for monadic first-order logic over $\mathcal{F}$. So for any formula $\varphi(t, P_1, \dots, P_k)$ of monadic first-order logic with one free variable t and unary predicates $P_1, \dots, P_k$, we may fix a propositional temporal formula $\overline{\varphi}(p_1, \dots, p_k)$ such that for every first-order structure $\mathfrak{M}$ based on a flow of time $\mathfrak{F} = \langle W, < \rangle \in \mathcal{F}$, and every valuation $\mathfrak{V}$ in $\mathfrak{F}$ with $\mathfrak{V}(p_i) = P_i^{\mathfrak{M}}$, we have

$$\langle \langle \mathfrak{F}, \mathfrak{V} \rangle, w \rangle \models \overline{\varphi} \text{ iff } \mathfrak{M} \models \varphi[w/t], \text{ for all } w \in W.$$

For $\psi \in \mathcal{TS}_{1t}$ with a free temporal variable t, if any, and $\psi' \in \mathcal{TL}$, we say that ψ' *expresses* ψ if the translation $(\psi')^\dagger$ of ψ' is equivalent to ψ in any first-order temporal model based on a flow of time in $\mathcal{F}$. Suppose now that $\chi = \chi(t, Q_1, \dots, Q_k) \in \mathcal{TS}_{1t}$. We prove that for every subformula ψ of χ with at most one free temporal variable, there is a $\mathcal{TL}$-formula $\widehat{\psi}$ that expresses ψ. The proof is by induction on ψ.

Case 1: ψ is atomic. If $\psi = t < t$, then put $\widehat{\psi} = \bot$. If $\psi = Q_i(t, x_1, \dots, x_n)$, then put $\widehat{\psi} = P_i(x_1, \dots, x_n)$.

Case 2: $\psi = \forall x \psi_1$. By the induction hypothesis, there exists $\widehat{\psi_1}$ that expresses ψ_1. But then, $\widehat{\psi} = \forall x \widehat{\psi_1}$ expresses ψ.

Case 3: otherwise. Let $\psi_1, \dots, \psi_l$ be a list of all subformulas of ψ of the form either $Q_i(t', y_1, \dots, y_n)$ or $\forall z \psi'$ that have an occurrence in ψ that is not within the scope of a domain quantifier $\forall y$. Since $\psi \in \mathcal{TS}_{1t}$, every ψ_i of the form $\forall z \psi_i'$ has at most one free temporal variable. Thus, by the induction hypothesis, there exists $\widehat{\psi_i} \in \mathcal{TL}$ that expresses ψ_i, for each $i \le l$.

Now replace in ψ every occurrence of a $\psi_i(t')$ that is not within the scope of a $\forall y$ by a predicate symbol $Q_{\psi_i}(t')$ of the monadic first-order logic. Denote the resulting monadic first-order formula by $\psi'(t, Q_{\psi_1}, \dots, Q_{\psi_l})$. Take the propositional formula $\overline{\psi'}(q_{\psi_1}, \dots, q_{\psi_l})$, and in it, replace every propositional variable q_{ψ_i} by $\widehat{\psi_i}$. The resulting formula $\widehat{\psi}$ clearly expresses ψ.

This completes the induction. So there is a $\mathcal{TL}$-formula $\widehat{\chi}$ expressing χ, proving the former claim of the theorem. To prove the latter, it is enough to observe that if $\psi \in \mathcal{TS}_1$ then $\widehat{\psi} \in \mathcal{TL}_1$. ❑

Remark 66. Clearly, the $\mathcal{TS}$-sentence

$$\exists t_1 \exists t_2 (t_1 < t_2 \wedge \forall x (P(t_1, x) \leftrightarrow P(t_2, x)))$$

is not in $\mathcal{TS}_{1t}$. By results of [27, 3, 2], it cannot be expressed in $\mathcal{TL}$ over the flow of time $\langle \mathbb{Q}, < \rangle$ nor over the class of all finite linear flows. It follows from Theorem 65 that over these flows, it is not equivalent to any $\mathcal{TS}_{1t}$-sentence.

