Monodic epistemic predicate logic

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Abstract

We consider the monodic formulas of common knowledge predicate logic, which allow applications of epistemic operators to formulas with at most one free variable. We provide finite axiomatizations of the monodic fragment of the most important common knowledge predicate logics (the full logics are known to be not recursively enumerable) and single out a number of their decidable fragments. On the other hand, it is proved that the addition of the equality symbol to the monodic fragment makes it not recursively enumerable.

1 Introduction

Ever since it became common knowledge that intelligent behaviour of an agent is based not only on her knowledge about the world but also on knowledge about both her own and other agents' knowledge, logical formalisms designed for reasoning about knowledge have attracted attention in artificial intelligence, computer science, economic theory, and philosophy (cf. e.g. the books [5, 16, 13] and the seminal works [8, 1]). In all these areas, one of the most successful approaches is to supply classical—propositional or first-order—logic with an explicit epistemic operator K_i for each agent i under consideration. $K_i\varphi$ means that *agent i knows* (*or believes*) φ, $K_1K_2\varphi$ says then that *agent 1 knows that agent 2 knows* φ, and the schema of positive introspection $K_i\psi \to K_iK_i\psi$ states that *agent i knows what she knows.* In the first-order case this language is capable of formalizing the distinction between 'knowing that' and 'knowing what' (i.e., modalities *de dicto* and *de re*): the formula $K_i\exists x\ \mathsf{name}(x,y)$ stands for '*i knows that y has a name*,' while $\exists x K_i\mathsf{name}(x,y)$ means '*i knows a name of y*.'

There can be different interpretations of the knowledge operators (e.g. with or without positive or negative introspection), and for many of them transparent axiomatic representations have been found (cf. e.g. [7, 5]). On the other hand, the possible worlds semantics [8] provided a framework to interpret this language: in a world w agent i knows φ if and only if φ holds in all worlds that i regards possible in w (the difference between various understandings of K_i is reflected by different accessibility relations among the worlds).

The situation becomes much more complicated when—in order to describe the behavior of *multi-agent* systems—we extend the language with one more modal operator, C, to capture the *common knowledge* of a group of agents. Such an operator was required for analyzing conventions [14], coordinizations in multi-agent systems [5], common sense reasoning [15], agreement [1, 2], etc.[1] Although the intended meaning of the *common knowledge operator* involves infinity: $C\varphi$ stands for the infinite conjunction of the form

$$K_1\varphi \wedge K_1K_2\varphi \wedge K_2K_1K_2\varphi \wedge \dots,$$

both natural possible worlds semantics and clear inductive axiomatizations have been found for *propositional* common knowledge logics [7]. (The new operator, however, considerably increases the computational complexity of these logics—from PSPACE to EXPTIME; consult [5].)

But real problems arise when we try to combine the common knowledge operator with the first-order quantifiers. First, no common knowledge predicate logic with both a finitary (or at least recursive) axiomatization *and* a reasonable semantics has ever been constructed! And second, the common knowledge predicate logics determined by the standard possible worlds semantics *are known* to be not recursively axiomatizable (and so not recursively enumerable) [17]. Thus, similar to second-order logic or first-order temporal logic, it is impossible to characterize common knowledge predicate logics syntactically. In some sense this means that neither we nor the Turing machine have the capacity of understanding the interaction between common knowledge and quantifiers. Moreover, this is true of even very small fragments of the logics, say, the monadic or two-variable fragments (see [17]).

Does it mean that we should completely abandon the idea of using common knowledge predicate logic? Still there exist manageable fragments with non-trivial interaction between the common knowledge operator and quantifiers.

A promising approach to singling out non-trivial decidable fragments of first-order modal and temporal logics has been proposed in [9, 20]. The idea is to restrict attention to the class of *monodic*[2] formulas which allow applications of modal or temporal operators only to formulas with at most one free variable. In the epistemic context, monodicity means, in particular, that

- we have the full expressive power of first-order logic as far as we do not apply epistemic operators to open formulas;
- we can reason about agents' knowledge of properties, for instance,

  $$\forall x \ (C\,\mathsf{loves}(John, x) \vee C\,\neg\mathsf{loves}(John, x))$$

  ('for every object x, it is a common knowledge whether John loves x'); however, we are not permitted to reason about agents' knowledge of relations, say

  $$\forall x, y \ (C\,\mathsf{loves}(x, y) \vee C\,\neg\mathsf{loves}(x, y))$$

  ('for all pairs x, y, it is a common knowledge whether x loves y').

The main aim of this paper is to show that the monodic fragment of common knowledge predicate logic turns out to be quite manageable. First, we show that for almost all interesting interpretations of the operators K_i the *monodic fragment* of the valid formulas (*without equality*) can be finitely axiomatized. Moreover, we observe that a number of natural subclasses of the monodic fragment, say, with only monadic predicates or two variables, are decidable. On the other hand, it is proved that the addition of the equality symbol to the monodic fragment makes it not recursively enumerable.

[1] An alternative approach adds infinitary operators to the language, see [10, 11].

[2] *Monody* is a composition with only one melodic line.

2 First-order logics of common knowledge

The logics we deal with in this paper are all based on the language we call $\mathcal{CL}$, which extends the standard first-order language (without equality) with a number of epistemic operators, including the operator expressing common knowledge. The alphabet of $\mathcal{CL}$ consists of:

- *predicate symbols* $P_0, P_1, \ldots,$
- *individual variables* $x_0, x_1, \ldots,$
- *individual constants* $c_0, c_1, \ldots,$
- the *booleans* $\wedge$, $\neg$,
- the *universal quantifier* $\forall x$ for each individual variable x,
- a finite number of *knowledge operators* $K_1, \ldots, K_n$, $n \geq 1$, and
- the *common knowledge operator* C.

We assume that the set of predicate symbols is non-empty and that each of them is equipped with some fixed arity; 0-ary predicates are called *propositional variables* and denoted by $p_0, p_1, \ldots$. The individual variables together with the individual constants form the set of *$\mathcal{CL}$-terms*. The set of *$\mathcal{CL}$-formulas* is defined as follows:

- if P is an n-ary predicate symbol and $\tau_1, \ldots, \tau_n$ are terms, then $P(\tau_1, \ldots, \tau_n)$ is a formula;
- if φ and ψ are formulas, then so are $\varphi \wedge \psi$ and $\neg\varphi$;
- if φ is a formula and x a variable, then $\forall x \varphi$ is a formula;
- if φ is a formula and $i \leq n$, then $K_i\varphi$ and $C\varphi$ are formulas.

Throughout the paper we make use of the following abbreviations: $\top$, $\bot$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$, and $\exists x \varphi$, which are defined as usual, as well as $E\varphi$ ('*everyone knows* φ') which stands for $K_1\varphi \wedge \cdots \wedge K_n\varphi$.

