Axiomatizing distance logics

Oliver Kutz[1], Holger Sturm[2], Nobu-Yuki Suzuki[3], Frank Wolter[1], and Michael Zakharyaschev[4]

[1] Computer Science Institute, Intelligent Systems Department, University of Leipzig, Augustus-Platz 10–11, 04109 Leipzig, Germany;
[2] Fachbereich Philosophie, Universität Konstanz, 78457 Konstanz, Germany;
[3] Department of Mathematics, Faculty of Science Shizuoka University, Ohya 836, Shizuoka 422–8529, Japan;
[4] Department of Computer Science, King's College, Strand, London WC2R 2LS, U.K.

Emails: {kutz,wolter}@informatik.uni-leipzig.de, holger.sturm@uni-konstanz.de, smnsuzu@ipz.shizuoka.ac.jp, and mz@dcs.kcl.ac.uk

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Abstract

In [8, 6] we introduced a family of 'modal' languages intended for talking about distances. These languages are interpreted in 'distance spaces' which satisfy some (or all) of the standard axioms of metric spaces. Among other things, we singled out decidable logics of distance spaces and proved expressive completeness results relating classical and modal languages. The aim of this paper is to axiomatize the modal fragments of the semantically defined distance logics of [6] and give a new proof of their decidability.

1 Introduction

Logics of distance spaces were conceived in [8] and [6] as knowledge representation formalisms aimed to bring a numerical, quantitative concept of distance into the conventional qualitative representation and reasoning. The logics allow for two kinds of 'distance expressions.' First, there are explicit facts of the form

$$\boldsymbol{\delta}(c_1, c_2) = a \quad \text{or} \quad \boldsymbol{\delta}(c_1, c_2) < a,$$

saying that the distance between the objects represented by location constants c_1 and c_2 is equal to a or, respectively, less than a, where a is some non-negative real number. Second, we have necessity-like operators $\mathsf{A}^{\leq a}$ and $\mathsf{A}^{>a}$ with the intended meaning 'everywhere in the neighborhood of radius a' and 'everywhere outside the neighborhood of radius a,' and their 'possibility-like' duals $\mathsf{E}^{\leq a}$ and $\mathsf{E}^{>a}$. For example, the formulas

$$\boldsymbol{\delta}(house, college) \leq 10,$$
$$house \in \mathsf{A}^{\leq 7}\mathsf{E}^{\leq 2}\mathsf{public\_transport}$$

mean that the distance between the house and the college is not more than 10 units (say, miles) and that whenever you are not more than 7 miles away from home, there is a bus stop or a tube station within a distance of 2 miles.

Distance logics are interpreted in so-called *distance spaces* which are pairs of the form $D = \langle W, d \rangle$, where W is a non-empty set (of points) and d a function from $W \times W$ into the set $\mathbb{R}^+$ (of non-negative real numbers) satisfying the natural axiom

$$d(x, y) = 0 \text{ iff } x = y \tag{1}$$

for all $x, y \in W$. The value $d(x, y)$ is called the *distance* from the point x to the point y. The more familiar *metric spaces* also satisfy two more axioms

$$d(x, z) \leq d(x, y) + d(y, z), \tag{2}$$
$$d(x, y) = d(y, x) \tag{3}$$

for all $x, y, z \in W$.

The distance logics of [8, 6] were defined purely semantically, which is usually enough for the purpose of knowledge representation if reasoning algorithms are provided. In this paper we address the logical problem of finding corresponding axiomatic systems and give a partial solution to the problem by presenting a Hilbert-style axiomatization of the 'modal fragments' of distance logics (containing no occurrences of predicates like $\boldsymbol{\delta}(c_1, c_2) = a$ and $\boldsymbol{\delta}(c_1, c_2) < a$).

2 Logics of distance spaces

We begin by introducing a family of propositional languages $\mathcal{L}(M)$ parametrized by subsets $M \subseteq \mathbb{R}^+$ of non-negative real numbers that are assumed to contain 0 and be closed under addition. Let us call such sets of reals *parameter sets*.

Definition 1 (syntax). Suppose $M \subseteq \mathbb{R}^+$ is a parameter set. The alphabet of the language $\mathcal{L}(M)$ consists of a denumerably infinite list $\{p_i : i < \omega\}$ of propositional variables, the Boolean connectives $\wedge$ and $\neg$, and two lists $\{\mathsf{A}^{\leq a} : a \in M\}$ and $\{\mathsf{A}^{>a} : a \in M\}$ of (unary) modal operators depending on M. The set of well-formed formulas of this language is constructed in the standard way; it will be identified with $\mathcal{L}(M)$.