For a class $\mathcal{H}$ of flows of time, denote by $TS(\mathcal{H})$ the set of all $\mathcal{TS}$-sentences that are true in all models based on frames in $\mathcal{H}$, and by $TS_{fin}(\mathcal{H})$ the set of $\mathcal{TS}$-sentences true in all models based on frames in $\mathcal{H}$ and having finite domains. Given a set $\mathcal{TL}' \subseteq \mathcal{TL}_1$, let

$$\mathcal{TS}' = \{\varphi \in \mathcal{TS}_1 : \widehat{\varphi} \in \mathcal{TL}'\},$$

where $\widehat{\varphi}$ is as defined in the proof of Theorem 65. Since $\widehat{\varphi}$ is constructed effectively from φ (see [26]), as an immediate consequence of Lemma 63 and Theorem 65 we obtain the following:

Corollary 67. *Suppose that every $\mathfrak{F} \in \mathcal{H}$ is dedekind-complete, and that $\mathcal{TL}' \subseteq \mathcal{TL}_1$. If the fragment $TL(\mathcal{H}) \cap \mathcal{TL}'$ is decidable, then the fragment $TS(\mathcal{H}) \cap \mathcal{TS}'$ is decidable. If the fragment $TL_{fin}(\mathcal{H}) \cap \mathcal{TL}'$ is decidable, then the fragment $TS_{fin}(\mathcal{H}) \cap \mathcal{TS}'$ is decidable.*

8.2 Two-variable fragment

We remind the reader that the language $\mathcal{TL}_1^2$ contains all monodic $\mathcal{TL}$-formulas with at most two variables. Let $\mathcal{TS}_1^2$ be the sublanguage of $\mathcal{TS}_1$ whose formulas contain at most two domain variables. Clearly, $\mathcal{TS}_1^2 = \{\varphi \in \mathcal{TS}_1 : \widehat{\varphi} \in \mathcal{TL}_1^2\}$. Below, $\mathcal{F}$ will denote any of the classes of flows of time mentioned in the formulation of Theorem 15—that is,

1. $\{\langle \mathbb{N}, < \rangle\}$,
2. $\{\langle \mathbb{Z}, < \rangle\}$,
3. $\{\langle \mathbb{Q}, < \rangle\}$,
4. the class of all finite strict linear orders,
5. any first-order-definable class of strict linear orders.

$\mathcal{F}^+$ will range over these and $\{\langle \mathbb{R}, < \rangle\}$. $\mathcal{G}$ will be one of $\langle \mathbb{N}, < \rangle$, $\langle \mathbb{Z}, < \rangle$, and the class of all finite strict linear orders, and $\mathcal{G}^+$ will range over these and $\{\langle \mathbb{R}, < \rangle\}$.

Theorem 68. *The fragments $TL(\mathcal{F}) \cap \mathcal{TL}_1^2$, $TL_{fin}(\mathcal{F}^+) \cap \mathcal{TL}_1^2$, $TS(\mathcal{G}) \cap \mathcal{TS}_1^2$, and $TS_{fin}(\mathcal{G}^+) \cap \mathcal{TS}_1^2$ are decidable.*

Proof The $\mathcal{L}$-formula $\alpha_{\mathfrak{C}}$, corresponding to a state candidate $\mathfrak{C}$ for a formula $\varphi \in \mathcal{TL}_1^2$, contains at most two individual variables. As is well known (see [41, 33]), the satisfiability problem for such formulas is decidable. Moreover, as the two variable fragment of $\mathcal{L}$ has the finite model property, the finite satisfiability is decidable as well. All that remains is to use the criteria of Theorems 15, 26, and 36 and Corollaries 37 and 67. ❑

As $\mathcal{TL}_1^2$ contains the set $\mathcal{TL}^1$ of $\mathcal{TL}$-formulas with at most one variable, $\mathcal{TS}_1^2$ contains the set $\mathcal{TS}_1^1$ of $\mathcal{TS}_1$-formulas with at most one domain variable, and $\mathcal{TS}_1^1 = \{\varphi \in \mathcal{TS}_1 : \widehat{\varphi} \in \mathcal{TL}^1\}$, we also have:

Corollary 69. *The fragments $TL(\mathcal{F}) \cap \mathcal{TL}^1$, $TL_{fin}(\mathcal{F}^+) \cap \mathcal{TL}^1$, $TS(\mathcal{G}) \cap \mathcal{TS}_1^1$, and $TS_{fin}(\mathcal{G}^+) \cap \mathcal{TS}_1^1$ are decidable.*

Remark 70. It is worth noting that the set of formulas $\mathcal{TL}^1$ corresponds to the propositional language $\mathcal{L}_{\mathcal{S},\mathcal{U},\Box}$ with the temporal operators $\mathcal{S}$, $\mathcal{U}$ and the modal (epistemic) operator $\Box$. Indeed, we may define a translation T from $\mathcal{L}_{\mathcal{S},\mathcal{U},\Box}$ onto $\mathcal{TL}^1$ by taking, for a fixed individual variable x,

$$\begin{array}{rcl}
\mathsf{T}(p_i) & = & P_i(x), \\
\mathsf{T}(\varphi \wedge \psi) & = & \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi), \\
\mathsf{T}(\neg\varphi) & = & \neg\mathsf{T}(\varphi), \\
\mathsf{T}(\varphi\mathcal{U}\psi) & = & \mathsf{T}(\varphi)\mathcal{U}\mathsf{T}(\psi), \\
\mathsf{T}(\varphi\mathcal{S}\psi) & = & \mathsf{T}(\varphi)\mathcal{S}\mathsf{T}(\psi), \\
\mathsf{T}(\Box\varphi) & = & \forall x\ \mathsf{T}(\varphi).
\end{array}$$

Recall that the *product* $L \times \mathbf{S5}$ of a propositional temporal logic L, determined by a class $\mathcal{H}$ of linear orders $\langle W, < \rangle$, and $\mathbf{S5}$ is the set of all formulas in $\mathcal{L}_{\mathcal{S},\mathcal{U},\Box}$ that are valid in frames of the form $\langle W \times V, \overline{<}, \sim \rangle$, where $\langle W, < \rangle \in \mathcal{H}$, V is a non-empty set, $\langle w, v \rangle \,\overline{<}\, \langle w', v' \rangle$ iff $v = v'$ and $w < w'$, and $\sim$ is an equivalence relation on $W \times V$ defined by $\langle w, v \rangle \sim \langle w', v' \rangle$ iff $w = w'$. For more information on products of modal logics, we refer the reader to [18].

It is easy to see that a formula $\varphi \in \mathcal{L}_{\mathcal{S},\mathcal{U},\Box}$ belongs to $L \times \mathbf{S5}$ iff $\mathsf{T}(\varphi)$ is valid in all first-order temporal models based on linear orders $\langle W, < \rangle$ validating L. Thus we obtain, for example, that $L(\mathbb{N}) \times \mathbf{S5}$ is decidable, where $L(\mathbb{N})$ denotes the propositional temporal logic determined by $\langle \mathbb{N}, < \rangle$. Observe that this logic coincides with the temporal-epistemic logic from [15] of one agent who doesn't forget, doesn't learn, and who knows time (the decidability of which is of course known already).

We do not know whether the logic $L(\mathbb{N}) \times_f \mathbf{S5}$, determined by the class of frames of the form $\langle \mathbb{N} \times V, \overline{<}, \sim \rangle$ with finite V, has been considered in the literature. This logic, the propositional version of $TL_{fin}(\mathbb{N}) \cap \mathcal{TL}^1$, is different from the temporal-epistemic logic $L(\mathbb{N}) \times \mathbf{S5}$ (the proof is similar to that of Theorem 25) and corresponds to the assumption that there are only finitely many possible runs of the multi-agent system.