The language $\mathcal{CL}$ is interpreted in *first-order Kripke models* which are structures of the form $\mathfrak{M} = \langle \mathfrak{F}, D, I \rangle$, where $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$ is the underlying *Kripke frame* ($W \neq \emptyset$ is a set of worlds and the R_i are binary relations on W), D is a nonempty set, the *domain* of $\mathfrak{M}$, and I a function associating with every world $w \in W$ a first-order structure

$$I(w) = \left\langle D, P_0^{I(w)}, \ldots, c_0^{I(w)}, \ldots \right\rangle,$$

in which $P_i^{I(w)}$ is a predicate on D of the same arity as P_i (for a propositional variable p_i, the predicate $p_i^{I(w)}$ is either $\top$ or $\bot$), and $c_i^{I(w)}$ is an element in D such that $c_i^{I(u)} = c_i^{I(v)}$ for any $u, v \in W$. The latter means that constants are treated as rigid designators in the sense that they designate the same object in every world. To simplify notation we will omit the superscript I and write P_i^w, p_i^w, c_i^w, etc., if I is clear from the context.

Remark 1. Note that we assume domains to be constant. Axiomatizations for the case of expanding or varying domains can easily be obtained from our results.

An *assignment* in D is a function $\mathfrak{a}$ from the set *var* of variables to D. The value $\tau^{\mathfrak{M},\mathfrak{a}}$ (or simply $\tau^{\mathfrak{a}}$ if understood) of a term τ under $\mathfrak{a}$ in $\mathfrak{M}$ is $\mathfrak{a}(\tau)$, if τ is a variable, and τ^w otherwise, where w is some (any) world in W. The *truth-relation* $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ (or simply $w \models^{\mathfrak{a}} \varphi$) in the model $\mathfrak{M}$ in the world w under the assignment $\mathfrak{a}$ is defined inductively as follows:

- $w \models^{\mathfrak{a}} P_i(\tau_1, \dots, \tau_n)$ iff $(\tau_1^{\mathfrak{a}}, \dots, \tau_n^{\mathfrak{a}}) \in P_i^w$; this fact will also be written as $P_i^w(\tau_1^{\mathfrak{a}}, \dots, \tau_n^{\mathfrak{a}})$, $I(w) \models^{\mathfrak{a}} P_i(\tau_1, \dots, \tau_n)$, or $I(w) \models P_i[\tau_1^{\mathfrak{a}}, \dots, \tau_n^{\mathfrak{a}}]$;
- $w \models^{\mathfrak{a}} \psi \land \chi$ iff $w \models^{\mathfrak{a}} \psi$ and $w \models^{\mathfrak{a}} \chi$;
- $w \models^{\mathfrak{a}} \neg\psi$ iff $w \not\models^{\mathfrak{a}} \psi$;
- $w \models^{\mathfrak{a}} \forall x \psi(x, y_1, \dots, y_n)$ iff $w \models^{\mathfrak{b}} \psi(x, y_1, \dots, y_n)$ for every assignment $\mathfrak{b}$ in D that may differ from $\mathfrak{a}$ only on x;
- $w \models^{\mathfrak{a}} K_i\psi$ iff $v \models^{\mathfrak{a}} \psi$ for all $v \in W$ such that wR_iv;
- $w \models^{\mathfrak{a}} C\psi$ iff $v \models^{\mathfrak{a}} \psi$ for all v such that $w(\bigcup_{i\leq n} R_i)^+v$, where the superscript $^+$ means taking the transitive closure of $\bigcup_{i\leq n} R_i$.

For a set of formulas Γ, a model $\mathfrak{M}$, a world w and an assignment $\mathfrak{a}$, we write $w \models^{\mathfrak{a}} \Gamma$ to say that $w \models^{\mathfrak{a}} \varphi$ for every $\varphi \in \Gamma$. In this case Γ is said to be *satisfied* in $\mathfrak{M}$. By $\mathfrak{F} \models \Gamma$ we mean that Γ is *valid* in $\mathfrak{F}$, i.e., $(\mathfrak{M}, w) \models^{\mathfrak{a}} \Gamma$ holds for every model $\mathfrak{M}$ based on $\mathfrak{F}$, every assignment $\mathfrak{a}$ in it, and every world w in $\mathfrak{F}$.

Different epistemic logics correspond to different classes of frames. Usually these classes are determined by combinations of the following properties: reflexivity (denoted by r), transitivity (t), seriality (s), and euclideanness (e). We denote by F^{r} the class of all reflexive frames, by F^{re} the class of all reflexive and euclidean frames (i.e., the class of frames with equivalence relations), etc. F^{a} is the class of all frames.

For a class F of frames, we define $L(\mathsf{F})$, the *logic of* F, to be the set of all $\mathcal{CL}$-formulas that are valid in all $\mathfrak{F} \in \mathsf{F}$. Here is a list of standard logics of common knowledge: $\mathbf{K}_n^C = L(\mathsf{F}^{\mathsf{a}})$, $\mathbf{T}_n^C = L(\mathsf{F}^{\mathsf{r}})$, $\mathbf{KD}_n^C = L(\mathsf{F}^{\mathsf{s}})$, $\mathbf{K4}_n^C = L(\mathsf{F}^{\mathsf{t}})$, $\mathbf{S4}_n^C = L(\mathsf{F}^{\mathsf{rt}})$, $\mathbf{KD45}_n^C = L(\mathsf{F}^{\mathsf{ste}})$, $\mathbf{S5}_n^C = L(\mathsf{F}^{\mathsf{re}})$.

3 Axiomatizing the monodic fragment

As was shown in [17], none of the logics listed above is recursively axiomatizable. Moreover, the restriction of these logics to such 'orthodox' fragments as the monadic or two-variable formulas does not bring a relief: they are still not recursively enumerable. By analyzing proofs of these 'negative' results, one can observe that all of them make use of formulas asserting that some agents know relations between two objects. On the other hand, the results of [18] establishing decidability of epistemic description logics (in which epistemic operators are applicable only to unary predicates) give some hope that the fragment without such formulas can be more manageable.

Definition 2 (monodic formulas). Denote by $\mathcal{CL}_1$ the set of all $\mathcal{CL}$-formulas φ such that any subformula of φ of the form $K_i\psi$ or $C\psi$ has at most one free variable. Such formulas will be called *monodic*. For a class F of frames, let $L_1(\mathsf{F}) = L(\mathsf{F}) \cap \mathcal{CL}_1$. In other words, $L_1(\mathsf{F})$ is the *monodic fragment* of the logic $L(\mathsf{F})$.

From now on all formulas are assumed to be monodic.

In this section we give axiomatizations of the monodic fragments of the epistemic logics defined above. (These axiomatizations are first-order extensions of those in [7].) To begin with, we axiomatize the monodic fragment of $\mathbf{K}_n^C$, i.e., $L_1(\mathsf{F}^{\mathsf{a}})$. This axiomatic system, denoted by K_n^C, has the following axiom schemata and inference rules:

Axiom schemata (over formulas in $\mathcal{CL}_1$):

- the set of axiom schemata from some axiomatization of classical first-order logic,
- $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$, for $i \leq n$,
- $C\varphi \to E(\varphi \wedge C\varphi)$,
- $K_i \forall x \psi \leftrightarrow \forall x K_i \psi$.

Inference rules (over formulas in $\mathcal{CL}_1$):

- the rules of classical first-order logic,
- $\dfrac{\varphi}{K_i\varphi}$, for $i \leq n$,
- $\dfrac{\varphi \to E(\psi \wedge \varphi)}{\varphi \to C\psi}$.

