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ADVANCED MODAL LOGIC

This chapter is a continuation of the preceding one, and we begin it at the place where the authors of *Basic Modal Logic* left us about fifteen years ago. Concluding his historical overview, Krister Segerberg wrote: "Where we stand today is difficult to say. Is the picture beginning to break up, or is it just the contemporary observer's perennial problem of putting his own time into perspective?" So, where did modal logic of the 1970s stand? Where does it stand now? Modal logicians working in philosophy, computer science, artificial intelligence, linguistics or some other fields would probably give different answers to these questions. Our interpretation of the history of modal logic and view on its future is based upon understanding it as part of mathematical logic.

Modal logicians of the First Wave constructed and studied modal systems trying to formalize a few kinds of necessity-like and possibility-like operators. The industrialization of the Second Wave began with the discovery of a deep connection between modal logics on the one hand and relational and algebraic structures on the other, which opened the door for creating many new systems of both artificial and natural origin. Other disciplines—the foundations of mathematics, computer science, artificial intelligence, etc.—brought (or rediscovered[1]) more. "This framework has had enormous influence, not only just on the logic of necessity and possibility, but in other areas as well. In particular, the ideas in this approach have been applied to develop formalisms for describing many other kinds of structures and processes in computer science, giving the subject applications that would have probably surprised the subject's founders and early detractors alike" [Barwise and Moss 1996]. Even two or three mathematical objects may lead to useful generalizations. It is no wonder then that this huge family of logics gave rise to an abstract notion (or rather notions) of a modal logic, which in turn put forward the problem of developing a general theory for it.

Big classes of modal systems were considered already in the 1950s, say extensions of **S5** [Scroggs 1951] or **S4** [Dummett and Lemmon 1959]. Completeness theorems of Lemmon and Scott [1977],[2] Bull [1966b] and Segerberg [1971] demonstrated that many logics, formerly investigated "piecewise", have in fact very much in common and can be treated by the same methods. A need for a uniting theory became obvious. "There are two main lacunae in recent work on modal logic: a lack of general results and a lack of negative results. This or that logic is shown to have such and such a property, but very little is known about the scope or bounds of the property.

[1] One of the celebrities in modal logic—the Gödel–Löb provability logic **GL**—was first introduced by Segerberg [1971] as an "artificial" system under the name **K4W**.

[2] This book was written in 1966.

Thus there are numerous results on completeness, decidability, finite model property, compactness, etc., but very few general or negative results", wrote Fine [1974c]. The creation of duality theory between relational and algebraic semantics ([Lemmon 1966a,b], [Goldblatt 1976a,b]), originated actually by Jónsson and Tarski [1951], the establishment of the connection between modal logics and varieties of modal algebras ([Kuznetsov 1971], Maksimova and Rybakov [1974], [Blok 1976]), and between modal and first and higher order languages ([Fine 1975b], [van Benthem 1983]) added those mathematical ingredients that were necessary to distinguish modal logic as a separate branch of mathematical logic.

On the other hand, various particular systems became subjects of more special disciplines, like provability logic, deontic logic, tense logic, etc., which has found reflection in the corresponding chapters of this Handbook.

In the 1980s and 1990s modal logic was developing both "in width" and "in depth", which made it more difficult for us to select material for this chapter. The expansion "in width" has brought in sight new interesting types of modal operators, thus demonstrating again the great expressive power of propositional modal languages. They include, for instance, polyadic operators, graded modalities, the fixed point and difference operators. We hope the corresponding systems will be considered in detail elsewhere in the *Handbook*; in this chapter they are briefly discussed in the appendix, where the reader can find enough references.

Instead of trying to cover the whole variety of existing types of modal operators, we decided to restrict attention mainly to the classes of normal (and quasi-normal) uni- and polymodal logics and follow "in depth" the way taken by Bull and Segerberg in *Basic Modal Logic*, the more so that this corresponds to our own scientific interests.

Having gone over from considering individual modal systems to big classes of them, we are certainly interested in developing general methods suitable for handling modal logics *en masse*. This somewhat changes the standard set of tools for dealing with logics and gives rise to new directions of research. First, we are almost completely deprived of proof-theoretic methods like Gentzen-style systems or natural deduction. Although proof theory has been developed for a number of important modal logics, it can hardly be extended to reasonably representative families. (Proof theory is discussed in the chapter *Sequent systems for modal logics* in a later volume of this *Handbook*; some references to recent results can be found in the appendix.)

In fact, modern modal logic is primarily based upon the frame-theoretic and algebraic approaches. The link connecting syntactical representations of logics and their semantics is general completeness theory which stems from the pioneering results of Bull [1966b], Fine [1974c], Sahlqvist [1975], Goldblatt and Thomason [1974]. Completeness theorems are usually the first step in understanding various properties of logics, especially those that have semantic or algebraic equivalents. A classical example is Maksimova's

[1979] investigation of the interpolation property of normal modal logics containing **S4**, or decidability results based on completeness with respect to "good" classes of frames. Completeness theory provides means for axiomatizing logics determined by given frame classes and characterizes those of them that are modal axiomatic.

Standard families of modal logics are endowed with the lattice structure induced by the set-theoretic inclusion. This gives rise to another line of studies in modal logic, addressing questions like "what are co-atoms in the lattice?" (i.e., what are maximal consistent logics in the family?), "are there infinite ascending chains?" (i.e., are all logics in the family finitely axiomatizable?), etc. From the algebraic standpoint a lattice of logics corresponds to a lattice of subvarieties of some fixed variety of modal algebras, which opens a way for a fruitful interface with a well-developed field in universal algebra.

A striking connection between "geometrical" properties of modal formulas, completeness, axiomatizability and $\bigcap$-prime elements in the lattice of modal logics was discovered by Jankov [1963, 1969], Blok [1978, 1980b] and Rautenberg [1979]. These observations gave an impetus to a project of constructing frame-theoretic languages which are able to characterize the "geometry" and "topology" of frames for modal logics ([Zakharyaschev 1984, 1992], [Wolter 1996c]) and thereby provide new tools for proving their properties and clarifying the structure of their lattices.

One more interesting direction of studies, arising only when we deal with big classes of logics, concerns the algorithmic problem of recognizing properties of (finitely axiomatizable) logics. Having undecidable finitely axiomatizable logics in a given class ([Thomason 1975a], [Shehtman 1978c]), it is tempting to conjecture that non-trivial properties of logics in this class are undecidable. However, unlike Rice's Theorem in recursion theory, some important properties turn out to be decidable, witness the decidability of interpolation above **S4** ([Maksimova 1979]). The machinery for proving the undecidability of various properties (e.g. Kripke completeness and decidability) was developed in [Thomason 1982] and [Chagrov 1990b,c].

Thomason [1982] proved the undecidability of Kripke completeness first in the class of polymodal logics and then transferred it to that of unimodal ones. In fact, Thomason's embedding turns out to be an isomorphism from the lattice of logics with n necessity operators onto an interval in the lattice of unimodal logics, preserving many standard properties [Kracht and Wolter 1999]. Such embeddings are interesting not only from the theoretical point of view but can also serve as a vehicle for reducing the study of one class of logics to another. Perhaps the best known example of such a reduction is the Gödel translation of intuitionistic logic and its extensions into normal modal logics above **S4** ([Maksimova and Rybakov 1974], [Blok 1976], [Esakia 1979a,b]). We will take advantage of this translation to give a brief survey of results in the field of superintuitionistic logics which actually were always studied in parallel with modal logics (see also Section 5 in *Intuitionistic Logic*).

