On the complexity of description logics with modal operators

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Introduction Description logics are intended for knowledge representation and reasoning in systems of artificial intelligence (see e.g. [5; 4; 1] and [7] for more references). An apparent general requirement to such logics is "to be sufficiently expressive and effective." However, the concrete balance between the expressive power and complexity depends on the application domain the representation formalism is designed for. There is a wide spectrum of description logics, from relatively weak ones, like $\mathcal{ALER}$, the (un)satisfaction problem for which is NP-complete, more complex $\mathcal{ALC}$ which is PSPACE-complete (see [6]), to very expressive ones, like $\mathcal{CIQ}$ of De Giacomo and Lenzerini [9] and De Giacomo [8], which is EXPTIME-complete.

The conventional description logics were designed to represent knowledge only about static application domains. To capture various dynamic features, for instance, intensional knowledge (in multi-agent systems), dependence on time or actions (in distributed systems), description logics are combined with suitable "modal" (propositional) logics, say epistemic, temporal, or dynamic. Again, there is a variety of possible combinations (see e.g [12; 11; 2; 3]). Some of them are rather simple and do not increase substantially the complexity (for example, the temporal description logic of Schild [12] is EXPTIME-complete); others are too expressive and undecidable (e.g. the multi-dimensional description logic of Baader and Ohlbach [3]).

An optimal compromise between the expressive power and decidability was found in the series of papers [13; 14; 15; 16], where various expressive and yet decidable description logics with epistemic, temporal, and dynamic operators were constructed. However, the complexity of the satisfaction problem in these logics has remained unclear.

The aim of this paper is to show that the modal description logics of [14] based on arbitrary frames, **S5**-frames (for knowledge), and **KD45**-frames (for

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beliefs) are NEXPTIME-complete (no matter whether the underlying description logic is $\mathcal{ALC}$ or $\mathcal{CIQ}$), while many other logics of the papers in the series are NEXPTIME-hard.

Modal description logics We begin by defining the modal concept description language $\mathcal{ALC}_{\mathcal{M}}$ and its semantics. The primitive symbols of $\mathcal{ALC}_{\mathcal{M}}$ are: *concept names* $C_0, C_1, \ldots$, *role names* $R_0, R_1, \ldots$, and *object names* $a_0, a_1, \ldots$. Starting from these we can form compound concepts and formulas using the following constructs. Suppose R is a role name and C, D are concepts (for the basis of our inductive definition we assume concept names to be atomic concepts). Then $\top$, $C \wedge D$, $\neg C$, $\exists R.C$, and $\Diamond C$ are *concepts. Atomic formulas* are expressions of the form $\top$, $C = D$, $a : C$, and aRb, where a, b are object names. If φ and ψ are formulas then so are $\varphi \wedge \psi$, $\neg\varphi$, and $\Diamond\varphi$.

The pure description part of this language is $\mathcal{ALC}$. By adding the constructs for the formation of the inversion R^-, union $R \vee S$, composition $R \circ S$, transitive reflexive closure R^*, test $C?$ and number restrictions $\exists_{\geq n} B.C$ (B a role name or its converse), we can extend it to $\mathcal{CIQ}$. The corresponding modal description language is denoted then by $\mathcal{CIQ}_{\mathcal{M}}$.

A *model* of $\mathcal{ALC}_{\mathcal{M}}$ based on a frame $\mathfrak{F} = \langle W, \lhd \rangle$ is a pair $\mathfrak{M} = \langle \mathfrak{F}, I \rangle$ in which I is a function associating with each $w \in W$ a structure

$$I(w) = \left\langle \Delta, R_0^{I,w}, \ldots, C_0^{I,w}, \ldots, a_0^{I,w}, \ldots \right\rangle,$$

where Δ is a nonempty set of objects, the *domain* of $\mathfrak{M}$, $R_i^{I,w}$ are binary relations on Δ, $C_i^{I,w}$ subsets of Δ, and $a_i^{I,w}$ are objects in Δ such that $a_i^{I,w} = a_i^{I,v}$, for any $v, w \in W$. (Note that we adopt technically the most difficult constant domain assumption.)

