Satisfiability problem in description logics with modal operators

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Abstract

The paper considers the standard concept description language $\mathcal{ALC}$ augmented with various kinds of modal operators which can be applied to concepts and axioms. The main aim is to develop methods of proving decidability of the satisfiability problem for this language and apply them to description logics with most important temporal and epistemic operators, thereby obtaining satisfiability checking algorithms for these logics. We deal with the possible world semantics under the constant domain assumption and show that the expanding and varying domain assumptions are reducible to it. Models with both finite and arbitrary constant domains are investigated. We begin by considering description logics with only one modal operator and then prove a general transfer theorem which makes it possible to lift the obtained results to many systems of polymodal description logic.

1 INTRODUCTION

Description (or terminological) logics have been developed and used[1] as a formalism for representing knowledge about static application domains. Having stemmed from real working systems like KL-ONE and its successors, they proved to be a successful compromise between expressibility and effectiveness.

In a description logic system, the knowledge of an application domain is represented in the form of conceptual and assertional axioms. The former introduce the relevant terminology—complex concepts defined in terms of atomic ones and binary relations (roles) between objects with the help of certain constructors. And the latter describe facts about some concrete objects in the domain in terms of concept and role instances. Although the existing description languages provide a wide choice of constructors (see e.g. Baader et al., 1990, Donini et al., 1996), usually they are intended to represent only static knowledge and are not able to express various dynamic aspects such as time-dependence, beliefs of different agents, obligations, etc., which are regarded to be important ingredients in modeling intelligent agents.

For example, in every standard description language we can define a concept "good car" as, say, a car with an airconditioner:

$$\mathit{good\ car} = \mathit{car} \wedge \exists \mathit{part}.\mathit{airconditioner}. \qquad (1)$$

However, we have no means to represent the subtler knowledge that only John believes (1) to be the case, while Mary does not think so:

$$[\mathit{John\ believes}](1) \wedge \neg[\mathit{Mary\ believes}](1).$$

Nor can we express the fact that (1) holds now but in future the notion of a good car may change (since, for instance, all cars will have airconditioners):

$$(1) \wedge \langle \mathit{eventually} \rangle \neg(1).$$

A way to bridge this gap seems to be quite clear. One can simply combine a description language with a suitable modal language treating belief, temporal, deontic or some other intensional operators. But one has to be careful, for such a combination may ruin the balance between expressibility and effectiveness, as it happened with too powerful pure description languages (see e.g. Schmidt-Schauß, 1989 or Patel-Schneider, 1989).

[1] See e.g. (Brachman and Schmolze 1985), (Borgida et al. 1989), (Baader and Hollunder 1991), and (Donini et al. 1996) for more references.

There is a number of parameters that determine the design of a modal extension of a given description language.

(I) First, modal operators can be applied to different kinds of well-formed expressions of the description language.

One can apply them only to conceptual and assertional axioms thereby forming new axioms of the form:

$$[John\ believes](good\ car = car \wedge \exists part.airconditioner),$$

$$[Mary\ believes]\,\langle eventually\rangle\,(John\ is\ rich).$$

Modal operators can be applied to concepts in order to form new ones:

$$[John\ believes]\ expensive$$

(i.e., the concept of all objects John believes to be expensive) or

$$human\ being \wedge \exists child.[Mary\ believes]\ \langle eventually\rangle\ good\ student$$

(i.e., the concept of all human beings with a child which Mary believes to be eventually a good student).

By allowing applications of modal operators to both concepts and axioms we obtain expressions of the form

$$[John\ believes](good\ car = [Mary\ believes]\ good\ car)$$

(i.e., John believes that a car is good if and only if Mary thinks so).

Finally, one can supplement the options above with modal operators applicable to roles. For example, using the temporal operator $[always]$ (in future) and the role *loves*, we can form the new role $[always]\ loves$ (which is understood as a relation between objects x and y that holds if and only if x will always love y) to say

$$John : \exists[always]\ loves.woman$$

(i.e., John will always love the very same woman (but perhaps not only her), which is not the same as $John : [always]\exists loves.woman$).

(II) All these languages are interpreted with the help of the possible world semantics in which the accessibility relations between worlds treat the modal operators,[2] and the worlds themselves consist of domains in which the concepts, role names and object names of the description component are interpreted.

The properties of the modal operators are determined by the conditions we impose on the corresponding accessibility relations. For example, by imposing no condition at all we obtain what is known as the minimal normal modal logic **K**—although of definite theoretical interest, it does not have the properties required to model operators like $[agent\ A\ knows]$, $\langle eventually\rangle$, etc. Transitivity of the accessibility relation for agent A's knowledge means what is called the positive introspection (A knows what he knows), Euclideannes corresponds to the negative introspection (A knows what he does not know), reflexivity reflects that only true facts are known to A (for more information and further references consult e.g. Halpern and Moses, 1992). In the temporal case, depending on the application domain we may assume time to be linear and discrete (i.e., the usual strict ordering of the natural numbers), or branching, or dense, etc. (see van Benthem, 1996).

(III) Another important parameter is the number of modal operators we need in our language and, respectively, the number of the corresponding accessibility relations. If we deal with multi-agent epistemic logic then every agent A gives rise to the operator $[agent\ A\ believes]$. If we also want to capture the development of beliefs in time, we should add the corresponding temporal operator. Note that certain combinations of "harmless" modalities may result in a logic of extremely high complexity (see e.g. Spaan, 1993).

(IV) When connecting worlds—that is ordinary models of the pure description language—by accessibility relations, we are facing the problem of connecting their objects. Depending on the particular application, we may assume worlds to have arbitrary domains (the *varying domain assumption*), or we may assume that the domain of a world accessible from a world w contains the domain of w (the *expanding domain assumption*), or that all the worlds share the same domain (the *constant domain assumption*). Consider, for instance, the following axioms:

$$\neg[agent\ A\ knows](unicorn = \bot),$$

$$([agent\ A\ knows]\neg unicorn) = \top.$$

The former means that agent A does not know that unicorns do not exist, while according to the latter, for every existing object, A knows that it is not a unicorn. Such a situation can be modeled under the expanding domain assumption, but these two formulas cannot be simultaneously satisfied in a model with constant domains.

[2] E.g. $[agent\ A\ believes]\ \varphi$ is regarded to be true in a world w iff φ is true in all the worlds agent A considers to be possible in w or, in other words, accessible from w via the relation interpreting agent A's beliefs.

(V) Following (Calvanese 1996), one can distinguish between models with finite and infinite domains. In many applications of pure description logics finite domains are preferable: after all the real world a knowledge base is talking about is finite. (For instance, when the domain consists of employees of a company then certainly we should assume it to be finite.) However, if we are dealing with time and temporal operators, it is natural to assume that with time passing potentially infinitely many different objects may appear in the application domain of the knowledge base. Note that the finite domain assumption does not mean that models are finite.

(VI) Finally, one should take into account the difference between *rigid* and *non-rigid designators*. In our context, the former are the object names interpreted by the same objects in every world in the model under consideration, while the latter are those whose interpretation is not fixed. Again the choice between these depends on the application domain: if the knowledge base is talking about employees of a company then the name *John Smith* should probably denote the same person no matter what world we consider, while *President of the company* may refer to different persons in different worlds. For a more detailed discussion consult e.g. (Fitting 1993) or (Kripke 1980).

The following kinds of description modal logics have been studied in the literature. Laux (1994) constructed a multi-agent logic of belief in which the belief operators apply only to axioms, the accessibility relations are transitive, serial and Euclidean, domains are constant and of arbitrary size, and designators are rigid. Schild (1993) introduced description logics with temporal operators applicable only to concepts and interpreted in models with linear and branching discrete unbounded time under the constant domain assumption and rigid designators. Baader and Laux (1995) consider a language in which modal operators can be applied to both axioms and concepts; they are interpreted in models with arbitrary accessibility relations under the expanding domain assumption. Baader and Ohlbach (1995) use modal operators as role constructors, but exclude object names and assertions from the language.

The languages of Schild (1993) and Laux (1994) present no serious technical difficulties: the satisfiability problem for both of them is reducible to the satisfiability problem in the well-known propositional modal logic (in the former case this was observed by Schild himself and the latter is treated by Theorem 7 below). On the other hand, the unrestricted use of modal operators to form new roles may lead to undecidable logics even under very natural conditions for the other parameters, as was proved by Baader and Ohlbach (1995).