Listed above are the most important general directions in mathematical modal logic we are going to concentrate on in this chapter. They, of course, do not cover the whole discipline. Other topics, for instance, modal systems with quantifiers, the relationship between the propositional modal language and the first (or higher) order classical language, or proof theory are considered in other chapters of this *Handbook*.

It should be emphasized once again that the reader will find no discussions of particular modal systems in this chapter. Modal logic is presented here as a mathematical theory analyzing big families of logics and thereby providing us with powerful methods for handling concrete ones. (In some cases we illustrate technically complex methods by considering concrete logics; for instance Rybakov's [1994] technique of proving the decidability of the admissibility problem for inference rules is explained only for **GL**.)

1 UNIMODAL LOGICS

We begin by considering normal modal logics with one necessity operator, which were introduced in Section 6 of *Basic Modal Logic*. Recall that each such logic is a set of modal formulas (in the language with the primitive connectives $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\Box$) containing all classical tautologies, the modal axiom

$$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q),$$

and closed under substitution, modus ponens and necessitation $\varphi/\Box\varphi$.

1.1 The lattice NExt**K**

First let us have a look at the class of normal modal logics from a purely syntactic point of view. Given a normal modal logic L_0, we denote by $\mathrm{NExt}L_0$ the family of its **n**ormal **ext**ensions. $\mathrm{NExt}\mathbf{K}$ is thus the class of all normal modal logics. Each logic L in $\mathrm{NExt}L_0$ can be obtained by adding to L_0 a set of modal formulas Γ and taking the closure under the inference rules mentioned above; in symbols this is denoted by

$$L = L_0 \oplus \Gamma.$$

Formulas in Γ are called *additional* (or *extra*) *axioms of* L *over* L_0. Formulas φ and ψ are said to be *deductively equivalent* in $\mathrm{NExt}L_0$ if $L_0 \oplus \varphi = L_0 \oplus \psi$. For instance, $\Box p \rightarrow p$ and $p \rightarrow \Diamond p$ are deductively equivalent in $\mathrm{NExt}\mathbf{K}$, both axiomatizing $\mathbf{T}$, however $(\Box p \rightarrow p) \leftrightarrow (p \rightarrow \Diamond p) \notin \mathbf{K}$. (For more information on the relation between these formulas see [Chellas and Segerberg 1994] and [Williamson 1994].)

We distinguish between two kinds of derivations from assumptions in a logic $L \in \mathrm{NExt}\mathbf{K}$. For a formula φ and a set of formulas Γ, we write $\Gamma \vdash_L \varphi$ if there is a derivation of φ from formulas in L and Γ with the help of only modus ponens. In this case the standard deduction theorem—$\Gamma, \psi \vdash_L \varphi$ iff $\Gamma \vdash_L \psi \to \varphi$—holds. The fact of derivability of φ from Γ in L using both modus ponens and necessitation is denoted by $\Gamma \vdash_L^* \varphi$; in such a case we say that φ is *globally derivable*[3] from Γ in L. For this kind of derivation we have the following variant of the deduction theorem which is proved by induction on the length of derivations in the same manner as for classical logic.

THEOREM 1 (Deduction). *For every logic $L \in \mathrm{NExt}\mathbf{K}$, all formulas φ and ψ, and all sets of formulas Γ,*

$$\Gamma, \psi \vdash_L^* \varphi \textit{ iff } \exists m \geq 0\ \Gamma \vdash_L^* \Box^{\leq m}\psi \to \varphi,$$

where $\Box^{\leq m}\psi = \Box^0\psi \wedge \cdots \wedge \Box^m\psi$ and $\Box^n\psi$ is ψ prefixed by n boxes.

It is to be noted that in general no upper bound for m can be computed even for a decidable L (see Theorem 194). However, if the formula

$$\boldsymbol{tra}_n = \Box^{\leq n}p \to \Box^{n+1}p$$

is in L—such an L is called *n-transitive*—then we can clearly take $m = n$. In particular, for every $L \in \mathrm{NExt}\mathbf{K4}$, $\Gamma, \psi \vdash_L^* \varphi$ iff $\Gamma \vdash_L^* \Box^+\psi \to \varphi$, where $\Box^+\psi = \psi \wedge \Box\psi$. Moreover, a sort of conversion of this observation holds.

THEOREM 2. *The following conditions are equivalent for every logic L in $\mathrm{NExt}\mathbf{K}$:*

(i) *L is n-transitive, for some $n < \omega$;*

(ii) *there exists a formula $\chi(p,q)$ such that, for any φ, ψ and Γ,*

$$\Gamma, \psi \vdash_L^* \varphi \textit{ iff } \Gamma \vdash_L^* \chi(\psi, \varphi).$$

Proof. The implication (i) $\Rightarrow$ (ii) is clear. To prove the converse, observe first that $\chi(p,q) \vdash_L^* \chi(p,q)$ and so $\chi(p,q), p \vdash_L^* q$. By Theorem 1, we then have $\chi(p,q) \vdash_L^* \Box^{\leq n}p \to q$, for some n. Let $q = \Box^{n+1}p$. Then

$$\chi(p, \Box^{n+1}p) \vdash_L^* \Box^{\leq n}p \to \Box^{n+1}p.$$

And since $p \vdash_L^* \Box^{n+1}p$, $\chi(p, \Box^{n+1}p) \in L$. Consequently, $\boldsymbol{tra}_n \in L$. ■

Remark. Note also that (i) is equivalent to the algebraic condition: the variety of modal algebras for L has equationally definable principal congruences. For more information on this and close results consult [Blok and Pigozzi 1982].

[3]This name is motivated by the semantical characterization of $\vdash_L^*$ to be given in Theorem 19.

The *sum* $L_1 \oplus L_2$ and *intersection* $L_1 \cap L_2$ of logics $L_1, L_2 \in \mathrm{NExt}L_0$ are clearly logics in $\mathrm{NExt}L_0$ as well. The former can be axiomatized simply by joining the axioms of L_1 and L_2. To axiomatize the latter we require the following definition. Given two formulas $\varphi(p_1, \dots, p_n)$ and $\psi(p_1, \dots, p_m)$ (whose variables are in the lists $p_1, \dots, p_n$ and $p_1, \dots, p_m$, respectively), denote by $\varphi \underline{\vee} \psi$ the formula $\varphi(p_1, \dots, p_n) \vee \psi(p_{n+1}, \dots, p_{n+m})$.

THEOREM 3. *Let $L_1 = L_0 \oplus \{\varphi_i : i \in I\}$ and $L_2 = L_0 \oplus \{\psi_j : j \in J\}$. Then*

$$L_1 \cap L_2 = L_0 \oplus \{\Box^m \varphi_i \,\underline{\vee}\, \Box^n \psi_j : i \in I,\ j \in J,\ m, n \geq 0\}.$$

Proof. Denote by L the logic in the right-hand side of the equality to be established and suppose that $\chi \in L_1 \cap L_2$. Then for some $m, n \geq 0$ and some finite I' and J' such that all φ'_i and ψ'_j, for $i \in I'$, $j \in J'$, are substitution instances of some $\varphi_{i'}$ and $\psi_{j'}$, for $i' \in I$, $j' \in J$, we have

$$\Box^{\leq m} \bigwedge_{i \in I'} \varphi'_i \to \chi \in L_0, \quad \Box^{\leq n} \bigwedge_{j \in J'} \psi'_j \to \chi \in L_0,$$

from which

$$\bigwedge_{\substack{i \in I', j \in J' \\ 0 \leq k, l \leq m+n}} (\Box^k \varphi'_i \vee \Box^l \psi'_j) \to \chi \in L_0$$

and so $\chi \in L$ because $\Box^k \varphi'_i \vee \Box^l \psi'_j$ is a substitution instance of $\Box^k \varphi_{i'} \underline{\vee} \Box^l \psi_{j'}$. Thus, $L_1 \cap L_2 \subseteq L$. The converse inclusion is obvious. ∎

Although the sum of logics differs in general from their union, these two operations have a few common important properties.