The monodic fragments of the remaining logics are axiomatized by adding to K_n^C the corresponding standard axiom schemata:

A_D: $K_i\varphi \to \neg K_i \neg\varphi$, $i \leq n$,

A_T: $K_i\varphi \to \varphi$, $i \leq n$,

A_4: $K_i\varphi \to K_i K_i \varphi$, $i \leq n$,

A_5: $\neg K_i\varphi \to K_i \neg K_i \varphi$, $i \leq n$.

Namely, T_n^C, KD_n^C and $K4_n^C$ as the axiomatic systems obtained by adding to K_n^C the schemata A_T, A_D, and A_4, respectively. $S4_n^C$ is $K4_n^C$ plus A_T. $KD45_n^C$ is $K4_n^C$ extended by A_D and A_5, and $S5_n^C$ is $KD45_n^C$ plus A_T.

Given an axiomatic system S, we denote by $\vdash_S$ its consequence relation. Our aim now is to prove that the defined systems indeed axiomatize the monodic fragments of our common knowledge logics. That is, we are going to show that for every monodic formula φ, we have $\vdash_{K_n^C} \varphi$ iff $\varphi \in \mathbf{K}_n^C$ iff φ is valid in all frames from F^a, and similar claims for the other logics.

The easy 'only if' part of these claims, i.e., correctness, follows from well-known results (consult e.g. [5, 3]) and the almost obvious fact that the rule $\varphi \to E(\psi \wedge \varphi)/\varphi \to C\psi$ preserves validity. The 'if' part, i.e., completeness, is much more complicated. It will be proved in the next section.

4 Completeness

Given a set Γ of $\mathcal{CL}_1$-formulas, we denote by $con(\Gamma)$ and $sub(\Gamma)$ the sets of all constants and subformulas of formulas in Γ, respectively; $sub_C(\Gamma)$ is defined as:

$$sub_C(\Gamma) = sub(\Gamma) \cup \{E(\psi \wedge C\psi), \psi \wedge C\psi, K_i(\psi \wedge C\psi) : C\psi \in sub(\Gamma), i \leq n\}.$$

Let $sub_C^{\neg}(\Gamma) = \{\neg\psi : \psi \in sub_C(\Gamma)\}$ and let $sub_n(\Gamma)$ be the subset of $sub_C^{\neg}(\Gamma)$ containing only formulas with $\leq n$ free variables. For instance, $sub_0(\Gamma)$ denotes the set of sentences in $sub_C^{\neg}(\Gamma)$. (Note that $sub_n(\Gamma)$ is not necessarily closed under subformulas and that modulo equivalence we may assume that $sub_n(\Gamma)$ is closed under $\neg$.) In what follows we will not be distinguishing between a finite set Γ of formulas and the conjunction $\bigwedge \Gamma$ of formulas in it.

Let x be a variable not occurring in Γ. Put

$$sub_x(\Gamma) = \{\psi\{x/y\} : \psi(y) \in sub_1(\Gamma)\} \cup \{\neg K_i \neg \bot, K_i \neg \bot, \neg \bot, \bot : i \leq n\}$$

For the rest of this section we fix an arbitrary $\mathcal{CL}_1$-sentence φ.

Definition 3 (type). By a *type* for φ we mean a boolean-saturated subset t of $sub_x(\varphi)$, i.e.,

- $\psi \wedge \chi \in t$ iff $\psi \in t$ and $\chi \in t$, for every $\psi \wedge \chi \in sub_x\varphi$;
- $\neg\psi \in t$ iff $\psi \notin t$, for every $\psi \in sub_x\varphi$.

We say that two types t and t' *agree on* $sub_0(\varphi)$ if $t \cap sub_0(\varphi) = t' \cap sub_0(\varphi)$. Given a type t for φ and a constant $c \in con(\varphi)$, the pair $\langle t, c\rangle$ will be called an *indexed type* for φ (indexed by c) and denoted by $t_c(x)$ or simply t_c.

Definition 4 (state candidate). Suppose T is a set of types for φ that agree on $sub_0(\varphi)$, and $T^{con} = \{\langle t, c\rangle : c \in con(\varphi)\}$ a set of indexed types such that $\{t : \langle t, c\rangle \in T^{con}\} \subseteq T$. The pair $\mathfrak{C} = \langle T, T^{con}\rangle$ is called then a *state candidate* for φ. A *pointed state candidate* for φ is the pair $\mathfrak{P} = \langle \mathfrak{C}, t\rangle$, where t is a type in T, called the *point* of $\mathfrak{P}$. With $\mathfrak{C}$ and $\mathfrak{P}$ we associate the formulas

$$\alpha_{\mathfrak{C}} = \bigwedge_{t \in T} \exists x\, t(x) \wedge \forall x \bigvee_{t \in T} t(x) \wedge \bigwedge_{\langle t,c\rangle \in T^{con}} t(c), \qquad \beta_{\mathfrak{P}} = \alpha_{\mathfrak{C}} \wedge t.$$

In what follows S ranges over the axiomatic systems introduced in Section 3. We remind the reader that a formula χ is said to be S-consistent if $\not\vdash_S \neg\chi$.

Definition 5 (suitable pairs). (1) A pair (t_1, t_2) of types for φ is called *i-suitable for S*, $i \leq n$, if the formula $t_1 \wedge \neg K_i \neg t_2$ is S-consistent.

(2) A pair $(\mathfrak{C}_1, \mathfrak{C}_2)$ of state candidates is *i-suitable for S*, $i \leq n$, if $\alpha_{\mathfrak{C}_1} \wedge \neg K_i \neg \alpha_{\mathfrak{C}_2}$ is S-consistent.

(3) A pair $(\mathfrak{P}_1, \mathfrak{P}_2)$ of pointed state candidates is *i-suitable for S*, $i \leq n$, if $\beta_{\mathfrak{P}_1} \wedge \neg K_i \neg \beta_{\mathfrak{P}_2}$ is S-consistent. In this case we write $\mathfrak{P}_1 \prec_i \mathfrak{P}_2$.

Lemma 6. (i) *For every finite S-consistent set Ψ of $\mathcal{CL}_1$-formulas, there is a pointed state candidate $\mathfrak{P} = (\mathfrak{C}, t)$ for φ such that $\bigwedge \Psi \wedge \beta_{\mathfrak{P}}$ is S-consistent. Moreover, if $\psi \in sub_x(\varphi)$ and $\psi \in \Psi$, then $\psi \in t$.*

(ii) *Suppose Ψ is a finite set of $\mathcal{CL}_1$-formulas and $\neg K_i \theta$ is a formula in $sub_x\varphi$ such that $\bigwedge \Psi \wedge \neg K_i \theta$ is S-consistent. Then there exists a pointed state candidate $\mathfrak{P} = (\mathfrak{C}, t)$ for φ such that $\neg\theta \in t$ and $\bigwedge \Psi \wedge \neg K_i \neg \beta_{\mathfrak{P}}$ is S-consistent.*

Proof (i) Denote by β_φ the disjunction of all formulas $\beta_{\mathfrak{P}}$, $\mathfrak{P}$ a pointed state candidate for φ. As β_φ is classically valid, it is provable in S, whence $\bigwedge \Psi \wedge \beta_\varphi$ is S-consistent. It follows that there is a disjunct $\beta_{\mathfrak{P}}$ of β_φ such that $\bigwedge \Psi \wedge \beta_{\mathfrak{P}}$ is S-consistent. Now, if $\psi \in \Psi \cap sub_x(\varphi)$ and t is the point of $\mathfrak{P}$, then $\psi \in t$, for otherwise $\neg\psi \in t$, which is a contradiction.