8.3 Monadic fragment

One more interesting fragment of $\mathcal{TL}$ is the set $\mathcal{TL}^{mo}$ of monadic temporal formulas. The corresponding fragment $\mathcal{TS}^{mo}$ consists of those $\mathcal{TS}$-formulas involving only predicate symbols of sort 'temporal $\times$ domain' or 'temporal'. As was shown in Section 2, the fragments $\mathcal{TL}^2 \cap \mathcal{TL}^{mo} \cap TL(\mathbb{N})$ and $\mathcal{TL}^2 \cap \mathcal{TL}^{mo} \cap TL_{fin}(\mathbb{N})$ are undecidable. However, this is not the case for the languages $\mathcal{TL}_1^{mo} = \mathcal{TL}_1 \cap \mathcal{TL}^{mo}$ and $\mathcal{TS}_1^{mo} = \mathcal{TS}_1 \cap \mathcal{TS}^{mo}$. For then, the formula $\alpha_{\mathfrak{C}}$, corresponding to a state candidate $\mathfrak{C}$ for $\varphi \in \mathcal{TL}_1^{mo}$, is a monadic $\mathcal{L}$-formula, and as is well-known (see [29]), the monadic fragment of first-order logic is decidable and has the finite model property. This yields:

Theorem 71. *The fragments $TL(\mathcal{F}) \cap \mathcal{TL}_1^{mo}$, $TL_{fin}(\mathcal{F}^+) \cap \mathcal{TL}_1^{mo}$, $TS(\mathcal{G}) \cap \mathcal{TS}_1^{mo}$, and $TS_{fin}(\mathcal{G}^+) \cap \mathcal{TS}_1^{mo}$ are decidable.*

8.4 Guarded fragment

Let us consider now the following natural generalization of the first-order guarded formulas of [5].

Definition 72 (guarded fragment). Denote by $\mathcal{TGF}$ the smallest set of $\mathcal{TL}$-formulas such that

- every atomic formula is in $\mathcal{TGF}$;
- if φ and ψ are in $\mathcal{TGF}$, then so are $\varphi \wedge \psi$, $\neg\varphi$, $\varphi\mathcal{S}\psi$, and $\varphi\mathcal{U}\psi$;
- if $\overline{x}, \overline{y}$ are tuples of variables, $G(\overline{x}, \overline{y})$ is atomic, $\varphi(\overline{x}, \overline{y}) \in \mathcal{TGF}$, and every free variable occurring in $\varphi(\overline{x}, \overline{y})$ occurs in $G(\overline{x}, \overline{y})$ as well, then $\forall \overline{y}(G(\overline{x}, \overline{y}) \rightarrow \varphi(\overline{x}, \overline{y}))$ is in $\mathcal{TGF}$.

The set $\mathcal{TGF}$ is called the *guarded fragment* of the first-order temporal language. We write $\mathcal{GF}$ for the guarded fragment $\mathcal{L} \cap \mathcal{TGF}$ of the first-order language $\mathcal{L}$.

Note that unlike the guarded fragment $\mathcal{GF}$ of classical first-order logic, which is known to be decidable (see [5]), the temporal guarded fragment interpreted in time structures $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$ turns out to be even not recursively enumerable.

Theorem 73. *Let $\mathfrak{F}$ be either $\langle \mathbb{N}, < \rangle$ or $\langle \mathbb{Z}, < \rangle$. Then $TL(\mathfrak{F}) \cap \mathcal{TL}^2 \cap \mathcal{TGF}$ is not recursively enumerable.*

Proof The proof is similar to that of Theorem 2. We simply write down the required formula $\varphi_{\mathfrak{T}}$ for a given set of tiles $\mathfrak{T} = \{\mathsf{t}_0, \ldots, \mathsf{t}_n\}$.