THEOREM 4. *The operation $\oplus$ is idempotent, commutative, associative and distributes over $\cap$; the operation $\cap$ distributes over (infinite) sums, i.e.,*

$$L \cap \bigoplus_{i \in I} L_i = \bigoplus_{i \in I} (L \cap L_i).$$

It follows that $\langle \mathrm{NExt}L_0, \oplus, \cap \rangle$ is a complete distributive lattice, with L_0 and the inconsistent logic, i.e., the set **For** of all modal formulas, being its zero and unit elements, respectively, and the set-theoretic $\subseteq$ its corresponding lattice order. Note, however, that $\oplus$ does not in general distribute over infinite intersections of logics. For otherwise we would have

$$(\mathbf{K} \oplus \neg\Box\bot) \oplus \bigcap_{1 \leq n < \omega} (\mathbf{K} \oplus \Box^n \bot) = \bigcap_{1 \leq n < \omega} (\mathbf{K} \oplus \neg\Box\bot \oplus \Box^n \bot),$$

which is a contradiction, since the logic in the left-hand side is consistent ($\mathbf{D}$, to be more precise), while that in the right-hand side is not.

If we are interested in finding a simple (in one sense or another) syntactic representation of a logic $L \in \mathrm{NExt}L_0$, we can distinguish *finite*, *recursive* and *independent axiomatizations of L over L_0*. The former two notions mean that $L = L_0 \oplus \Gamma$, for some finite or, respectively, recursive Γ, and a set of axioms Γ is independent over L_0 if $L \neq L_0 \oplus \Delta$ for any proper subset Δ of Γ. In the case when L_0 is $\mathbf{K}$ or any other finitely axiomatizable over $\mathbf{K}$ logic, we may omit mentioning L_0 and say simply that L is finitely (recursively, independently) axiomatizable.

It is fairly easy to see that L is not finitely axiomatizable over L_0 iff there is an infinite sequence of logics $L_1 \subset L_2 \subset \dots$ in $\mathrm{NExt}L_0$ such that $L = \bigoplus_{i>0} L_i$. This observation is known as *Tarski's criterion*. (It is worth noting that finite axiomatizability is not preserved under $\cap$. For example, using Tarski's criterion, one can show that $\mathbf{D} \cap (\mathbf{K} \oplus \Box p \vee \Box\neg p)$ is not finitely axiomatizable.) The recursive axiomatizability of a logic L, as was observed by Craig [1953], is equivalent to the recursive enumerability of L. As for independent axiomatizability, an interesting necessary condition can be derived from [Kleyman 1984]. Suppose a normal modal logic L_1 has an independent axiomatization. Then, for every finitely axiomatizable normal modal logic $L_2 \subset L_1$, the interval of logics

$$[L_2, L_1] = \{L \in \mathrm{NExt}\mathbf{K} : L_2 \subseteq L \subseteq L_1\}$$

contains an immediate predecessor of L_1. Using this condition Chagrov and Zakharyaschev [1995a] constructed various logics in $\mathrm{NExt}\mathbf{K4}$, $\mathrm{NExt}\mathbf{S4}$ and $\mathrm{NExt}\mathbf{Grz}$ without independent axiomatizations.

To understand the structure of the lattice $\mathrm{NExt}L_0$ it may be useful to look for a set Γ of formulas which is *complete* in the sense that its formulas are able to axiomatize all logics in the class, and *independent* in the sense that it contains no complete proper subsets. Such a set (if it exists) may be called an *axiomatic basis* of $\mathrm{NExt}L_0$. The existence of an axiomatic basis depends on whether every logic in the class can be represented as the sum of "indecomposable" logics. A logic $L \in \mathrm{NExt}L_0$ is said to be $\bigoplus$*–irreducible* in $\mathrm{NExt}L_0$ if for any family $\{L_i : i \in I\}$ of logics in $\mathrm{NExt}L_0$, $L = \bigoplus_{i\in I} L_i$ implies $L = L_i$ for some $i \in I$. L is $\bigoplus$*–prime* if for any family $\{L_i : i \in I\}$, $L \subseteq \bigoplus_{i\in I} L_i$ only if there is $i \in I$ such that $L \subseteq L_i$. It is not hard to see (using Theorem 4) that a logic is $\bigoplus$–irreducible iff it is $\bigoplus$–prime. This does not hold, however, for the dual notions of $\bigcap$–irreducible and $\bigcap$–prime logics. We have only one implication in general: if L is $\bigcap$–prime (i.e., $\bigcap_{i\in I} L_i \subseteq L$ only if $L_i \subseteq L$, for some $i \in I$) then it is $\bigcap$–irreducible (i.e., $L = \bigcap_{i\in I} L_i$ only if $L = L_i$, for some $i \in I$). A formula φ is said to be *prime* in $\mathrm{NExt}L_0$ if $L_0 \oplus \varphi$ is $\bigoplus$–prime in $\mathrm{NExt}L_0$.

PROPOSITION 5. *Suppose a set of formulas Γ is complete for $\mathrm{NExt}L_0$ and contains no distinct deductively equivalent in $\mathrm{NExt}L_0$ formulas. Then Γ is an axiomatic basis for $\mathrm{NExt}L_0$ iff every formula in Γ is prime.*

Although the definitions above seem to be quite simple, in practice it is not so easy to understand whether a given logic is $\bigoplus$– or $\bigcap$–prime, at least at the syntactical level. However, these notions turn out to be closely related to the following lattice-theoretic concept of splitting for which in the next section we shall provide a semantic characterization.

A pair (L_1, L_2) of logics in $\mathrm{NExt}L_0$ is called a *splitting pair* in $\mathrm{NExt}L_0$ if it divides the lattice $\mathrm{NExt}L_0$ into two disjoint parts: the filter $\mathrm{NExt}L_2$ and the ideal $[L_0, L_1]$. In this case we also say that L_1 *splits* and L_2 *cosplits* $\mathrm{NExt}L_0$.

THEOREM 6. *A logic L_1 splits $\mathrm{NExt}L_0$ iff it is $\bigcap$–prime in $\mathrm{NExt}L_0$, and L_2 cosplits $\mathrm{NExt}L_0$ iff it is $\bigoplus$–prime in $\mathrm{NExt}L_0$. Moreover, the following conditions are equivalent:*

(i) *(L_1, L_2) is a splitting pair in $\mathrm{NExt}L_0$;*
(ii) *L_1 is $\bigcap$–prime in $\mathrm{NExt}L_0$ and $L_2 = \bigcap\{L \in \mathrm{NExt}L_0 : L \not\subseteq L_1\}$;*
(iii) *L_2 is $\bigoplus$–prime in $\mathrm{NExt}L_0$ and $L_1 = \bigoplus\{L \in \mathrm{NExt}L_0 : L \not\supseteq L_2\}$.*

Splittings were first introduced in lattice theory by Whitman [1943] and McKenzie [1972] (see also [Day 1977], [Jipsen and Rose 1993]). Jankov [1963, 1968b, 1969], Blok [1976] and Rautenberg [1977] started using splittings in non-classical logic.