Other Booleans as well as the dual modal operators $\mathsf{E}^{\leq a}$ and $\mathsf{E}^{>a}$ are defined as abbreviations (e.g., $\mathsf{E}^{\leq} = \neg\mathsf{A}^{\leq a}\neg$, $\mathsf{E}^{>a} = \neg\mathsf{A}^{>a}\neg$). We use lower case Latin letters $p, q, r, \ldots$ to denote propositional variables, lower case Greek letters $\chi, \varphi, \psi, \ldots$ to denote formulas, and upper case Greek letters $\Delta, \Sigma, \Theta, \ldots$ to denote sets of formulas.

Definition 2 (semantics). An $\mathcal{L}(M)$-*model* is a structure of the form:

$$\mathfrak{B} = \left\langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \ldots \right\rangle, \tag{4}$$

where $\langle W, d\rangle$ is a distance space and the $p_i^{\mathfrak{B}}$ are subsets of W. The *truth-relation* $\langle \mathfrak{B}, w\rangle \models \varphi$, for an $\mathcal{L}(M)$-formula φ and a point $w \in W$, is defined inductively as follows:

- $\langle \mathfrak{B}, w\rangle \models p$ iff $w \in p^{\mathfrak{B}}$;
- $\langle \mathfrak{B}, w\rangle \models \varphi \wedge \psi$ iff $\langle \mathfrak{B}, w\rangle \models \varphi$ and $\langle \mathfrak{B}, w\rangle \models \psi$;
- $\langle \mathfrak{B}, w\rangle \models \neg\varphi$ iff $\langle \mathfrak{B}, w\rangle \not\models \varphi$;
- $\langle \mathfrak{B}, w\rangle \models \mathsf{A}^{\leq a}\varphi$ iff $\langle \mathfrak{B}, u\rangle \models \varphi$ for all $u \in W$ with $d(w, u) \leq a$;
- $\langle \mathfrak{B}, w\rangle \models \mathsf{A}^{>a}\varphi$ iff $\langle \mathfrak{B}, u\rangle \models \varphi$ for all $u \in W$ with $d(w, u) > a$.

Note that our language contains standard modal operators like

- the universal modality $\Box\varphi = \mathsf{A}^{\leq a}\varphi \wedge \mathsf{A}^{>a}\varphi$,
- the difference operator $D\varphi = \mathsf{E}^{>0}\varphi$

which allow for the definition of nominals [2, 5].

As usual, a formula φ is said to be *valid in a model*, if it is true at every point of the model; φ is *valid in a distance space* D, if it is valid in every model based on D. Finally, φ is *valid in a class* $\mathcal{C}$ of models (or distance spaces), if it is valid in every model (respectively, distance space) of $\mathcal{C}$.

As in [6] we use the following notation:

- $\mathcal{D}$ denotes the class of all distance spaces,
- $\mathcal{D}_{tr}$ denotes the class of all distance spaces satisfying (2),
- $\mathcal{D}_{sym}$ denotes the class of all distance spaces satisfying (3), and
- $\mathcal{M}$ stands for the class of all metric spaces.

Now, given a parameter set $M \subseteq \mathbb{R}^+$, we define the *distance logic of* $\mathcal{D}$ (and M) as the set $\mathcal{MS}(M)$ of all $\mathcal{L}(M)$-formulas that are valid in all distance spaces. Similarly, $\mathcal{MS}^t(M)$ is the logic of $\mathcal{D}_{tr}$, $\mathcal{MS}^s(M)$ is the logic of $\mathcal{D}_{sym}$, and $\mathcal{MS}^m(M)$ the logic of all metric spaces.

