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Common knowledge and quantification

Holger Sturm[1], Frank Wolter[1], Michael Zakharyaschev[2]

[1] Institut für Informatik, Universität Leipzig, Augustus-Platz 10-11, 04109 Leipzig, Germany
[2] Department of Computer Science, King's College, Strand, London WC2R 2LS, U.K.

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Abstract The paper consists of two parts. The first one is a concise introduction to epistemic (both propositional and predicate) logic with common knowledge operator. As the full predicate logics of common knowledge are not even recursively enumerable, in the second part we introduce and investigate the monodic fragment of these logics which allows applications of the epistemic operators to formulas with at most one free variable. We provide the monodic fragments of the most important common knowledge predicate logics with finite Hilbert-style axiomatizations, prove their completeness, and single out a number of decidable subfragments. On the other hand, we show that the addition of equality to the monodic fragment makes it not recursively enumerable.

1 Introduction

Ever since it became common knowledge that the intelligent behavior of an agent is based not only on her knowledge about the world but also on the knowledge about both her own and other agents' knowledge, logical formalisms designed for reasoning about knowledge have attracted attention in artificial intelligence, computer science, economic theory, and philosophy (see e.g. the books [7,21,18] and the seminal works [11,2]). In all these areas, one of the most successful approaches is to supply classical—propositional or first-order—logic with an explicit epistemic operator K_i for each agent i under consideration. $K_i\varphi$ means that *agent i knows* (or *believes*) φ, $K_1K_2\varphi$ says then that *agent 1 knows that agent 2 knows* φ.

There can be different explications of the knowledge operators. For example, the schema $K_i\psi \rightarrow K_iK_i\psi$, known as *positive introspection*, states that *agent i knows what she knows*, while the schema $\neg K_i\psi \rightarrow K_i\neg K_i\psi$, known as *negative introspection*, says that *agent i knows what she does not know*. To accept one of these schemata or both as valid means to assume the corresponding cognitive capabilities of the agents.

Many of such explications have been characterized by means of axiomatic systems (see e.g. [10,7]). On the other hand, the possible worlds semantics [11] provided a framework to interpret the epistemic language formally: in a world w agent i knows φ if and only if φ holds in all worlds that i regards as possible in w (the difference between various understandings of K_i is reflected by different accessibility relations among the worlds).

The situation becomes much more complicated when—in order to describe the behavior of *multi-agent* systems—we have to capture the *common knowledge* of a group of agents. Common knowledge that a proposition φ holds does not only mean that all the agents in the group know that φ holds, but also that all the agents know that all the agents know that φ is true, that all the agents know that all the agents know that all the agents know that φ is true, and so on. This concept of common knowledge is required for analyzing conventions [19], coordinizations in multi-agent systems [7], common sense reasoning [20], agreement [2,3], etc.

There are different approaches to representing common knowledge by means of logic formalisms. Perhaps the most intuitive one is to supply the language described above with infinitary conjunctions: given an infinite set Γ of formulas, we are allowed to form the formula $\bigwedge \Gamma$. For

$$\Gamma = \{K_1\varphi, K_2\varphi, K_2K_1\varphi, K_1K_2\varphi, K_2K_1K_2\varphi, \dots\}$$

the formula $\bigwedge \Gamma$ means then 'it is common knowledge of the group consisting of agents 1 and 2 that φ holds.' The resulting infinitary logic was investigated in detail by Kaneko and Nagashima [15,16]].

Another approach to formalizing common knowledge tries to avoid infinite connectives. Of course, one can simply add to the basic epistemic language an explicit unary operator C with the intended meaning 'it is common knowledge (of the group of agents) that .... ' The problem is to axiomatize this operator and supply it with a reasonable semantics without using infinite conjunctions. This problem was successfully solved in the *propositional* case: both natural possible worlds semantics and clear inductive axiomatizations have been found for propositional common knowledge logics [10]. (The new operator, however, considerably increases the computational complexity of these logics—from PSPACE to EXPTIME; consult [7].) Moreover, the infinitary and the finitary approaches proved to be closely related: [14] constructs a faithful embedding of the propositional common knowledge logics into infinitary ones.

It turned out to be much harder to combine the common knowledge operator with the first-order quantifiers. First, no common knowledge predicate logic with both a finitary (or at least recursive) axiomatization and a reasonable semantics has ever been constructed. And second, the common knowledge predicate logics determined by the standard possible worlds semantics proved to be not recursively axiomatizable (and so not recursively enumerable) [22]. Thus, similar to arithmetic or second-order logic, it is impossible to characterize these semantically defined logics by means of effective proof systems. In a sense this means that neither we nor the Turing machine have the capacity of understanding the interaction between common knowledge and quantifiers. Moreover, this is true of even very small fragments of the logics, say, the monadic (in which only unary predicates are allowed) or two-variable fragments (see [22]).

These negative results might be regarded as an argument against the finitary approach to formalizing common knowledge in predicate logic: if there is no effective (and hence no finitary) way to characterize common knowledge predicate logic syntactically, then why should we consider finite languages at all? However, the example of second-order logic gives a weighty counter-argument: even logics of monstrous complexity may contain useful 'manageable' and expressive fragments, witness the monadic second-order logic with its multiple applications (see e.g. [4] and references therein). The main aim of this paper is to show that common knowledge predicate logics also contain effective—in fact, finitely axiomatizable or even decidable—and yet expressive fragments with interesting interactions between the common knowledge operator and quantifiers.

A promising approach to singling out such kind of fragments of first-order modal and temporal logics has been proposed in [12,25]. The idea is to restrict attention to the class of *monodic*[1] formulas which allow applications of modal or temporal operators only to formulas with at most one free variable. In the epistemic context, monodicity means, in particular, that:

- we have the full expressive power of common knowledge predicate logic as far as we apply epistemic operators only to sentences (i.e., formulas without free variables);
- monodic formulas are capable of formalizing the distinction between 'knowing that' and 'knowing what' (i.e., modalities *de dicto* and *de re*): the monodic formula $K_i \exists x \ \mathsf{name}(x, a)$ says '*agent i knows that a has a name*,' while the monodic formula $\exists x K_i \mathsf{name}(x, a)$ means '*i knows the name of a*;'
- we can reason about agents' knowledge of properties, for instance,

  $$\forall x \ (C \,\mathsf{ancestor}(John, x) \lor C \,\neg\mathsf{ancestor}(John, x))$$

  ('for every object x, it is common knowledge whether John is an ancestor of x'); however, we are not permitted to reason about agents' knowledge of relations, say

  $$\forall x, y \ (C \,\mathsf{ancestor}(x, y) \lor C \,\neg\mathsf{ancestor}(x, y))$$

  ('for all pairs x, y, it is common knowledge whether x is an ancestor of y').

In this paper

[1] *Monody* is a composition with only one melodic line.

we provide finite axiomatizations of the monodic fragments of almost all interesting common knowledge predicate logics *without equality*. Moreover, we observe that a number of natural subclasses of the monodic fragment, say, with only monadic predicates or two variables, are decidable. On the other hand, it is proved that the addition of equality to the monodic fragment makes it not recursively enumerable.

The last 'negative' result together with those of [22] show that the monodic fragment (without equality) is an optimal compromise between expressiveness and effectiveness: even seemingly small extensions of this fragment are not recursively enumerable any more. Thus, this paper reveals how much interaction between the common knowledge operator and quantifiers can be characterized by means of a finitary proof system.

Although the monodic fragment considerably extends the expressive power of propositional common knowledge logic, one should remember that it can be used only when the application domain does not need complex forms of quantification into epistemic contexts. To see the limitations, suppose that we have a first-order formula $\mathsf{Nash}_G(x_1, \ldots, x_n)$ which is true if and only if $\langle x_1, \ldots, x_n \rangle$ is a Nash-Equilibrium for a game G with n players.[2] Consider now the formulas

$$C \exists x_1 \ldots \exists x_n \mathsf{Nash}_G(x_1, \ldots, x_n),$$

$$\exists x_1 \ldots \exists x_n C \mathsf{Nash}_G(x_1, \ldots, x_n).$$

The former formula states that it is common knowledge that there exists a Nash-Equilibrium for G; it belongs to the monodic fragment. The latter one says that at least one Nash-Equilibrium for G is common knowledge; it belongs to the monodic fragment only in the trivial case of G having a single player. Thus, in the monodic fragment we cannot formalize the difference between common knowledge that a Nash-Equilibrium exists and common knowledge of a specific Nash-Equilibrium (for a discussion of this topic in both frameworks—infinitary epistemic predicate logic and full common knowledge predicate logic—consult [15,16,22]

The paper is organized in the following way. Sections 2 and 3 can be regarded as a brief introduction to the propositional and first-order logics of common knowledge; they were written with the aim to keep the paper self-contained and make it readable for non-specialists. In Section 4 we define the monodic fragments of the logics introduced in Section 3 and provide axiomatic systems for them. Section 5 shows that these axiomatic systems are complete with respect to their intended semantics. More precisely, we prove for each axiomatic system S that a monodic formula φ is provable in S if and only if it is valid in the class of Kripke frames for S. In Section 6 we show how to extract decidable fragments of the monodic part of

[2] Such a formula was constructed in [15, 22].

common knowledge logics. The language of the logics we deal with in Sections 3–6 does not contain equality. In Section 7 we consider the language with equality and prove that it makes the monodic fragment not recursively enumerable.

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2 Propositional logics of common knowledge

We begin by introducing the propositional part of the common knowledge logics we deal with in this paper.[3] Its language, called here $\mathcal{CL}$, is obtained by extending the standard language of propositional logic with the epistemic operators K_i ('agent i knows') and C ('it is common knowledge that'). More precisely, the alphabet of $\mathcal{CL}$ consists of

- *propositional variables* $p_0, p_1, \dots,$
- the *boolean connectives* $\wedge$, $\neg$,
- a finite number of *knowledge operators* $K_1, \dots, K_n$, $n \geq 1$, and
- the *common knowledge operator* C.

The set of *$\mathcal{CL}$-formulas* is defined inductively as follows:

- every propositional variable is a formula (an *atomic formula* or an *atom*, to be more precise);
- if φ and ψ are formulas, then so are $(\varphi \wedge \psi)$ and $(\neg\varphi)$;
- if φ is a formula and $i \leq n$, then $(K_i\varphi)$ and $(C\varphi)$ are formulas.

(Thus $\mathcal{CL}$ is a usual propositional polymodal language with $n+1$ unary modal operators K_i and C.) Throughout the paper we use the following standard abbreviations and conventions on reading formulas:

$$\begin{aligned}
\varphi \vee \psi &= \neg((\neg\varphi) \wedge (\neg\psi)),\\
\varphi \to \psi &= (\neg\varphi) \vee \psi,\\
\varphi \leftrightarrow \psi &= (\varphi \to \psi) \wedge (\psi \to \varphi),\\
\top &= p_0 \vee (\neg p_0),\\
\bot &= p_0 \wedge (\neg p_0).
\end{aligned}$$

The unary logical connectives ($\neg$, K_i, and C) are assumed to bind formulas stronger than the binary ones, and among the latter $\wedge$ and $\vee$ bind stronger than $\to$ and $\leftrightarrow$. So $Cp_0 \to \neg p_1 \wedge K_1 p_2$ should be read as $(Cp_0) \to ((\neg p_1) \wedge (K_1 p_2))$.

Epistemic logics are intended to codify correct principles and rules of reasoning about what people (and other agents) know and believe. *Logics*,

[3] For a more detailed exposition of propositional epistemic logic and its applications in game theory we refer the reader to "Introduction to epistemic logics and their game theoretic applications" by M. Kaneko in this issue.

in contrast to theories, are mainly concerned with truth by meaning, not by fact: they are supposed to single out those formulas (from the underlying language) that are true independently from the world. In this sense, epistemic logics describe our conceptual knowledge with respect to the epistemic notions involved or, to put it another way, they help to fix the meaning of these notions.

Formally, a propositional epistemic logic is just a certain subset of the set of $\mathcal{CL}$-formulas. It should contain those and only those formulas that are regarded to be valid no matter what truth-values are assigned to their propositional variables. For example, it is usually assumed that epistemic logics contain at least all tautologies of classical propositional logic. Other formulas in the logic reflect a particular understanding of the operators K_i. For instance, when we treat K_i as 'agent i knows,' it is reasonable to accept formulas of the form $K_i\varphi \to \varphi$ saying that only true things can be known. However, if K_i is supposed to mean 'agent i believes,' these formulas would hardly be acceptable.

There are two main approaches to characterizing a logic in the above sense. The syntactic approach deals with representations of logics in the form of *axiomatic systems* which consist of

- a language (in our case, $\mathcal{CL}$ and its extensions),
- a set of axioms (typically given by axiom schemata),
- and a set of inference rules.