A few standard normal modal logics are listed in Table 1. Note that our notations are somewhat different from those used in *Basic Modal logic.* ($\mathbf{A}^*$ was introduced by Artemov; see [Shavrukov 1991]. The formulas B_n bounding depth of frames are defined in Section 15 of *Basic Modal Logic.*)

1.2 Semantics

The algebraic counterpart of a logic $L \in \mathrm{NExt}\mathbf{K}$ is the variety of modal algebras validating L (for definitions consult Section 10 of *Basic Modal Logic*). Conversely, each variety (equationally definable class) $\mathcal{V}$ of modal algebras determines the normal modal logic $\mathrm{Log}\mathcal{V} = \{\varphi : \forall \mathfrak{A} \in \mathcal{V}\ \mathfrak{A} \models \varphi\}$. Thus we arrive at a dual isomorphism between the lattice $\mathrm{NExt}\mathbf{K}$ and the lattice of varieties of modal algebras, which makes it possible to exploit the apparatus of universal algebra for studying modal logics.

It is often more convenient, however, to deal not with modal algebras directly but with their relational representations discovered by Jónsson and Tarski [1951] and now known as general frames. Each *general frame* $\mathfrak{F} = \langle W, R, P\rangle$ is a hybrid of the usual Kripke frame $\langle W, R\rangle$ and the modal algebra

$$\mathfrak{F}^+ = \langle P, \emptyset, W, -, \cap, \cup, \Box, \Diamond\rangle$$

in which the operations $\Box$ and $\Diamond$ are uniquely determined by the accessibility relation R: for every $X \in P \subseteq 2^W$,

$$\Box X = \{x \in W : \forall y\ (xRy \to y \in X)\}, \quad \Diamond X = -\Box - X.$$

Table 1. A list of standard normal modal logics.

$\mathbf{D}$	$=$	$\mathbf{K} \oplus \Box p \to \Diamond p$
$\mathbf{T}$	$=$	$\mathbf{K} \oplus \Box p \to p$
$\mathbf{KB}$	$=$	$\mathbf{K} \oplus p \to \Box\Diamond p$
$\mathbf{K4}$	$=$	$\mathbf{K} \oplus \Box p \to \Box\Box p$
$\mathbf{K5}$	$=$	$\mathbf{K} \oplus \Diamond\Box p \to \Box p$
$\mathbf{Alt}_n$	$=$	$\mathbf{K} \oplus \Box p_1 \vee \Box(p_1 \to p_2) \vee \cdots \vee \Box(p_1 \wedge \cdots \wedge p_n \to p_{n+1})$
$\mathbf{D4}$	$=$	$\mathbf{K4} \oplus \Diamond\top$
$\mathbf{S4}$	$=$	$\mathbf{K4} \oplus \Box p \to p$
$\mathbf{GL}$	$=$	$\mathbf{K4} \oplus \Box(\Box p \to p) \to \Box p$
$\mathbf{Grz}$	$=$	$\mathbf{K} \oplus \Box(\Box(p \to \Box p) \to p) \to p$
$\mathbf{K4.1}$	$=$	$\mathbf{K4} \oplus \Box\Diamond p \to \Diamond\Box p$
$\mathbf{K4.2}$	$=$	$\mathbf{K4} \oplus \Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$
$\mathbf{K4.3}$	$=$	$\mathbf{K4} \oplus \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$
$\mathbf{S4.1}$	$=$	$\mathbf{S4} \oplus \Box\Diamond p \to \Diamond\Box p$
$\mathbf{S4.2}$	$=$	$\mathbf{S4} \oplus \Diamond\Box p \to \Box\Diamond p$
$\mathbf{S4.3}$	$=$	$\mathbf{S4} \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$
$\mathbf{Triv}$	$=$	$\mathbf{K4} \oplus \Box p \leftrightarrow p$
$\mathbf{Verum}$	$=$	$\mathbf{K4} \oplus \Box p$
$\mathbf{S5}$	$=$	$\mathbf{S4} \oplus p \to \Box\Diamond p$
$\mathbf{K4B}$	$=$	$\mathbf{K4} \oplus p \to \Box\Diamond p$
$\mathbf{A}^*$	$=$	$\mathbf{GL} \oplus \Box\Box p \to \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$
$\mathbf{Dum}$	$=$	$\mathbf{S4} \oplus \Box(\Box(p \to \Box p) \to p) \to (\Diamond\Box p \to p)$
$\mathbf{K4BW}_n$	$=$	$\mathbf{K4} \oplus \bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0 \leq i \neq j \leq n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j))$
$\mathbf{K4BD}_n$	$=$	$\mathbf{K4} \oplus B_n$
$\mathbf{K4}_{n,m}$	$=$	$\mathbf{K4} \oplus \Box^n p \to \Box^m p$, for $1 \leq m < n$

So, using general frames we can take advantage of both relational and algebraic semantics. To simplify notation, we denote general frames of the form $\mathfrak{F} = \langle W, R, 2^W \rangle$ by $\mathfrak{F} = \langle W, R \rangle$. Such frames will be called *Kripke frames.* Given a class of frames $\mathcal{C}$, we write Log$\mathcal{C}$ to denote the logic determined by $\mathcal{C}$, i.e., the set of formulas that are valid in all frames in $\mathcal{C}$; it is called the *logic of* $\mathcal{C}$. If $\mathcal{C}$ consists of a single frame $\mathfrak{F}$, we write simply Log$\mathfrak{F}$.

Basic facts about duality between frames and algebras can be found in the chapters *Basic Modal Logic* and *Correspondence Theory.* Here we remind the reader of the definitions that will be important in what follows.

A frame $\mathfrak{G} = \langle V, S, Q \rangle$ is said to be a *generated subframe* of a frame $\mathfrak{F} = \langle W, R, P \rangle$ if $V \subseteq W$ is *upward closed* in $\mathfrak{F}$, i.e., $x \in V$ and xRy imply $y \in V$, $S = R \upharpoonright V$ and $Q = \{X \cap V : X \in P\}$. The smallest generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ containing a set $X \subseteq W$ is called the *subframe generated by* X. A frame $\mathfrak{F}$ is *rooted* if there is $x \in W$—a *root* of $\mathfrak{F}$—such that the subframe of $\mathfrak{F}$ generated by $\{x\}$ is $\mathfrak{F}$ itself.

A map f from W onto V is a *reduction* (or *p-morphism*) of a frame $\mathfrak{F} = \langle W, R, P \rangle$ to $\mathfrak{G} = \langle V, S, Q \rangle$ if the following three conditions are satisfied for all $x, y \in W$ and $X \in Q$

(R1) xRy implies $f(x)Sf(y)$;

(R2) $f(x)Sf(y)$ implies $\exists z \in W\ (xRz \wedge f(z) = f(y))$;

(R3) $f^{-1}(X) \in P$.

The operations of reduction and generating subframes are relational counterparts of the algebraic operations of forming subalgebras and homomorphic images, respectively, and so preserve validity.