(ii) If $\bigwedge \Psi \wedge \neg K_i\theta$ is S-consistent, then so is $\bigwedge \Psi \wedge \neg K_i \neg(\neg\theta \wedge \beta_\varphi)$. It follows that there is a pointed state candidate $\mathfrak{P}$ with point t such that $\bigwedge \Psi \wedge \neg K_i \neg(\neg\theta \wedge \beta_{\mathfrak{P}})$ is S-consistent. Clearly, $\neg\theta \in t$, and we are done. ❑

Note that Lemma 6 will hold true if we replace x by some constant c.

Lemma 7. (i) *If a pair $(\mathfrak{C}_1, \mathfrak{C}_2)$ of state candidates for φ is i-suitable for S, $i \leq n$, then:*

1. *for every $t \in T_1$ there exists a $t' \in T_2$ such that (t, t') is i-suitable for S;*
2. *for every $t' \in T_2$ there exists a $t \in T_1$ such that (t, t') is i-suitable for S.*

(ii) *Suppose that a pair of types (t, t') is i-suitable for S. Then:*

1. *$\psi \in t'$ whenever $K_i\psi \in t$;*
2. *if $A_4 \in S$, then $K_i\psi \in t'$ whenever $K_i\psi \in t$;*
3. *if $\{D, A_5\} \subseteq S$, then $K_i\psi \in t$ whenever $K_i\psi \in t'$;*
4. *if $\{D, A_4, A_5\} \subseteq S$ or $\{T, A_5\} \subseteq S$, then $K_i\psi \in t$ iff $K_i\psi \in t'$.*

(iii) *Suppose (t, t') is i-suitable for S. Then $C\psi \in t$ implies $C\psi \in t'$. If $\{A_4, A_5\} \subseteq S$, then $C\psi \in t$ iff $C\psi \in t'$.*

Proof (i) Suppose that $t \in T_1$ but there is no $t' \in T_2$ for which (t, t') is i-suitable for S. This means that $\vdash_S t \to K_i \neg t'$, for each $t' \in T_2$, and so $\vdash_S t \to K_i \neg \bigvee_{t' \in T_2} t'$. Then we have $\vdash_S \exists x t \to K_i \exists x \neg \bigvee_{t' \in T_2} t'$. Since $\vdash_S \exists x \neg \bigvee_{t' \in T_2} t' \to \neg\alpha_{\mathfrak{C}_2}$ and $\vdash_S \alpha_{\mathfrak{C}_1} \to \exists x t$, we finally obtain $\vdash_S \alpha_{\mathfrak{C}_1} \to K_i \neg\alpha_{\mathfrak{C}_2}$, contrary to S-consistency of $\alpha_{\mathfrak{C}_1} \wedge \neg K_i \neg \alpha_{\mathfrak{C}_2}$. Claim (i.2) is proved in a similar way.

(ii) Suppose that $K_i\psi \in t$ but $\psi \notin t'$. Then $\neg\psi \in t'$, $\vdash_S K_i\psi \to K_i\neg t'$, and so $t \wedge \neg K_i \neg t'$ is S-inconsistent, which is a contradiction.

Now suppose that S contains A_4, $K_i\psi \in t$, but $K_i\psi \notin t'$. Then $\neg K_i\psi \in t'$. Hence $\vdash_S K_i K_i \psi \to K_i \neg t'$. It follows from A_4 that $\vdash_S K_i\psi \to K_i\neg t'$, and so $K_i\psi \wedge \neg K_i \neg t'$ is S-inconsistent, contrary to S-consistency of (t, t'). Claims (i.3) and (i.4) are proved analogously.

(iii) Suppose $C\psi \in t$. Then $E(\psi \wedge C\psi) \in t$ and so $K_i(\psi \wedge C\psi) \in t$, for $i \leq n$. By (ii.1), $\psi \wedge C\psi \in t'$, from which $C\psi \in t'$.

If $C\psi \in t'$ then, as we know, $K_i(\psi \wedge C\psi) \in t'$, for $i \leq n$. So if $\{A_4, A_5\} \subseteq S$, then we have by (ii.4), $\psi \wedge C\psi \in t$, and so $C\psi \in t$. ❑

Definition 8 (basic tree). Let $\mathfrak{T} = \langle W, <_1, \ldots, <_n \rangle$ be a structure with pairwise disjoint binary relations $<_i$ on W such that $\left\langle W, \bigcup_{i \leq n} <_i \right\rangle$ is an intransitive tree.[3] By a *basic tree* for φ we mean the pair $\langle \mathfrak{T}, \sigma \rangle$, where σ is a map associating with every $w \in W$ a state candidate $\sigma(w) = \langle T_w, T_c^w \rangle$ for φ. A basic tree is called a *basic S-tree* if $\alpha_{\sigma(w)}$ is S-consistent, for every $w \in W$, and the pair $(\sigma(w_1), \sigma(w_2))$ is i-suitable for S whenever $w_1 <_i w_2$.

[3]We remind the reader that $\mathfrak{G} = \langle W, \lhd \rangle$ is an intransitive *tree* if (i) $\mathfrak{G}$ is *rooted*, i.e., there is $w_0 \in W$ (a *root* of $\mathfrak{G}$) such that $w_0 \lhd^* w$ for every $w \in W$, where $\lhd^*$ is the transitive and reflexive closure of $\lhd$, (ii) for every $w \in W$, the set $\{v \in W : v \lhd^* w\}$ is finite and linearly ordered by $\lhd^*$, (iii) every world v in $\mathfrak{G}$, save its root, has precisely one predecessor, i.e., $|\{u \in W : u \lhd v\}| = 1$, and (iv) the root w_0 is *irreflexive*, i.e., $\neg w_0 \lhd w_0$.

Definition 9 (run). A *run* r in a basic S-tree $\langle \mathfrak{T}, \sigma \rangle$ is a map associating with every $w \in W$ a point $r(w) \in T_w$ such that

- the pair $(r(w_1), r(w_2))$ is i-suitable for S whenever $w_1 <_i w_2$;
- if $\neg K_i \psi \in r(w)$ then $\psi \notin r(w')$ for some $w' >_i w$;
- if $\neg C\psi \in r(w)$ then $\psi \notin r(w')$ for some w' such that $w(\bigcup_{i \leq n} <_i)^+ w'$.

Definition 10 (quasimodel). A basic S-tree $\langle \mathfrak{T}, \sigma \rangle$ is called an *S-quasimodel* for φ if

- for all $w \in W$ and $t \in T_w$ ($\sigma(w) = \langle T_w, T_c^w \rangle$), there exists a run r in $\mathfrak{T}$ such that $r(w) = t$;
- for every constant c, the function r_c defined by $r_c(w) = t$, for $\langle t, c \rangle \in T_w^{con}$, $w \in W$, is a run in $\langle \mathfrak{T}, \sigma \rangle$.

We say φ is *satisfied* in $\langle \mathfrak{T}, \sigma \rangle$ if there exists $w \in W$ such that $\alpha_{\sigma(w)} \wedge \varphi$ is S-consistent.