The language of Baader and Laux (1995) appears to be sufficiently expressive and yet manageable. However, it was analyzed only in the abstract case of **K**-type modalities. More interesting for applications are modal operators with explicit temporal or epistemic interpretations to which the decision procedure of Baader and Laux is not extended. Besides, their technique works only under the expanding domain assumption. In general, the case of constant domains turns out to be much harder. First, there are description logics lacking the finite model property under the constant domain assumption but enjoying it if expanding domains are allowed (see Remark 10 below). And second, one can actually reduce the case of expanding or varying domains to that of constant domains (see Theorem 6).

Baader and Laux (1995) did not consider specially models with finite domains. Actually, in their case there is no need to distinguish between the variants of finite and infinite domains: as will be shown below, the sets of formulas satisfiable in models with arbitrary accessibility relations are the same no matter which of the two variants is adopted. However, these sets become different if we consider linear temporal models or models whose accessibility relations are reflexive and transitive (see Theorem 9). A similar situation arises in pure description logic when one extends the expressive power in such a way that the resultant logic does not have the finite model property (see e.g. De Giacomo and Lenzerini, 1994). In this case the set of formulas satisfiable in finite domains does not coincide with the set of formulas satisfiable in arbitrary domains.

The aim of this paper is to develop methods of proving decidability of the satisfiability problem for the description language with modal operators and apply them to most important systems. We will consider modal description logics with the following parameters.

1. The modal operators can be applied to concepts and axioms, but not to roles.
2. The language is interpreted in models with the accessibility relations satisfying most conditions of the standard nomenclature for the belief and temporal operators (in modal logic they correspond to the systems K, S5, KD45, S4, S4.3, GL, GL.3 and the tense logic of discrete linear unbounded time).
3. We begin by considering description logics with

only one modal operator and then prove a general transfer theorem which makes it possible to lift the obtained results to many systems of polymodal description logic.

4. We adopt the constant domain assumption and show that the varying domain assumption as well as the expanding domain assumption are reducible to it.

5. Both finite and arbitrary constant domains are considered.

6. Designators are assumed to be rigid.

(The standard way of proving decidability in modal logic by using a variant of the filtration technique does not work for the logics under consideration. First, the filtration of worlds often conflicts with the constant domain assumption (which is not the case when expanding domains are allowed). And second, not all our logics enjoy the finite model property.)

Although our underlying description language is the standard $\mathcal{ALC}$, the obtained results can be extended to languages with more expressive power, for instance, to $\mathcal{ALC}$ enriched with number restrictions or transitive reflexive closure.

2 SYNTAX AND SEMANTICS

Definition 1 (alphabet) The *primitive symbols* of the modal concept description language $\mathcal{ALC}_{\mathcal{M}}$ are:

– concept names: $C_0, C_1, \dots$;

– role names: $R_0, R_1, \dots$;

– object names: $a_0, a_1, \dots$;

– the booleans (say, $\wedge$, $\neg$, $\top$), modal operators $\Diamond_0, \Diamond_1, \dots$, and the relativized existential quantifier $\exists R_i$, for every role name R_i.

Other standard logical connectives are defined in the usual way. For instance, $C \to D$ is an abbreviation for $\neg(C \wedge \neg D)$, $\bot$ for $\neg\top$, and $\Box_i$ for $\neg\Diamond_i\neg$.

Definition 2 (concept) Concepts are defined inductively as follows: all concept names as well as $\top$ are (*atomic*) *concepts*, and if C, D are concepts, R is a role name, and $\Diamond_i$ a modal operator in our language then $C \wedge D$, $\neg C$, $\Diamond_i C$, $\exists R.C$ are *concepts*.

Definition 3 (formula) Let C and D be concepts, R a role name and a, b object names. Then expressions of the form $C = D$, aRb, $a : C$ are (*atomic*) *formulas*. If φ and ψ are formulas then so are $\Diamond_i\varphi$, $\neg\varphi$, and $\varphi\wedge\psi$.

Note that in the definition above we did not impose any restriction on the form of conceptual and assertional axioms. (Baader and Laux (1995) consider, for instance, only atomic formulas prefixed by sequences of modal operators.) This will have no affect on our decidability results as far as we do not touch on the complexity of the decision algorithms.

By $md(\varphi)$, the *modal depth* of a formula φ, we mean the length of the longest chain of nested modal operators in φ (including those in the concepts occurring in φ); $\Box^{\leq m}\varphi$ is the conjunction of all distinct formulas which are obtained by prefixing to φ a sequence of $\leq m$ operators $\Box_0, \Box_1, \dots$ (in arbitrary order). For instance,

$$\Box^{\leq 2}\varphi = \varphi \wedge \Box_0\varphi \wedge \Box_1\varphi \wedge \dots \wedge \Box_0\Box_1\varphi \wedge \Box_1\Box_0\varphi \wedge \dots$$

Denote by $con\varphi$, $rol\varphi$ and $ob\varphi$ the sets of all concepts, role names and object names occurring in φ, respectively; $sub\varphi$ is the set of all subformulas in φ.

We remind the reader that models of a pure modal language are based on Kripke frames, structures of the form $\mathfrak{F} = \langle W, \lhd_0, \lhd_1, \dots\rangle$ in which each $\lhd_i$ is a binary (accessibility) relation on the set of worlds W. What is going on inside the worlds is of no importance. Models of $\mathcal{ALC}_{\mathcal{M}}$ are also constructed on Kripke frames; however, in this case their worlds are models of $\mathcal{ALC}$.

Definition 4 (model) A *model* of $\mathcal{ALC}_{\mathcal{M}}$ based on a frame $\mathfrak{F} = \langle W, \lhd_0, \lhd_1, \dots\rangle$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, I\rangle$ in which I is a function associating with each $w \in W$ a structure

$$I(w) = \left\langle \Delta^{I(w)}, R_0^{I(w)}, \dots, C_0^{I(w)}, \dots, a_0^{I(w)}, \dots \right\rangle,$$

where $\Delta^{I(w)}$ is a non-empty set of objects, the *domain* of w, $R_i^{I(w)}$ are binary relations on $\Delta^{I(w)}$, $C_i^{I(w)}$ subsets of $\Delta^{I(w)}$, and $a_i^{I(w)}$ are objects in $\Delta^{I(w)}$ such that $a_i^{I(w)} = a_i^{I(v)}$, for any $v, w \in W$.

One can distinguish between three types of models: those with *constant*, *expanding*, and *varying domains*. In models with constant domains $\Delta^{I(v)} = \Delta^{I(w)}$, for all $v, w \in W$. In models with expanding domains $\Delta^{I(v)} \subseteq \Delta^{I(w)}$ whenever $v \lhd_i w$, for some i. And models with varying domains are just arbitrary models.

Definition 5 (satisfaction) For a model $\mathfrak{M} = \langle \mathfrak{F}, I\rangle$ and a world w in it, the *value* $C^{I(w)}$ of a concept C in w and the *truth-relation* $(\mathfrak{M}, w) \models \varphi$ (or simply $w \models \varphi$, if $\mathfrak{M}$ is understood) are defined inductively in the following way:

1. $\top^{I(w)} = \Delta^{I(w)}$ and $C^{I(w)} = C_i^{I(w)}$, for $C = C_i$;

2. $(C \wedge D)^{I(w)} = C^{I(w)} \cap D^{I(w)}$;

3. $(\neg C)^{I(w)} = \Delta^{I(w)} - C^{I(w)}$;

4. $x \in (\Diamond_i C)^{I(w)}$ iff $\exists v \rhd_i w\ x \in C^{I(v)}$;

5. $x \in (\exists R_i.C)^{I(w)}$ iff $\exists y \in C^{I(w)}\ xR_i^{I(w)}y$;

6. $w \models C = D$ iff $C^{I(w)} = D^{I(w)}$;

7. $w \models a : C$ iff $a^{I(w)} \in C^{I(w)}$;

8. $w \models aR_ib$ iff $a^{I(w)}R_i^{I(w)}b^{I(w)}$;

9. $w \models \Diamond_i\varphi$ iff $\exists v \rhd_i w\ v \models \varphi$;

10. $w \models \varphi \wedge \psi$ iff $w \models \varphi$ and $w \models \psi$;

11. $w \models \neg\varphi$ iff $w \not\models \varphi$.