A frame $\mathfrak{F} = \langle W, R, P \rangle$ is *differentiated* if, for any $x, y \in W$,

$$x = y \text{ iff } \forall X \in P\ (x \in X \leftrightarrow y \in X).$$

$\mathfrak{F}$ is *tight* if

$$xRy \text{ iff } \forall X \in P\ (x \in \Box X \rightarrow y \in X).$$

Those frames that are both differentiated and tight are called *refined.* A frame $\mathfrak{F}$ is said to be *compact* if every subset $\mathcal{X}$ of P with the finite intersection property (i.e., with $\bigcap \mathcal{X}' \neq \emptyset$ for any finite subset $\mathcal{X}'$ of $\mathcal{X}$) has non-empty intersection. Finally, refined and compact frames are called *descriptive.* A characteristic property of a descriptive $\mathfrak{F}$ is that it is isomorphic to its bidual $(\mathfrak{F}^+)_+$. The classes of all differentiated, tight, refined and descriptive frames will be denoted by $\mathcal{DF}$, $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{D}$, respectively.

When representing frames in the form of diagrams, we denote by $\bullet$ irreflexive points, by $\circ$ reflexive ones, and by $\boxed{\circ\,\circ}$ two-point clusters. An arrow from x to y means that y is accessible from x. If the accessibility relation is transitive, we draw arrows only to the immediate successors of x.

EXAMPLE 7. (Van Benthem 1979) Let $\mathfrak{F} = \langle W, R, P \rangle$ be the frame whose underlying Kripke frame is shown in Fig. 1 ($\omega + 1$ sees only ω and the

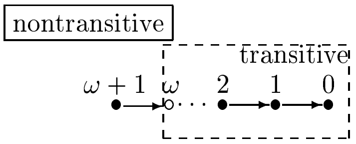


Figure 1.

subframe generated by ω is transitive) and $X \subseteq W$ is in P iff either X is finite and $\omega \notin X$ or X is cofinite in W and $\omega \in X$. It is easy to see that P is closed under $\cap$, $-$ and $\Diamond$. Clearly, $\mathfrak{F}$ is refined. Suppose $\mathcal{X}$ is a subset of P with the finite intersection property. If $\mathcal{X}$ contains a finite set then obviously $\bigcap \mathcal{X} \neq \emptyset$. And if $\mathcal{X}$ consists of only infinite sets then $\omega \in \bigcap \mathcal{X}$. Thus, $\mathfrak{F}$ is descriptive.

A frame $\mathfrak{F}$ is said to be *$\varkappa$-generated*, $\varkappa$ a cardinal, if its dual $\mathfrak{F}^+$ is a $\varkappa$-generated algebra.[4] Each modal logic L is determined by the free finitely generated algebras in the corresponding variety, i.e., by the Tarski–Lindenbaum (or canonical) algebras $\mathfrak{A}_L(n)$ for L in the language with $n < \omega$ variables. Their duals are denoted by $\mathfrak{F}_L(n) = \langle W_L(n), R_L(n), P_L(n)\rangle$ and called the *universal frames of rank n for L*. Analogous notation and terminology will be used for the free algebras $\mathfrak{A}_L(\varkappa)$ with $\varkappa$ generators. Note that $\langle W_L(\varkappa), R_L(\varkappa)\rangle$ is (isomorphic to) the canonical Kripke frame for L with $\varkappa$ variables (defined in Section 11 of *Basic Modal Logic*) and $P_L(\varkappa)$ is the collection of the truth-sets of formulas in the corresponding canonical model. Unless otherwise stated, we will assume in what follows that the language of the logics under consideration contains ω variables.

An important property of the universal frame of rank $\varkappa$ for L is that every descriptive $\varkappa'$-generated frame for L, $\varkappa' \leq \varkappa$, is a generated subframe of $\mathfrak{F}_L(\varkappa)$. Thus, the more information about universal frames for L we have, the deeper our knowledge about the structure of arbitrary frames for L and thereby about L itself.

Although in general universal frames for modal logics are very complicated, considerable progress was made in clarifying the structure of the upper part (points of finite depth) of the universal frames of finite rank for logics in NExt**K4**. The studies in this direction were started actually by Segerberg [1971]. Shehtman [1978a] presented a general method of constructing the universal frames of finite rank for logics in NExt**S4** with the finite model property. Later similar results were obtained by other authors; see e.g. [Bellissima 1985]. The structure of free finitely generated algebras for **S4** was investigated by Blok [1976].

Let us try to understand first the constitution of an arbitrary transitive

[4] An algebra is said to be *$\varkappa$-generated* if it contains a set X of cardinality $\leq \varkappa$ such that the closure of X under the algebra's operations coincides with its universe.

refined frame $\mathfrak{F} = \langle W, R, P\rangle$ with n generators $G_1, \dots, G_n \in P$. Define $\mathfrak{V}$ to be the valuation of the set of variables $\Sigma = \{p_1, \dots, p_n\}$ in $\mathfrak{F}$ such that $x \models p_i$ iff $x \in G_i$. Say that points x and y are Σ-*equivalent*, $x \sim_\Sigma y$ in symbols, if the same variables in Σ are true at them; for $X, Y \subseteq W$ we write $X \sim_\Sigma Y$ if every point in X is Σ-equivalent to some point in Y and vice versa. Let $d(\mathfrak{F})$ denote the *depth*[5] of $\mathfrak{F}$; if $\mathfrak{F}$ is of infinite depth, we write $d(\mathfrak{F}) = \infty$. For $d < d(\mathfrak{F})$, $W^{=d}$ and $W^{>d}$ are the sets of all points in $\mathfrak{F}$ of depth d and $> d$, respectively; $W^{<d}$, $W^{\leq d}$, etc. are defined analogously. $\mathfrak{F}^{\leq d}$ is the subframe of $\mathfrak{F}$ generated by $W^{\leq d}$. The set of all successors (predecessors) of points in a set $X \subseteq W$ is denoted by $X{\uparrow}$ (respectively, $X{\downarrow}$); in the transitive case $X\underline{\uparrow} = X{\uparrow} \cup X$ and $X\overline{\downarrow} = X{\downarrow} \cup X$ are then the upward and downward closure operations. A set X is said to be a *cover* for a set Y in $\mathfrak{F}$ if $Y \subseteq X\overline{\downarrow}$. A point x is called an *atom* in $\mathfrak{F}$ if $\{x\} \in P$.

THEOREM 8. *Suppose* $\mathfrak{F} = \langle W, R, P\rangle$ *is a transitive refined* n*-generated frame, for some* $n < \omega$. *Then*

(i) *each cluster in* $\mathfrak{F}$ *contains* $\leq 2^n$ *points;*

(ii) *for every finite* $d \leq d(\mathfrak{F})$, $W^{=d}$ *is a cover for* $W^{\geq d}$ *and contains at most* $c_n(d)$ *distinct clusters, where*

$$c_n(1) = 2^n + 2^{2^n} - 1, \quad c_n(m+1) = c_n(1) \cdot 2^{c_n(1)+\dots+c_n(m)};$$

(iii) *every point of finite depth in* $\mathfrak{F}$ *is an atom.*

Proof. (i) follows from the differentiatedness of $\mathfrak{F}$ and the obvious fact that precisely the same formulas (in $p_1, \dots, p_n$) are true under $\mathfrak{V}$ at Σ-equivalent points in the same cluster.

The proof of (ii) proceeds by induction on d. Let $x \in W^{>d}$. Since $\mathfrak{F}$ is transitive and $W^{\leq d}$ is finite (by the induction hypothesis), there exists a non-empty upward closed in $W^{>d}$ set X (i.e., $X = X\underline{\uparrow} \cap W^{>d}$) such that $x \in X\overline{\downarrow}$, points in X see exactly the same points of depth $\leq d$ and either

(1) $\quad \forall u, v \in X\, \exists w \in u{\uparrow} \cap X \;\; w \sim_\Sigma v$

or

(2) $\quad \forall u, v \in X \;\; (u \sim_\Sigma v \wedge \neg uRv)$.