Given a model $\mathfrak{M}$ and a world w in it, we define the *value* $C^{I,w}$ of a concept C in w and the *truth-relation* $(\mathfrak{M}, w) \models \varphi$ (or simply $w \models \varphi$, if $\mathfrak{M}$ is understood) by taking: $\top^{I,w} = \Delta$ and $C^{I,w} = C_i^{I,w}$, for $C = C_i$; $(C \wedge D)^{I,w} = C^{I,w} \cap D^{I,w}$; $(\neg C)^{I,w} = \Delta - C^{I,w}$; $x \in (\Diamond C)^{I,w}$ iff $\exists v \rhd w\ x \in C^{I,v}$; $x \in (\exists R.C)^{I,w}$ iff $\exists y \in C^{I,w}\ xR^{I,w}y$; $w \models C = D$ iff $C^{I,w} = D^{I,w}$; $w \models a : C$ iff $a^{I,w} \in C^{I,w}$; $w \models aRb$ iff $a^{I,w} R^{I,w} b^{I,w}$; $w \models \Diamond\varphi$ iff $\exists v \rhd w\ v \models \varphi$; $w \models \varphi \wedge \psi$ iff $w \models \varphi$ and $w \models \psi$; $w \models \neg\varphi$ iff $w \not\models \varphi$.

A formula φ is *satisfiable* in a class of models $\mathcal{M}$ if there is a model $\mathfrak{M} \in \mathcal{M}$ and a world w in $\mathfrak{M}$ such that $w \models \varphi$. First we will be interested in three classes of models (and so three types of modal operators): $\mathcal{K}$, the class of all models, $\mathcal{S}5$, the class of models based on frames with the universal relations, i.e., $u \lhd v$ for all u and v (this class is often regarded to be a good model for explicit knowledge), $\mathcal{KD}45$, the class of transitive, serial ($\forall u \exists v\ u \lhd v$) and Euclidean ($u \lhd v \wedge u \lhd w \to v \lhd w$) models (this class is often regarded to be a good model for explicit beliefs that are not necessarily true).

Complexity The following theorem characterizes the complexity of the satisfaction problem (and thereby the complexity of many other reasoning tasks as well) in the above mentioned classes of models.

Theorem 1. *The satisfaction problem for $\mathcal{ALC}_{\mathcal{M}}$- and $\mathcal{CIQ}_{\mathcal{M}}$-formulas in each of the classes $\mathcal{K}$, $\mathcal{S}5$, and $\mathcal{KD}45$ is NEXPTIME-complete.*

Proof The upper bound follows from [14] and [8]. We will show here only the lower bound for the satisfaction problem in $\mathcal{S}5$ by reducing to it the $n \times n$ tiling problem, n given in binary, which is known to be NEXPTIME-complete. Namely, for a set $\mathcal{T} = \{t_1, \ldots, t_s\}$ of tiles and $n < \omega$, we construct an $\mathcal{ALC}_{\mathcal{M}}$-formula φ of length $O(n^2 + s^2)$ such that φ is satisfied in an $\mathcal{ALC}_{\mathcal{M}}$-model from $\mathcal{S}5$ iff $\mathcal{T}$ tiles $2^n \times 2^n$.

To encode the $2^n \times 2^n$ grid, we define 2^{2n} concepts B_{ij}, $0 \leq i, j < 2^n$, using $2n$ concept names $C_0, \ldots, C_{2n-1}$ and a role name R. Let ψ_1 be the conjunction of the following formulas (that are similar to those of [10], p. 371):

$$\exists R.\top = \top, \quad \neg((\neg C_0 \wedge \cdots \wedge \neg C_{2n-1}) = \bot),$$
$$\bigwedge_{i=0}^{2n-1} (\bigwedge_{j=0}^{i-1} C_j \rightarrow (C_i \rightarrow \forall R.\neg C_i) \wedge (\neg C_i \rightarrow \forall R.C_i) = \top),$$
$$\bigwedge_{i=0}^{2n-1} (\bigvee_{j=0}^{i-1} \neg C_j \rightarrow (C_i \rightarrow \forall R.C_i) \wedge (\neg C_i \rightarrow \forall R.\neg C_i) = \top).$$

For any $i, j \in \{0, \ldots, 2^n - 1\}$ written in binary as $(d_{2n-1}, \ldots, d_n)$ and $(d_{n-1}, \ldots, d_0)$, respectively, we put $B_{ij} = C_0^{d_0} \wedge \cdots \wedge C_{2n-1}^{d_{2n-1}}$, where C^d is C if $d = 1$ and $\neg C$ otherwise. If ψ_1 is satisfied in an $\mathcal{ALC}$-model then the sets B_{ij} in this model are nonempty, pairwise disjoint and cover the domain of the model.