Proposition 3. *The sets $\mathcal{MS}(M)$, $\mathcal{MS}^s(M)$, $\mathcal{MS}^t(M)$ and $\mathcal{MS}^m(M)$ are all normal multi-modal logics.*

Proof. Let $\mathcal{C}$ be one of the above classes of distance spaces. It easily follows from the definition of the truth-relation that (i) all propositional tautologies are valid in $\mathcal{C}$, (ii) the K-axioms $\mathsf{A}^{\leq a}(\varphi \to \psi) \to (\mathsf{A}^{\leq a}\varphi \to \mathsf{A}^{\leq a}\psi)$ and $\mathsf{A}^{>a}(\varphi \to \psi) \to (\mathsf{A}^{>a}\varphi \to \mathsf{A}^{>a}\psi)$ are valid in $\mathcal{C}$ for any $a \in M$, and (iii) that the rules of substitution, modus ponens and necessitation (i.e., $\varphi/\mathsf{A}^{\leq a}\varphi$ and $\varphi/\mathsf{A}^{>a}\varphi$) preserve validity. ❏

3 Axiomatizations

We will now present Hilbert-style axiomatizations of the logics $\mathcal{MS}(M)$, $\mathcal{MS}^s(M)$, $\mathcal{MS}^t(M)$, and $\mathcal{MS}^m(M)$ for any given parameter set $M \subseteq \mathbb{R}^+$. The corresponding axiomatic systems will be denoted by $\mathsf{MS}(M)$, $\mathsf{MS}^s(M)$, $\mathsf{MS}^t(M)$, and $\mathsf{MS}^m(M)$.

Set $\Box_a\varphi = \mathsf{A}^{\leq a}\varphi \wedge \mathsf{A}^{>a}\varphi$. Accordingly, the dual modal operator $\Diamond_a\varphi$ is an abbreviation for $\mathsf{E}^{\leq a}\varphi \vee \mathsf{E}^{>a}\varphi$. (Since $\mathsf{A}^{\leq a}$ and $\mathsf{A}^{>a}$ are both normal modal operators, the operator $\Box_a$ is normal, as well.)

Let $\mathsf{MS}(M)$ be the axiomatic system with the following axiom schemata and inference rules:

Axiom schemata:

(CL) the axiom schemata of classical propositional calculus

($\mathsf{K}'_{\mathsf{A}^{\leq}}$) $\mathsf{A}^{\leq a}(\varphi \to \psi) \to (\mathsf{A}^{\leq a}\varphi \to \mathsf{A}^{\leq b}\psi)$ $(a, b \in M,\ a \geq b)$

($\mathsf{K}'_{\mathsf{A}^{>}}$) $\mathsf{A}^{>a}(\varphi \to \psi) \to (\mathsf{A}^{>a}\varphi \to \mathsf{A}^{>b}\psi)$ $(a, b \in M,\ a \leq b)$

($\mathsf{T}_{\mathsf{A}^{\leq 0}}$) $\mathsf{A}^{\leq 0}\varphi \to \varphi$

($\mathrm{T}^c_{\mathsf{A}^{\leq 0}}$) $\varphi \to \mathsf{A}^{\leq 0}\varphi$

(U1) $\Box_0\varphi \to \Box_a\varphi$ $(a \in M)$

(U2) $\Box_a\varphi \to \Box_0\varphi$ $(a \in M)$

($4_\Box$) $\Box_a\varphi \to \Box_a\Box_a\varphi$ $(a \in M)$

($\mathsf{B}_\Box$) $\varphi \to \Box_a\Diamond_a\varphi$ $(a \in M)$

Inference rules: the inference rules are *modus ponens* and *necessitation* for both $\mathsf{A}^{\leq a}$ and $\mathsf{A}^{>a}$ and every $a \in M$, namely

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ \text{(MP)} \qquad \frac{\varphi}{\mathsf{A}^{\leq a}\varphi}\ \text{(RN1)} \qquad \frac{\varphi}{\mathsf{A}^{>a}\varphi}\ \text{(RN2)} \quad (a \in M)$$

The intuitive meaning of the axioms should be clear. We only note that the operator $\Box_a$ can be regarded as an analogue of the universal modality of [5].

The following formulas (which will be used in the proof of Theorem 8) are clearly theorems of $\mathsf{MS}(M)$, for any $\varphi \in \mathcal{L}(M)$:

$$(\mathsf{Eq}_\Box) \qquad \Box_a\varphi \leftrightarrow \Box_b\varphi \qquad (a, b \in M)$$

$$(\mathsf{Mo}_{\mathsf{A}^\leq}) \qquad \mathsf{A}^{\leq b}\varphi \to \mathsf{A}^{\leq a}\varphi \qquad (a, b \in M, a \leq b)$$

$$(\mathsf{Mo}_{\mathsf{A}^>}) \qquad \mathsf{A}^{>a}\varphi \to \mathsf{A}^{>b}\varphi \qquad (a, b \in M, a \leq b)$$

$$(\mathsf{T}_\Box) \qquad \Box_a\varphi \to \varphi \qquad (a \in M)$$

The proof is left to the reader as an easy exercise.