Theorem 11. *If φ is satisfiable in an S-quasimodel for φ, then φ is satisfiable in a model based on a frame for S.*

Proof For every monodic formula $\psi(x)$ of the form $K_i\chi(x)$ or $C\chi(x)$ with one free variable x, we reserve a unary predicate $P_\psi(x)$. Likewise, for every sentence $\psi = K_i\chi$ or $\psi = C\chi$ we fix a propositional variable p_ψ. $P_\psi(x)$ and p_ψ will be called the *surrogates* for $\psi(x)$ and ψ.

Given a monodic formula ψ, we denote by $\overline{\psi}$ the formula that results from ψ by replacing all subformulas of the form $K_i\chi(x)$, $K_i\chi$, $C\chi(x)$, and $C\chi$, which are not within the scope of another epistemic operator, with their surrogates. Thus, $\overline{\psi}$ contains no occurrences of epistemic operators, i.e., it is a purely first-order formula; we will call $\overline{\psi}$ the *$\mathcal{L}$-reduct* of ψ. For a set of $\mathcal{CL}_1$-formulas Γ, let $\overline{\Gamma} = \{\overline{\psi} : \psi \in \Gamma\}$.

Now suppose φ is satisfied in an S-quasimodel $\langle \mathfrak{T}, \sigma \rangle$, $\mathfrak{T} = \langle W, <_1, \ldots, <_n \rangle$. So there is $w^* \in W$ such that $\varphi \wedge \alpha_{\sigma(w^*)}$ is S-consistent. It follows that the $\mathcal{L}$-reduct $\overline{\varphi} \wedge \overline{\alpha}_{\sigma(w^*)}$ is consistent with respect to classical first-order logic. Moreover, by Definition 8, $\alpha_{\sigma(w)}$ is S-consistent and $\overline{\alpha}_{\sigma(w)}$ is first-order consistent, for every $w \in W$. So, for each $w \in W$, we can find a structure $I(w) \models \overline{\alpha}_{\sigma(w)}$. We may also assume that $I(w^*) \models^{\mathfrak{a}^*} \overline{\varphi}$, for some assignment $\mathfrak{a}^*$.

Take a cardinal $\kappa \geq \aleph_0$ exceeding the cardinality of the set Ω of all runs in $\langle \mathfrak{T}, \sigma \rangle$ and put

$$D = \{(r, \xi) : r \in \Omega, \xi < \kappa\}.$$

Without loss of generality we can assume that D is the domain of the first-order structures $I(w)$ satisfying the $\overline{\alpha}_{\sigma(w)}$, that $c^w = (r_c, 0)$, and that

$$r(w) = \{\psi \in sub_x\varphi : I(w) \models \psi[(r, \xi)]\}, \tag{1}$$

for all runs r and $\xi < \kappa$. (Note that the underlying first-order language does not contain equality; for details see [9], Lemma 9.)

Let us now define the underlying frame $\mathfrak{F}$ of the model we are constructing. Its set of worlds is W. The accessibility relations R_i depend on S. Namely, we define R_i to be

- $<_i$ if $S = K_n^C$ or $S = KD_n^C$;

- $<_i \cup \{\langle w, w\rangle : w \in W\}$ if $S = T_n^C$;
- the transitive closure of $<_i$ if $S = K4_n^C$;
- the reflexive and transitive closure of $<_i$ if $S = S4_n^C$;
- $<_i^+ \cup \{\langle w, w'\rangle : \exists v \in W (v <_i^+ w \,\&\, v <_i^+ w' \,\&\, \neg\exists u\ u <_i v)\}$ if $S = KD45_n^C$;
- the reflexive, symmetric and transitive closure of $<_i$ if $S = S5$.

Note that for $S = KD_n^C$ the R_i are serial because in this case every S-consistent type for φ contains at least one formula of the form $\neg K_i\psi$. For $S = KD45_n^C$ the R_i are clearly serial. Suppose $w_1 R_i w$ and $w R_i w_2$. If w_1 has no $<_i$-predecessor, then $w_1 <_i^+ w_2$ and so $w_1 R_i w_2$. Otherwise, there are v_j, for $j = 1, 2$, such that $v_j <_i^+ w_j$, $v_j <_i^+ w$ and the v_j have no $<_i$-predecessors. Since $\mathfrak{T}$ is an irreflexive tree, we get $v_1 = v_2$. Thus $v_1 <_i^+ w_1$ and $v_1 <_i^+ w_2$. By the definition of R_i, it follows that $w_1 R_i w_2$. Hence R_i is transitive. Similarly one can show that R_i is euclidean.

Thus we have the model $\mathfrak{M} = \langle \mathfrak{F}, R_1, \ldots, R_n, I\rangle$. By induction on the construction of $\psi \in sub(\varphi)$ we will show now that for every assignment $\mathfrak{a}$

$$I(w) \models^{\mathfrak{a}} \overline{\psi} \text{ iff } (\mathfrak{M}, w) \models^{\mathfrak{a}} \psi.$$

The basis of induction, i.e., the case where $\psi = P_i(\tau_1, \ldots, \tau_m)$ is clear; for then $\psi = \overline{\psi}$. The induction step for $\psi = \psi_1 \wedge \psi_2$, $\psi = \neg\psi_1$, and $\psi = \forall x \psi_1$ follows by the induction hypothesis from the equations $\overline{\psi_1 \wedge \psi_2} = \overline{\psi_1} \wedge \overline{\psi_2}$, $\overline{\neg\psi_1} = \neg\overline{\psi_1}$, $\overline{\forall x \psi_1} = \forall x \overline{\psi_1}$. Let $\psi = K_i\chi(y)$ and assume that $\mathfrak{a}(y) = \langle r, \xi\rangle$. (If ψ is a sentence, y is any variable.) We then have:

$$\begin{aligned}
I(w) \models^{\mathfrak{a}} \overline{K_i\chi(y)} &\Leftrightarrow_1 I(w) \models^{\mathfrak{a}} P_{K_i\chi}(y)\\
&\Leftrightarrow_2 K_i\chi(x) \in r(w)\\
&\Leftrightarrow_3 \forall v\ (wR_i v \rightarrow \chi(x) \in r(v))\\
&\Leftrightarrow_4 \forall v\ (wR_i v \rightarrow I(v) \models^{\mathfrak{a}} \overline{\chi}(y))\\
&\Leftrightarrow_5 \forall v\ (wR_i v \rightarrow (\mathfrak{M}, v) \models^{\mathfrak{a}} \chi(y))\\
&\Leftrightarrow_6 (\mathfrak{M}, w) \models^{\mathfrak{a}} K_i\chi(y).
\end{aligned}$$

Equivalence $\Leftrightarrow_1$ holds by the definition of $\overline{\psi}$; $\Leftrightarrow_2$ and $\Leftrightarrow_4$ are consequences of (1). The induction hypothesis yields $\Leftrightarrow_5$, and $\Leftrightarrow_6$ holds by definition. The only non-trivial case is $\Leftrightarrow_3$.

($\Rightarrow_3$) Suppose $K_i\chi \in r(w)$ and $wR_i w'$. So if $w <_i w'$ the claim follows by Definition 9. If $S = K$ or $S = D$ then we are done, because $R_i = <_i$.