The logic generated by such an axiomatic system S is the set of formulas that can be *proved* from the axioms of S by means of its inference rules. More precisely, a formula φ is *provable* in S ($\vdash_S \varphi$, for short) if there is a finite sequence $\varphi_0, \ldots, \varphi_n$ of formulas in the language of S such that $\varphi_n = \varphi$ and, for each $i \leq n$, either φ_i is an axiom of S or it is obtained from some previous formulas in the sequence by applying one of the inference rules of S. Let Γ be a set of formulas. We say that φ is *derivable* from Γ in S ($\Gamma \vdash_S \varphi$ in symbols) if either $\vdash_S \varphi$ or there are $\psi_1, \ldots, \psi_m \in \Gamma$ such that $\vdash_S \psi_1 \wedge \cdots \wedge \psi_m \to \varphi$.

The following axiom schemata and rules will be part of all our axiomatic systems.

Axiom schemata:

Propositional Calculus: the set of axiom schemata from some axiomatization of classical propositional logic,

Distribution: $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$, for $i \leq n$.

Inference rules:

Modus Ponens: $\dfrac{\varphi \to \psi,\ \varphi}{\psi}$,

Necessitation: $\dfrac{\varphi}{K_i\varphi}$, for $i \leq n$.

Let $\mathcal{CL}^-$ be the sublanguage of $\mathcal{CL}$ which consists of all those $\mathcal{CL}$-formulas that contain no occurrences of the common knowledge operator C. The axiomatic system in the language $\mathcal{CL}^-$ determined by the axioms and inference rules formulated above will be denoted by $\boldsymbol{K}_n$ (it generates the minimal normal modal logic with n necessity operators). By extending $\boldsymbol{K}_n$ with one or more axiom schemata from the following list (where $1 \leq i \leq n$) we obtain the other propositional epistemic logics we deal with in what follows :

A_D: $K_i\varphi \to \neg K_i\neg\varphi$,
A_T: $K_i\varphi \to \varphi$,
A_4: $K_i\varphi \to K_iK_i\varphi$,
A_5: $\neg K_i\varphi \to K_i\neg K_i\varphi$.

Namely, $\boldsymbol{T}_n$ is obtained from $\boldsymbol{K}_n$ by adding the axiom schema A_T. $\boldsymbol{KD}_n$ and $\boldsymbol{K4}_n$ are defined as the axiomatic systems obtained by adding to $\boldsymbol{K}_n$ the schemata A_D, and A_4, respectively. $\boldsymbol{S4}_n$ is $\boldsymbol{K4}_n$ plus A_T. $\boldsymbol{KD45}_n$ is $\boldsymbol{K4}_n$ extended by A_D and A_5, and $\boldsymbol{S5}_n$ is $\boldsymbol{KD45}_n$ plus A_T.

Now let us turn to the semantical approach to characterizing logics. Semantics is supposed to interpret syntactical objects of a formal language in certain mathematical structures (say, algebraic structures or another formal language). Classical propositional logic is usually interpreted in the boolean algebra with two elements (truth and falsehood): these elements are regarded as possible values of the propositional variables, while the truth-values of compound formulas are computed starting from those of the atomic ones in accordance with the standard truth-tables. Each assignment of truth-values to the atoms describes a 'possible state of affairs' or a 'possible world' the language is talking about. A formula is classically valid if and only if it is true in all such worlds.

In the epistemic case we need to interpret the knowledge operators K_i as well. Being at a state of affairs or 'living' in a world w and knowing φ, agent i can regard as possible (in w) only those worlds where φ is true. Conversely, if φ is true in all worlds that i considers possible in a world w, then it is natural to evaluate $K_i\varphi$ as true in w. This is the intuition behind the Kripke (or the possible world) semantics of epistemic logics (cf. e.g. [7]).

More formally, a *Kripke frame* is a structure of the form $\mathfrak{F} = \langle W, R_1, \ldots, R_n\rangle$ where W is a non-empty set (of possible worlds) and R_i, for each $i \leq n$, is a binary relation on W (uR_iv iff agent i regards v as possible in u). A *Kripke model* of the language $\mathcal{CL}^-$ based on the frame $\mathfrak{F}$ is a pair $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{V}\rangle$, where $\mathfrak{V}$, a *valuation*, is a function which associates with each propositional variable p the set of worlds in W in which p is supposed to be true.

The *truth-relation* $(\mathfrak{M}, w) \models \varphi$, read as '$\varphi$ is true in the world w of the model $\mathfrak{M}$,' is defined inductively as follows:

- $(\mathfrak{M}, w) \models p_i$ iff $w \in \mathfrak{V}(p_i)$, for each propositional variable p_i;
- $(\mathfrak{M}, w) \models \psi \wedge \chi$ iff $(\mathfrak{M}, w) \models \psi$ and $(\mathfrak{M}, w) \models \chi$;
- $(\mathfrak{M}, w) \models \neg\psi$ iff $(\mathfrak{M}, w) \not\models \psi$;

– $(\mathfrak{M}, w) \models K_i\psi$ iff $(\mathfrak{M}, v) \models \psi$ for all $v \in W$ such that wR_iv.

A formula φ is said to be *valid* in a frame $\mathfrak{F}$ ($\mathfrak{F} \models \varphi$ in symbols) if $(\mathfrak{M}, w) \models \varphi$ for all Kripke models $\mathfrak{M}$ based on $\mathfrak{F}$ and all worlds w in $\mathfrak{F}$. φ is *valid* in a class F of frames (in symbols $\mathsf{F} \models \varphi$) if φ is valid in all $\mathfrak{F} \in \mathsf{F}$.

The reader can readily check that the set of $\mathcal{CL}^-$-formulas that are valid in any given class of frames contains all axioms of $\boldsymbol{K}_n$ and is closed under its inference rules. Another important observation is that frames validating axioms A_D, A_T, A_4, and A_5 satisfy, respectively, the following properties:

- *seriality* (denoted by s), i.e., for every world w and every $i \leq n$, there exists a world v such wR_iv;
- *reflexivity* (r), i.e., wR_iw for every $i \leq n$ and every w;
- *transitivity* (t), i.e., wR_iv and vR_iu imply wR_iu, for all worlds w, v, u and every $i \leq n$;
- *euclideanness* (e), i.e., wR_iv and wR_iu imply vR_iu, for all w, v, u and every $i \leq n$.

Denote by F^{r} the class of all reflexive frames, by F^{re} the class of all reflexive and euclidean frames (i.e., the class of frames with equivalence relations), etc. F^{a} is the class of all frames. The next lemma formalizes the observation above.

Lemma 1 *For every frame $\mathfrak{F}$ the following equivalences hold:*
(1) $\mathfrak{F} \models A_D$ *iff* $\mathfrak{F}$ *is serial;*
(2) $\mathfrak{F} \models A_T$ *iff* $\mathfrak{F}$ *is reflexive;*
(3) $\mathfrak{F} \models A_4$ *iff* $\mathfrak{F}$ *is transitive;*
(4) $\mathfrak{F} \models A_5$ *iff* $\mathfrak{F}$ *is euclidean.*

Let F be a class of frames and S an axiomatic system in a language $\mathcal{L}$. S is said to be *sound* with respect to F if, for every $\mathcal{L}$-formula φ, $\vdash_S \varphi$ implies $\mathsf{F} \models \varphi$. We say that $\mathfrak{F}$ is a *frame for S* if S is sound with respect to $\{\mathfrak{F}\}$. S is *complete* with respect to F if, for all formulas φ, $\mathsf{F} \models \varphi$ implies $\vdash_S \varphi$. If S is both sound and complete for F then the logic generated by S coincides with the set of formulas that are valid in F; we call this set the *logic* of F (in the language $\mathcal{L}$) and denote it by $L_{\mathcal{L}}(\mathsf{F})$. In other words, S *axiomatizes* $L_{\mathcal{L}}(\mathsf{F})$.

The following theorem establishes soundness and completeness of the axiomatic systems above with respect to the classes of their frames (for proofs consult [10] or any textbook on modal logic, e.g. [5]).

Theorem 1 *Every $S \in \{\boldsymbol{K}_n, \boldsymbol{T}_n, \boldsymbol{KD}_n, \boldsymbol{K4}_n, \boldsymbol{S4}_n, \boldsymbol{KD45}_n, \boldsymbol{S5}_n\}$ is sound and complete with respect to the class of frames for S, i.e., $\boldsymbol{K}_n$ axiomatizes $L_{\mathcal{CL}^-}(\mathsf{F}^{\mathsf{a}})$, $\boldsymbol{T}_n$ axiomatizes $L_{\mathcal{CL}^-}(\mathsf{F}^{\mathsf{r}})$, etc.*

Let us return to the full language $\mathcal{CL}$ and see how to interpret the common knowledge operator C. The formula $C\varphi$—'it is common knowledge (in the group of agents) that φ'—can be understood also as 'everybody knows

φ, everybody knows that everybody knows φ, everybody knows that everybody knows that everybody knows φ, etc.' That everybody knows φ is represented by the formula

$$E\varphi = K_1\varphi \wedge \cdots \wedge K_n\varphi,$$

and so $C\varphi$ corresponds to the infinite conjunction

$$E^1\varphi \wedge E^2\varphi \wedge \cdots \wedge E^n\varphi \wedge \ldots,$$

where $E^1\varphi = E\varphi$ and $E^{n+1}\varphi = EE^n\varphi$.

Now, given a Kripke model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ (with $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$) and a world $w \in W$, we see that $(\mathfrak{M}, w) \models E\varphi$ iff $(\mathfrak{M}, v) \models \varphi$ for every $v \in W$ such that $w(\bigcup_{i \leq n} R_i)v$; $(\mathfrak{M}, w) \models E^2\varphi$ iff $(\mathfrak{M}, v) \models \varphi$ for every $v \in W$ such that $w(\bigcup_{i \leq n} R_i)u(\bigcup_{i \leq n} R_i)v$ for some world u, etc.

For a binary relation R on W, denote by R^+ the *transitive closure* of R, i.e., uR^+v holds iff there is a sequence of worlds

$$u = u_0Ru_1R \ldots Ru_nRu_{n+1} = v$$

for some $n \geq 0$.

Thus we arrive at the following truth-condition for the common knowledge operator:

– $(\mathfrak{M}, w) \models C\psi$ iff $(\mathfrak{M}, v) \models \psi$ for all v such that $w(\bigcup_{i \leq n} R_i)^+v$.

To axiomatize epistemic logics with the common knowledge operator, we need the axiom schema

A_C: $C\varphi \to E(\varphi \wedge C\varphi)$,

and the inference rule

induction:

$$\frac{\varphi \to E(\psi \wedge \varphi)}{\varphi \to C\psi}.$$

The minimal axiomatic system $\boldsymbol{K}_n^C$ of common knowledge is defined by

– the language $\mathcal{CL}$,
– the inference rules: modus ponens, necessitation, and induction, and
– the axiom schemata: propositional calculus, distribution, and A_C.

The axiomatic systems $\boldsymbol{T}_n^C$, $\boldsymbol{KD}_n^C$, $\boldsymbol{K4}_n^C$, $\boldsymbol{S4}_n^C$, $\boldsymbol{KD45}_n^C$, and $\boldsymbol{S5}_n^C$ are obtained by adding to $\boldsymbol{K}_n^C$ the missing axioms of $\boldsymbol{T}_n$, $\boldsymbol{KD}_n$, etc.

Theorem 2 (Halpern and Moses 1992) *Every system S in the list $\boldsymbol{K}_n^C$, $\boldsymbol{T}_n^C$, $\boldsymbol{KD}_n^C$, $\boldsymbol{K4}_n^C$, $\boldsymbol{S4}_n^C$, $\boldsymbol{KD45}_n^C$, $\boldsymbol{S5}_n^C$ is sound and complete with respect to the class of frames for S, i.e., $\boldsymbol{K}_n^C$ axiomatizes $L_{\mathcal{CL}}(\mathsf{F^a})$, $\boldsymbol{T}_n^C$ axiomatizes $L_{\mathcal{CL}}(\mathsf{F^r})$, etc.*

3 First-order logics of common knowledge

To be able to express in our formalism statements like 'it is common knowledge that John loves somebody' or 'it is common knowledge whom John loves,' we combine $\mathcal{CL}$ with the language of classical first-order logic. The resulting first-order language of common knowledge is denoted by $\mathcal{QCL}$. The alphabet of $\mathcal{QCL}$ consists of:

- *predicate symbols* $P_0, P_1, \dots$,
- a set $var = \{x_0, x_1, \dots\}$ of *individual variables*,
- *individual constants* $c_0, c_1, \dots$,
- the *booleans* $\wedge$, $\neg$,
- the *universal quantifier* $\forall x$ for each individual variable x,
- a finite number of *knowledge operators* $K_1, \dots, K_n$, $n \geq 1$, and
- the *common knowledge operator* C.

We assume that the set of predicate symbols is non-empty and that each of them is of some fixed arity; 0-ary predicates are identified with propositional variables and denoted by $p_0, p_1, \dots$. Note that $\mathcal{QCL}$ contains neither function symbols nor equality.

Individual variables and constants are called *$\mathcal{QCL}$-terms*. *$\mathcal{QCL}$-formulas* are defined inductively as follows:

- if P is an n-ary predicate symbol and $\tau_1, \dots, \tau_n$ are terms, then $P(\tau_1, \dots, \tau_n)$ is an atomic formula;
- if φ and ψ are formulas, then so are $\varphi \wedge \psi$ and $\neg\varphi$;
- if φ is a formula and x an individual variable, then $\forall x\varphi$ is a formula;
- if φ is a formula and $i \leq n$, then $K_i\varphi$ and $C\varphi$ are formulas.