Such a set X is called d-*cyclic*; it is *nondegenerate* if (1) holds and *degenerate* otherwise. One can readily show that the same formulas are true at Σ-equivalent points in X. Since $\mathfrak{F}$ is refined, X is then a cluster of depth $d+1$. Thus, $W^{>d} \subseteq W^{=d+1}\overline{\downarrow}$. The upper bound for the number of distinct clusters of depth $d+1$ follows from the differentiatedness of $\mathfrak{F}$ and the definition of d-cyclic sets.

[5]In Section 15 of *Basic Modal Logic* $d(\mathfrak{F})$ was called the rank of $\mathfrak{F}$.

To establish (iii), for every point x of depth $d+1$ one can construct by induction on d a formula (expressing the definition of the d-cyclic set containing x) which is true in $\mathfrak{F}$ under $\mathfrak{V}$ only at x. For details consult [Chagrov and Zakharyaschev 1997]. ■

It is fairly easy now to construct the (generated) subframe $\mathfrak{F}_{\mathbf{K4}}^{<\infty}(n)$ of the universal frame of rank n for **K4** consisting of finite depth points. Indeed, $\mathfrak{F}_{\mathbf{K4}}(n)$ is n-generated, refined and so has the form as described in Theorem 8. On the other hand, it is universal and contains any n-generated descriptive frame as a generated subframe, which means roughly that it contains all possible points of finite depth that can exist in n-generated refined frames.

More precisely, assuming that each point is assigned the set of variables in Σ that are true at it, we begin constructing a frame $\mathfrak{G}_{\mathbf{K4}}(n)$ by putting at depth 1 in it 2^n non-Σ-equivalent degenerate clusters and $2^{2^n}-1$ non-Σ-equivalent non-degenerate clusters with $\leq 2^n$ non-Σ-equivalent points. Suppose that $\mathfrak{G}_{\mathbf{K4}}^{\leq d}(n)$ is already constructed. Then for every antichain $\mathfrak{a}$ of clusters in $\mathfrak{G}_{\mathbf{K4}}^{\leq d}(n)$ containing at least one cluster of depth d and different from a singleton with a non-degenerate cluster, we add to $\mathfrak{G}_{\mathbf{K4}}^{\leq d}(n)$ copies of all $2^n + 2^{2^n} - 1$ clusters of depth 1 so that they would be inaccessible from each other and could see only the clusters in $\mathfrak{a}$ and their successors. And for every singleton $\mathfrak{a} = \{C\}$ with a non-degenerate cluster C, we add to $\mathfrak{G}_{\mathbf{K4}}^{\leq d}(n)$ copies of those clusters of depth 1 which are not Σ-equivalent to any subset of C (otherwise the frame will not be refined) so that again they would be mutually inaccessible and could see only C and its successors in $\mathfrak{G}_{\mathbf{K4}}^{\leq d}(n)$.

Let $\mathfrak{N}_{\mathbf{K4}}(n) = \langle \mathfrak{G}_{\mathbf{K4}}(n), \mathfrak{U}_{\mathbf{K4}}(n)\rangle$ be the resulting model (the relational component of $\mathfrak{G}_{\mathbf{K4}}(n)$ is completely determined by the construction and its set of possible values is the collection of the truth-sets of formulas in $\mathfrak{G}_{\mathbf{K4}}(n)$ under $\mathfrak{U}_{\mathbf{K4}}(n)$). It is not hard to show that $\mathfrak{G}_{\mathbf{K4}}(n)$ is atomic. Moreover, for every point x in this frame one can construct a formula $\varphi(p_1, \ldots, p_n)$ such that $x \not\models \varphi$ and, for any frame $\mathfrak{F}$, $\mathfrak{F} \not\models \varphi$ iff there is a generated subframe of $\mathfrak{F}$ reducible to the subframe of $\mathfrak{G}_{\mathbf{K4}}(n)$ generated by x. It follows in particular that $\mathfrak{G}_{\mathbf{K4}}(n)$ is refined. Thus, every $\mathfrak{G}_{\mathbf{K4}}^{\leq d}(n)$ is a generated subframe of $\mathfrak{F}_{\mathbf{K4}}(n)$. On the other hand, by Theorem 8, $\mathfrak{F}_{\mathbf{K4}}(n)$ contains no clusters of depth $\leq d$ different from those in $\mathfrak{G}_{\mathbf{K4}}^{\leq d}(n)$ and so $\mathfrak{F}_{\mathbf{K4}}^{<\infty}(n)$ is isomorphic to $\mathfrak{G}_{\mathbf{K4}}(n)$. It worth noting also that, since **K4** has the finite model property, it is characterized by $\mathfrak{F}_{\mathbf{K4}}^{<\infty}(n)$, and so $\mathfrak{F}_{\mathbf{K4}}(n)$ is isomorphic to the bidual of $\mathfrak{F}_{\mathbf{K4}}^{<\infty}(n)$.

The universal frame $\mathfrak{F}_L(n)$ for an arbitrary consistent logic L in NExt**K4** is a generated subframe of $\mathfrak{F}_{\mathbf{K4}}(n)$. It can be constructed by removing from $\mathfrak{F}_{\mathbf{K4}}(n)$ those points at which some formulas in L are refuted (under $\mathfrak{V}_{\mathbf{K4}}(n)$). For example, $\mathfrak{F}_{\mathbf{S4}}^{<\infty}(n)$ is obtained by removing from $\mathfrak{F}_{\mathbf{K4}}^{<\infty}(n)$ all irreflexive points and their predecessors. In other words, $\mathfrak{F}_{\mathbf{S4}}^{<\infty}(n)$ can

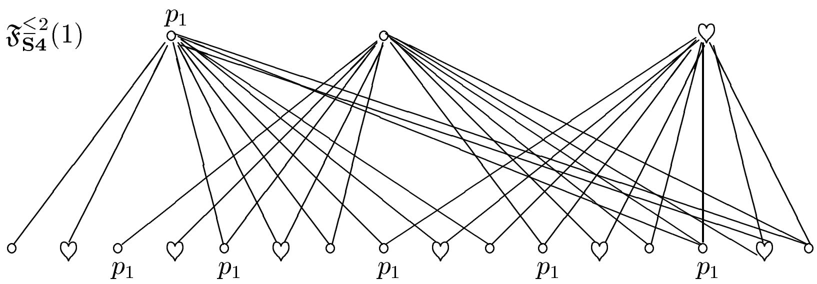


Figure 2.

be constructed in the same way as $\mathfrak{F}^{<\infty}_{\mathbf{K4}}(n)$ but using only non-degenerate clusters. $\mathfrak{F}^{\leq 2}_{\mathbf{S4}}(1)$ (the corresponding model, to be more exact) is shown in Fig. 2, where $\heartsuit$ denotes the cluster with two points at one of which p_1 is true. To construct $\mathfrak{F}^{<\infty}_{\mathbf{Grz}}(n)$ and $\mathfrak{F}^{<\infty}_{\mathbf{GL}}(n)$, we take only simple clusters and degenerate clusters, respectively.

In general, this method of constructing universal frames does not work for logics with nontransitive frames. However, using the fact that $\mathbf{K}$ is characterized by the class of finite intransitive irreflexive trees (see Section 13 of *Basic Modal Logic*), in the same manner as above one can construct an intransitive irreflexive model characterizing $\mathbf{K}$ and such that $\mathfrak{F}_{\mathbf{K}}(n)$ is isomorphic to the bidual of the frame associated with this model.