A formula φ is *satisfiable* in a class of models $\mathcal{M}$ if there is a model $\mathfrak{M} \in \mathcal{M}$ and a world w in $\mathfrak{M}$ such that $w \models \varphi$.

In this paper our main concern is to find out whether there exist algorithms for checking satisfiability of formulas in several important classes of models. Other standard inference problems (concept satisfiability, subsumption, instance checking, consistency) are reducible to the satisfiability problem. The entailment problem can also be reduced to it, at least for the classes of models considered below.[3]

With every class $\mathcal{C}$ of Kripke frames (the number of accessibility relations in which corresponds to the number of modal operators in $\mathcal{ALC}_\mathcal{M}$) we associate the classes $\mathcal{M}(\mathcal{C})$, $\mathcal{M}^e(\mathcal{C})$, and $\mathcal{M}^v(\mathcal{C})$ of all models of $\mathcal{ALC}_\mathcal{M}$ based on frames in $\mathcal{C}$ and having constant, expanding and varying domains, respectively; $\mathcal{M}_{fin}(\mathcal{C})$ will denote the class of models based on frames in $\mathcal{C}$ and having constant finite domains. The set of formulas satisfiable in a class of models $\mathcal{M}$ will be denoted by $Sat\mathcal{M}$.

We will use special names for certain classes of frames with one accessibility relation. Namely,

- $\mathcal{K}$ will stand for the class of all frames (with arbitrary accessibility relations),
- $\mathcal{GL}$ for the class of transitive frames without infinite ascending chains (in other words, transitive Noetherian frames),
- $\mathcal{GL}.3$ for the class of transitive Noetherian frames which are linear (i.e., $u \lhd v \vee v \lhd u \vee u = v$),
- $\mathcal{S}5$ will stand for the class of frames with the universal relations, i.e., $u \lhd v$ for all u and v (this class is often regarded to be a good model for explicit knowledge),
- $\mathcal{S}4$ for the class of frames with transitive reflexive relations (i.e., quasi-ordered frames),
- $\mathcal{S}4.3$ for the class of linear quasi-ordered frames,
- $\mathcal{KD}45$ will stand for the class of transitive, serial ($\forall u \exists v\ u \lhd v$) and Euclidean ($u \lhd v \wedge u \lhd w \to v \lhd w$) frames (this class is often regarded to be a good model for explicit beliefs that are not necessarily true), and
- $\mathcal{N}$ for the frame $\langle \mathbb{N}, < \rangle$, where $\mathbb{N}$ is the set of natural numbers.

By $\mathcal{K}_n$ ($\mathcal{GL}_n$, etc.) we denote the classes of frames with n arbitrary (respectively, n transitive Noetherian, etc.) accessibility relations.

Our strategy is to consider first the unimodal case ($n = 1$) and then lift the obtained results to the polymodal one by proving a general transfer theorem for independent joins of logics.

Let us start, however, with two simple observations. First, it turns out that the satisfiability problem for models with expanding and varying domains can be reduced to the satisfiability problem for models with constant domains. To show this, we introduce a concept ex the intended meaning of which is to contain in each world precisely those objects that are assumed to exist (under the expanding or varying domain assumption) in this world. By relativizing all concepts and formulas to the concept ex, one can simulate varying and expanding domains using constant ones.

Theorem 6 *If $Sat\mathcal{M}(\mathcal{C})$ is decidable, $\mathcal{C}$ a class of frames, then $Sat\mathcal{M}^e(\mathcal{C})$ and $Sat\mathcal{M}^v(\mathcal{C})$ are also decidable.*

Proof Given a formula φ, let ex be a concept name which does not occur in φ. By induction on the construction of a concept C we define its relativization

[3] This is clear for the local consequence—$\Gamma \models_\mathcal{M} \varphi$ iff $(\mathfrak{M}, w) \models \Gamma \Rightarrow (\mathfrak{M}, w) \models \varphi$, for every $\mathfrak{M} \in \mathcal{M}$ and every world w in $\mathfrak{M}$—in this case $\Gamma \models_\mathcal{M} \varphi$ iff $\neg(\bigwedge \Gamma \to \varphi)$ is not satisfiable in $\mathcal{M}$. For the global consequence—$\Gamma \models^*_\mathcal{M} \varphi$ iff $\mathfrak{M} \models \Gamma \Rightarrow \mathfrak{M} \models \varphi$, for every $\mathfrak{M} \in \mathcal{M}$—we have $\Gamma \models^*_\mathcal{M} \varphi$ iff $\neg(\bigwedge \Gamma \wedge \Box \bigwedge \Gamma \to \varphi)$ is not satisfiable in $\mathcal{M}$ when models in $\mathcal{M}$ are transitive, and the class of all models is treated similarly to Theorem 3.57 of (Chagrov and Zakharyaschev 1997).

$C{\downarrow}\mathsf{ex}$:

$$\begin{aligned} C_i{\downarrow}\mathsf{ex} &= C_i \wedge \mathsf{ex},\ C_i \text{ a concept name},\\ (C \wedge D){\downarrow}\mathsf{ex} &= (C{\downarrow}\mathsf{ex}) \wedge (D{\downarrow}\mathsf{ex}),\\ (\neg C){\downarrow}\mathsf{ex} &= \mathsf{ex} \wedge \neg(C{\downarrow}\mathsf{ex}),\\ (\exists R.C){\downarrow}\mathsf{ex} &= \mathsf{ex} \wedge \exists R.(C{\downarrow}\mathsf{ex}),\\ (\Diamond_i C){\downarrow}\mathsf{ex} &= \mathsf{ex} \wedge \Diamond_i(C{\downarrow}\mathsf{ex}). \end{aligned}$$

The relativization of φ is defined inductively as follows:

$$\begin{aligned} (aRb){\downarrow}\mathsf{ex} &= aRb \wedge (a : \mathsf{ex}) \wedge (b : \mathsf{ex}),\\ (a : C){\downarrow}\mathsf{ex} &= a : (C{\downarrow}\mathsf{ex}),\\ (C = D){\downarrow}\mathsf{ex} &= ((C{\downarrow}\mathsf{ex}) = (D{\downarrow}\mathsf{ex})),\\ (\neg\varphi){\downarrow}\mathsf{ex} &= \neg(\varphi{\downarrow}\mathsf{ex}),\\ (\varphi \wedge \psi){\downarrow}\mathsf{ex} &= (\varphi{\downarrow}\mathsf{ex}) \wedge (\psi{\downarrow}\mathsf{ex}),\\ (\Diamond_i\varphi){\downarrow}\mathsf{ex} &= \Diamond_i(\varphi{\downarrow}\mathsf{ex}). \end{aligned}$$

Suppose now that $\mathfrak{F} = \langle W, \lhd_0, \dots \rangle$ is a frame and $m = md(\varphi)$. Then φ is satisfiable in a model based on $\mathfrak{F}$ and having varying domains iff the formula

$$\varphi' = \varphi{\downarrow}\mathsf{ex} \wedge \Box^{\le m}(\neg(\mathsf{ex} = \bot) \wedge \bigwedge_{a \in ob\varphi} a : \mathsf{ex})$$

is satisfiable in a model based on $\mathfrak{F}$ and having constant domains. Indeed, assuming that φ is satisfied in a model $\mathfrak{M} = \langle \mathfrak{F}, I \rangle$ with varying domains and that

$$I(w) = \left\langle \Delta^{I(w)}, R_0^{I(w)}, \dots, C_0^{I(w)}, \dots, a_0^{I(w)}, \dots \right\rangle,$$

for $w \in W$, we construct a model $\mathfrak{N} = \langle \mathfrak{F}, J \rangle$ with constant domains by defining $J(w)$ as

$$\langle \bigcup_{w \in W} \Delta^{I(w)}, R_0^{I(w)}, \dots, C_0^{I(w)}, \dots, \mathsf{ex}^{J(w)}, a_0^{I(w)}, \dots \rangle,$$

where $\mathsf{ex}^{J(w)} = \Delta^{I(w)}$. It is readily checked by induction that for any $\psi \in sub\varphi$ and any $w \in W$, $(\mathfrak{M}, w) \models \psi$ iff $(\mathfrak{N}, w) \models \psi{\downarrow}\mathsf{ex}$. Thus φ' is satisfied in $\mathfrak{N}$.