To axiomatize $\mathcal{MS}^s(M)$, $\mathcal{MS}^t(M)$, and $\mathcal{MS}(M)$, we require four extra axiom schemata:

$$(\mathsf{B}_{\mathsf{A}^\leq}) \qquad \varphi \to \mathsf{A}^{\leq a}\mathsf{E}^{\leq a}\varphi \qquad (a \in M)$$

$$(\mathsf{B}_{\mathsf{A}^>}) \qquad \varphi \to \mathsf{A}^{>a}\mathsf{E}^{>a}\varphi \qquad (a \in M)$$

$$(\mathsf{Tr1}) \qquad \mathsf{A}^{\leq a+b}\varphi \to \mathsf{A}^{\leq a}\mathsf{A}^{\leq b}\varphi \qquad (a, b \in M)$$

$$(\mathsf{Tr2}) \qquad \mathsf{E}^{\leq a}\mathsf{A}^{>b}\varphi \to \mathsf{A}^{>a+b}\varphi \qquad (a, b \in M)$$

Denote by $\mathsf{MS}^s(M)$ the extension of $\mathsf{MS}(M)$ with schemata $(\mathsf{B}_{\mathsf{A}^\leq})$ and $(\mathsf{B}_{\mathsf{A}^>})$; $\mathsf{MS}^t(M)$ is the extension of $\mathsf{MS}(M)$ with schemata (Tr1) and (Tr2); finally, $\mathsf{MS}^m(M)$ is obtained by adding all four schemata to $\mathsf{MS}(M)$. For an $\mathcal{L}(M)$-formula φ, we write $\vdash_{\mathsf{MS}(M)} \varphi$, $\vdash_{\mathsf{MS}^s(M)} \varphi$, etc. if φ is a theorem of $\mathsf{MS}(M)$, $\mathsf{MS}^s(M)$, etc. To simplify notation, we will usually omit M and write MS, MS^s, $\vdash_{\mathsf{MS}} \varphi$, $\vdash_{\mathsf{MS}^s} \varphi$, etc.

The main result of this paper is the following:

Theorem 4 (completeness). *For every $\mathcal{L}(M)$-formula φ,*

1. $\vdash_{\mathsf{MS}} \varphi$ *iff* $\varphi \in \mathcal{MS}$;
2. $\vdash_{\mathsf{MS}^s} \varphi$ *iff* $\varphi \in \mathcal{MS}^s$;
3. $\vdash_{\mathsf{MS}^t} \varphi$ *iff* $\varphi \in \mathcal{MS}^t$;
4. $\vdash_{\mathsf{MS}^m} \varphi$ *iff* $\varphi \in \mathcal{MS}^m$.

We begin the proof of this theorem by establishing the soundness of the axiomatic systems.

Lemma 5 (soundness). *Let M be any of the axiomatic systems mentioned in Theorem 4 and $\mathcal{M}$ the corresponding logic. Then for every $\mathcal{L}(M)$-formula φ,*

$$\vdash_{\mathsf{M}} \varphi \quad \textit{implies} \quad \varphi \in \mathcal{M}.$$

Proof. (1) Let us start with the system MS and the class of all distance spaces as intended models. The validity of the generalized K-axioms ($\mathsf{K}'_{\mathsf{A}\leq}$) and ($\mathsf{K}'_{\mathsf{A}>}$) follows from the semantic definition of the modal operators (in the case of $a = b$) and the definition of distance spaces. For suppose that $a > b$ and $\langle \mathfrak{B}, w\rangle \models \mathsf{A}^{\leq a}(\varphi \to \psi) \wedge \mathsf{A}^{\leq a}\varphi \wedge \mathsf{E}^{\leq b}\neg\psi$. Then there exists a $u \in W$ such that $d(w,u) \leq b$ and $\langle \mathfrak{B}, u\rangle \not\models \psi$. But since $\leq$ is the usual linear order on $\mathbb{R}$, we have $d(w,u) \leq a$, and hence $\langle \mathfrak{B}, u\rangle \models (\varphi \to \psi) \wedge \varphi$, which is a contradiction. The other K-axiom is considered analogously.

The validity of the remaining axioms follows immediately from the definitions (note that $\langle \mathfrak{B}, w\rangle \models \Box_a\varphi$ means that φ is valid in $\mathfrak{B}$), and it should be clear that validity is preserved under the inference rules.