Let $S = KD45$. By the definition of R_i, we have either $w <_i^+ w'$ or there exists v such that $v <_i^+ w$, $v <_i^+ w'$ and $\neg\exists u\ u <_i v$. The former case is easy; we leave it to the reader and consider the latter one here. We have some $m \in \omega$ and worlds $v_0, \ldots, v_{m+1}$ such that $v_0 = v$, $v_{m+1} = w$ and $v_j <_i v_{j+1}$ for every $j \leq m$. By Definition 9, $(r(v_j), r(v_{j+1}))$ is i-suitable for S whenever $j \leq m$. By Lemma 7, $K_i\chi(x) \in r(v_0) = r(v)$. Similarly, using $v <_i^+ w'$ we obtain worlds $u_0, \ldots, u_{l+1}$ such that $u_0 = v$, $u_{l+1} = w'$ and $u_j <_i u_{j+1}$ for every $j \leq l$. Again, $(r(u_j), r(u_{j+1}))$ is i-suitable for S whenever $j \leq l$, and by Lemma 7, $K_i\chi(x) \in r(u_l)$. Using the same lemma once again, we obtain $\chi(x) \in r(u_{l+1}) = r(w')$. ($\Leftarrow_3$) is an immediate consequence of Definition 9. Other cases for S are treated analogously.

Finally, let $\psi = C\chi(y)$ and $\mathfrak{a}(y) = \langle r, \xi \rangle$. Then we have the following chain of equivalences:

$$
\begin{aligned}
I(w) \models^{\mathfrak{a}} \overline{C\chi(y)} &\Leftrightarrow_1 I(w) \models^{\mathfrak{a}} P_{C\chi}(y) \\
&\Leftrightarrow_2 C\chi(x) \in r(w) \\
&\Leftrightarrow_3 \forall v \ (w(\bigcup_{i \leq n} R_i)^+ v \to \chi(x) \in r(v)) \\
&\Leftrightarrow_4 \forall v \ (w(\bigcup_{i \leq n} R_i)^+ v \to I(v) \models^{\mathfrak{a}} \overline{\chi}(y)) \\
&\Leftrightarrow_5 \forall v \ (w(\bigcup_{i \leq n} R_i)^+ v \to (\mathfrak{M}, v) \models^{\mathfrak{a}} \chi(y)) \\
&\Leftrightarrow_6 (\mathfrak{M}, w) \models^{\mathfrak{a}} C\chi(y).
\end{aligned}
$$

As before, the only non-trivial one is $\Leftrightarrow_3$. ($\Leftarrow_3$) follows from Definition 9. For the other direction, suppose $C\chi(x) \in r(w)$ and $w(\bigcup_{i \leq n} R_i)^+ v$. Then there are $v_0, \ldots, v_{m+1}$, for some $m \in \omega$, such that $v_0 = w$, $v_{m+1} = v$ and for all $j \leq m$ there is some $i_j \leq n$ with $v_j R_{i_j} v_{j+1}$. By induction on j one can show that $C\chi \in r(v_j)$ for all $j \leq m$. (The basis of induction is clear, and the induction step is similar to the proof ($\Rightarrow_3$) above.) Thus we have $C\chi \in r(v_m)$. By the axiom $C\chi \to E(\chi \wedge C\chi)$, we obtain $E(\chi \wedge C\chi) \in r(v_m)$. Hence, $K_{i_m}(\chi \wedge C\chi) \in r(v_m)$, and so, in view of Lemma 7, $\chi \in r(v_{m+1}) = r(v)$. ❑

Thus, to prove completeness of our axiom system S, it suffices to construct an S-quasimodel satisfying φ whenever φ is S-consistent.

Lemma 12. *Let $\mathfrak{P} = \langle \mathfrak{C}, t \rangle$ be a pointed state candidate for φ such that $\beta_{\mathfrak{P}}$ is S-consistent.*

(i) *If $\neg K_i \psi \in t$, then there exists $\mathfrak{P}' = \langle \mathfrak{C}', t' \rangle$ such that $\mathfrak{P} \prec_i \mathfrak{P}'$ and $\psi \notin t'$. Moreover, if $\langle t, c \rangle \in \mathfrak{C}$ for some constant c, then we can choose $\mathfrak{P}' = \langle \mathfrak{C}', t' \rangle$ with $\mathfrak{P} \prec_i \mathfrak{P}'$ and $\psi \notin t'$ so that $\langle t', c \rangle \in \mathfrak{C}'$.*

(ii) *If $\neg C\psi \in t$, then there are pointed state candidates $\mathfrak{P}_j = \langle \mathfrak{C}_j, t_j \rangle$, $j \leq k$, with $\mathfrak{P}_0 = \mathfrak{P}$ and*

$$\mathfrak{P}_0 \prec_{i_1} \mathfrak{P}_1 \prec_{i_2} \cdots \prec_{i_k} \mathfrak{P}_k,$$

for some $i_1, \ldots, i_k \leq n$, such that $\neg\psi \in t_k$. Moreover, if $\langle t, c \rangle \in \mathfrak{C}$, then we can choose such a sequence with $\langle t_j, c \rangle \in \mathfrak{C}_j$ for all $j \leq k$.

Proof (i) follows from Lemma 6. So let us prove (ii). Suppose that such a sequence does not exist. Let $\mathcal{T}$ be the minimal set of pointed state candidates such that

- $\mathfrak{P} \in \mathcal{T}$,
- if $\mathfrak{D}_1 \in \mathcal{T}$ and $\mathfrak{D}_1 \prec_i \mathfrak{D}_2$ for some i, then $\mathfrak{D}_2 \in \mathcal{T}$.

Let $\vartheta = \bigvee_{\mathfrak{D} \in \mathcal{T}} \beta_{\mathfrak{D}}$. Then $\vdash_S \vartheta \to K_i \vartheta$, for all $i \leq n$. Indeed, suppose otherwise. Then $\vartheta \wedge \neg K_i \vartheta$ is S-consistent for some $i \leq n$. But then, by Lemma 6, $\vartheta \wedge \neg K_i \neg \beta_{\mathfrak{P}'}$ is S-consistent for some pointed state candidate $\mathfrak{P}' \notin \mathcal{T}$. This, however, contradicts the definition of $\mathcal{T}$, since we would have $\mathfrak{D} \prec_i \mathfrak{P}'$ for some disjunct $\beta_{\mathfrak{D}}$ of ϑ. Hence $\vdash_S \vartheta \to E\vartheta$. Clearly, $\psi \in s$ for every $(\mathfrak{D}, s) \in \mathcal{T}$, for otherwise we could construct a sequence satisfying condition (ii). Thus, $\vdash_S \vartheta \to \psi$ and so $\vdash_S \vartheta \to E(\psi \wedge \vartheta)$. By the inference rule for C we obtain $\vdash_S \vartheta \to C\psi$, and so $\vdash_S \beta_{\mathfrak{P}}(x) \to C\psi$, since $\mathfrak{P} \in \mathcal{T}$. But then $\beta_{\mathfrak{P}}$ is S-inconsistent, which is a contradiction. ❑

We are in a position now to prove the main result of this section.