If convenient, we call *$\mathcal{QCL}$-terms* just terms and *$\mathcal{QCL}$-formulas* formulas.

In this language the statements formulated at the beginning of this section can be represented by the formulas:

$$C\exists x\ \mathsf{loves}(John, x) \quad \text{and} \quad \exists x C\ \mathsf{loves}(John, x),$$

where $\exists x\varphi$ is an abbreviation for $\neg\forall x\neg\varphi$.

An occurrence of a variable x in a formula φ is called *free* if it is not in the scope of the quantifier $\forall x$. The set of all variables having free occurrences in φ is denoted by $\mathsf{free}(\varphi)$; these are *free variables* of φ. If $\mathsf{free}(\varphi) = \emptyset$, then φ is said to be a *closed formula* or a *sentence*.

Given a formula φ, a term τ and a variable x, we denote by $\varphi\{\tau/x\}$ the formula that results from φ by replacing each free occurrence of x in φ with an occurrence τ. A variable y is said to be *free for x* in φ if x has no free occurrence in φ which lies within the scope of the quantifier $\forall y$, i.e., there exists no subformula in φ of the form $\forall y\psi$ with $x \in \mathsf{free}(\psi)$.

The possible world semantics for $\mathcal{QCL}$ can be constructed as a hybrid of the propositional Kripke semantics for $\mathcal{CL}$ and classical first-order models. Namely, we will interpret $\mathcal{QCL}$ in *first-order Kripke models* which are structures of the form $\mathfrak{M} = \langle \mathfrak{F}, D, I\rangle$, where $\mathfrak{F} = \langle W, R_1, \dots, R_n\rangle$ is a Kripke frame, D a non-empty set, the *domain* of $\mathfrak{M}$, and I is a function associating with every world $w \in W$ a first-order structure

$$I(w) = \left\langle D, P_0^{I(w)}, \dots, c_0^{I(w)}, \dots \right\rangle,$$

in which $P_i^{I(w)}$ is an n_i-ary relation on D, n_i being the arity of P_i (for a propositional variable p_i, the predicate $p_i^{I(w)}$ is either $\top$ or $\bot$), and $c_i^{I(w)}$ is an element in D such that $c_i^{I(u)} = c_i^{I(v)}$ for any $u, v \in W$. To simplify notation, we will omit the superscript I and write P_i^w, p_i^w, c_i^w, etc., if I is clear from the context. Thus, we can think of each world w in $\mathfrak{M}$ as having an inner structure, viz. $I(w)$.

Remarks. (i) The condition $c_i^{I(u)} = c_i^{I(v)}$, for all $u, v \in W$, means that constants are treated as *rigid designators* in the sense that they designate the same object in every world. Intuitively, this means that it is *common knowledge* which object a constant designates. In many application domains this assumption is justified because one is interested not in formalizing knowledge about names (alias constants), but rather knowledge about facts.

(ii) Note also that we assume the first-order structures $I(w)$, $w \in W$, to have the same domain D. This means that it is *common knowledge* which objects exist. Again, this is a natural assumption for all those application domains that deal not with knowledge about the existence of certain objects but facts.

An *assignment* in D is a function $\mathfrak{a}$ from the set *var* of individual variables to D. The value $\tau^{\mathfrak{M},\mathfrak{a}}$ (or simply $\tau^{\mathfrak{a}}$ if understood) of a term τ under the assignment $\mathfrak{a}$ in $\mathfrak{M}$ is $\mathfrak{a}(\tau)$, if τ is a variable, and τ^w otherwise, where w is some (any) world in W. The *truth-relation* $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ (or simply $w \models^{\mathfrak{a}} \varphi$) is defined inductively as follows:

- $w \models^{\mathfrak{a}} P_i(\tau_1, \dots, \tau_n)$ iff $\langle \tau_1^{\mathfrak{a}}, \dots, \tau_n^{\mathfrak{a}} \rangle \in P_i^w$; this fact will also be written as $I(w) \models^{\mathfrak{a}} P_i(\tau_1, \dots, \tau_n)$;
- $w \models^{\mathfrak{a}} \psi \wedge \chi$ iff $w \models^{\mathfrak{a}} \psi$ and $w \models^{\mathfrak{a}} \chi$;
- $w \models^{\mathfrak{a}} \neg\psi$ iff $w \not\models^{\mathfrak{a}} \psi$;
- $w \models^{\mathfrak{a}} \forall x \psi(x, y_1, \dots, y_n)$ iff $w \models^{\mathfrak{b}} \psi(x, y_1, \dots, y_n)$ for every assignment $\mathfrak{b}$ in D that may differ from $\mathfrak{a}$ only on x;
- $w \models^{\mathfrak{a}} K_i\psi$ iff $v \models^{\mathfrak{a}} \psi$ for all $v \in W$ such that wR_iv;
- $w \models^{\mathfrak{a}} C\psi$ iff $v \models^{\mathfrak{a}} \psi$ for all v such that $w(\bigcup_{i \le n} R_i)^+ v$.

If $\mathsf{free}(\varphi) \subseteq \{x_1, \dots, x_m\}$ and $\mathfrak{a}(x_i) = d_i$ for $i \le m$, then instead of $w \models^{\mathfrak{a}} \varphi$ we will sometimes write $w \models \varphi[d_1, \dots, d_m]$.

Given a set Γ of formulas, a model $\mathfrak{M}$, a world w and an assignment $\mathfrak{a}$, we write $(\mathfrak{M}, w) \models^{\mathfrak{a}} \Gamma$ to say that $(\mathfrak{M}, w) \models^{\mathfrak{a}} \varphi$ for every $\varphi \in \Gamma$. In this case Γ is said to be *satisfied* in $\mathfrak{M}$. By writing $(\mathfrak{M}, w) \models \Gamma$, we mean that $(\mathfrak{M}, w) \models^{\mathfrak{a}} \Gamma$ for any assignment $\mathfrak{a}$. Clearly, $(\mathfrak{M}, w) \models \Gamma$ iff $(\mathfrak{M}, w) \models^{\mathfrak{a}} \Gamma$ for some (any) assignment $\mathfrak{a}$, whenever Γ is a set of sentences. $\mathfrak{F} \models \Gamma$ means that Γ is *valid* in $\mathfrak{F}$, i.e., $(\mathfrak{M}, w) \models \Gamma$ holds for every model $\mathfrak{M}$ based on $\mathfrak{F}$ and every world w in $\mathfrak{F}$.

The way we combined the propositional logic of common knowledge and classical first-order logic was rather straightforward, so one might expect that it should not be too difficult to find axiomatic systems for the first-order common knowledge logics of the standard classes F^{a}, F^{r}, etc. of frames. This is indeed the case if we deal only with the fragment $\mathcal{QCL}^-$ of $\mathcal{QCL}$ consisting of formulas without the common knowledge operator.

Denote by $\boldsymbol{QK}_n$ the axiomatic system in the language $\mathcal{QCL}^-$ whose axiom schemata and inference rules are those of $\boldsymbol{K}_n$, the standard classical schemata

$Q1$: $\forall x\varphi \to \varphi\{\tau/x\}$, where τ is either a constant or a variable that is free for x in φ,

$Q2$: $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$, where $x \notin \mathsf{free}(\varphi)$,

the *Barcan formula*

BF: $K_i\forall x\varphi \leftrightarrow \forall x K_i\varphi$,[4]

and the

generalization rule: $\dfrac{\varphi}{\forall x\varphi}$.

By extending $\boldsymbol{QK}_n$ with the missing schemata of $\boldsymbol{T}_n$, $\boldsymbol{KD}_n$, etc. we obtain the axiomatic systems $\boldsymbol{QT}_n$, $\boldsymbol{QKD}_n$, $\boldsymbol{QK4}_n$, $\boldsymbol{QS4}_n$, $\boldsymbol{QKD45}_n$, and $\boldsymbol{QS5}_n$.

Theorem 3 *Each system S in the list $\boldsymbol{QK}_n$, $\boldsymbol{QT}_n$, $\boldsymbol{QKD}_n$, $\boldsymbol{QK4}_n$, $\boldsymbol{QS4}_n$, $\boldsymbol{QKD45}_n$, $\boldsymbol{QS5}_n$ is sound and complete with respect to the class of frames for S, i.e., $\boldsymbol{QK}_n$ axiomatizes $L_{\mathcal{QCL}^-}(\mathsf{F}^{\mathsf{a}})$, $\boldsymbol{QT}_n$ axiomatizes $L_{\mathcal{QCL}^-}(\mathsf{F}^{\mathsf{r}})$, etc.*[5]

The situation drastically changes when we consider the full language $\mathcal{QCL}$. The following 'non-axiomatizability theorem' was proved in [22]:

Theorem 4 *Let F be one of F^{a}, F^{r}, F^{t}, F^{s}, F^{e}, F^{rt}, F^{re}, F^{se}. Then the logic $L_{\mathcal{QCL}}(\mathsf{F}) = \{\varphi \in \mathcal{QCL} : \mathsf{F} \models \varphi\}$ is not recursively enumerable. In particular, $L_{\mathcal{QCL}}(\mathsf{F})$ cannot be axiomatized by a system with a finite set of axiom schemata and inference rules. Moreover, even*

- *the set of formulas in $L_{\mathcal{QCL}}(\mathsf{F})$ with at most two variables, and*
- *the set of formulas in $L_{\mathcal{QCL}}(\mathsf{F})$ with only unary predicates*

are not recursively enumerable, and so not finitely axiomatizable.

The problem now is to localize the origin of the 'bad algorithmic behavior' of first-order logics of common knowledge and to single out maximally possible 'well-behaved' fragments.

[4] BF requires models to have constant domains.

[5] For proofs and information concerning the history of these results consult e.g. [13] or [8].

4 The monodic fragment

Two observations give a key to a solution to this problem. The first one is that the proof of Theorem 4 essentially uses formulas asserting that some agents know *relations between two objects*. On the other hand, Wolter and Zakharyaschev [23] prove decidability of various epistemic description logics in which epistemic operators are applicable only to unary predicates, i.e., formulas in one free variable. These observations lead to the following fragment of $\mathcal{QCL}$.

Definition 1 (monodic formulas) *Denote by $\mathcal{QCL}_1$ the set of all $\mathcal{QCL}$-formulas φ such that any subformula of φ of the form $K_i\psi$ or $C\psi$ has at most one free variable. Such formulas will be called* monodic. *Note that the set of monodic formulas is closed under subformulas.*

For a class F of Kripke frames, let $L_1(\mathsf{F}) = L_{\mathcal{QCL}}(\mathsf{F}) \cap \mathcal{QCL}_1$. In other words, $L_1(\mathsf{F})$ is the monodic fragment *of the logic $L_{\mathcal{QCL}}(\mathsf{F})$.*

Our aim now is to show that the monodic fragments of the first-order logics of common knowledge defined above can be axiomatized by putting together the axioms and inference rules of their propositional parts, those of classical first-order logic, and the Barcan formula. More precisely, let S be a propositional axiomatic system in the language $\mathcal{CL}$. Then the system $\boldsymbol{Q}_1S$ (*quantified monodic* S) is determined by

- the language $\mathcal{QCL}_1$,
- the axiom schemata of S plus $Q1$, $Q2$, and BF,
- the inference rules of S and the rule of generalization.

Theorem 5 *Suppose S is one of the systems $\boldsymbol{Q}_1\boldsymbol{K}_n^C$, $\boldsymbol{Q}_1\boldsymbol{T}_n^C$, $\boldsymbol{Q}_1\boldsymbol{KD}_n^C$, $\boldsymbol{Q}_1\boldsymbol{K4}_n^C$, $\boldsymbol{Q}_1\boldsymbol{S4}_n^C$, $\boldsymbol{Q}_1\boldsymbol{KD45}_n^C$, $\boldsymbol{Q}_1\boldsymbol{S5}_n^C$. Then S is sound and complete with respect to the class F of Kripke frames for S, i.e., for every monodic formula φ, we have*

$$\vdash_S \varphi \quad \text{iff} \quad \mathsf{F} \models \varphi \quad \text{iff} \quad \varphi \in L_1(\mathsf{F}).$$

The soundness part, i.e., that $\vdash_S \varphi$ implies $\mathsf{F} \models \varphi$, is quite straightforward and easy. It suffices to show that all $\mathcal{QCL}_1$-instances of the axiom schemata of S are valid in F and that the set of formulas valid in F is closed under the inference rules of S. We leave this to the reader as an exercise.

It is much harder to prove the completeness part: $\varphi \in L_1(\mathsf{F})$ implies $\vdash_S \varphi$. This will be done in the next section. Here we only observe that if we regard every subformula of φ of the form $K_i\psi$ or $C\psi$, which is not in the scope of another epistemic operator, as a fresh predicate of at most one variable and if (under this understanding) φ is valid in all classical first-order models, then $\vdash_S \varphi$. To make this observation more precise, we require the following definition.