Conversely, suppose φ' is satisfied in a world v in a model $\mathfrak{N} = \langle \mathfrak{F}, J \rangle$ with constant domains and that

$$J(w) = \left\langle \Delta, R_0^{J(w)}, \dots, C_0^{J(w)}, \dots, \mathsf{ex}^{J(w)}, a_0^{J(w)}, \dots \right\rangle,$$

for $w \in W$. Consider the model $\mathfrak{M} = \langle \mathfrak{F}, I \rangle$ in which

$$I(w) = \left\langle \mathsf{ex}^{J(w)}, R_0^{I(w)}, \dots, C_0^{I(w)}, \dots, a_0^{J(w)}, \dots \right\rangle,$$

where $R_i^{I(w)}$ and $C_i^{I(w)}$ are the restrictions of $R_i^{J(w)}$ and $C_i^{J(w)}$ to $\mathsf{ex}^{J(w)}$, respectively, for every w accessible from v by $\le m$ steps[4], and $I(w) = J(w)$ for all the other worlds w in $\mathfrak{F}$. Since $(\mathfrak{N}, v) \models \Box^{\le m}\neg(\mathsf{ex} = \bot)$, the domains of worlds in $\mathfrak{M}$ are not empty. One can show by induction that for every $\psi \in sub\varphi$, $(\mathfrak{N}, v) \models \psi{\downarrow}\mathsf{ex}$ iff $(\mathfrak{M}, v) \models \psi$ (here we use the fact, well-known in modal logic, that the truth-value of φ in v depends only on the worlds accessible by $\le m$ steps from v).

The case of expanding domains is considered analogously by adding to φ' under $\Box^{\le m}$ one more conjunct $(\mathsf{ex} \to \Box^{\le 1}\mathsf{ex}) = \top$. ❏

[4] I.e., $v \lhd_i v_1 \lhd_j \cdots \lhd_k v_{n-1} \lhd_l v_n$ for some $n \le m$ and some $i, j, \dots, k, l$.

Theorem 6 gives us grounds for restricting attention only to models with constant domains. So in the remaining part of the paper we adopt the constant domain assumption.

Our second observation concerns the satisfiability problem for the set $\mathcal{ALC}_{\mathcal{M}}^-$ of formulas in $\mathcal{ALC}_{\mathcal{M}}$ which contain no concepts of the form $\Diamond_i C$. By extending the technique of Finger and Gabbay (1992) to modal description logics in which modal operators apply only to formulas one can prove the following:

Theorem 7 *If the modal logic characterized by a class of frames $\mathcal{C}$ is decidable, then the sets of formulas in $\mathcal{ALC}_{\mathcal{M}}^-$ that are satisfiable in the classes $\mathcal{M}(\mathcal{C})$ and $\mathcal{M}_{fin}(\mathcal{C})$ coincide and the satisfiability problem for them is decidable.*

Note also that this restricted language does not feel any difference between constant and non-constant domains.

From now on till Section 6 we will be considering the concept description language $\mathcal{ALC}_{\mathcal{M}}$ with only one modal operator $\Diamond$ (and its dual $\Box$).

3 LOGICS WITHOUT THE FINITE MODEL PROPERTY

In pure modal logic, the classes of frames introduced in Section 2 determine decidable logics, which is usually established by proving their finite model property (see e.g. Chagrov and Zakharyaschev, 1997). However, this way of proving decidability does not go through for all corresponding modal description logics.

Definition 8 (FMP) Say that the set $Sat\mathcal{M}(\mathcal{C})$, $\mathcal{C}$ a class of frames, has the *finite model property* (FMP, for short) if every formula in $Sat\mathcal{M}(\mathcal{C})$ is satisfiable in a finite model in $\mathcal{M}(\mathcal{C})$. $Sat\mathcal{M}(\mathcal{C})$ has the *bounded FMP* (BMP, for short) if there is an effective function $f : \mathbb{N} \mapsto \mathbb{N}$ such that every formula φ in $Sat\mathcal{M}(\mathcal{C})$ is satisfiable in a model from $\mathcal{M}(\mathcal{C})$ with at most $f(|\varphi|)$

worlds and objects, $|\varphi|$ the length (say, the number of symbols) of φ.

It should be clear that if $Sat\mathcal{M}(\mathcal{C})$ has BMP and the set of finite frames in $\mathcal{C}$ is recursive then there ia an algorithm deciding whether a given formula is satisfiable in $\mathcal{M}(\mathcal{C})$. If $Sat\mathcal{M}(\mathcal{C})$ has FMP then clearly $Sat\mathcal{M}(\mathcal{C}) = Sat\mathcal{M}_{fin}(\mathcal{C})$. But the converse does not hold in general as follows from the existence of pure modal logics without FMP.

Theorem 9 *For any class* $\mathcal{C} \in \{\mathcal{S}4, \mathcal{S}4.3, \mathcal{N}, \mathcal{GL}.3\}$, $Sat\mathcal{M}(\mathcal{C}) \supsetneqq Sat\mathcal{M}_{fin}(\mathcal{C})$.

Proof For a formula ψ, let $\Box^+\psi = \psi \wedge \Box\psi$ and let φ_1 be the conjunction of the following formulas:

$$a : C, \quad \Box^+((C \to \Box C) = \top),$$

$$\Box^+(\exists R.\neg C = \top), \quad \Box^+((\neg C \to \Diamond C) = \top).$$

One can readily check that φ_1 is satisfied in the models $\langle\langle \mathbb{N}, <\rangle, I\rangle$ and $\langle\langle \mathbb{N}, \leq\rangle, I\rangle$ in which, for every $n \in \mathbb{N}$,

$$I(n) = \langle \mathbb{N}, R^n, C^n, a^n\rangle,$$

where $R^n = \mathbb{N} \times \mathbb{N}$, $C^n = \{0, \ldots, n\}$, $a^n = 0$. It is not hard to see, however, that φ_1 cannot be satisfied in any model based on $\langle \mathbb{N}, <\rangle$ or on a frame in $\mathcal{S}4$ and having a finite domain. It follows that $Sat\mathcal{M}(\mathcal{C}) \supsetneqq Sat\mathcal{M}_{fin}(\mathcal{C})$, for $\mathcal{C} \in \{\mathcal{S}4, \mathcal{S}4.3, \mathcal{N}\}$.

Now take φ_2 to be the conjunction of the following three formulas:

$$\Diamond(C \wedge \neg\Diamond C) = \top, \ \Box((C \to \exists R.\neg C) = \top),$$

$$\Box((C \to \neg\exists R.\Diamond C) = \top).$$

It is readily checked that φ_2 is true at the root of the model $\langle\langle W, \lhd\rangle, I\rangle$ in which $W = \{0, 1, \ldots, \omega\}$, $i \lhd j$ iff $i > j$, for $i, j \in W$ (so the frame $\langle W, \lhd\rangle$ is transitive, linear and Noetherian), and for every $n \in W$,

$$I(n) = \left\langle \mathbb{N}, R^{I(n)}, C^{I(n)} \right\rangle,$$

where $C^{I(n)} = \{n\}$, for $n < \omega$, $C^{I(\omega)} = \emptyset$, and $0R^{I(n)}1R^{I(n)}2R^{I(n)}\ldots$. However, φ_2 is not satisfiable in any transitive linear Noetherian model with a finite domain. ❑

Remark 10 It is of interest to note that (i) φ_2 is satisfied in a non-linear Noetherian model with only three worlds and two objects (see Theorem 24), (ii) $Sat\mathcal{M}^e(\mathcal{GL}.3)$ has FMP (only two worlds are enough to satisfy φ_2 under the expanding domain assumption), and (iii) $Sat\mathcal{M}_{fin}(\mathcal{GL}.3)$ has BMP (see Theorem 25).