(2) Now assume that the distance function d is symmetric and consider axiom ($\mathsf{B}_{\mathsf{A}\leq}$). Suppose that $\langle \mathfrak{B}, w\rangle \not\models \varphi \to \mathsf{A}^{\leq a}\mathsf{E}^{\leq a}\varphi$. Then $\langle \mathfrak{B}, w\rangle \models \varphi$ and there is a point u with $d(w,u) \leq a$ such that $\langle \mathfrak{B}, u\rangle \models \mathsf{A}^{\leq a}\neg\varphi$. Since d is symmetric, we have $d(u,w) \leq a$, and hence $\langle \mathfrak{B}, w\rangle \models \neg\varphi$, which is a contradiction. The validity of axiom schema ($\mathsf{B}_{\mathsf{A}>}$) in symmetric distance spaces is shown in a similar manner.

(3) Suppose that the distance function d satisfies the triangular inequality (2) and $\langle \mathfrak{B}, w\rangle \models \mathsf{A}^{\leq a+b}\varphi$. Take any points u, v such that $d(w,u) \leq a$ and $d(u,v) \leq b$. By (2), we have $d(w,v) \leq d(w,u) + d(u,v) \leq a + b$. Therefore, $\langle \mathfrak{B}, v\rangle \models \varphi$, and so $\langle \mathfrak{B}, w\rangle \models \mathsf{A}^{\leq a}\mathsf{A}^{\leq b}\varphi$, which shows the validity of ($\mathsf{Tr1}$) in triangular spaces. To show the validity of ($\mathsf{Tr2}$), assume that $\langle \mathfrak{B}, w\rangle \models \mathsf{E}^{\leq a}\mathsf{A}^{>b}\varphi$, i.e., that there is a u with $d(w,u) \leq a$ such that $\langle \mathfrak{B}, u\rangle \models \mathsf{A}^{>b}\varphi$. Take any point v such that $d(w,v) > a + b$. We then have $a + d(u,v) \geq d(w,u) + d(u,v) \geq d(w,v) > a + b$, from which $d(u,v) > b$, and hence $\langle \mathfrak{B}, w\rangle \models \mathsf{A}^{>a+b}\varphi$.

(4) The case of metric spaces is a consequence of (1), (2) and (3). ❑

To prove completeness, we will use a representation of distance spaces in the form of relational structures.

4 Frame representation

Let $M \subseteq \mathbb{R}^+$ be a parameter set. An *M-frame* is a structure

$$\mathfrak{f} = \langle W, (R_a)_{a\in M}, (R_{\overline{a}})_{a\in M}\rangle \tag{5}$$

which consists of a set W of possible worlds, henceforth called *points*, and two families $(R_a)_{a\in M}$ and $(R_{\overline{a}})_{a\in M}$ of binary relations on W. The intended meaning of uR_av is 'the distance from u to v is at most a' and that of $uR_{\overline{a}}v$ is 'the distance from u to v is more than a.' An *M-model based on* $\mathfrak{f}$ is a structure of the form

$$\mathfrak{M} = \left\langle \mathfrak{f}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots \right\rangle,$$

where the $p_i^{\mathfrak{M}}$ are subsets of W. The notions of truth and validity in M-models and M-frames are standard. For instance,

$$\langle \mathfrak{M}, w\rangle \models \mathsf{A}^{\leq a}\varphi \quad \text{iff} \quad \langle \mathfrak{M}, u\rangle \models \varphi \text{ for all } u \in W \text{ such that } wR_a u,$$
$$\langle \mathfrak{M}, w\rangle \models \mathsf{A}^{> a}\varphi \quad \text{iff} \quad \langle \mathfrak{M}, u\rangle \models \varphi \text{ for all } u \in W \text{ such that } wR_{\overline{a}} u.$$

The following definition singles out those M-frames that reflect properties of distance spaces.

Definition 6 (standard frames). An M-frame $\mathfrak{f}$ of the form (5) is called $\mathcal{D}$-*standard*, if it meets the following requirements:

(S1) $R_a \cup R_{\overline{a}} = W \times W$;

(S2) $R_a \cap R_{\overline{a}} = \emptyset$;

(S3) if $uR_a v$ and $a \leq b$, then $uR_b v$;

(S3′) if $uR_{\overline{a}} v$ and $a \geq b$, then $uR_{\overline{b}} v$;

(S4) for all $u, v \in W$, we have $uR_0 v$ iff $u = v$.