Definition 2 (surrogate and reduct) *Let $\varphi \in \mathcal{QCL}_1$. For every monodic formula $\psi(y)$ of the form $K_i\chi(y)$ or $C\chi(y)$ with one free variable y, we reserve a unary predicate $P_\psi(y)$ that does not occur in φ. Likewise, for every sentence $\psi = K_i\chi$ or $\psi = C\chi$ we fix a propositional variable p_ψ not occurring in φ. $P_\psi(y)$ and p_ψ will be called the* surrogates *for $\psi(y)$ and ψ (relative to φ).*

Now, given a monodic formula ϑ all predicate symbols in which occur in φ, we denote by $\overline{\vartheta}$ the formula that results from ϑ by replacing all subformulas of the form $K_i\chi(y)$, $K_i\chi$, $C\chi(y)$, and $C\chi$, which are not within the scope of another epistemic operator, with their surrogates. Thus, $\overline{\vartheta}$ contains no occurrences of epistemic operators at all, i.e., it is a purely first-order formula. We will call $\overline{\vartheta}$ the (classical) reduct *of ϑ.*

Lemma 2 *Let S be an axiomatic system in the language $\mathcal{QCL}$ containing all axiom schemata and inference rules of classical first-order logic, and let ϑ be a monodic formula. If $\overline{\vartheta}$ is valid in all classical first-order models, then $\vdash_S \vartheta$.*

Proof By Gödel's completeness theorem for classical first-order logic, we have $\vdash_S \overline{\vartheta}$. By replacing in a proof of $\overline{\vartheta}$ in S all occurrences of surrogate predicates P_ψ and surrogate variables p_χ with ψ and χ, respectively, we obtain a proof of ϑ in S.

5 Completeness

In this section we prove the completeness part of Theorem 5, i.e., that $\varphi \in L_1(\mathsf{F})$ implies $\vdash_S \varphi$. Suppose otherwise. Then φ is valid in all first-order Kripke models based on frames in F, but $\not\vdash_S \varphi$. To derive a contradiction, we need to show that $\neg\varphi$ is satisfied in a model based on a frame in F.

Recall that a formula ψ is called *consistent* with S (or *S-consistent*) if $\not\vdash_S \neg\psi$. Thus, the assumption $\not\vdash_S \varphi$ means that $\neg\varphi$ is S-consistent. So we shall prove our theorem if we manage to show that

– *every S-consistent monodic formula is satisfiable in a model based on a frame in F.*

The standard technique of obtaining such results in modal, in particular epistemic, logic is by constructing the *canonical model* for S. In the propositional case the canonical model $\mathfrak{M}^S = \langle \mathfrak{F}^S, \mathfrak{V}^S \rangle$ for S is based on the frame $\mathfrak{F}^S = \langle W^S, R^S \rangle$, where W^S is the set of all maximal S-consistent sets of formulas (in the language of S), uR^Sv iff $\{\psi : K\psi \in u\} \subseteq v$, and $w \in \mathfrak{V}^S(p)$ iff $p \in w$. By induction on ψ one shows then the following truth-lemma: for all formulas ψ and worlds w in the canonical model, we have $\psi \in w$ iff $w \models \psi$. The proof is based on a modal version of Lindenbaum's lemma which consists of two parts:

1. every S-consistent set is contained in a maximal S-consistent set;

2. if a maximal S-consistent set u contains a formula of the form $\neg K\psi$, then there is a maximal S-consistent set v such that $uR^S v$ and $\neg\psi \in v$.

As φ is S-consistent, it follows from (1) that φ is contained in some maximal S-consistent set $w \in W^S$. By the truth-lemma, we then obtain $(\mathfrak{M}^S, w) \models \varphi$. So, to complete the proof, it remains to show that $\mathfrak{F}^S$ is a frame for S.

The method of canonical models works perfectly well in the case of Theorem 1. However, it does not work for logics with the common knowledge operator because they lack the *compactness property*: there exists an S-consistent set of formulas which is not satisfiable in a model for S. (Take, for instance, the set

$$\Sigma = \{E^n p : n \geq 0\} \cup \{\neg Cp\}.$$

Every finite subset of Σ is satisfiable in a model for S, and so Σ is S-consistent, but there is no model satisfying the whole Σ.) It should be clear, however, that the method of canonical models goes through only for compact logics.

Sometimes the problem of compactness can be avoided by restricting attention to some finite set $F(\varphi)$ of formulas containing a given φ (e.g. the set of all subformulas in φ, their negations, and perhaps something else). In this case, the worlds of the restricted canonical model are maximal S-consistent subsets of $F(\varphi)$. Using this idea one can prove Theorem 2.

Instead of taking the (restricted) canonical model one can try to construct a model satisfying φ 'step-by-step:' first take an S-consistent set $\Sigma_0 \subseteq F(\varphi)$ containing φ, then hook to it via R_i, for every $\neg K_i\psi \in \Sigma_0$, an S-consistent set $\Sigma_1 \subseteq F(\varphi)$ such that $\neg\psi \in \Sigma_1$ and $\chi \in \Sigma_1$, for all $K_i\chi \in \Sigma_0$, and so forth. It is this way that we will follow in our completeness proof below.

The first-order part of the language brings extra complications. One of them is that to satisfy a first-order formula we may need an *infinite* structure. This means, in particular, that we should be ready to deal with first-order Kripke models with *infinite domains.* We will handle such models by codifying them in the form of *quasimodels* which have finite domains representing *types* of objects from real models.

To explain the idea behind the notion of quasimodel, let us begin with the following observation. Suppose that a suitable set of (monodic) formulas $F(\varphi)$ has been fixed and suppose that $\mathfrak{M}$ is a first-order Kripke model. For a world w in $\mathfrak{M}$ and an individual d in its domain, the *type of d realized in* w is defined to be the set of formulas $\psi \in F(\varphi)$ with ≤ 1 free variable such that $(\mathfrak{M}, w) \models \psi[d]$. Since $F(\varphi)$ is finite, there are only a finite number of distinct types. And since all our formulas are monodic, these types provide enough information about the 'modal contents' of w. Sets of types will be called below (Definition 4) world candidates. That a world candidate $\mathfrak{C}$ is realized in a world w can be expressed by saying that a certain sentence $\alpha_{\mathfrak{C}}$ is satisfied in the first-order structure $I(w)$. Thus, the worlds in $\mathfrak{M}$ can be represented by the world candidates they realize.

However, by representing worlds as world candidates we loose the information about the domain of $\mathfrak{M}$. To restore it, we should remember, for each individual d in the domain, which types are satisfied by d in each world. This can be done with the help of *runs* (Definition 7), which are functions r associating with each world w a type $r(w)$. Thus, an individual d can be represented by the run coming through the types realized by d. Another function of runs is to take care of the modal part of the language: every run r must satisfy the following condition: $K_i\psi \in r(w)$ iff $\psi \in r(w')$ for all worlds w' such that wR_iw' and all $K_i\psi \in F(\varphi)$.

Now, a *quasimodel* for φ (see Definition 8) can be thought of as a frame whose worlds are realizable world candidates such that every type t in every such world candidate lies on the route of at least one run. Theorem 6 shows that if a monodic formula φ is satisfiable in a quasimodel based on a frame for S then φ is satisfiable in a first-order Kripke model based on the same frame. So to prove the completeness of S it suffices to show that

– *every S-consistent monodic formula φ is satisfiable in a quasimodel for φ based on a frame in* F.

This is done in Theorem 7. Here is a rough idea of the proof. As was said above, we construct a quasimodel satisfying φ in a step-by-step manner. At the first step, we take some world candidate $\mathfrak{C}$ such that $\varphi \wedge \alpha_{\mathfrak{C}}$ is S-consistent, and declare it to be the root of the quasimodel to be constructed. (The existence of such a $\mathfrak{C}$ follows actually from Lemma 2.) Suppose now that the quasimodel has already been constructed up to step $m \geq 1$. Then, for every world candidate $\mathfrak{C}$ introduced at step m, every type t in it, and every $\neg K_i\psi \in t$, we select a realizable world candidate $\mathfrak{C}'$ with a type t' containing $\neg\psi$ such that $\mathfrak{C}'$ can be hooked to $\mathfrak{C}$ via R_i, with t' and t lying on the route of the same run. In the case of $\neg C\psi \in t$ we may need to hook to $\mathfrak{C}$ a sequence of several realizable world candidates. That such world candidates always exist is guaranteed by Lemma 6. 'In the limit' this construction will provide us with the required quasimodel.

Now we present the proof in full detail. Given a set Γ of $\mathcal{QCL}_1$-formulas, we denote by $con(\Gamma)$ and $sub(\Gamma)$ the sets of all constants and all subformulas of formulas in Γ, respectively, and put

$$sub_C(\Gamma) = sub(\Gamma) \cup \{E(\psi \wedge C\psi), \psi \wedge C\psi, K_i(\psi \wedge C\psi) : C\psi \in sub(\Gamma),\ i = 1, \dots, n\}.$$

Let

$$sub_C^{\neg}(\Gamma) = sub_C(\Gamma) \cup \{\neg\psi : \psi \in sub_C(\Gamma)\}$$

and let $sub_n(\Gamma)$ be the subset of $sub_C^{\neg}(\Gamma)$ containing only formulas with $\leq n$ free variables. For example, $sub_0(\Gamma)$ is the set of all sentences in $sub_C^{\neg}(\Gamma)$. Note that $sub_n(\Gamma)$ is not necessarily closed under subformulas and that modulo equivalence $\vartheta \leftrightarrow \neg\neg\vartheta$ we may assume that $sub_n(\Gamma)$ is closed under $\neg$. If Γ is a singleton set, say $\Gamma = \{\psi\}$, we write $con(\psi)$ instead of $con(\{\psi\})$, $sub(\psi)$ instead of $sub(\{\psi\})$, etc. In what follows we will not be distinguishing between a finite set Γ of formulas and the conjunction $\bigwedge \Gamma$ of formulas in it.

Let x be a variable not occurring in Γ. Put

$$sub_x(\Gamma) = \{\psi\{x/y\} : \psi(y) \in sub_1(\Gamma)\} \cup \{\neg K_i\bot, K_i\bot, \neg\bot, \bot : i \leq n\}$$

All formulas in $sub_x(\Gamma)$ have at most one free variable, and that variable is x. If Γ is finite, then $sub_x(\Gamma)$ is finite as well.

For the remaining part of this section we fix an arbitrarily given $\mathcal{QCL}_1$-sentence φ.

Definition 3 (type) *By a* type *for φ we mean a boolean-saturated subset t of $sub_x(\varphi)$, i.e.,*

– *$\psi \wedge \chi \in t$ iff $\psi \in t$ and $\chi \in t$, for every $\psi \wedge \chi \in sub_x(\varphi)$;*
– *$\neg\psi \in t$ iff $\psi \notin t$, for every $\neg\psi \in sub_x(\varphi)$.*

We say that two types t and t' agree on $sub_0(\varphi)$ *if $t \cap sub_0(\varphi) = t' \cap sub_0(\varphi)$. Given a type t for φ and a constant $c \in con(\varphi)$, the pair $\langle t, c\rangle$ will be called an* indexed type *for φ (indexed by c) and denoted by $t_c(x)$ or simply t_c.*

Definition 4 (world candidate) *Suppose T is a (non-empty) set of types for φ that agree on $sub_0(\varphi)$, and T^{con} is a set containing, for each $c \in con(\varphi)$, one and only one indexed type t_c such that $t \in T$. Then the pair $\mathfrak{C} = \langle T, T^{con}\rangle$ is called a* world candidate *for φ. A* pointed world candidate *for φ is a pair $\mathfrak{P} = \langle \mathfrak{C}, t\rangle$, where t is a type in T called the* point *of $\mathfrak{P}$.*

Given a world candidate $\mathfrak{C} = \langle T, T^{con}\rangle$ and a pointed world candidate $\mathfrak{P} = \langle \mathfrak{C}, t\rangle$, we put

$$\alpha_{\mathfrak{C}} = \bigwedge_{t\in T} \exists x\; t(x) \wedge \forall x \bigvee_{t\in T} t(x) \wedge \bigwedge_{\langle t,c\rangle\in T^{con}} t(c),$$

$$\beta_{\mathfrak{P}} = \alpha_{\mathfrak{C}} \wedge t.$$

Say that $\mathfrak{C}$ (or $\mathfrak{P}$) is *S-consistent* if the formula $\alpha_{\mathfrak{C}}$ (respectively, $\beta_{\mathfrak{P}}$) is S-consistent, i.e., $\not\vdash_S \neg\alpha_{\mathfrak{C}}$ ($\not\vdash_S \neg\beta_{\mathfrak{P}}$). Here and below S ranges over the axiomatic systems defined in Section 4.