Theorem 13. *If S is one of the axiomatic systems defined above and φ an S-consistent monodic formula, then φ is satisfiable in a model based on a frame for S.*

Proof In view of Theorem 11, it suffices to construct an S-quasimodel satisfying φ. By Lemma 6, we can find a state candidate $\mathfrak{C}$ such that $\varphi \wedge \alpha_{\mathfrak{C}}$ is S-consistent. We are going to construct the required quasimodel as the limit of a sequence

$$(\mathfrak{T}_m, \sigma_m) = (\langle W_m, <_1^m, \dots, <_n^m \rangle, \sigma_m),$$

of basic S-trees, $m \in \omega$.

Let $W_0 = \{w^*\}$ for some point w^* and let $\sigma_0(w^*) = \mathfrak{C}$. Suppose now that $\mathfrak{T}_m$ has been already defined. For every $w \in W_m - W_{m-1}$ we shall construct a number of new points 'saturating' $\sigma_m(w)$ $(W_{-1} = \emptyset)$. Let $\mathfrak{D} = \sigma_m(w)$, $\mathfrak{D} = \langle T, T^{con} \rangle$. Pick some $t \in T$ and do the following:

(a) For every $\chi = \neg K_i \psi \in t$ we take *two* points a_χ and b_χ, add them to W_m, put $w <_i^{m+1} a_\chi$, $w <_i^{m+1} b_\chi$, and $\sigma_{m+1}(a_\chi) = \sigma_{m+1}(b_\chi) = \mathfrak{D}'$, for some $\mathfrak{D}'$ underlying a pointed state candidate $\mathfrak{P}' = (\mathfrak{D}', t')$ with $(\mathfrak{D}, t) \prec_i \mathfrak{P}'$ and $\psi \notin t'(x)$. That such a $\mathfrak{P}'$ exists is guaranteed by Lemma 12. If $\langle t, c \rangle \in \sigma(w)$ for some constant c, then we take for t' the type s with $\langle s, c \rangle \in \mathfrak{D}'$.

(b) For every $\chi = \neg C\psi \in t$ we take *two* sequences $a_\chi^1, \dots, a_\chi^k$ and $b_\chi^1, \dots, b_\chi^k$ and put

$$w <_{i_1}^{m+1} a_\chi^1 <_{i_2}^{m+1} \cdots <_{i_k}^{m+1} a_\chi^k, \qquad w <_{i_1}^{m+1} b_\chi^1 <_{i_2}^{m+1} \cdots <_{i_k}^{m+1} b_\chi^k,$$

and

$$\sigma_{m+1}(a_\chi^j) = \sigma_{m+1}(b_\chi^j) = \mathfrak{D}^j, \text{ for all } j \leq k,$$

where $(\mathfrak{D}^j, t^j)$ is a sequence of pointed state candidates with

$$(\mathfrak{D}, t) \prec_{i_1} (\mathfrak{D}^1, t^1) \prec_{i_2} \cdots \prec_{i_k} (\mathfrak{D}^k, t^k)$$

and $\neg\psi \in t^k$. Again Lemma 12 ensures the existence of such a sequence. If $\langle t, c \rangle \in \sigma(w)$ for some constant c, then we take a sequence with $\langle t^j, c \rangle \in \mathfrak{D}^j$ for all $j \leq k$.

In the same manner we consider the remaining types in T and then the remaining worlds $v \in W_m - W_{m-1}$. W_{m+1} is defined as the (disjoint) union of W_m and the constructed new points. The relations $<_i^{m+1}$ coincide with $<_i^m$ on W_m. For the new points their extension is defined above. The function σ_{m+1} coincides with σ_m on W_m and is defined above for the new worlds. Thus we have constructed $\mathfrak{T}_{m+1}$.

Finally, put $(\mathfrak{T}, \sigma) = (\langle W, <_1^m, \dots, <_n^m \rangle, \sigma)$, where

$$W = \bigcup_{m<\omega} W_m, \quad <_i = \bigcup_{m<\omega} <_i^m, \quad \sigma = \bigcup_{m<\omega} \sigma_m.$$

It remains to show that $(\mathfrak{T}, \sigma)$ is an S-quasimodel. It should be clear that the functions r_c are runs. So it suffices to show that, for all $w \in W$ and $t \in \sigma(w)$, there exists a run r with $r(w) = t$.

First, using Lemma 7 we find a sequence

$$w^* = w_0 <_{i_1} w_1 <_{i_2} \cdots <_{i_k} w_k = w$$

and types $t_j \in \sigma(w_j)$, $0 \leq j \leq k$, such that $t_k = t$ and $t_j(x) \wedge \neg K_{i_{j+1}} \neg t_{j+1}(x)$ is S-consistent for all $j < k$.

Let $r(w_j) = t_j$ and $V_0 = \{w_0, \ldots, w_{k+1}\}$. Define by induction an increasing chain of sets $V_i \supseteq W_i$ with $V_i - W_i \subseteq V_0$, on which we define r. Suppose V_n is defined. For every $w \in W_n - W_{n-1}$ with $r(w) = t$ we do the following:

- If $\neg K_i \psi \in t$, then take $v \in W_{n+1} - V_n$ with $w <_i v$ and $t' \in \sigma(v)$ such that $t \wedge \neg K_i \neg t'$ is S-consistent and $\psi \notin t'$. This can be done because we always took *two* saturating worlds in the construction above. Put $r(v) = t'$.
- If $\neg C\psi \in t$, then take a sequence $v_1, \ldots, v_k$ from $W_{n+1} - V_n$ such that $w <_{i_0} v_1$, $v_1 <_{i_1} \cdots <_{i_{k-1}} v_{i_k}$, and types $t_j \in \sigma(v_j)$, $1 \leq j \leq k$, such that
  - (t, t_1) is i_0-suitable for S,
  - (t_j, t_{j+1}) is i_j-suitable for S, $1 \leq j < k$,
  - $\psi \notin t_k$.

  Again, this can be done since we always took two saturating sequences. Put $r(v_j) = t_j$ for all $1 \leq j \leq k$.

Finally, we have to define r for all $v \in W_{n+1}$ where r was not defined above. This can be done recursively as follows. Suppose $r(v)$ is not defined yet for some $v \in W_{n+1}$. If r is defined already for the (unique) v' such that $v' <_i v$, then take a $t \in \sigma(v)$ such that $(r(v'), t)$ is i-suitable for S and put $r(v) = t$ (t exists by Lemma 7). Otherwise consider first v' itself.

It is now straightforward to see that r is a run. ❑

As a consequence we obtain:

Theorem 14. *Let $S \in \{K_n^C, T_n^C, KD_n^C, K4_n^C, S4_n^C, KD45_n^C, S5_n^C\}$ and let F be the class of frames for S. Then for every monodic formula φ, we have $\vdash_S \varphi$ iff $\varphi \in L_1(\mathsf{F})$.*

5 Decidability

Another important algorithmic feature of the monodic formulas is that if, roughly speaking, we restrict the underlying purely first-order formulas to a decidable class, then the resulting monodic fragments of the epistemic logics under consideration will also be decidable. In particular, we have the following:

Theorem 15. *Let F be any of the frame classes mentioned at the end of Section 2. Then the following fragments are decidable:*

- *the monadic fragment of $L_1(\mathsf{F})$,*
- *the two-variable fragment of $L_1(\mathsf{F})$,*
- *the guarded fragment of $L_1(\mathsf{F})$.*

(Note, however, that the guarded fragment of $L(\mathsf{F}^{\mathsf{a}})$ is undecidable.) For more details and an idea of the proof the reader is referred to [9, 20]. Actually, no non-trivial decidable fragments of epistemic predicate logics have been constructed before.