4 DECIDABILITY WITHOUT BMP

Our aim in this section is to present an algorithm for checking satisfiability in models based on $\langle \mathbb{N}, <\rangle$. Let us fix an arbitrary formula φ.

Definition 11 (quasiworld) A *quasiworld* for φ is a structure of the form

$$\mathfrak{w} = \langle X, R_0^{\mathfrak{w}}, \ldots, C_0^{\mathfrak{w}}, \ldots, (\Diamond D_0)^{\mathfrak{w}}, \ldots, a_0^{\mathfrak{w}}, \ldots\rangle,$$

where X is a finite set, $R_i^{\mathfrak{w}} \subseteq X \times X$ for every $R_i \in rol\varphi$, $C_i^{\mathfrak{w}} \subseteq X$ for every concept name C_i in φ, $(\Diamond D_i)^{\mathfrak{w}} \subseteq X$ for every $\Diamond D_i \in con\varphi$, and $a_i^{\mathfrak{w}} \in X$ for every $a_i \in ob\varphi$. The *value* $C^{\mathfrak{w}}$ of a concept $C \in con\varphi$ in $\mathfrak{w}$ is computed as in Definition 5, but with item 4 replaced by the following: $C^{\mathfrak{w}} = (\Diamond D_i)^{\mathfrak{w}}$, for $C = \Diamond D_i$.

Now consider a structure

$$\mathfrak{m} = \langle \mathfrak{w}_1, \ldots, \mathfrak{w}_k | \mathfrak{w}_{k+1}, \ldots, \mathfrak{w}_l\rangle, \tag{2}$$

in which $\mathfrak{w}_i$, $1 \leq i \leq l$, are quasiworlds for φ with domains X_i. Define a function $h : \mathbb{N} \mapsto \{1, \ldots, l\}$ by taking $h(i) = i$ for $1 \leq i \leq l$ and $h(l + m) = k + 1 + \mathrm{mod}_{l-k}(m - 1)$, for $m > 0$; that is h returns the sequence $1, \ldots, k, k + 1, \ldots, l, k + 1, \ldots, l, \ldots$.

Definition 12 (run) A *run* in $\mathfrak{m}$ is any sequence $r = x_1, x_2, \ldots$ such that $x_i \in X_{h(i)}$ and, for every concept $\Diamond C \in con\varphi$ and every $i < \omega$,

(a) $x_i \in (\Diamond C)^{\mathfrak{w}_{h(i)}}$ iff $x_j \in C^{\mathfrak{w}_{h(j)}}$ for some $j > i$.

The ith element of a run r will be denoted by $r(i)$, the quasiworld $\mathfrak{w}_{h(i)}$ by $\mathfrak{w}(i)$ and its domain by $X(i)$ (thus $r(i) \in X(i)$). Any two elements $x = r(i)$ and $y = r(i + 1)$ of a run r satisfy the following condition:

$$\forall \Diamond C \in con\varphi \ (x \in (\Diamond C)^{\mathfrak{w}(i)} \Leftrightarrow$$
$$y \in C^{\mathfrak{w}(i+1)} \cup (\Diamond C)^{\mathfrak{w}(i+1)}).$$

A pair $x \in X(i)$, $y \in X(i + 1)$ satisfying it will be called *suitable*.

Definition 13 (quasimodel) A structure $\mathfrak{m}$ of the form (2) is a *quasimodel* for φ if the following conditions hold:

(b) for every $a \in ob\varphi$, $r_a = a^{\mathfrak{w}(1)}, a^{\mathfrak{w}(2)}, \ldots$ is a run in $\mathfrak{m}$;

(c) for every $i < \omega$ and every $x \in X(i)$, there is a run r in $\mathfrak{m}$ such that $r(i) = x$.

Example 14 The structure $\mathfrak{m} = \langle \ |\mathfrak{w}\rangle$ in which

$$\mathfrak{w} = \langle X, R^{\mathfrak{w}}, C^{\mathfrak{w}}, (\Diamond C)^{\mathfrak{w}}, (\Diamond\neg C)^{\mathfrak{w}}, a^{\mathfrak{w}}\rangle,$$

where $X = \{x,y,z\}$, $R^{\mathfrak{w}} = X \times X$, $C^{\mathfrak{w}} = \{x\}$, $(\Diamond C)^{\mathfrak{w}} = X$, $(\Diamond\neg C)^{\mathfrak{w}} = \{z\}$, and $a^{\mathfrak{w}} = x$ is a quasimodel for the formula φ_1 constructed in the proof of Theorem 9. The sequences

$$r_1 = x,x,x,x,x,\ldots,\quad r_2 = y,x,x,x,x,\ldots,$$

$$r_3 = z,y,x,x,x,\ldots,\quad r_4 = z,z,y,x,x,\ldots$$

are runs in $\mathfrak{m}$, while $r = z,z,z,z,z,\ldots$ is not a run because $z \in (\Diamond C)^{\mathfrak{m}(1)}$ but $z \notin C^{\mathfrak{m}(i)}$ for any $i < \omega$.

It is worth noting that given a structure of the form (2), we can effectively decide whether it is a quasimodel for φ. For we have the following:

Lemma 15 *A structure $\langle \mathfrak{w}_1,\ldots,\mathfrak{w}_k|\mathfrak{w}_{k+1},\ldots,\mathfrak{w}_l\rangle$, in which all $\mathfrak{w}_i$ are quasiworlds for φ, is a quasimodel for φ iff*

(i) *for every $i \le l$ and every $y \in X(i+1)$ there exists $x \in X(i)$ such that the pair x, y is suitable; in particular, for any $a \in ob\varphi$, every pair of adjacent elements in the sequence $a^{\mathfrak{w}(1)},\ldots,a^{\mathfrak{w}(l+1)}$ is suitable;*

(ii) *for every $i \le l$ and every $x_0 \in X(i)$ there is*

$$n \le k + |con\varphi| \cdot (l-k) \cdot |X_{k+1}| \cdot \ldots \cdot |X_l|$$

and there are objects $x_j \in X(i+j)$, for $j = 1,\ldots,n$, such that

$$\forall \Diamond C \in con\varphi\ (x_0 \in (\Diamond C)^{\mathfrak{w}(i)} \Rightarrow \exists m \in \{1,\ldots,n\}\ x_{i+m} \in C^{\mathfrak{w}(i+m)}), \quad (3)$$

with every pair x_j, x_{j+1}, for $0 \le j < n$, being suitable; in particular, for every $a \in ob\varphi$ and every $i \le l$,

$$\forall \Diamond C \in con\varphi\ (a^{\mathfrak{w}(i)} \in (\Diamond C)^{\mathfrak{w}(i)} \Rightarrow \exists m \le l-k\ a^{\mathfrak{w}(i+m)} \in C^{\mathfrak{w}(i+m)}).$$

Proof ($\Leftarrow$) To construct a run through $x_m \in X(m)$, $m < \omega$, we first take objects $x_i \in X(i)$, for $i < m$, such that every pair of adjacent elements in the sequence $x_1,\ldots,x_m$ is suitable—this can be done by (i). Then using (ii) we select a sequence $x_m,\ldots,x_{m+n}$ such that every pair of adjacent elements in it is suitable and $x_m \in (\Diamond C)^{\mathfrak{w}(m)}$ only if $x_{m+i} \in C^{\mathfrak{w}(m+i)}$ for some $i \le n$. After that we select by (ii) such a sequence starting from $x_{m+n} \in X(m+n)$, and so on. It is readily seen that the resulting sequence $x_1,\ldots,x_m,\ldots,x_{m+n},\ldots$ is a run in $\mathfrak{m}$.

($\Rightarrow$) That (i) holds follows immediately from (b), (c) and the definition of a run. To prove (ii), notice first that since some run in $\mathfrak{m}$ comes through $x_0 \in X(i)$, there is a sequence $x_j \in X(i+j)$, $j = 1,\ldots,n$, satisfying (3) and containing only suitable pairs x_j, x_{j+1}. So the problem is to bound n by the constant mentioned in the formulation of the lemma. And this can be done by deleting certain redundant segments from the sequence $x_0,\ldots,x_n$ using the obvious fact that to reach x_j from x_i, $k+1 \le i < j \le n$, (via suitable pairs of objects) one needs not more than $(l-k)\cdot|X_{k+1}|\cdot\ldots\cdot|X_l|$ elements. ❏

The truth-relation $\mathfrak{w}(i) \models \psi$ in a quasimodel $\mathfrak{m}$ is computed in the same way as in Definition 5, but with item 9 replaced by the following: $\mathfrak{w}(i) \models \Diamond\psi$ iff $\mathfrak{w}(j) \models \psi$ for some $j > i$.