A $\mathcal{D}$-standard frame $\mathfrak{f}$ is called $\mathcal{D}_{sym}$-*standard* if it additionally satisfies

(S5) $uR_a v$ iff $vR_a u$;

(S5)′ $uR_{\overline{a}} v$ iff $vR_{\overline{a}} u$.

A $\mathcal{D}$-standard frame $\mathfrak{f}$ is $\mathcal{D}_{tr}$-*standard* if it satisfies the conditions

(S6) if $uR_a v$ and $vR_b w$ then $uR_{a+b} w$;

(S7) if $uR_a v$ and $uR_{\overline{a+b}} w$ then $vR_{\overline{b}} w$.

A frame satisfying all of these properties is called an $\mathcal{M}$-*standard* or a *metric frame*. We denote by $\mathcal{F}$, $\mathcal{F}_{sym}$, $\mathcal{F}_{tr}$, $\mathcal{F}_{met}$ the classes of $\mathcal{D}$-, $\mathcal{D}_{sym}$-, $\mathcal{D}_{tr}$-, and $\mathcal{M}$-standard frames, respectively.

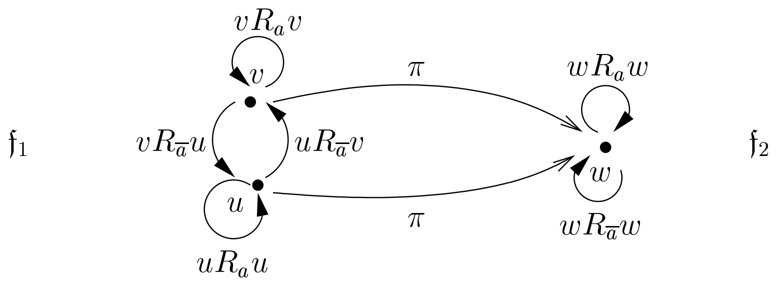
Observe that if both (S1) and (S2) hold, then (S3) is equivalent to (S3)′, (S5) is equivalent to (S5)′, and (S7) follows from (S6). The reason why we need these 'redundant' conditions is that (S2) is not definable in our language, namely we have the following:

Proposition 7. *There is no set Φ of $\mathcal{L}(M)$-formulas such that, for all M-frames $\mathfrak{f}$, we would have*

$$\mathfrak{f} \models \Phi \quad \textit{iff} \quad \mathfrak{f} \textit{ satisfies condition } (\mathrm{S2}).$$

Proof. Suppose otherwise, i.e., $\mathfrak{f} \models \Phi$ iff $\mathfrak{f}$ satisfies (S2), for some set Φ of $\mathcal{L}(M)$-formulas. The M-frame $\mathfrak{f}_1$ in the picture below (where a ranges over M) clearly satisfies (S2), and so $\mathfrak{f}_1 \models \Phi$. The depicted map π is obviously a p-morphism from $\mathfrak{f}_1$ onto the M-frame $\mathfrak{f}_2$. Then we must have $\mathfrak{f}_2 \models \Phi$, which is a contradiction because $\mathfrak{f}_2$ does not satisfy (S2). ❑

In fact, $\mathfrak{f}_1$ satisfies all the properties (S1)–(S7), which means that none of the classes $\mathcal{F}$, $\mathcal{F}_{sym}$, $\mathcal{F}_{tr}$, and $\mathcal{F}_{met}$ is $\mathcal{L}(M)$-definable.



5 Completeness

Now, continuing with the proof of Theorem 4, we next show that our axiomatic systems are sound and complete with respect to the classes (of finite members of) $\mathcal{F}$, $\mathcal{F}_{sym}$, $\mathcal{F}_{tr}$, and $\mathcal{F}_{met}$. We will combine the standard method of canonical models and Sahlqvist's theorem (for details see e.g. [1]) with the duplication and filtration technique of [6], which extends the corresponding technique developed in [3, 4].

Theorem 8 (frame completeness). *For every $\mathcal{L}(M)$-formula φ we have:*

1. $\vdash_{\mathsf{MS}} \varphi$ *iff for all finite* $\mathfrak{f} \in \mathcal{F}$: $\mathfrak{f} \models \varphi$;
2. $\vdash_{\mathsf{MS}^s} \varphi$ *iff for all finite* $\mathfrak{f} \in \mathcal{F}_{sym}$: $\mathfrak{f} \models \varphi$;
3. $\vdash_{\mathsf{MS}^t} \varphi$ *iff for all finite* $\mathfrak{f} \in \mathcal{F}_{tr}$: $\mathfrak{f} \models \varphi$;
4. $\vdash_{\mathsf{MS}^m} \varphi$ *iff for all finite* $\mathfrak{f} \in \mathcal{F}_{met}$: $\mathfrak{f} \models \varphi$.