Let us consider now the semantical meaning of splittings. In view of the following observation we focus attention only on splittings by the logics of finite rooted frames.

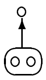
THEOREM 9. *If L_1 splits* $\mathrm{NExt}L_0$ *and L_0 has the finite model property then $L_1 = \mathrm{Log}\mathfrak{F}$, for some finite rooted frame $\mathfrak{F}$ validating L_0.*

Proof. Since L_2 in the splitting pair (L_1, L_2) is a proper extension of L_0, there is a finite frame $\mathfrak{G}$ such that $\mathfrak{G} \models L_0$ and $\mathfrak{G} \not\models L_2$. It follows that $\mathrm{Log}\mathfrak{G} \subseteq L_1$. As we shall see later (Corollary 86), every extension of a tabular logic is also tabular. So $L_1 = \mathrm{Log}\mathfrak{F}$ for some finite $\mathfrak{F} \models L_0$. And since L_1 is $\bigcap$–prime, $\mathfrak{F}$ must be rooted. ■

We say that a frame $\mathfrak{F}$ *splits* $\mathrm{NExt}L_0$ if $\mathrm{Log}\mathfrak{F}$ splits $\mathrm{NExt}L_0$. The logic L_2 of the splitting pair $(\mathrm{Log}\mathfrak{F}, L_2)$ is denoted by $L_0/\mathfrak{F}$ and called the *splitting* of $\mathrm{NExt}L_0$ by $\mathfrak{F}$. This notation reflects the fact that L_2 is the smallest logic in $\mathrm{NExt}L_0$ which is not validated by $\mathfrak{F}$.

EXAMPLE 10. We show that $\mathbf{D} = \mathbf{K}/\bullet$. Recall that $\mathbf{D} = \mathbf{K} \oplus \Diamond\top$ is characterized by the class of serial frames (in which every point has a successor). So if $\bullet \models L$ then $L \subseteq \mathrm{Log}\bullet$; otherwise no frame for L has a dead end, which means that $\Diamond\top \in L$ and $\mathbf{D} \subseteq L$. The inconsistent logic **For** can be represented as $\mathbf{D}/\circ$.

To illustrate some applications of splittings we require a few definitions. Given $L \in \mathrm{NExt}L_0$, we say that the *axiomatization problem* for L above L_0 is decidable if the set $\{\varphi : L_0 \oplus \varphi = L\}$ is recursive. L is *strictly Kripke complete* above L_0 if no other logic in $\mathrm{NExt}L_0$ has exactly the same Kripke frames as L. If all frames in a set $\mathcal{F}$ split $\mathrm{NExt}L_0$, we call the logic $\bigoplus\{L_0/\mathfrak{F} : \mathfrak{F} \in \mathcal{F}\}$ the *union-splitting* of $\mathrm{NExt}L_0$ and denote it by $L_0/\mathcal{F}$.

EXAMPLE 11. **Grz** is not a splitting of $\mathrm{NExt}\mathbf{S4}$. However, it is a union-splitting: $\mathbf{Grz} = \mathbf{S4}/\{$,$\}$. $\mathbf{S4.1} = \mathbf{S4}/$. A frame may split the lattice $\mathrm{NExt}L_0/\mathcal{F}$ but not $\mathrm{NExt}L_0$: e.g. $\circ$ splits $\mathrm{NExt}\mathbf{K}/\bullet$ but does not split $\mathrm{NExt}\mathbf{K}$.

THEOREM 12. *Suppose* $L \in \mathrm{NExt}L_0$ *and* $L = (\dots(L_0/\mathcal{F}_1)/\dots)/\mathcal{F}_n$, *for a sequence* $\mathcal{F}_1, \dots, \mathcal{F}_n$ *of sets of finite rooted frames.*

(i) *If* $\mathcal{F} = \bigcup_{i=1}^{n} \mathcal{F}_i$ *is finite and* L *is decidable then the axiomatization problem for* L *above* L_0 *is decidable. More precisely,*

$$\{\varphi : L_0 \oplus \varphi = L\} = \{\varphi \in L : \forall \mathfrak{F} \in \mathcal{F}\ \mathfrak{F} \not\models \varphi\}.$$

(ii) *If* L *is Kripke complete then* L *is strictly Kripke complete above* L_0.

(iii) *The immediate predecessors of* L *in* $\mathrm{NExt}L_0$ *are precisely the logics* $L \cap \mathrm{Log}\mathfrak{F}$, *for* $\mathfrak{F} \in \mathcal{F}$ *such that* $\mathfrak{F}$ *is not a reduct of a generated subframe of another frame in* $\mathcal{F}$.

Proof. (i) is left to the reader as an easy exercise.

(ii) Let L' be a logic in $\mathrm{NExt}L_0$ with the same Kripke frames as L. Then obviously $L' \subseteq L$. On the other hand, the frames in $\mathcal{F}$ do not validate L' and so $L \subseteq L'$.

(iii) If L' is an immediate predecessor of L in $\mathrm{NExt}L_0$ then $\mathfrak{F} \models L'$, for some $\mathfrak{F} \in \mathcal{F}$. Therefore, $L' \subseteq L \cap \mathrm{Log}\mathfrak{F} \subset L$ and so $L' = L \cap \mathrm{Log}\mathfrak{F}$. Suppose now that $\mathfrak{F}$ is not a reduct of a generated subframe of another frame in $\mathcal{F}$ and $L \cap \mathrm{Log}\mathfrak{F} \subseteq L' \subset L$. Then $L' \subseteq \mathrm{Log}\mathfrak{F}'$ for some $\mathfrak{F}' \in \mathcal{F}$, and hence $\mathfrak{F}' = \mathfrak{F}$, $L' = L \cap \mathrm{Log}\mathfrak{F}$. ■

As follows from Theorem 12 and Example 10, **For** has exactly two immediate predecessors in $\mathrm{NExt}\mathbf{K}$: $\mathbf{Verum} = \mathrm{Log}\bullet$ and $\mathbf{Triv} = \mathrm{Log}\circ$ (and each consistent normal modal logic is contained in one of them). This result is known as Makinson's [1971] Theorem. Moreover, the axiomatization problem for **For** is decidable, i.e., there is an algorithm which decides, given a formula φ whether $\mathbf{K} \oplus \varphi$ is consistent. Likewise, since $\mathbf{D} = \mathbf{K} \oplus \Diamond\top$ is decidable, there is an algorithm recognizing, given φ, whether $\mathbf{D} = \mathbf{K} \oplus \varphi$. We shall see later in Section 4.4 that in fact not so many properties of logics are decidable (e.g. the axiomatization problem for $\mathbf{K} \oplus \neg\Diamond\top$ is undecidable; see Theorem 207) and that Theorem 12 (i) provides the main method for proving decidability results of this type.