Given a quasimodel $\mathfrak{m}$ for φ of the form (2), we can construct a standard model $\mathfrak{M} = \langle\langle\mathbb{N},<\rangle, I\rangle$ in the following way. Its domain Δ consists of all runs in $\mathfrak{m}$ and, for every $n \in \mathbb{N}$,

$$I(n) = \left\langle \Delta, R_0^{I(n)},\ldots,C_0^{I(n)},\ldots,r_{a_0},\ldots \right\rangle,$$

where $rR_i^{I(n)}r'$ iff $r(n)R_i^{\mathfrak{w}(n)}r'(n)$, and $r \in C_i^{I(n)}$ iff $r(n) \in C_i^{\mathfrak{w}(n)}$. By a straightforward induction one can show that for all $C \in con\varphi$, $\psi \in sub\varphi$, $n \in \mathbb{N}$ and $r \in \Delta$, we have $r \in C^{I(n)}$ iff $r(n) \in C^{\mathfrak{w}(n)}$, and $n \models \psi$ iff $\mathfrak{w}(n) \models \psi$ (condition (a) ensures that $r \in (\Diamond D)^{I(n)}$ iff $r(n) \in (\Diamond D)^{\mathfrak{w}(n)}$ and condition (c) guarantees that $r \in (\exists R_i.D)^{I(n)}$ iff $r(n) \in (\exists R_i.D)^{\mathfrak{w}(n)}$). Thus, a formula φ is satisfiable in $\mathcal{M}(\mathcal{N})$ whenever φ is satisfiable in some quasimodel for φ.

To prove the converse, for a model $\mathfrak{M} = \langle\langle\mathbb{N},<\rangle, I\rangle$ satisfying φ and having a domain Δ, we construct a quasimodel representing $\mathfrak{M}$ modulo φ.

Definition 16 (types) The *type* of an object x in a world w of $\mathfrak{M}$ (relative to φ) is the set

$$t_w^{\mathfrak{M}}(x) = \{C \in con\varphi : x \in C^{I(w)}\}.$$

The *type* of w in $\mathfrak{M}$ (relative to φ) is the triple

$$T^{\mathfrak{M}}(w) = \langle\{t_w^{\mathfrak{M}}(x) : x \in \Delta\}, \{\psi \in sub\varphi : w \models \varphi\}, \{\left\langle a, t_w^{\mathfrak{M}}(a^{I(w)})\right\rangle : a \in ob\varphi\}\rangle.$$

We will omit the superscript $\mathfrak{M}$ and write simply $t_w(x)$ and $T(w)$ if understood.

Every model contains at most $2^{|con\varphi|}$ objects of pairwise distinct types in every world and at most

$$\sharp(\varphi) = 2^{2^{|con\varphi|}} \cdot 2^{|sub\varphi|} \cdot |ob\varphi| \cdot 2^{|con\varphi|}$$

worlds having pairwise distinct types.

With every world i in $\mathfrak{M}$ we associate the quasiworld

$$\mathfrak{w}_i = \left\langle X_i, R_0^{\mathfrak{w}_i}, \dots, C_0^{\mathfrak{w}_i}, \dots, (\Diamond D_0)^{\mathfrak{w}_i}, \dots, a_0, \dots \right\rangle,$$

where X_i contains the objects $a \in ob\varphi$ from Δ[5] and also one representative $z \notin ob\varphi$ from each class $[x]_i = \{y \in \Delta : t_i(x) = t_i(y)\}$, if such z exists (so $|X_i| \leq \flat(\varphi) = 2^{|con\varphi|} + |ob\varphi|$), $xR_j^{\mathfrak{w}_i}y$ iff either one of x, y is not in $ob\varphi$ and $x'R_j^{I(i)}y'$ for some $x' \in [x]_i$, $y' \in [y]_i$, or $x, y \in ob\varphi$ and $xR_j^{I(i)}y$, $x \in C_j^{\mathfrak{w}_i}$ iff $x \in C_j^{I(i)}$, and $x \in (\Diamond D_j)^{\mathfrak{w}_i}$ iff $x \in (\Diamond D_j)^{I(i)}$.

Consider the structure $\mathfrak{m} = \langle \mathfrak{w}_1, \dots, \mathfrak{w}_k | \mathfrak{w}_{k+1}, \dots, \mathfrak{w}_l \rangle$ in which each $\mathfrak{w}_i$ is the quasiworld associated with the world i in $\mathfrak{M}$, $1 \leq i \leq l$, k is the minimal number such that $T(k+1)$ occurs infinitely often in the sequence $T(k+1), T(k+2), \dots$, and l is the minimal number such that $T(k+1) = T(l+1)$ and the following conditions (d) and (e) hold:

(d) $\forall \Diamond C \in con\varphi\ \forall a \in ob\varphi\ (a \in (\Diamond C)^{\mathfrak{w}_{k+1}} \Leftrightarrow \exists i \in \{k+2, \dots, l\}\ a \in C^{\mathfrak{w}_i})$,

(e) for every $x_{k+1} \in X_{k+1}$ there are $x_{k+i} \in X_{k+i}$, $i = 2, \dots, l-k$, such that every pair x_{k+j}, x_{k+j+1} is suitable and

$$\forall \Diamond C \in con\varphi\ (x_{k+1} \in (\Diamond C)^{\mathfrak{w}_{k+1}} \Leftrightarrow \exists i \in \{k+2, \dots, l\}\ x_i \in C^{\mathfrak{w}_i}).$$

By Lemma 15, $\mathfrak{m}$ is a quasimodel for φ. Indeed, (i) follows from the fact that every pair $x \in [y]_i$, $x' \in [y]_{i+1}$, for $y \in \Delta$, is suitable. And to show (ii) it suffices to take $n = 2l - k - i$ and the sequence

$$x_i \in X_i, \dots, x_l \in X_l, x_{l+1} \in X_{k+1}, \dots, x_{2l} \in X_l$$

such that every pair of adjacent elements in $x_i, \dots, x_l, x_{l+1}$ is suitable and $x_{l+1}, \dots, x_{2l}$ satisfies (e). It is easily checked by induction that $\mathfrak{m}$ satisfies φ.

We show now that by deleting some quasiworlds from $\mathfrak{m}$ one can construct a quasimodel satisfying φ and containing not more than some effectively computed number of quasiworlds.

In the "linear" part $\mathfrak{w}_1, \dots, \mathfrak{w}_k$ of $\mathfrak{m}$ we delete all the quasiworlds $\mathfrak{w}_{i+1}, \dots, \mathfrak{w}_j$ such that $T(i) = T(j)$, for $i < j \leq k$. By Lemma 15, the resulting structure is again a quasimodel satisfying φ. Thus we may assume that $T(i) \neq T(j)$ whenever $1 \leq i \neq j \leq k$, and so $k \leq \sharp(\varphi)$.

Let us consider now the "cyclic" part $\mathfrak{w}_{k+1}, \dots, \mathfrak{w}_l$. For every $x \in X_{k+1}$, fix a sequence $s_x = x_{k+2}, \dots, x_l$ satisfying (e) (for $x = a \in ob\varphi$ we take $s_x = a^{\mathfrak{w}_{k+2}}, \dots, a^{\mathfrak{w}_l}$) and put $s_x(i) = x_i$, $i \in \{k+2, \dots, l\}$. There are at most $\flat(\varphi)$ sequences s_x satisfying (e). For each of them, say s_x, we mark (at most $|con\varphi|$) numbers $m, \dots, m'$ in the set $\{k+2, \dots, l\}$ such that $s_x(n) \in C^{\mathfrak{w}_n}$, for some $n \in \{m, \dots, m'\}$, whenever $x \in (\Diamond C)^{\mathfrak{w}_{k+1}}$. Let $m_1 < \dots < m_n$ be all marked numbers for all $x \in X_{k+1}$. We will keep the quasiworlds $\mathfrak{w}_{k+1}, \mathfrak{w}_{m_1}, \dots, \mathfrak{w}_{m_n}$ in our quasimodel. (Note that $n \leq |con\varphi| \cdot \flat(\varphi)$.) And if for $i \in \{k+2, \dots, l\} - \{m_1, \dots, m_n\}$ there is $j > i$ such that $T(i) = T(j)$ and

$$\{i, i+1, \dots, j-1\} \cap \{k+1, m_1, \dots, m_n\} = \emptyset.$$

then we delete all the quasiworlds $\mathfrak{w}_i, \dots, \mathfrak{w}_{j-1}$ from $\mathfrak{m}$. The number of the remaining quasiworlds in the "cyclic" part does not exceed $|con\varphi| \cdot \flat(\varphi) \cdot \sharp(\varphi)$. Using Lemma 15 one can readily see that the resulting structure is a quasimodel satisfying φ.