Definition 5 (suitable pairs) *(1) A pair (t_1, t_2) of types for φ is called* i-suitable for S, *$i \leq n$, if the formula $t_1 \wedge \neg K_i \neg t_2$ is S-consistent.*

(2) A pair $(\mathfrak{C}_1, \mathfrak{C}_2)$ of world candidates is i-suitable for S, *$i \leq n$, if $\alpha_{\mathfrak{C}_1} \wedge \neg K_i \neg \alpha_{\mathfrak{C}_2}$ is S-consistent.*

(3) A pair $\mathfrak{P}_1 = \langle \mathfrak{C}_1, t_1\rangle$, $\mathfrak{P}_2 = \langle \mathfrak{C}_2, t_2\rangle$ of pointed world candidates is i-suitable for S, *$i \leq n$, if $\beta_{\mathfrak{P}_1} \wedge \neg K_i \neg \beta_{\mathfrak{P}_2}$ is S-consistent and $\langle t_2, c\rangle \in T_2^{con}$ whenever $\langle t_1, c\rangle \in T_1^{con}$, where $\mathfrak{C}_1 = \langle T_1, T_1^{con}\rangle$ and $\mathfrak{C}_2 = \langle T_2, T_2^{con}\rangle$. In this case we write $\mathfrak{P}_1 \prec_i \mathfrak{P}_2$.*

Lemma 3 (i) *If a pair $(\mathfrak{C}_1, \mathfrak{C}_2)$ of world candidates for φ is i-suitable for S, $i \leq n$, then:*

(i.1) *for every* $t \in T_1$ *there exists a* $t' \in T_2$ *such that* (t, t') *is* i*-suitable for* S*;*
(i.2) *for every* $t' \in T_2$ *there exists a* $t \in T_1$ *such that* (t, t') *is* i*-suitable for* S*;*
(i.3) *if* $\langle t, c\rangle \in T_1^{con}$ *and* $\langle t', c\rangle \in T_2^{con}$*, then the pair* (t, t') *is* i*-suitable for* S*.*

(ii) *Suppose that a pair of types* (t, t') *is* i*-suitable for* S*. Then the following holds:*

(ii.1) $\psi \in t'$ *whenever* $K_i\psi \in t$*;*
(ii.2) *if* S *contains* A_4*, then* $K_i\psi \in t'$ *whenever* $K_i\psi \in t$*;*
(ii.3) *if* S *contains* A_5*, then* $K_i\psi \in t$ *whenever* $K_i\psi \in t'$*;*
(ii.4) *if* S *contains* A_4, A_5 *or* S *contains* A_T, A_5*, then* $K_i\psi \in t$ *iff* $K_i\psi \in t'$*.*

(iii) *Suppose* (t, t') *is* i*-suitable for* S*. Then* $C\psi \in t$ *implies* $C\psi \in t'$*. If* S *contains* A_T *and* A_5*, then* $C\psi \in t$ *iff* $C\psi \in t'$*.*

Proof (i.1) We first show that for every ψ with $K_i\psi \in \mathcal{QCL}_1$ we have

$$\vdash_S \exists x K_i\psi \to K_i\exists x\psi. \tag{1}$$

By the law of contraposition, this is equivalent to $\vdash_S \neg K_i\exists x\psi \to \neg\exists x K_i\psi$. By the definition of $\exists$, $\vdash_S \neg K_i\exists x\psi \to \neg K_i\neg\forall x\neg\psi$. By $Q1$, $\vdash_S \forall x\neg\psi \to \neg\psi\{y/x\}$ where y is a fresh variable. Hence $\vdash_S \neg K_i\neg\forall x\neg\psi \to \neg K_i\neg\neg\psi\{y/x\}$, and so

$$\vdash_S \neg K_i\exists x\psi \to \neg K_i\psi\{y/x\}.$$

By $Q2$, $\vdash_S \neg K_i\exists x\psi \to \forall y\neg K_i\psi\{y/x\}$. As $\vdash_S \neg\exists x K_i\psi \leftrightarrow \forall y\neg K_i\psi\{y/x\}$, we then obtain $\vdash_S \neg K_i\exists x\psi \to \neg\exists x K_i\psi$ and hence (1).

Suppose now that $t \in T_1$, but there is no $t' \in T_2$ for which (t, t') is i-suitable for S. This means that $\vdash_S t \to K_i\neg t'$ for each $t' \in T_2$, and so

$$\vdash_S t \to K_i\neg \bigvee_{t'\in T_2} t'.$$

Then we have

$$\vdash_S \exists x t \to \exists x K_i\neg \bigvee_{t'\in T_2} t',$$

from which, by (1),

$$\vdash_S \exists x t \to K_i\exists x\neg \bigvee_{t'\in T_2} t'.$$

Since

$$\vdash_S \exists x\neg \bigvee_{t'\in T_2} t' \to \neg\alpha_{\mathfrak{C}_2} \quad \text{and} \quad \vdash_S \alpha_{\mathfrak{C}_1} \to \exists x t,$$

we finally obtain $\vdash_S \alpha_{\mathfrak{C}_1} \to K_i\neg\alpha_{\mathfrak{C}_2}$, contrary to $\alpha_{\mathfrak{C}_1} \wedge \neg K_i\neg\alpha_{\mathfrak{C}_2}$ being S-consistent.

(i.2) Now suppose that $t' \in T_2$ but there is no $t \in T_1$ for which (t, t') is i-suitable for S. This means that

$$\vdash_S \bigvee_{t\in T_1} t \to K_i\neg t'.$$

Hence

$$\vdash_S \forall x \bigvee_{t \in T_1} t \to \forall x K_i \neg t'$$

and, by the Barcan formula,

$$\vdash_S \forall x \bigvee_{t \in T_1} t \to K_i \forall x \neg t',$$

contrary to $\alpha_{\mathfrak{C}_1} \wedge \neg K_i \neg \alpha_{\mathfrak{C}_2}$ being S-consistent.

(i.3) Suppose $\langle t, c\rangle \in T_1^{con}$ and $\langle t', c\rangle \in T_2^{con}$. As the pair $(\mathfrak{C}_1, \mathfrak{C}_2)$ is i-suitable, $\alpha_{\mathfrak{C}_1} \wedge \neg K_i \neg \alpha_{\mathfrak{C}_2}$ is S-consistent. Hence $t\{c/x\} \wedge \neg K_i \neg t'\{c/x\}$ is S-consistent as well, which means that (t_1, t_2) is i-suitable for S.

(ii.1) Suppose that $K_i \psi \in t$ but $\psi \notin t'$. Then $\neg\psi \in t'$ and $\vdash_S \psi \to \neg t'$. Hence $\vdash_S K_i\psi \to K_i \neg t'$, and so $t \wedge \neg K_i \neg t'$ is S-inconsistent, which is a contradiction.

(ii.2) Now suppose that S contains A_4, $K_i\psi \in t$, but $K_i \psi \notin t'$. Then we have $\neg K_i \psi \in t'$. Hence $\vdash_S K_i\psi \to \neg t'$ and so $\vdash_S K_i K_i \psi \to K_i \neg t'$. It follows from A_4 that $\vdash_S K_i \psi \to K_i \neg t'$, which means that $K_i\psi \wedge \neg K_i \neg t'$ is S-inconsistent, contrary to (t, t') being i-suitable for S. Claims (ii.3) and (ii.4) are proved analogously.

(iii) Let $C\psi \in t$. By the definition, both $E(\psi \wedge C\psi)$ and $K_i(\psi \wedge C\psi)$ are in $sub_x(\varphi)$ for every $i \leq n$. As t is boolean-saturated and S-consistent and in view of A_C, we have then $E(\psi \wedge C\psi) \in t$ and so $K_i(\psi \wedge C\psi) \in t$, for $i \leq n$. By (ii.1), $\psi \wedge C\psi \in t'$, from which $C\psi \in t'$.

If $C\psi \in t'$ then, as we know, $K_i(\psi \wedge C\psi) \in t'$, for $i \leq n$. If S contains A_T and A_5, then we have $K_i(\psi \wedge C\psi) \in t$ by (ii.3), hence $\psi \wedge C\psi \in t$, and so $C\psi \in t$.

Before defining the notion of quasimodel, we remind the reader that a frame $\mathfrak{G} = \langle W, \lhd\rangle$ is said to be an *intransitive tree* if (i) $\mathfrak{G}$ is *rooted*, i.e., there is $w_0 \in W$ (a *root* of $\mathfrak{G}$) such that $w_0 \lhd^* w$ for every $w \in W$, where $\lhd^*$ is the transitive and reflexive closure of $\lhd$, (ii) for every $w \in W$, the set $\{v \in W : v \lhd^* w\}$ is finite and linearly ordered by $\lhd^*$, (iii) every world v in $\mathfrak{G}$, save its root, has precisely one predecessor, i.e., $|\{u \in W : u \lhd v\}| = 1$, and (iv) the root w_0 is *irreflexive*, i.e., $\neg w_0 \lhd w_0$.

Definition 6 (basic tree) *Let $\mathfrak{T} = \langle W, <_1, \dots, <_n\rangle$ be a frame with pairwise disjoint binary relations $<_i$ on W such that $\left\langle W, \bigcup_{i \leq n} <_i \right\rangle$ is an intransitive tree, and let σ be a map associating with every $w \in W$ a world candidate $\sigma(w) = \langle T_w, T_c^w\rangle$ for φ. The pair $\langle \mathfrak{T}, \sigma\rangle$ is called then a* basic tree *for φ. The basic tree $\langle \mathfrak{T}, \sigma\rangle$ is called a* basic S-tree *if $\alpha_{\sigma(w)}$ is S-consistent for every $w \in W$, and the pair $(\sigma(w_1), \sigma(w_2))$ is i-suitable for S whenever $w_1 <_i w_2$.*

Definition 7 (run) *A* run *r in a basic S-tree $\langle \mathfrak{T}, \sigma\rangle$ is a map associating with every $w \in W$ a type $r(w) \in T_w$ such that*

- *the pair $(r(w_1), r(w_2))$ is i-suitable for S whenever $w_1 <_i w_2$;*

- *if $\neg K_i\psi \in r(w)$ then $\psi \notin r(w')$ for some $w' >_i w$;*
- *if $\neg C\psi \in r(w)$ then $\psi \notin r(w')$ for some w' such that $w(\bigcup_{i\leq n} <_i)^+ w'$.*

Definition 8 (quasimodel) *Let $\langle \mathfrak{T}, \sigma \rangle$ be a basic S-tree for φ and let $\sigma(w) = \langle T_w, T_c^w \rangle$. $\langle \mathfrak{T}, \sigma \rangle$ is called an S-*quasimodel *for φ if*

- *for all $w \in W$ and $t \in T_w$, there exists a run r in $\langle \mathfrak{T}, \sigma \rangle$ such that $r(w) = t$;*
- *for every constant c, the function r_c defined by $r_c(w) = t$, for $\langle t, c \rangle \in T_c^w$, $w \in W$, is a run in $\langle \mathfrak{T}, \sigma \rangle$.*

We say that φ is satisfied *in $\langle \mathfrak{T}, \sigma \rangle$ if there exists $w \in W$ such that $\alpha_{\sigma(w)} \wedge \varphi$ is S-consistent.*

To prove Theorem 6 saying that φ is satisfiable in a model for S whenever it is satisfiable in an S-quasimodel, we require the following lemma from classical model theory ($|X|$ is the cardinality of X).

Lemma 4 *Suppose $J' = \left\langle D', P_o^{J'}, \ldots, c_0^{J'}, \ldots \right\rangle$ is a first-order structure with $|D'| \leq \aleph_0$ and let κ be an infinite cardinal. Then there exists a first-order structure J such that for every formula $\psi(y_1, \ldots, y_n)$,*

$$J' \models \exists y_1 \ldots \exists y_n \psi \quad \textit{iff} \quad |\{\langle d_1, \ldots, d_n \rangle : J \models \psi[d_1, \ldots, d_n]\}| = \kappa.$$

Proof Define $J = \langle D, P_0^J, \ldots, c_0^J, \ldots \rangle$ by taking

- $D = \{\langle d, \xi \rangle : d \in D',\ \xi < \kappa\}$,
- $P_i^J = \{\langle \langle d_1, \xi_1 \rangle, \ldots, \langle d_m, \xi_m \rangle \rangle : \langle d_1, \ldots, d_m \rangle \in P_i^{J'}\}$,
- $c_i^J = \left\langle c_i^{J'}, 0 \right\rangle$.

By induction on the construction of ψ one can easily show that, for all tuples $d_1, \ldots, d_m \in D$ and all $\xi_1, \ldots, \xi_m < \kappa$,

$$J' \models \psi[d_1, \ldots, d_m] \quad \text{iff} \quad J \models \psi[\langle d_1, \xi_1 \rangle, \ldots, \langle d_m, \xi_m \rangle].$$

The claim of the lemma follows immediately.

Theorem 6 *If φ is satisfiable in an S-quasimodel for φ, then φ is satisfiable in a model based on a frame for S.*

Proof Suppose φ is satisfied in an S-quasimodel $\langle \mathfrak{T}, \sigma \rangle$, $\mathfrak{T} = \langle W, <_1, \ldots, <_n \rangle$. Then there is $w^* \in W$ such that $\varphi \wedge \alpha_{\sigma(w^*)}$ is S-consistent. By Lemma 2 it follows that the reduct $\overline{\varphi} \wedge \overline{\alpha}_{\sigma(w^*)}$ is consistent with classical first-order logic. Moreover, by Definition 6 $\alpha_{\sigma(w)}$ is S-consistent, and so $\overline{\alpha}_{\sigma(w)}$ is first-order consistent for every $w \in W$. So, for each $w \in W$ we can find a first-order structure $I(w)$ such that $I(w) \models \overline{\alpha}_{\sigma(w)}$ and $I(w^*) \models \overline{\varphi}$.