It maybe also of interest to note that these decidability results make it possible to construct various decidable description logics with common knowledge and other epistemic operators applicable to concepts and formulas (but not to roles; see [19]). Weaker epistemic description logics were proposed in [6, 12].

6 Adding equality

In this section we show that the addition of equality to the language of monodic formulas restores the 'status quo,' namely, that all the fragments considered above become non-enumerable. Let $\mathcal{CL}_1^=$ be the language $\mathcal{CL}_1$ extended with the equality symbol interpreted in first-order structures as identity.

Theorem 16. *Let* F *be any of the frame classes defined at the end of Section 2. Then the logic* $L_1(\mathsf{F})$ *in the language* $\mathcal{CL}_1^=$ *is not recursively enumerable.*

Proof Define ψ to be the conjunction of the following $\mathcal{CL}_1^=$-formulas:

$$\psi_1 = \exists x P(x) \wedge \forall x \forall y\ (P(x) \wedge P(y) \to x = y),$$
$$\psi_2 = C\forall x \forall y\ (\neg K_i \neg P(x) \wedge \neg K_i \neg P(y) \wedge \neg P(x) \wedge \neg P(y) \to x = y),\ \text{for } i \leq 2,$$
$$\psi_3 = \neg C \neg \forall x\ (P(x) \leftrightarrow \neg C \neg P(x)),$$
$$\psi_4 = \forall x\ (\neg C \neg P(x) \to C \neg C \neg P(x)),$$
$$\psi_5 = \forall x\ (Q(x) \leftrightarrow \neg C \neg P(x)).$$

First we notice that for all models $\mathfrak{M} = \langle \mathfrak{F}, D, I \rangle$ with $\mathfrak{F} = \langle W, R_1, \dots, R_n \rangle$ and all $w \in W$, if $(\mathfrak{M}, w) \models \psi$ then $Q^{I(w)}$ is finite. Indeed, by ψ_1, the set $P^{I(w)}$ is a singleton. From ψ_3 we get some w' for which $w(\bigcup_{i \leq 1} R_i)^+ w'$ and $w' \models \forall x\ (P(x) \leftrightarrow \neg C \neg P(x))$. Hence there are $w_0, \dots, w_{m+1} \in W$ such that $w_0 = w$, $w_{m+1} = w'$ and for every $j \leq m$ there is some $i_j \leq 2$ for which $w_j R_{i_j} w_{j+1}$ holds. In view of ψ_2, $|P^{I(w_{j+1})} - P^{I(w_j)}| \leq 1$ for every $j \leq m$, which yields $|P^{I(w')}| \leq m + 1$. Thus it remains to show that $Q^{I(w)} \subseteq P^{I(w')}$. Suppose $a \in Q^{I(w)}$. Then, by ψ_5, $w \models \neg C \neg P[a]$ and by ψ_4 we obtain $w \models C \neg C \neg P[a]$, from which $w' \models P[a]$.

Second, we show that for every first-order sentence θ containing neither P nor Q the following are equivalent:

(a) θ is true in all *finite* first-order structures;

(b) $\psi \to \theta^Q$ is valid in all frames in F.

(Here θ^Q is the *relativisation of* θ to Q, i.e., $\theta^Q = \theta$ if θ is atomic, ^{Q} commutes with the booleans, and $(\forall x \theta_1)^Q = \forall x\ (Q(x) \to \theta_1^Q)$.)

(a) $\Rightarrow$ (b). Suppose there is a model $\mathfrak{M}$ and a world w in it such that $w \models \psi$ but $w \not\models \theta^Q$. Define a finite first-order structure J with domain $E = Q^{I(w)}$ and predicates $P_k^J = P_k^{I(w)} \cap E$. It can be easily shown by induction that for every formula χ and every assignment $\mathfrak{a}$ in E, we have $J \models^{\mathfrak{a}} \chi$ iff $w \models^{\mathfrak{a}} \chi^Q$. In particular, $J \not\models \theta$.

(a) $\Leftarrow$ (b). Let us show first that for every natural number $m > 0$, there are $\mathfrak{M}_m = \langle \mathfrak{F}_m, D_m, I_m \rangle$ based on a frame $\mathfrak{F}_m \in \mathsf{F}^{\mathsf{re}}$ and w in $\mathfrak{F}_m$ such that $|W| = m$ and $w \models \psi$. Indeed, put $W_m = \{w_1, \ldots, w_m\}$, $w_i \neq w_j$ whenever $i \neq j$, $D_m = \mathbb{N}$, and

$$R_1 = \{\langle w_k, w_{k+1} \rangle, \langle w_{k+1}, w_k \rangle : k < m \ \& \ \exists l \ k = 2l+1\} \cup \{\langle w_k, w_k \rangle : k \leq m\},$$
$$R_2 = \{\langle w_k, w_{k+1} \rangle, \langle w_{k+1}, w_k \rangle : k < m \ \& \ \exists l \ k = 2l\} \cup \{\langle w_k, w_k \rangle : k \leq m\}.$$

Finally, for each $k \leq m$, put $P^{I(w_k)} = \{0, \ldots, k-1\}$ and $Q^{I(w_k)} = \{0, \ldots, m-1\}$ (see Fig. 1). It is easy to see that the model is reflexive and euclidean, and $w_1 \models \psi$.

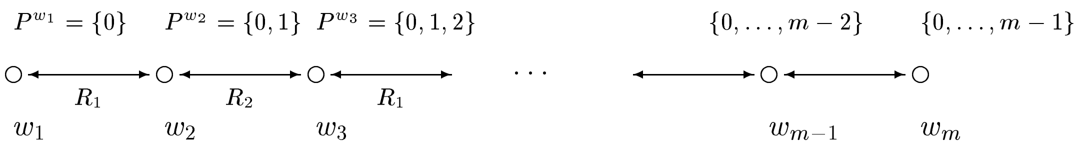

Figure 1:

Now, to complete the proof, suppose that there is a finite first-order structure J with domain D such that $|D| = m$ and $J \not\models \theta$. Take the model $\mathfrak{M}_m$ and the world $w \not\models \psi$ constructed above. Without loss of generality we can assume that $Q^{I_m(w)} = D$. Now expand $\mathfrak{M}_m$ to a model $\mathfrak{M}'_m$ by interpreting each predicate symbol P_i as follows: $P_i^{I'_m(w')} = P_i^J$, for each $w' \in W_m$. Now, for every first-order formula χ (without P, Q) and every assignment $\mathfrak{a}$ in D, we have $J \models^{\mathfrak{a}} \chi$ iff $(\mathfrak{M}'_m, w) \models^{\mathfrak{a}} \chi^Q$. Therefore, $(\mathfrak{M}'_m, w) \not\models \theta^Q$, and so $\psi \to \theta^Q$ is not valid in F^{re} (which is contained in all our frame classes).

It remains to recall that, by Trakhtenbrot's theorem (see [4]), the set of first-order sentences that are valid in finite structures is not recursively enumerable. ❑

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