Proof. ($\Rightarrow$) The soundness part is easy and left to the reader.

($\Leftarrow$) Let M be any of the axiomatic systems mentioned in the theorem and $\mathfrak{M}$ its canonical model based on the canonical frame $\mathfrak{f}$. As all axioms of M are Sahlqvist formulas, by Sahlqvist's theorem we have $\mathfrak{f} \models \mathsf{M}$. It is not hard to see that $\mathfrak{f}$ satisfies all the corresponding properties of M, except perhaps (S1) and (S2). (For instance, conditions (S3) and (S3)$'$ are first-order equivalents of ($\mathsf{Mo}_{\mathsf{A}\leq}$) and ($\mathsf{Mo}_{\mathsf{A}>}$).)

Suppose now that $\nvdash_{\mathsf{M}} \varphi$. Then there exists a point Θ in $\mathfrak{f}$ such that $\langle \mathfrak{M}, \Theta\rangle \not\models \varphi$. Take the submodel $\mathfrak{M}_\Theta$ of $\mathfrak{M}$ generated by Θ. Then clearly $\langle \mathfrak{M}_\Theta, \Theta\rangle \not\models \varphi$ and the underlying frame $\mathfrak{f}_\Theta = \langle W, (R_a)_{a\in M}, (R_{\overline{a}})_{a\in M}\rangle$ of $\mathfrak{M}_\Theta$ satisfies all the properties mentioned above. We claim that $\mathfrak{f}_\Theta$ satisfies (S1) as well. Indeed, by ($\mathsf{4}_\Box$), ($\mathsf{B}_\Box$) and ($\mathsf{T}_\Box$), for every $a \in M$, $\Box_a$ is an $\mathsf{S5}$-box interpreted by the relation $R_a \cup R_{\overline{a}}$. It follows that the $R_a \cup R_{\overline{a}}$ are equivalence relations on W. By ($\mathsf{Eq}_\Box$), we also have

$$R_a \cup R_{\overline{a}} = R_b \cup R_{\overline{b}}$$

for all $a, b \in M$. And since $\mathfrak{f}_\Theta$ is rooted, we can conclude that $R_a \cup R_{\overline{a}}$ is the universal relation on W, i.e., $R_a \cup R_{\overline{a}} = W \times W$, as required.

It remains to transform $\mathfrak{M}_\Theta$ into a finite model $\mathfrak{M}'_\Theta$ which still refutes φ and has all the corresponding properties of M – now including (S2). The required construction is rather complex: a finite filtration of $\mathfrak{M}_\Theta$ is manipulated by duplicating certain points to obtain a finite model validating (S2). A detailed description of the construction can be found in [6] (Lemmas 16–17, 26–27). ❑

We are in a position now to complete the proof of Theorem 4.

Lemma 9 (completeness). *Let* M *be any of the axiomatic systems mentioned in Theorem 4 and* $\mathcal{M}$ *the corresponding logic. Then for every* $\mathcal{L}(M)$*-formula* φ*, we have*

$$\vdash_{\mathsf{M}} \varphi \quad \textit{whenever} \quad \varphi \in \mathcal{M}.$$

Proof. Suppose otherwise, i.e., $\varphi \in \mathcal{M}$ but $\not\vdash_{\mathsf{M}} \varphi$. By Theorem 8 we then have a model refuting φ based on a corresponding finite standard M-frame $\mathfrak{f}$. It remains to transform $\mathfrak{f}$ into a distance space for $\mathcal{M}$ which also refutes φ. That this can be done was proved already in [6]. However, in order to keep the paper reasonably self-contained we repeat the argument for the case that $\mathsf{M} = \mathsf{MS}$ and $\mathcal{M} = \mathcal{MS}$. Let $\mathfrak{M} = \langle \mathfrak{f}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots \rangle$ be a model based on a metric frame $\mathfrak{f}$ such that $\mathfrak{M} \not\models \varphi$. We need to construct a model $\mathfrak{B} = \langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \dots \rangle$ based on a metric space $\langle W, d \rangle$ such that $\mathfrak{B} \not\models \varphi$. Let $\gamma > 0$ be the smallest natural number which is properly greater than any number in

$$M_0 = \{a \in \mathbb{R} : a \text{ appears in } \varphi \}$$

and set

$$M_1 = \{a_1 + \ldots + a_n < \gamma : a_i \in M_0, n < \omega\} \cup \{\gamma\} \cup \{0\}.$$