Let R be a binary predicate and $P_0, \ldots, P_n$, Q unary ones. Define $\varphi_{\mathfrak{T}}$ to be the conjunction of the following formulas:

$$
\begin{array}{l}
\exists x(Q(x) \wedge \Box\Diamond P_0(x)),\\
\forall x(Q(x) \rightarrow \exists y(R(x,y) \wedge Q(y))),\\
\Box^+\forall x(Q(x) \rightarrow \bigcirc Q(x)),\\
\forall x, y(R(x,y) \rightarrow \Box R(x,y)),\\
\Box^+\forall x(Q(x) \rightarrow \bigvee_{i=0}^{n} P_i(x) \wedge \bigwedge_{i \neq j}(P_i(x) \rightarrow \neg P_j(x))),\\
\Box^+\forall x(P_i(x) \rightarrow \forall y(R(x,y) \rightarrow \bigvee_{up(\mathsf{t}_i)=down(\mathsf{t}_j)} P_j(y))),\\
\Box^+\forall x(P_i(x) \rightarrow \bigcirc \bigvee_{right(\mathsf{t}_i)=left(\mathsf{t}_j)} P_j(x)).
\end{array}
$$

Clearly, $\varphi_{\mathfrak{T}}$ belongs to $\mathcal{TL}^2 \cap \mathcal{TGF}$. It is readily seen that $\varphi_{\mathfrak{T}}$ is satisfiable in $\mathfrak{F}$ iff there is a recurrent tiling of $\mathbb{N} \times \mathbb{N}$ by $\mathfrak{T}$. ❑

We may define the guarded fragment $\mathcal{SGF}$ of $\mathcal{TS}$, as follows: every atomic formula is in $\mathcal{SGF}$, $\mathcal{SGF}$ is closed under the boolean connectives and temporal quantification $\forall t$, and if $\overline{x}, \overline{y}$ are tuples of variables, $G(t, \overline{x}, \overline{y})$ is atomic, $\varphi(t, \overline{x}, \overline{y}) \in \mathcal{SGF}$, and every free domain variable of φ occurs in $G(t, \overline{x}, \overline{y})$, then $\forall \overline{y}(G(t, \overline{x}, \overline{y}) \rightarrow \varphi(t, \overline{x}, \overline{y})) \in \mathcal{SGF}$.

Let $\mathcal{TGF}_1 = \mathcal{TGF} \cap \mathcal{TL}_1$, and $\mathcal{SGF}_1 = \mathcal{SGF} \cap \mathcal{TS}_1$.

Theorem 74. *The fragments $TL(\mathcal{F}) \cap \mathcal{TGF}_1$, $TL_{fin}(\mathcal{F}^+) \cap \mathcal{TGF}_1$, $TS(\mathcal{G}) \cap \mathcal{SGF}_1$, and $TS_{fin}(\mathcal{G}^+) \cap \mathcal{SGF}_1$ are decidable.*

Proof By Theorems 15, 26, and 36, and Corollary 37, the result for the TL-classes in the theorem may be established by showing that given $\varphi \in \mathcal{TGF}_1$, it is decidable whether a given state candidate for φ is (finitely) realizable. It is evident from the proof of Theorem 65 that

$\mathcal{TGF}_1$ is expressively complete for $\mathcal{SGF}_1$ over $\mathcal{G}, \mathcal{G}^+$. So the result for the TS-classes in the theorem follows from this and Corollary 67.

So let $\varphi \in \mathcal{TGF}_1$ and let $\mathfrak{C} = \langle T, T^{con} \rangle$ be a state candidate for φ. By Lemma 8, to decide whether $\mathfrak{C}$ is (finitely) realizable it suffices to show that it is decidable whether the $\mathcal{L}$-sentence

$$\alpha_{\mathfrak{C}} \quad = \quad \bigwedge_{t \in T} \exists x\, \overline{t}(x) \quad \wedge \quad \forall x \bigvee_{t \in T} \overline{t}(x) \quad \wedge \quad \bigwedge_{\langle t,c \rangle \in T^{con}} \overline{t}(c)$$

has a (finite) model.