Take a cardinal $\kappa \geq \aleph_0$ exceeding the cardinality of the set Ω of all runs in $\langle \mathfrak{T}, \sigma \rangle$ and put

$$D = \{\langle r, \xi \rangle : r \in \Omega,\ \xi < \kappa\}.$$

Claim: *we can assume that D is the domain of $I(w)$, $c^{I(w)} = \langle r_c, 0\rangle$, for all $c \in con(\varphi)$, and that*

$$r(w) = \{\psi \in sub_x(\varphi) : I(w) \models \overline{\psi}[\langle r, \xi\rangle]\}, \qquad (2)$$

for every run r and every cardinal $\xi < \kappa$.

Indeed, let $\sigma(w) = \langle T, T^{con}\rangle$ and let J' be a first-order structure such that $J' \models \overline{\alpha}_{\sigma(w)}$. By the Löwenheim–Skolem theorem, we can assume that the domain of J' is of cardinality $\leq \aleph_0$. So, by Lemma 4 we have a first-order structure $J = \left\langle D', P_0^J, \dots, c_0^J, \dots \right\rangle$ such that $J \models \overline{\alpha}_{\sigma(w)}$ and for all first-order formulas $\psi(y)$, $J \models \exists y \psi$ implies $|\{d \in D' : J \models \psi[d]\}| = \kappa$.

Denote by $\overline{t}^J$ the set $\{d \in D' : J \models \overline{t}[d]\}$. Then we have:

- $|\overline{t}^J| = \kappa$ for every $t \in T$,
- $\overline{t}^J = \emptyset$ for every type $t \notin T$,
- $\overline{t}_1^J \cap \overline{t}_2^J = \emptyset$ whenever $t_1 \neq t_2$.

For each $t \in T$, take a bijective function f_t from $\{\langle r, \xi\rangle : r(w) = t, r \in \Omega,\ \xi < \kappa\}$ onto $\overline{t}^J$ such that $f_t(\langle r_c, 0\rangle) = c^J$ for $c \in con(\varphi)$. (Such a function exists because $|\Omega| < \kappa$.) Then put $f = \bigcup_{t \in T} f_t$. It should be clear that f is a bijection from D onto D'. Moreover, it follows from the definition of f that

$$J \models \overline{t}[f(\langle r, \xi\rangle)] \quad \text{iff} \quad r(w) = t$$

for all $t \in T$ and $\langle r, \xi\rangle \in D$.

Now we define a new model $I(w)$ based on D by taking

- $P_i^{I(w)} = \{\langle\langle r_1, \xi_1\rangle, \dots, \langle r_m, \xi_m\rangle\rangle : \langle f(\langle r_1, \xi_1\rangle), \dots, f(\langle r_m, \xi_m\rangle)\rangle \in P_i^J\}$,
- $c_i^{I(w)} = \langle r_{c_i}, 0\rangle$.

It is not difficult to check that f is an isomorphism between $I(w)$ and J. Hence, for all $t \in T$ and all $\langle r, \xi\rangle \in D$,

$$I(w) \models \overline{t}[\langle r, \xi\rangle] \quad \text{iff} \quad J \models \overline{t}[f(\langle r, \xi\rangle)]$$

and so

$$I(w) \models \overline{t}[\langle r, \xi\rangle] \quad \text{iff} \quad r(w) = t,$$

which proves the claim.

Let us now define the frame $\mathfrak{F}$ underlying the model we are constructing. Its set of worlds is W. The accessibility relations R_i depend on S. Namely, we define R_i to be

- $<_i$ if $S = \boldsymbol{Q}_1\boldsymbol{K}_n^C$ or $S = \boldsymbol{Q}_1\boldsymbol{KD}_n^C$;
- $<_i \cup \{\langle w, w\rangle : w \in W\}$ if $S = \boldsymbol{Q}_1\boldsymbol{T}_n^C$;
- the transitive closure of $<_i$ if $S = \boldsymbol{Q}_1\boldsymbol{K4}_n^C$;
- the reflexive and transitive closure of $<_i$ if $S = \boldsymbol{Q}_1\boldsymbol{S4}_n^C$;
- $<_i^+ \cup \{\langle w, w'\rangle : \exists v \in W(v <_i^+ w \,\&\, v <_i^+ w' \,\&\, \neg\exists u\ u <_i v)\}$ if $S = \boldsymbol{Q}_1\boldsymbol{KD45}_n^C$;
- the reflexive, symmetric and transitive closure of $<_i$ if $S = \boldsymbol{Q}_1\boldsymbol{S5}_n^C$.

Note that for $S = \boldsymbol{Q}_1\boldsymbol{KD}_n^C$ the R_i are serial because in this case every S-consistent type for φ contains at least one formula of the form $\neg K_i\psi$, namely $\neg K_i\bot$, and so there is, by Definition 8, for every world w at least one world w' with $w <_i w'$. For $S = \boldsymbol{Q}_1\boldsymbol{KD45}_n^C$ the R_i are clearly serial. Suppose $w_1 R_i w$ and $w R_i w_2$. If w_1 has no $<_i$-predecessor, then $w_1 <_i^+ w_2$ and so $w_1 R_i w_2$. Otherwise there are v_j, for $j = 1, 2$, such that $v_j <_i^+ w_j$, $v_j <_i^+ w$ and the v_j have no $<_i$-predecessors. Since $\mathfrak{T}$ is an intransitive tree, we get $v_1 = v_2$. Thus $v_1 <_i^+ w_1$ and $v_1 <_i^+ w_2$. By the definition of R_i, it follows that $w_1 R_i w_2$. Hence R_i is transitive. Similarly one can show that R_i is euclidean.

Let $\mathfrak{M} = \langle \mathfrak{F}, D, I\rangle$. By induction on the construction of $\psi \in sub(\varphi)$ we show now that for every assignment $\mathfrak{a}$, we have

$$I(w) \models^{\mathfrak{a}} \overline{\psi} \quad \text{iff} \quad (\mathfrak{M}, w) \models^{\mathfrak{a}} \psi.$$

The basis of induction, i.e., the case when $\psi = P_i(\tau_1, \dots, \tau_m)$, is clear; for then $\psi = \overline{\psi}$. The induction step for $\psi = \psi_1 \wedge \psi_2$, $\psi = \neg\psi_1$, and $\psi = \forall y \psi_1$ follows by the induction hypothesis from the equations

$$\overline{\psi_1 \wedge \psi_2} = \overline{\psi_1} \wedge \overline{\psi_2}, \quad \overline{\neg\psi_1} = \neg\overline{\psi_1}, \quad \overline{\forall y \psi_1} = \forall y \overline{\psi_1}.$$

Let $\psi = K_i\chi(y)$ and $\mathfrak{a}(y) = \langle r, \xi\rangle$. (If ψ is a sentence, y is any variable.) We then have:

$$\begin{aligned} I(w) \models^{\mathfrak{a}} \overline{K_i\chi(y)} &\Leftrightarrow_1 I(w) \models^{\mathfrak{a}} P_{K_i\chi}(y) \\ &\Leftrightarrow_2 K_i\chi(x) \in r(w) \\ &\Leftrightarrow_3 \forall v\ (wR_iv \to \chi(x) \in r(v)) \\ &\Leftrightarrow_4 \forall v\ (wR_iv \to I(v) \models^{\mathfrak{a}} \overline{\chi}(y)) \\ &\Leftrightarrow_5 \forall v\ (wR_iv \to (\mathfrak{M}, v) \models^{\mathfrak{a}} \chi(y)) \\ &\Leftrightarrow_6 (\mathfrak{M}, w) \models^{\mathfrak{a}} K_i\chi(y). \end{aligned}$$

Equivalence $\Leftrightarrow_1$ holds by the definition of $\overline{\psi}$; $\Leftrightarrow_2$ and $\Leftrightarrow_4$ are consequences of (2). The induction hypothesis yields $\Leftrightarrow_5$, and $\Leftrightarrow_6$ holds by definition. The only non-trivial case is $\Leftrightarrow_3$.

($\Rightarrow_3$) Suppose $K_i\chi \in r(w)$ and wR_iw'. So if $w <_i w'$ the claim follows by Definition 7 and Lemma 3 (ii.1). If $S = \boldsymbol{Q}_1\boldsymbol{K}_n^C$ or $S = \boldsymbol{Q}_1\boldsymbol{KD}_n^C$ then we are done, because $R_i = <_i$.

Let $S = \boldsymbol{Q}_1\boldsymbol{KD45}_n^C$. By the definition of R_i, we have either $w <_i^+ w'$ or there exists v such that $v <_i^+ w$, $v <_i^+ w'$ and $\neg\exists u\ u <_i v$. The former case is easy; we leave it to the reader as an exercise. In the latter case, we some worlds $v_0, \dots, v_{m+1}$ such that $v_0 = v$, $v_{m+1} = w$ and $v_j <_i v_{j+1}$ for every $j \leq m$. By Definition 7, $(r(v_j), r(v_{j+1}))$ is i-suitable for S whenever $j \leq m$. By Lemma 3 (ii.3), $K_i\chi(x) \in r(v_0) = r(v)$. Similarly, using $v <_i^+ w'$ we obtain worlds $u_0, \dots, u_{l+1}$ such that $u_0 = v$, $u_{l+1} = w'$ and $u_j <_i u_{j+1}$ for every $j \leq l$. Again, $(r(u_j), r(u_{j+1}))$ is i-suitable for S whenever $j \leq l$, and by Lemma 3 (ii.2), $K_i\chi(x) \in r(u_l)$. Using the same lemma once again, this time (ii.1), we obtain $\chi(x) \in r(u_{l+1}) = r(w')$. Other cases for S are treated analogously.

($\Leftarrow_3$) is an immediate consequence of Definition 7.

Finally, let $\psi = C\chi(y)$ and $\mathfrak{a}(y) = \langle r, \xi \rangle$. Then we have the following chain of equivalences:

$$\begin{aligned}
I(w) \models^{\mathfrak{a}} \overline{C\chi(y)} &\Leftrightarrow_1 I(w) \models^{\mathfrak{a}} P_{C\chi}(y) \\
&\Leftrightarrow_2 C\chi(x) \in r(w) \\
&\Leftrightarrow_3 \forall v \ (w(\bigcup_{i \leq n} R_i)^+ v \to \chi(x) \in r(v)) \\
&\Leftrightarrow_4 \forall v \ (w(\bigcup_{i \leq n} R_i)^+ v \to I(v) \models^{\mathfrak{a}} \overline{\chi}(y)) \\
&\Leftrightarrow_5 \forall v \ (w(\bigcup_{i \leq n} R_i)^+ v \to (\mathfrak{M}, v) \models^{\mathfrak{a}} \chi(y)) \\
&\Leftrightarrow_6 (\mathfrak{M}, w) \models^{\mathfrak{a}} C\chi(y).
\end{aligned}$$

As before, the only non-trivial one is $\Leftrightarrow_3$. ($\Leftarrow_3$) follows from Definition 7. For the other direction, suppose $C\chi(x) \in r(w)$ and $w(\bigcup_{i \leq n} R_i)^+ v$. Then there are $v_0, \ldots, v_{m+1}$, for some $m \in \omega$, such that $v_0 = w$, $v_{m+1} = v$ and for all $j \leq m$ there is some $i_j \leq n$ with $v_j R_{i_j} v_{j+1}$. By induction on j one can show that $C\chi \in r(v_j)$ for all $j \leq m$. (The basis of induction is clear, and the induction step is similar to the proof ($\Rightarrow_3$) above.) Thus we have $C\chi \in r(v_m)$. By the axiom $C\chi \to E(\chi \wedge C\chi)$, we obtain $E(\chi \wedge C\chi) \in r(v_m)$. Hence, $K_{i_m}(\chi \wedge C\chi) \in r(v_m)$, and so, in view of Lemma 3 (ii.1), $\chi \in r(v_{m+1}) = r(v)$.

Thus, to prove completeness of our axiom system S, it suffices to construct an S-quasimodel satisfying φ whenever φ is S-consistent. This will be done in Theorem 7 the proof of which is based on the following two lemmas.