Define, for every pair of points $w, v \in W$:

$$d(w, v) := min\{\gamma, a : a \in M_1 \text{ and } wR_a v\}.$$

Since M_1 is easily shown to be finite, this defines a function $d : W \times W \to M_1 \subset \mathbb{R}^+$. We claim that $\langle W, d \rangle$ is a metric space.

(1) $d(w, v) = 0$ iff $min\{\gamma, a : a \in M_1 \text{ and } wR_a v\} = 0$ iff $wR_0 v$ iff $w = v$ by condition (S4).

(2) We have to show that $d(u, v) + d(v, w) \geq d(u, w)$. If $d(u, v) + d(v, w) \geq \gamma$ the inequality obtains, since $d(x, y) \leq \gamma$ for all x, y. Hence we can assume that $d(u, v) = a$, $d(v, w) = b$ for $a, b \in M_1 - \{\gamma\}$ and $a + b < \gamma$. Then, $a + b \in M_1$, $uR_a v$ and $vR_b w$, whence, by condition (S6), $uR_{a+b} w$, which implies that $d(u, w) \leq a + b$.

(3) First, assume $d(w, v) = \gamma$. Then for all $a \in M_1 - \{\gamma\}$: $\neg wR_a v$. But this is the case iff $\neg vR_a w$ for all $a \in M_1 - \{\gamma\}$, by condition (S5). Hence $d(w, v) = \gamma = d(v, w)$.

Second, for $a \neq \gamma : d(w,v) = a$ iff $wR_a v \wedge \forall b < a : \neg wR_b v$ iff $vR_a w \wedge \forall b < a : \neg vR_b w$ iff $d(v,w) = a$ by (S5).

It remains to show that φ can be falsified in the metric space $\langle W, d\rangle$. To this end define $\mathfrak{B} = \langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \ldots\rangle$ by putting $p_i^{\mathfrak{B}} = p_i^{\mathfrak{M}}$ for all $i < \omega$. By a straightforward induction one can prove

$$\langle \mathfrak{M}, w\rangle \models \psi \Leftrightarrow \langle \mathfrak{B}, w\rangle \models \psi$$

for all subformulas ψ of φ and all $w \in W$. Hence $\mathfrak{B}$ refutes φ. ❑

We immediately obtain from Theorem 8 and the proof above:

Corollary 10. *All the logics $\mathcal{MS}$, $\mathcal{MS}^s$, $\mathcal{MS}^t$, $\mathcal{MS}^m$ have the finite model property.*

It may be also worth noting that as a consequence of Theorem 4, Corollary 10 and the fact (established in [6]) that the satisfiability problem for $\mathcal{L}(M)$-formulas in finite distance spaces of a given size is decidable we immediately obtain the following:

Theorem 11. *All the logics $\mathcal{MS}(\mathbb{Q}^+)$, $\mathcal{MS}^s(\mathbb{Q}^+)$, $\mathcal{MS}^t(\mathbb{Q}^+)$, $\mathcal{MS}^m(\mathbb{Q}^+)$ are decidable.*

Proof. It suffices to observe that all these logics are recursively axiomatizable and use Harrop's theorem (see e.g. [1]). ❑

6 Outlook

As was noted in the introduction, logics of distance spaces were introduced and investigated primarily in view of their possible applications in knowledge representation and reasoning (for a more detailed discussion see [6]). In this respect the following directions of research appear to be of special interest:

- So far we have considered *arbitrary* metric and distance spaces. However, applications may require more specialised spaces, say, Euclidean spaces.

- Our decidability results obtained in this paper and in [8, 6] do not provide 'practical' decision procedures required in knowledge representation systems.

- Logics of distance spaces reflect only one aspect of possible application domains. We envisage these logics as components of more complex many-dimensional representation formalisms involving, for instance, also logics of time and space (see e.g. [9]). However, to construct such formalisms with a non-trivial interaction between dimensions, we need appropriate 'combination techniques' preserving good computational properties of the components (see e.g. [7]).

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