The formulas $\overline{t}(x)$, $\overline{t}(c)$ are in $\mathcal{GF}$, but $\alpha_{\mathfrak{C}}$ is not. However, we can transform it into a guarded sentence as follows. Let P be a new unary predicate. Observe that if $\psi \in \mathcal{GF}$ then the relativization ψ^P of ψ to P is logically equivalent to a $\mathcal{GF}$-formula. For atomic ψ, $\psi^P = \psi \in \mathcal{GF}$; the boolean cases are trivial; and for guarded $\psi(\overline{x}, \overline{y})$ and atomic $G(\overline{x}, \overline{y})$, $\big((\exists y_1, \dots, y_n(G(\overline{x}, \overline{y}) \wedge \psi)\big)^P$ is by definition $\exists y_1, \dots, y_n(\bigwedge_{1 \le i \le n} P(y_i) \wedge G(\overline{x}, \overline{y}) \wedge \psi^P)$, which is equivalent to $\exists y_1, \dots, y_n(G(\overline{x}, \overline{y}) \wedge (\bigwedge_{1 \le i \le n} P(y_i) \wedge \psi^P))$ and hence is (inductively) equivalent to a guarded formula. Now,

$$(\alpha_{\mathfrak{C}})^P \quad = \quad \bigwedge_{t \in T} \exists x (P(x) \wedge \overline{t}^P(x)) \quad \wedge \quad \forall x (P(x) \to \bigvee_{t \in T} \overline{t}^P(x)) \quad \wedge \quad \bigwedge_{\langle t,c \rangle \in T^{con}} \overline{t}^P(c),$$

and we see that, up to logical equivalence, $(\alpha_{\mathfrak{C}})^P \in \mathcal{GF}$.

By classical model theory, $\alpha_{\mathfrak{C}}$ has a (finite) model iff $(\alpha_{\mathfrak{C}})^P$ has a (respectively, finite) model. Since $(\alpha_{\mathfrak{C}})^P$ is logically equivalent to a $\mathcal{GF}$-sentence, and by results of [5, 21], $\mathcal{GF}$ is decidable and has the finite model property, we see that it is decidable whether $\alpha_{\mathfrak{C}}$ has a (finite) model, as required. ❑

8.5 Temporal description logics

The notion of quasimodel used in this paper is actually a generalization of the quasimodels introduced in [46] to prove the decidability of the satisfiability problems for the temporal description logic $\mathcal{CIQ}_{\mathcal{US}}$ (i.e., the description logic $\mathcal{CIQ}$ of De Giacomo and Lenzerini [20] extended with Since and Until) in models based on the time structures $\langle \mathbb{N}, < \rangle$ and $\langle \mathbb{Z}, < \rangle$. However, the satisfiability problems in $\langle \mathbb{Q}, < \rangle$ and arbitrary strict linear orders were left open in that paper. Using the embedding technique of Section 4, one can show that these problems are decidable too. Thus, we have:

Theorem 75. *There are algorithms that are capable of deciding whether a given $\mathcal{CIQ}_{\mathcal{US}}$-formula is satisfiable in:*

- $\langle \mathbb{N}, < \rangle$,
- $\langle \mathbb{Z}, < \rangle$,
- $\langle \mathbb{Q}, < \rangle$,
- *finite linear orders,*
- *arbitrary strict linear orders.*

Note, however, that $\mathcal{CIQ}$ (which is actually **CPDL** with qualified number restrictions or counting modalities) does not have the finite model property, and it is not known whether the finite model reasoning in it is decidable. So we cannot say whether the satisfiability problem for $\mathcal{CIQ}_{\mathcal{US}}$-formulas is decidable in models with finite domains. For more information on the connection between multi-dimensional description logics and first-order modal logic, see [47].

9 Open questions

We end the paper with some problems arising from the work above.

1. Do our results extend to the flow of time $\langle \mathbb{R}, < \rangle$ with arbitrary domains? Or with countable domains? (The logic here is different—see Theorem 25.)
2. Can our results be extended to first-order temporal logic with equality?
3. Or to logics over non-linear flows of time, such as historical necessity logics, and CTL^*?
4. What is the computational complexity of satisfiability of an arbitrary monodic formula φ over the flows of time considered earlier, given an oracle for determining if a state candidate for φ is realizable?
5. Are there other natural decidable (and expressive) fragments of $\mathcal{TL}$?

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