For each tile $t_i \in \mathcal{T}$ we introduce a concept name T_i. Its intended meaning is as follows: we will say that t_k covers an element (i, j) in the grid iff $B_{ij} \subseteq T_k$ (i.e., $B_{ij} \rightarrow T_k = \top$). The problem now is to guarantee that every element of the grid is covered by precisely one tile and that the colours of adjacent tiles match *without using too many formulas.* To this end we require $2n$ new concept names $Q_0, \ldots, Q_{2n-1}$; they will encode 2^{2n} worlds w_{ij}, $0 \leq i, j < 2^n$. The formula

$$\psi_2 = \bigwedge_{i=0}^{2n-1} (\Diamond C_i = \Box C_i) \wedge \bigwedge_{i=1}^{s} (\Diamond T_i = \Box T_i)$$

means that each C_i (T_j) contains the same objects in every world. Let B, B^r, B^u be three other concept names. B will coincide with B_{ij} in the world w_{ij} determined by the condition $w_{ij} \models Q_0^{d_0} \wedge \cdots \wedge Q_{2n-1}^{d_{2n-1}} = \top$, where $(d_{2n-1}, \ldots, d_n)$ and $(d_{n-1}, \ldots, d_0)$ are binary representations of i and j, respectively. This is ensured by the formula ψ_3:

$$(\Diamond B = \top) \wedge \Box(\bigwedge_{i=0}^{2n-1} ((Q_i = \top) \vee (Q_i = \bot)) \wedge (B = \bigwedge_{i=0}^{2n-1} ((C_i \wedge Q_i) \vee (\neg C_i \wedge \neg Q_i))).$$

And the conjunction ψ_4 of the formulas

$$\bigvee_{i=1}^{s} T_i = \top, \quad \bigwedge_{i \neq j} (T_i \wedge T_j = \bot), \quad \Box \bigwedge_{i=1}^{s} ((B \wedge T_i = \bot) \vee (B \wedge T_i = B))$$

says that every element of the grid is covered by precisely one tile.

If $B = B_{ij}$ in some world then $B^r = B_{i,j+1}$, $B^u = B_{i+1,j}$ in this world; this is forced by the formulas:

$$\psi_5 = \Box \bigwedge_{k=0}^{n-1} ((\neg Q_k \wedge \bigwedge_{j=0}^{k-1} Q_j) = \top \rightarrow$$
$$B^r = \bigwedge_{i=0}^{k-1} \neg C_i \wedge C_k \wedge \bigwedge_{i=k+1}^{2n-1} ((C_i \wedge Q_i) \vee (\neg C_i \wedge \neg Q_i))),$$

$$\psi_6 = \Box \bigwedge_{k=n}^{2n-1} ((\neg Q_k \wedge \bigwedge_{j=n}^{k-1} Q_j) = \top \rightarrow$$
$$B^u = \bigwedge_{i=n}^{k-1} \neg C_i \wedge C_k \wedge \bigwedge_{i \notin \{n,\dots,k\}} ((C_i \wedge Q_i) \vee (\neg C_i \wedge \neg Q_i))).$$

Now we are in a position to write down formulas which say that the colours of adjacent tiles match:

$$\psi_7 = \Box(\neg((\bigwedge_{i=0}^{n-1} Q_i) = \top) \rightarrow \bigwedge_{j=1}^{s} (B \subseteq T_j \rightarrow B^r \subseteq \bigvee_{\mathrm{right}(j)=\mathrm{left}(l)} T_l)),$$
$$\psi_8 = \Box(\neg((\bigwedge_{i=n}^{2n-1} Q_i) = \top) \rightarrow \bigwedge_{j=1}^{s} (B \subseteq T_j \rightarrow B^u \subseteq \bigvee_{\mathrm{up}(j)=\mathrm{down}(l)} T_l)).$$

One can show that $\varphi = \psi_1 \wedge \dots \wedge \psi_8$ is as required. ❑

For the classes $\mathcal{S}4$, $\mathcal{K}4$, $\mathcal{GL}$, $\mathcal{N}$ of all quasi-ordered models, transitive models, transitive Noetherian models (i.e., containing no infinite ascending chains), and models based on $\langle \mathbb{N}, < \rangle$, respectively, we have the following:

Theorem 2. *The satisfaction problem for $\mathcal{ALC}_{\mathcal{M}}$- and $\mathcal{CIQ}_{\mathcal{M}}$-formulas in each of the classes $\mathcal{S}4$, $\mathcal{K}4$, $\mathcal{GL}$, and $\mathcal{N}$ is NEXPTIME-hard.*

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