Lemma 5 (i) *For every finite S-consistent set Ψ of $\mathcal{QCL}_1$-formulas, there is a pointed world candidate $\mathfrak{P} = \langle \mathfrak{C}, t \rangle$ for φ such that $\Psi \wedge \beta_{\mathfrak{P}}$ is S-consistent.*

(ii) *Suppose Ψ is a finite set of $\mathcal{QCL}_1$-formulas and $\neg K_i \theta$ is a formula in $sub_x(\varphi)$ such that $\Psi \wedge \neg K_i \theta$ is S-consistent. Then there is a pointed world candidate $\mathfrak{P} = \langle \mathfrak{C}, t \rangle$ for φ such that $\neg\theta \in t$ and $\Psi \wedge \neg K_i \neg \beta_{\mathfrak{P}}$ is S-consistent.*

(iii) *Suppose Ψ is a finite set of $\mathcal{QCL}_1$-formulas, c a constant in $con(\varphi)$ and $\neg K_i \theta$ a formula in $sub_x(\varphi)$ such that $\Psi \wedge (\neg K_i \theta\{c/x\})$ is S-consistent. Then there exists a pointed world candidate $\mathfrak{P} = \langle \mathfrak{C}, t \rangle$ for φ with $\mathfrak{C} = \langle T, T^{con} \rangle$ such that $\neg\theta \in t$, $\Psi \wedge (\neg K_i \neg \beta_{\mathfrak{P}}\{c/x\})$ is S-consistent and $\langle t, c \rangle \in T^{con}$.*

Proof (i) Denote by β_φ the disjunction of all formulas $\beta_{\mathfrak{P}}$, $\mathfrak{P}$ a world candidate for φ. It is easy to see that $\overline{\beta}_\varphi$ is valid in all first-order models. Hence, by Lemma 2, β_φ is provable in S. So $\Psi \wedge \beta_\varphi$ is S-consistent. It follows that there is a disjunct $\beta_{\mathfrak{P}}$ of β_φ such that $\Psi \wedge \beta_{\mathfrak{P}}$ is S-consistent.

(ii) Suppose $\Psi \wedge \neg K_i\theta$ is S-consistent. We know that β_φ is provable in S. So $\Psi \wedge \neg K_i \neg(\neg\theta \wedge \beta_\varphi)$ is S-consistent. It follows that there is a pointed world candidate $\mathfrak{P}$ with point t such that $\Psi \wedge \neg K_i \neg(\neg\theta \wedge \beta_{\mathfrak{P}})$ is S-consistent. Clearly, $\neg\theta \in t$, and we are done.

(iii) is proved analogously.

Lemma 6 *Let $\mathfrak{P} = \langle \mathfrak{C}, t\rangle$ be a pointed world candidate for φ such that $\beta_{\mathfrak{P}}$ is S-consistent. Then the following holds:*

(i) *If $\neg K_i\psi \in t$, then there exists $\mathfrak{P}' = \langle \mathfrak{C}', t'\rangle$ such that $\mathfrak{P} \prec_i \mathfrak{P}'$ and $\psi \notin t'$.*

(ii) *If $\neg C\psi \in t$, then there are pointed world candidates $\mathfrak{P}_j = \langle \mathfrak{C}_j, t_j\rangle$, $j \leq k$, such that*

$$\mathfrak{P} = \mathfrak{P}_0 \prec_{i_1} \mathfrak{P}_1 \prec_{i_2} \cdots \prec_{i_k} \mathfrak{P}_k,$$

for some $i_1, \ldots, i_k \leq n$, and $\neg\psi \in t_k$.

Proof (i) Consider two cases. First, assume that there is no $c \in con(\varphi)$ with $\langle t, c\rangle \in T^{con}$, where $\mathfrak{C} = \langle T, T^{con}\rangle$. Clearly, $\beta_{\mathfrak{P}} \wedge \neg K_i\psi$ is S-consistent. Take $\Psi = \beta_{\mathfrak{P}}$. By Lemma 5 (ii), we have a pointed world candidate $\mathfrak{P}' = \langle \mathfrak{C}', t'\rangle$ such that $\beta_{\mathfrak{P}} \wedge \neg K_i \neg \beta_{\mathfrak{P}'}$ is S-consistent, that is $\mathfrak{P} \prec_i \mathfrak{P}'$, and $\neg\psi \in t'$.

Now let us suppose that the set of all $c \in con(\varphi)$ with $\langle t, c\rangle \in T^{con}$ is non-empty. Let c be some arbitrary member of this set. It is easy to see that $\beta_{\mathfrak{P}}\{c/x\} \wedge \neg K_i\psi\{c/x\}$ is S-consistent. Just note that $\beta_{\mathfrak{P}}\{c/x\} = \alpha_{\mathfrak{C}} \wedge t\{c/x\}$ and $t\{c/x\} \in T^{con}$. So we can apply Lemma 5 (iii) to $\Psi = \beta_{\mathfrak{P}}\{c/x\}$. The resulting pointed world candidate $\mathfrak{P}' = \langle \mathfrak{C}', t'\rangle$ is as required. In particular, $\langle t', c\rangle$ is an indexed type in $\mathfrak{C}'$.

(ii) Suppose that such a sequence does not exist. Let $\mathcal{T}$ be the minimal set of pointed world candidates such that

- $\mathfrak{P} \in \mathcal{T}$,
- if $\mathfrak{D}_1 \in \mathcal{T}$ and $\mathfrak{D}_1 \prec_i \mathfrak{D}_2$ for some i, then $\mathfrak{D}_2 \in \mathcal{T}$.

Note that $\mathcal{T}$ is finite. Let $\vartheta = \bigvee_{\mathfrak{D}\in\mathcal{T}} \beta_{\mathfrak{D}}$. Then we have $\vdash_S \vartheta \to K_i\vartheta$, for all $i \leq n$. Indeed, suppose otherwise. This means that $\vartheta \wedge \neg K_i\vartheta$ is S-consistent for some $i \leq n$. But then $\vartheta \wedge \neg K_i \neg\beta_{\mathfrak{P}'}$ is S-consistent for some pointed world candidate $\mathfrak{P}' \notin \mathcal{T}$, which contradicts the definition of $\mathcal{T}$, since $\mathfrak{D} \prec_i \mathfrak{P}'$ for some disjunct $\beta_{\mathfrak{D}}$ of ϑ. Therefore $\vdash_S \vartheta \to K_i\vartheta$ for all $i \leq n$, and so $\vdash_S \vartheta \to E\vartheta$.

By Lemma 6 (i), $E\psi \in s$ for every $(\mathfrak{C}^*, s) \in \mathcal{T}$, for otherwise we could construct a sequence satisfying (ii). Thus, by Lemma 5 (i), $\vdash_S \vartheta \to E\psi$. Together with $\vdash_S \vartheta \to E\vartheta$ this yields $\vdash_S \vartheta \to E(\psi \wedge \vartheta)$. By the inference rule for C we obtain $\vdash_S \vartheta \to C\psi$, and so $\vdash_S \beta_{\mathfrak{P}}(x) \to C\psi$, since $\mathfrak{P} \in \mathcal{T}$. But then $\beta_{\mathfrak{P}}$ is S-inconsistent, which is a contradiction.

We are in a position now to prove the main result of this section.

Theorem 7 *If S is one of the axiomatic systems defined above and φ is an S-consistent monodic sentence, then φ is satisfiable in a model based on a frame for S.*

Proof In view of Theorem 6, it suffices to construct an S-quasimodel satisfying φ. By Lemma 5 (i), we can find a world candidate $\mathfrak{C}^*$ such that $\varphi \wedge \alpha_{\mathfrak{C}^*}$ is S-consistent. Since φ is a sentence, φ is contained in each type from $\mathfrak{C}^*$. We are going to construct the required quasimodel as the limit of a sequence

$$\langle \mathfrak{T}_m, \sigma_m \rangle = \langle \langle W_m, <_1^m, \dots, <_n^m \rangle, \sigma_m \rangle,$$

of basic S-trees, $m = 0, 1, \dots$.

Let $W_0 = \{w^*\}$ and $\sigma_0(w^*) = \mathfrak{C}^*$. Suppose now that $\langle \mathfrak{T}_m, \sigma_m \rangle$ has already been defined. For every $w \in W_m - W_{m-1}$ we then construct a number of new points 'saturating' $\sigma_m(w)$ (where $W_{-1} = \emptyset$).

Let $\mathfrak{C} = \sigma_m(w)$, $\mathfrak{C} = \langle T, T^{con} \rangle$, and let $t \in T$. We then do the following.

(a) For every $\chi = \neg K_i \psi \in t$ we take *two* points a_χ and b_χ, add them to W_m, put $w <_i^{m+1} a_\chi$, $w <_i^{m+1} b_\chi$, and $\sigma_{m+1}(a_\chi) = \sigma_{m+1}(b_\chi) = \mathfrak{C}'$, for some $\mathfrak{P}' = \langle \mathfrak{C}', t' \rangle$ such that $\langle \mathfrak{C}, t \rangle \prec_i \mathfrak{P}'$ and $\psi \notin t'(x)$. That such a $\mathfrak{P}'$ exists is guaranteed by Lemma 6 (i). In particular, if $\langle t, c \rangle \in T^{con}$ for some constant c, then $\langle t', c \rangle$ is an indexed type in $\mathfrak{C}'$.

(b) For every $\chi = \neg C\psi \in t$ we take *two* sequences $a_\chi^1, \dots, a_\chi^k$ and $b_\chi^1, \dots, b_\chi^k$ and put

$$w <_{i_1}^{m+1} a_\chi^1 <_{i_2}^{m+1} \cdots <_{i_k}^{m+1} a_\chi^k, \qquad w <_{i_1}^{m+1} b_\chi^1 <_{i_2}^{m+1} \cdots <_{i_k}^{m+1} b_\chi^k,$$

and

$$\sigma_{m+1}(a_\chi^j) = \sigma_{m+1}(b_\chi^j) = \mathfrak{C}^j, \text{ for all } j \leq k,$$

where the $\langle \mathfrak{C}^j, t^j \rangle$ form a sequence of pointed world candidates such that

$$\langle \mathfrak{C}, t \rangle \prec_{i_1} \langle \mathfrak{C}^1, t^1 \rangle \prec_{i_2} \cdots \prec_{i_k} \langle \mathfrak{C}^k, t^k \rangle$$

and $\neg\psi \in t^k$. Lemma 6 (ii) ensures the existence of such a sequence. If $\langle t, c \rangle \in T^{con}$ for some constant c, then we have a sequence with $\langle t^j, c \rangle$ from $\mathfrak{C}^j$, for all $j \leq k$.

In the same manner we consider all the other types in T and all the other worlds $v \in W_m - W_{m-1}$. W_{m+1} is defined then as the (disjoint) union of W_m and the new points constructed by performing steps (a) and (b). The relations $<_i^{m+1}$ and the function σ_{m+1} coincide with, respectively, $<_i^m$ and σ_m on W_m and are defined by (a) and (b) for the new points. This gives us $\langle \mathfrak{T}_{m+1}, \sigma_{m+1} \rangle$.

Finally, we put $\langle \mathfrak{T}, \sigma \rangle = \langle \langle W, <_1^m, \dots, <_n^m \rangle, \sigma \rangle$, where

$$W = \bigcup_{m<\omega} W_m, \quad <_i = \bigcup_{m<\omega} <_i^m, \quad \sigma = \bigcup_{m<\omega} \sigma_m.$$

Let us prove that $\langle \mathfrak{T}, \sigma \rangle$ is an S-quasimodel.

By Lemma 3 (i.3), the functions r_c are runs. So it remains to show that, for every $w \in W$ and every type t in $\sigma(w)$, there exists a run r such that $r(w) = t$.

Using Lemma 3 (i.2) we find a sequence

$$w^* = w_0 <_{i_1} w_1 <_{i_2} \cdots <_{i_k} w_k = w$$

and types t_j in $\sigma(w_j)$, $0 \leq j \leq k$, such that $t_k = t$ and $t_j \wedge \neg K_{i_{j+1}} \neg t_{j+1}$ is S-consistent for all $j < k$.

Define r on the worlds w_j in $V = \{w_0, \ldots, w_k\}$ by taking $r(w_j) = t_j$. Now by induction we extend r to the sets $V \cup W_i$. Suppose we have defined r on $V \cup W_n$. Then for every $v \in W_n - W_{n-1}$ with $r(v) = t$ we do the following:

- If $\neg K_i \psi \in t$, then we take $v' \in W_{n+1} - (W_n \cup V)$ and t' in $\sigma(v')$ such that $v <_i v'$, $t \wedge \neg K_i \neg t'$ is S-consistent and $\psi \notin t'$. This can be done because we always took *two* saturating worlds in the construction above. Put $r(v') = t'$.
- If $\neg C\psi \in t$, then we take a sequence $v_1, \ldots, v_k$ from $W_{n+1} - (W_n \cup V)$ and types t_j in $\sigma(v_j)$, $1 \leq j \leq k$, such that

  $$v <_{i_0} v_1 <_{i_1} \cdots <_{i_{k-1}} v_{i_k}$$

  and
  - (t, t_1) is i_0-suitable for S,
  - (t_j, t_{j+1}) is i_j-suitable for S, $1 \leq j < k$,
  - $\psi \notin t_k$.

  Again this can be done because we always took two saturating sequences. Put $r(v_j) = t_j$ for all $1 \leq j \leq k$.

To define r on the remaining $v \in W_{n+1}$ we do inductively the following. Suppose r is not defined yet on v but defined on the unique v' such that $v' <_i v$. Then using Lemma 3 (i.1) we take a t in $\sigma(v)$ such that $(r(v'), t)$ is i-suitable for S and put $r(v) = t$.

It is now straightforward to see that the constructed r is a run.