Thus a formula φ is satisfiable in $\mathcal{M}(\mathcal{N})$ iff it is satisfiable in a quasimodel for φ of some effectively computable size. And the latter condition is effectively checked with the help of Lemma 15. Using similar (though technically more sophisticated) methods one can construct satisfiability checking algorithms for $\mathcal{M}(\mathcal{S}4)$, $\mathcal{M}(\mathcal{S}4.3)$ and $\mathcal{M}(\mathcal{GL}.3)$. Thus we obtain

Theorem 17 *The satisfiability problem for $\mathcal{M}(\mathcal{N})$, $\mathcal{M}(\mathcal{S}4)$, $\mathcal{M}(\mathcal{S}4.3)$ and $\mathcal{M}(\mathcal{GL}.3)$ is decidable.*

5 PROVING BMP

As we shall see in this section, all the sets $Sat\mathcal{M}(\mathcal{C})$, for $\mathcal{C} \in \{\mathcal{K}, \mathcal{KD}45, \mathcal{S}5, \mathcal{GL}\}$, have BMP. In principle, one can prove this by filtrating worlds through some suitable sets of formulas and duplicating certain objects in the filtrated worlds to comply with the constant domain assumption. It turns out, however, that actually the same result can be achieved by using the method of quasimodels we started developing above.

By quasimodels in this section we will mean certain frames of the form

$$\mathfrak{m} = \langle Q, \lhd \rangle \qquad (4)$$

in which Q is a set of quasiworlds for some formula φ and $\lhd$ a binary relation on Q. To give a precise definition we again require a notion of a run in $\mathfrak{m}$.

Definition 18 (run) A *run* in $\mathfrak{m} = \langle Q, \lhd \rangle$ is a set r which contains precisely one object from the domain $X_{\mathfrak{w}}$ of each quasiworld $\mathfrak{w} \in Q$—let us denote this object by $r(\mathfrak{w})$—and, for every $r(\mathfrak{u})$ and every $\Diamond C \in con\varphi$, we have $r(\mathfrak{u}) \in (\Diamond C)^{\mathfrak{u}}$ iff there is $r(\mathfrak{v}) \in C^{\mathfrak{v}}$ for some $\mathfrak{v} \rhd \mathfrak{u}$.

[5] Without loss of generality we may assume $a^{I(n)} = a$.

Definition 19 (quasimodel) A *quasimodel* for a formula φ is a frame $\mathfrak{m}$ of the form (4) such that

(f) for every $a \in ob\varphi$, $r_a = \{a^{\mathfrak{w}} : \mathfrak{w} \in Q\}$ is a run in $\mathfrak{m}$;

(g) every object in every quasiworld in $\mathfrak{m}$ belongs to some run in $\mathfrak{m}$.

The truth-relation $(\mathfrak{m}, \mathfrak{w}) \models \psi$ is defined similarly to Definition 5. Given a quasimodel $\mathfrak{m} = \langle Q, \lhd \rangle$ for φ, construct a standard model $\mathfrak{M} = \langle \mathfrak{m}, I \rangle$ by taking for each $\mathfrak{w} \in Q$

$$I(\mathfrak{w}) = \left\langle \Delta, R_0^{I(\mathfrak{w})}, \dots, C_0^{I(\mathfrak{w})}, \dots, a_0^{I(\mathfrak{w})}, \dots \right\rangle,$$

where Δ is the set of all runs in $\mathfrak{m}$, $rR_i^{I(\mathfrak{w})}r'$ iff $r(\mathfrak{w})R_i^{\mathfrak{w}}r'(\mathfrak{w})$, $r \in C_i^{I(\mathfrak{w})}$ iff $r(\mathfrak{w}) \in C_i^{\mathfrak{w}}$, and $a_i^{I(\mathfrak{w})} = r_{a_i}(\mathfrak{w})$. It is readily checked by induction that for all $C \in con\varphi$, $\psi \in sub\varphi$, $\mathfrak{w} \in Q$ and $r \in \Delta$, we have $r \in C^{I(\mathfrak{w})}$ iff $r(\mathfrak{w}) \in C^{\mathfrak{w}}$, and $(\mathfrak{M}, \mathfrak{w}) \models \psi$ iff $(\mathfrak{m}, \mathfrak{w}) \models \psi$.

$Sat\mathcal{M}(\mathcal{K})$**:** It is well known from modal logic (see e.g. Chagrov and Zakharyaschev, 1997) that every satisfiable purely modal formula φ can be satisfied in a finite intransitive tree of depth $\leq md(\varphi)$ and branching $\leq |sub\varphi|$. We remind the reader that a frame $\mathfrak{F} = \langle W, \lhd \rangle$ is called a *tree* if (i) $\mathfrak{F}$ is *rooted*, i.e., there is $w_0 \in W$ (a *root* of $\mathfrak{F}$) such that $w_0 \lhd^* w$ for every $w \in W$, where $\lhd^*$ is the transitive and reflexive closure of $\lhd$, and (ii) for every $w \in W$, the set $\{v \in W : v \lhd^* w\}$ is finite and linearly ordered by $\lhd^*$. The *depth* of a tree is the length of its longest branch. A tree $\mathfrak{F} = \langle W, \lhd \rangle$ is *intransitive* if every world v in $\mathfrak{F}$, save its root, has precisely one predecessor, i.e., $|\{u \in W : u \lhd v\}| = 1$, and the root w_0 is *irreflexive*, i.e., $\neg w_0 \lhd w_0$ (in fact, all worlds in an intransitive frame are irreflexive). Using the standard technique of modal logic one can prove the following

Lemma 20 *Every $\varphi \in Sat\mathcal{M}(\mathcal{K})$ is satisfiable in a model based on an intransitive tree of depth $\leq md(\varphi)$.*

Thus, to establish BMP of $Sat\mathcal{M}(\mathcal{K})$ it remains to show that trees of finite branching are enough to satisfy all formulas in $Sat\mathcal{M}(\mathcal{K})$ and to estimate the degree of branching.

Suppose a formula φ is satisfied in a model $\mathfrak{M} = \langle \mathfrak{F}, I \rangle$ based on an intransitive tree $\mathfrak{F} = \langle W, \lhd \rangle$ of depth $\leq md(\varphi)$ (but possibly with infinitely many branches). As in Section 4, with every world $w \in W$ we associate the quasiworld

$$\mathfrak{w} = \langle X_{\mathfrak{w}}, R_0^{\mathfrak{w}}, \dots, C_0^{\mathfrak{w}}, \dots, (\Diamond D_0)^{\mathfrak{w}}, \dots, a_0^{\mathfrak{w}}, \dots \rangle$$