This completes the proof of Theorem 5.[6]

6 Decidability

Another important algorithmic feature of the monodic formulas is that if, for some class $\mathcal{L}$ of monodic formulas, we have an algorithm that is capable of deciding, given a $\varphi \in \mathcal{L}$ and a world candidate $\mathfrak{C}$ for φ, whether $\overline{\alpha}_{\mathfrak{C}}$ is classically satisfiable, then the fragment $L_1(\mathsf{F}) \cap \mathcal{L}$ is decidable as well, where F is one of the frame classes defined in Section 3. In particular, we have the following:

[6] Note that Theorem 7 and hence Theorem 5 were proved only for *sentences*. To extend the latter to an arbitrary monodic formula $\varphi(y_1, \ldots, y_m)$ valid in all models for S, we consider the monodic sentence $\forall y_1 \ldots \forall y_m \varphi(y_1, \ldots, y_m)$: it is also valid in all models for S, and so we have $\vdash_S \forall y_1 \ldots \forall y_m \varphi(y_1, \ldots, y_m)$, from which by $Q1$ we obtain $\vdash_S \varphi(y_1, \ldots, y_m)$.

Theorem 8 *Let* F *be any of the classes of Kripke frames mentioned at the end of Section 3. Then the following fragments are decidable:*

- *the monadic fragment of* $L_1(\mathsf{F})$*,*
- *the two-variable fragment of* $L_1(\mathsf{F})$*,*
- *the guarded fragment of* $L_1(\mathsf{F})$*.*

Here, by the *guarded* $\mathcal{QLC}$*-formulas* we mean the formulas in the smallest class $\mathcal{G}$ of $\mathcal{QLC}$-formulas such that

- all atomic formulas are in $\mathcal{G}$;
- if φ and ψ are in $\mathcal{G}$, then so are $\varphi \wedge \psi$, $\neg\varphi$, $K_i\varphi$, and $C\varphi$;
- if $\overline{x}$, $\overline{y}$ are tuples of variables, $G(\overline{x}, \overline{y})$ is atomic, $\varphi(\overline{x}, \overline{y}) \in \mathcal{G}$, and every free variable occurring in $\varphi(\overline{x}, \overline{y})$ occurs in $G(\overline{x}, \overline{y})$ as well, then the formula $\forall \overline{y}(G(\overline{x}, \overline{y}) \to \varphi(\overline{x}, \overline{y}))$ is in $\mathcal{G}$.

Note that although the guarded fragment of classical first-order logic is decidable (see [1]), the guarded fragment of $L(\mathsf{F}^{\mathsf{a}})$ turns out to be undecidable.

For more details and an idea of the proof the reader is referred to [12,25]. Actually, no non-trivial decidable fragments of epistemic predicate logics have been constructed before.

It maybe also of interest to note that these decidability results make it possible to construct various decidable description logics with common knowledge and other epistemic operators applicable to concepts and formulas (but not to roles; see [24]). Weaker epistemic description logics were proposed in [9,17].

7 Adding equality

In this section we show that the addition of equality to the language of monodic formulas restores the 'status quo': the monodic fragments of the first-order logics of common knowledge with equality considered above are not recursively enumerable. In the proof we use of the following result (for a proof see [6]):

Theorem 9 (Trakhtenbrot's theorem) *The set of first-order sentences valid in all finite structures is not recursively enumerable.*

Denote by $\mathcal{QCL}_1^=$ the language $\mathcal{QCL}_1$ extended with the equality symbol which is interpreted in first-order structures as identity.

Theorem 10 *Let* F *be any of the classes of Kripke frames defined at the end of Section 3. Then the logic* $L_1(\mathsf{F})$ *in the language* $\mathcal{QCL}_1^=$ *is not recursively enumerable.*

Proof Define ψ to be the conjunction of the following $\mathcal{QCL}_1^=$-formulas:

$\psi_1 = \exists x P(x) \wedge \forall x \forall y\ (P(x) \wedge P(y) \rightarrow x = y)$,
$\psi_2 = C\forall x \forall y\ (\neg K_i \neg P(x) \wedge \neg K_i \neg P(y) \wedge \neg P(x) \wedge \neg P(y) \rightarrow x = y)$, for $i = 1, 2$,
$\psi_3 = \neg C \neg \forall x\ (P(x) \leftrightarrow \neg C \neg P(x))$,
$\psi_4 = \forall x\ (\neg C \neg P(x) \rightarrow C \neg C \neg P(x))$,
$\psi_5 = \forall x\ (Q(x) \leftrightarrow \neg C \neg P(x))$.

First, we notice that for all models $\mathfrak{M} = \langle \mathfrak{F}, D, I \rangle$ with $\mathfrak{F} = \langle W, R_1, \dots, R_n \rangle$ and all $w \in W$, if $(\mathfrak{M}, w) \models \psi$ then $Q^{I(w)}$ is finite. Indeed, by ψ_1, the set $P^{I(w)}$ is a singleton. Using ψ_3 we can find w' for which $w(\bigcup_{i \leq 1} R_i)^+ w'$ and

$$w' \models \forall x\ (P(x) \leftrightarrow \neg C \neg P(x)).$$

Hence there are $w_0, \dots, w_{m+1} \in W$ such that $w_0 = w$, $w_{m+1} = w'$ and for every $j \leq m$ there is some $i_j \leq 2$ for which $w_j R_{i_j} w_{j+1}$ holds. In view of ψ_2,

$$\left| P^{I(w_{j+1})} - P^{I(w_j)} \right| \leq 1$$

for every $j \leq m$, which yields $\left| P^{I(w')} \right| \leq m + 1$. Thus, it remains to show that $Q^{I(w)} \subseteq P^{I(w')}$. Suppose that $a \in Q^{I(w)}$. Then by ψ_5 $w \models \neg C \neg P[a]$ and by ψ_4 we obtain $w \models C \neg C \neg P[a]$, from which $w' \models P[a]$.

Second, we show that for every classical first-order sentence θ containing neither P nor Q the following conditions are equivalent:

(a) θ is true in all *finite* first-order structures;
(b) $\psi \rightarrow \theta^Q$ is valid in all first-order Kripke models based on frames in F.

(Here θ^Q is the *relativization of* θ to Q, i.e., $\theta^Q = \theta$ if θ is atomic, ^{Q} commutes with the booleans, and $(\forall x \theta_1)^Q = \forall x\ (Q(x) \rightarrow \theta_1^Q)$.)

(a) $\Rightarrow$ (b). Suppose there is a first-order Kripke model $\mathfrak{M}$ based on a frame in F and a world w in it such that $w \models \psi$ but $w \not\models \theta^Q$. Define a finite first-order structure J with the domain $E = Q^{I(w)}$ and predicates $P_k^J = P_k^{I(w)} \cap E$. It can be easily shown by induction that for every classical first-order formula χ containing neither P nor Q and every assignment $\mathfrak{a}$ in E, we have $J \models^{\mathfrak{a}} \chi$ iff $w \models^{\mathfrak{a}} \chi^Q$. In particular, $J \not\models \theta$.

(a) $\Leftarrow$ (b). Let us show first that for every natural number $m > 0$, there is a model $\mathfrak{M}_m = \langle \mathfrak{F}_m, D_m, I_m \rangle$ based on a frame $\mathfrak{F}_m \in \mathsf{F}^{\mathsf{re}}$ and w in $\mathfrak{F}_m$ such that $|W| = m$ and $w \models \psi$. Indeed, put $W_m = \{w_1, \dots, w_m\}$, $w_i \neq w_j$ whenever $i \neq j$, $D_m = \mathbb{N}$, and

$R_1 = \{\langle w_k, w_{k+1} \rangle, \langle w_{k+1}, w_k \rangle : k < m\ \&\ \exists l\ k = 2l + 1\} \cup \{\langle w_k, w_k \rangle : k \leq m\}$,
$R_2 = \{\langle w_k, w_{k+1} \rangle, \langle w_{k+1}, w_k \rangle : k < m\ \&\ \exists l\ k = 2l\} \cup \{\langle w_k, w_k \rangle : k \leq m\}$.

Finally, for each $k \leq m$, put $P^{I(w_k)} = \{0, \dots, k-1\}$ and $Q^{I(w_k)} = \{0, \dots, m-1\}$ (see Fig. 1). It is easy to see that the model is reflexive and euclidean, and $w_1 \models \psi$.

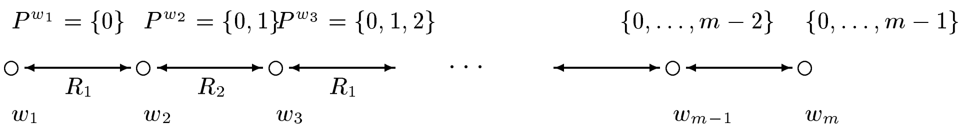

Fig. 1

To complete the proof, suppose that there is a finite first-order structure J with domain D such that $|D| = m$ and $J \not\models \theta$. Take the model $\mathfrak{M}_m$ and the world $w \models \psi$ constructed above. Without loss of generality we can assume that $Q^{I_m(w)} = D$. Expand $\mathfrak{M}_m$ by interpreting each predicate symbol P_i different from P and Q as P_i^J in all worlds in W_m. Now, for every first-order formula χ without occurrences of P and Q and for every assignment $\mathfrak{a}$ in D, we have $J \models^{\mathfrak{a}} \chi$ iff $(\mathfrak{M}_m, w) \models^{\mathfrak{a}} \chi^Q$. Therefore, $(\mathfrak{M}_m, w) \not\models \theta^Q$, and so $\psi \to \theta^Q$ is not valid in F^{re} (which is contained in all our classes of Kripke frames).

The claim of the theorem follows now immediately from Trakhtenbrot's theorem.

References

1. H. Andréka, J. van Benthem, and I. Németi. Modal languages and bounded fragments of predicate logic. *Journal of Philosophical Logic*, 27:217–274, 1998.
2. R.J. Aumann. Agreeing to disagree. *The Annals of Statistics*, 4:1236–1239, 1976.
3. M. Bacharach. The epistemic structure of a theory of game. *Theory and Decision*, 37:7–48, 1994.
4. E. Boerger, E. Graedel, and Y. Gurevich. *The classical decision problem*. Springer–Verlag, 1997.
5. A.V. Chagrov and M.V. Zakharyaschev. *Modal Logic*. Clarendon Press, Oxford, 1997.
6. H. Ebbinghaus and J. Flum. *Finite Model Theory*. Springer, 1995.
7. R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning about Knowledge*. MIT Press, 1995.
8. J.W. Garson. Quantification in modal logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, Volume II*, pages 249–307. Kluwer Academic Publishers, Dordrecht, 1984.
9. A. Gräber, H. Bürckert, and A. Laux. Terminological reasoning with knowledge an belief. In A. Laux and H. Wansing, editors, *Knowledge and Belief in Philosophy and Artificial Intelligence*, pages 29–61. Akademie Verlag, 1995.
10. J. Halpern and Yo. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54:319–379, 1992.
11. J. Hintikka. *Knowledge and Belief: An Introduction to the Logic of Two Notions*. Cornell University Press, 1962.
12. I. Hodkinson, F. Wolter, and M. Zakharyaschev. Decidable fragments of first-order temporal logics. *Annals of Pure and Applied Logic*, 2000.

13. G.E. Hughes and M.J.Cresswell. *A new Introduction to Modal Logic*. Routledge, London, 1996.
14. M. Kaneko. Common knowledge logic and game logic. *Journal of Symbolic Logic*, 64:685–700, 1999.
15. M. Kaneko and T. Nagashima. Game logic and its applications 1. *Studia Logica*, 57:325–354, 1996.
16. M. Kaneko and T. Nagashima. Game logic and its applications 2. *Studia Logica*, 58:273–303, 1997.
17. A. Laux. Beliefs in multi-agent worlds: a terminological approach. In *Proceedings of the 11th European Conference on Artificial Intelligence*, pages 299–303, Amsterdam, 1994.
18. W. Lenzen. Recent work in epistemic logic. *Acta Philosophica Fennica*, 30:1–219, 1978.
19. D. Lewis. *Convention. A Philosophical Study.* Harvard University Press, Cambridge, Massachusets, 1969.
20. J. McCarthy, M. Sato, T. Hayashi, and S. Igarishi. On the model theory of knowledge. Technical Report STAN-CS-78-657, Stanford University, 1979.
21. J.J. Meyer and W. van der Hoek. *Epistemic Logic for AI and Computer Science*. Cambrigde University Press, 1995.
22. F. Wolter. Fragments of first-order common knowledge logics. *Studia Logica*, 65:249 – 271, 2000.
23. F. Wolter and M. Zakharyaschev. Satisfiability problem in description logics with modal operators. In *Proceedings of the sixth Conference on Principles of Knowledge Representation and Reasoning*, Montreal, Canada, 1998. Morgan Kaufman.
24. F. Wolter and M. Zakharyaschev. Modal description logics: modalizing roles. *Fundamenta Informaticae*, 39:411–438, 1999.
25. F. Wolter and M. Zakharyaschev. Decidable fragments of first-order modal logics. *Journal of Symbolic Logic*, 2000.