for φ. (The associated quasiworlds will be denoted by the Gothic letters corresponding to the Roman letters denoting the worlds in $\mathfrak{F}$.) Let $\mathfrak{m} = \langle Q, \prec \rangle$ be the quasimodel for φ in which $Q = \{\mathfrak{w} : w \in W\}$ and $\mathfrak{u} \prec \mathfrak{v}$ iff $u \lhd v$. We are going to select (by induction) a subtree $\mathfrak{m}' = \langle Q', \prec' \rangle$ of $\mathfrak{m}$ which is also a quasimodel for φ and whose degree of branching is $\leq |con\varphi| \cdot \flat(\varphi) + |sub\varphi|$. The root of $\mathfrak{m}'$ is the root of $\mathfrak{m}$. Assume now that we have already selected a quasiworld $\mathfrak{v}$ for Q' and are looking for its successors. Consider an arbitrary object $x \in X_{\mathfrak{v}}$ and all the concepts $\Diamond D_i \in con\varphi$, for $i = 1, \dots, n$, such that $x \in (\Diamond D_i)^{\mathfrak{v}}$. By condition (g), there is a run r in $\mathfrak{m}$ containing x and such that $r(\mathfrak{u}_i) \in D_i^{\mathfrak{u}_i}$ for some $\mathfrak{u}_i \succ \mathfrak{v}$; if $x = a$, $a \in ob\varphi$, we use (f) instead of (g). Then we add $\mathfrak{u}_i$, for $i = 1, \dots, n$, to the quasimodel under construction as successors of $\mathfrak{v}$ and in the same manner consider the other objects in $X_{\mathfrak{v}}$. Also, for every $\Diamond\psi \in sub\varphi$ such that $\mathfrak{v} \models \Diamond\psi$, we add to our quasimodel one successor of $\mathfrak{v}$ in which ψ is true. Clearly, the total number of the added successors does not exceed $|con\varphi| \cdot \flat(\varphi) + |sub\varphi|$. To conclude the construction, we denote by Q' the set of all selected quasiworlds and define $\prec'$ to be the restriction of $\prec$ to Q. It is matter of routine to check that $\mathfrak{m}'$ is a quasimodel satisfying φ and

$$|Q'| \leq \sum_{n=0}^{md(\varphi)} (|con\varphi| \cdot \flat(\varphi) + |sub\varphi|)^n.$$

As a result we obtain the following

Theorem 21 *$Sat\mathcal{M}(\mathcal{K})$ has BMP and is decidable.*

$Sat\mathcal{M}(\mathcal{S}5)$**:** Let $\mathfrak{M} = \langle \mathfrak{F}, I \rangle$ be a model based on a frame $\mathfrak{F} = \langle W, W \times W \rangle$ and satisfying φ. In each class $[w] = \{v : T(w) = T(v)\}$ we select $\flat(\varphi)$ distinct representatives (if the number of worlds in $[w]$ is less than $\flat(\varphi)$ then we select all of them) and consider the structure $\mathfrak{m} = \langle Q, Q \times Q \rangle$ in which Q consists of the quasiworlds associated with those representatives (so $|Q| \leq \sharp(\varphi) \cdot \flat(\varphi)$). It is easily seen that $\mathfrak{m}$ is a quasimodel satisfying φ. Thus we have

Theorem 22 *$Sat\mathcal{M}(\mathcal{S}5)$ has BMP and is decidable.*

$Sat\mathcal{M}(\mathcal{KD}45)$**:** A frame in $\mathcal{KD}45$ is a non-degenerate cluster (i.e., a frame in $\mathcal{S}5$) possibly having one irreflexive predecessor (which in this case is the root of the frame). So, given a model $\mathfrak{M}$ based on such a frame and satisfying φ, we build a quasimodel $\mathfrak{m}$ for φ by taking the quasiworld associated with the root of $\mathfrak{M}$, if any (then it will be the irreflexive root of $\mathfrak{m}$), and the quasimodel for the cluster of $\mathfrak{M}$ constructed in precisely the same way as in the case of $Sat\mathcal{M}(\mathcal{S}5)$. This yields us

Theorem 23 *$Sat\mathcal{M}(\mathcal{KD}45)$ has BMP and is decidable.*

This technique can be also adopted to prove

Theorem 24 *$Sat\mathcal{M}(\mathcal{GL})$ has BMP and is decidable.*

We close this section with a decidability result under the finite constant domain assumption.

Theorem 25 *Let $\mathcal{C}$ be any of the following classes of frames: (i) all transitive frames, (ii) all transitive reflexive frames, (iii) all transitive linear frames, (iv) all transitive Noetherian linear frames, (v) any class of linear quasiorders. Then $Sat\mathcal{M}_{fin}(\mathcal{C})$ has BMP and is decidable. In particular decidable is $Sat\mathcal{M}_{fin}(\mathcal{C})$ for any $\mathcal{C} \in \{\mathcal{K}, \mathcal{S}5, \mathcal{S}4, \mathcal{KD}45, \mathcal{GL}.3, \mathcal{N}\}$ or $\mathcal{C} \subseteq \mathcal{S}4.3$.*

6 POLYMODAL DESCRIPTION LOGICS

In order to extend the results obtained in the previous section to polymodal description logics, we show that the decidability of satisfiability is preserved under fusions of frame classes. More precisely, for classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of frames of the form $\langle W_1, \lhd_1, \dots, \lhd_m \rangle$ and $\langle W_2, \lhd_{m+1}, \dots, \lhd_n \rangle$, respectively, the *fusion* $\mathcal{C}_1 \otimes \mathcal{C}_2$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ consists of all frames $\langle W, \lhd_1 \dots, \lhd_n \rangle$ such that $\langle W, \lhd_1, \dots, \lhd_m \rangle \in \mathcal{C}_1$, $\langle W, \lhd_{m+1}, \dots, \lhd_n \rangle \in \mathcal{C}_2$. For example, $\mathcal{S}5_{n+1} = \mathcal{S}5_n \otimes \mathcal{S}5_1$, for any $n \geq 1$. By extending the technique of Kracht and Wolter (1991) developed for pure modal logics (see also (Fine and Schurz 1996), (Gabbay 1996), (Wolter 1997)), one can prove the following:

Theorem 26 *For any two classes of frames $\mathcal{C}_1$, $\mathcal{C}_2$,*

(i) *if $Sat\mathcal{M}(\mathcal{C}_i)$ is decidable, for $i = 1, 2$, then $Sat\mathcal{M}(\mathcal{C}_1 \otimes \mathcal{C}_2)$ is decidable;*

(ii) *if $Sat\mathcal{M}_{fin}(\mathcal{C}_i)$ is decidable, for $i = 1, 2$, then $Sat\mathcal{M}_{fin}(\mathcal{C}_1 \otimes \mathcal{C}_2)$ is decidable;*

(iii) *if $Sat\mathcal{M}_{fin}(\mathcal{C}_i) = Sat\mathcal{M}(\mathcal{C}_i)$, for $i = 1, 2$, then $Sat\mathcal{M}_{fin}(\mathcal{C}_1 \otimes \mathcal{C}_2) = Sat\mathcal{M}(\mathcal{C}_1 \otimes \mathcal{C}_2)$.*

As a consequence we obtain

Theorem 27 *There exists a satisfiability checking algorithm for each of the following classes of models: $\mathcal{M}(\mathcal{K}_n)$, $\mathcal{M}(\mathcal{KD}45_n)$, $\mathcal{M}(\mathcal{S}5_n)$, $\mathcal{M}(\mathcal{GL}_n)$, $\mathcal{M}(\mathcal{S}4_n)$, where $n \geq 1$.*

7 CONCLUSION

In this paper we have shown the decidability of the satisfiability problem for most of the standard systems of epistemic and temporal description logics. It would be of interest, however, to analyze a number of more complex systems. For instance, in epistemic logic we did not touch on the common knowledge operator which is interpreted by the transitive and reflexive closure of the union of the accessibility relations for the individual agents. In the temporal case we studied only the simplest models with the operator "eventually". However, many applications require more expressive sets of modal operators (like "Next", "Until", "in the past", etc.).

This paper provides rather general methods of establishing decidability of modal description logics. It does not analyze specific systems and the corresponding decision algorithms in detail. But this will certainly be necessary in order to make modal description logics applicable. An important task is to develop reasonably efficient algorithms for checking satisfiability and to determine the complexity of the satisfiability problem.

Our decision algorithms can treat models with varying, expanding and constant domains. But they are oriented to rigid designators, and it is not clear how to extend the algorithms to cover non-rigid ones.

Also we would like to draw attention to some interesting mathematical problems concerning modal description logics. For instance, what is the connection between the decidability (or the finite model property) of the pure modal logic determined by a class of frames $\mathcal{C}$ and the decidability of $Sat\mathcal{M}(\mathcal{C})$? How does the decidability depend on the cardinality of domains?

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