To determine whether a finite rooted frame $\mathfrak{F} = \langle W, R\rangle$ splits $\mathrm{NExt}L_0$, we need the formulas defined below:

$$\begin{aligned}\Delta_{\mathfrak{F}} \;\;=\;\; & \{p_x \to \Diamond p_y : x, y \in W,\ xRy\} \cup \\ & \{p_x \to \neg\Diamond p_y : x, y \in W,\ \neg xRy\} \cup \\ & \{p_x \to \neg p_y : x, y \in W,\ x \neq y\},\end{aligned}$$

$$\sigma_{\mathfrak{F}} = \bigwedge \Delta_{\mathfrak{F}}, \quad \delta_{\mathfrak{F}} = \sigma_{\mathfrak{F}} \wedge \bigvee \{p_x : x \in W\}.$$

The meaning of $\delta_{\mathfrak{F}}$ is explained by the following lemma, in which

$$\Box^{<\omega}\varphi = \{\Box^n\varphi : n < \omega\}.$$

LEMMA 13. *For any finite $\mathfrak{F}$ with root r, the set of formulas $\{p_r\} \cup \Box^{<\omega}\delta_{\mathfrak{F}}$ is satisfiable in a frame $\mathfrak{G}$ iff there is a generated subframe $\mathfrak{H}$ of $\mathfrak{G}$ reducible to $\mathfrak{F}$. Moreover, if $\mathfrak{F}$ is cycle free (i.e., contains no path from a point to itself) then ω can be replaced by $n = d(\mathfrak{F}) + 1$.*

Proof. ($\Rightarrow$) Suppose $\{p_r\} \cup \Box^{<\omega}\delta_{\mathfrak{F}}$ is satisfied at a point u in $\mathfrak{G}$. It is not hard to check that the map f defined by $f(v) = x$ iff $v \models p_x$ is a reduction of the subframe $\mathfrak{H}$ of $\mathfrak{G}$ generated by u to $\mathfrak{F}$. If $\mathfrak{F}$ is cycle free and $\{p_r\} \cup \Box^{<\omega}\delta_{\mathfrak{F}}$ is satisfied at u then $d(\mathfrak{H}) = d(\mathfrak{F})$. For otherwise an ascending chain of $n+1$ points starts from u and so $\mathfrak{F}$ must contain a cycle.

($\Leftarrow$) Let f be a reduction of $\mathfrak{H}$ to $\mathfrak{F}$. Define a valuation in $\mathfrak{G}$ so that $v \models p_x$ iff $v \in f^{-1}(x)$. The reader can readily verify that under this valuation $\{p_r\} \cup \Box^{<\omega}\delta_{\mathfrak{F}}$ is true at any point in $f^{-1}(r)$. ∎

LEMMA 14. *For every logic $L \in \mathrm{NExt}\mathbf{K}$ and every finite rooted frame $\mathfrak{F}$, $\mathfrak{F} \models L$ iff $\forall n < \omega\ \Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r \notin L$.*

Proof. The implication ($\Rightarrow$) follows from Lemma 13. Suppose now that $\Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r \notin L$, for every $n < \omega$. Then the set $\{p_r\} \cup \Box^{<\omega}\delta_{\mathfrak{F}}$ is L-consistent and so it is satisfied in a frame $\mathfrak{G}$ for L. By Lemma 13, a generated subframe of $\mathfrak{G}$ is reducible to $\mathfrak{F}$, and hence $\mathfrak{F} \models L$. ∎

We are now in a position to characterize finite frames that split $\mathrm{NExt}L_0$ and to axiomatize splittings.

THEOREM 15. *Suppose $\mathfrak{F}$ is a finite frame with root r and $L_0 \in \mathrm{NExt}\mathbf{K}$. Then $\mathfrak{F}$ splits $\mathrm{NExt}L_0$ iff there is $n < \omega$ such that, for every frame $\mathfrak{G} \models L_0$, $\Box^{\leq n}\delta_{\mathfrak{F}} \wedge p_r$ is satisfiable in $\mathfrak{G}$ only if $\Box^{\leq m}\delta_{\mathfrak{F}} \wedge p_r$ is satisfiable in $\mathfrak{G}$ for every $m > n$. In this case $L_0/\mathfrak{F} = L_0 \oplus \Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r$.*

Proof. ($\Rightarrow$) Suppose otherwise and consider a sequence $\{\mathfrak{G}_n : n < \omega\}$ of frames for L_0 such that $\Box^{\leq n}\delta_{\mathfrak{F}} \wedge p_r$ is satisfiable in $\mathfrak{G}_n$ but $\Box^{\leq m}\delta_{\mathfrak{F}} \wedge p_r$ is not satisfied, for some $m > n$. By Lemma 14, the former condition implies $\bigcap_{n<\omega} \mathrm{Log}\mathfrak{G}_n \subseteq \mathrm{Log}\mathfrak{F}$, while the latter means that $\mathfrak{F} \not\models \mathrm{Log}\mathfrak{G}_n$, for every $n < \omega$, contrary to $\mathrm{Log}\mathfrak{F}$ being $\bigcap$–prime.

($\Leftarrow$) We show that $L_0/\mathfrak{F} = L_0 \oplus \Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r$. Suppose $L \not\subseteq \mathrm{Log}\mathfrak{F}$. Then, by Lemma 14, there is $m < \omega$ such that $\Box^{\leq m}\delta_{\mathfrak{F}} \to \neg p_r \in L$. It follows that $\Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r \in L$ and so $L_0 \oplus \Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r \subseteq L$. ■

For more general versions of this criterion consult [Kracht 1990] and [Wolter 1993].

COROLLARY 16 (Rautenberg 1980). *Suppose that $L_0 \in \mathrm{NExt}(\mathbf{K} \oplus \boldsymbol{tra}_n)$, for some $n < \omega$. Then every finite rooted frame $\mathfrak{F}$ for L_0 splits $\mathrm{NExt}L_0$ and $L_0/\mathfrak{F} = L_0 \oplus \Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r$.*

In particular, every transitive finite rooted frame splits NExt**K4**. This result may also be obtained using the fact that all finite subdirectly irreducible algebras split the lattice of subvarieties of a variety with equationally definable principal congruences (see [Blok and Pigozzi 1982]). However, not every frame splits NExt**K**.

THEOREM 17 (Blok 1978). *A finite rooted frame $\mathfrak{F}$ splits* NExt**K** *iff it is cycle free. In this case $\mathbf{K}/\mathfrak{F} = \mathbf{K} \oplus \Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r$, where $n = d(\mathfrak{F})$.*

Proof. That frames with cycles do not split NExt**K** follows from the fact that **K** is characterized by cycle free finite rooted frames. And the converse is an immediate consequence of Lemma 13 and Theorem 15. ■

An element $x \neq 0$ of a complete lattice $\mathfrak{L}$ is called an *atom* in $\mathfrak{L}$ if the zero element 0 in $\mathfrak{L}$ is the immediate predecessor of x, i.e., there is no y such that $0 < y < x$. Splittings turn out to be closely related to the existence of atoms in finitely generated free algebras; see [Blok 1976], [Bellissima 1984, 1991] and [Wolter 1997c]. We demonstrate the use of splittings by the following

THEOREM 18 (Blok 1980a). *The lattice* NExt**K** *has no atoms.*

Proof. If a logic L is an atom in NExt**K**, it is $\bigoplus$–prime. It follows that L cosplits NExt**K** and the logic $L' = \mathrm{Log}\mathfrak{F}$ in the splitting pair (L', L) has no proper predecessor that splits NExt**K**. Add a new irreflexive root to $\mathfrak{F}$. By Theorem 17, the resulting frame $\mathfrak{G}$ splits NExt**K**, and clearly $\mathrm{Log}\mathfrak{G} \subset \mathrm{Log}\mathfrak{F}$, which is a contradiction. ■

A logic is linked with its semantics via completeness theorems. The most general completeness theorem states that every consistent normal modal logic is characterized by the class of (descriptive) frames validating it. Or, if we want to characterize the consequence relations $\vdash_L$ and $\vdash_L^*$, we can use the following

THEOREM 19. (i) *For $L \in$ NExt$\mathbf{K}$, $\Gamma \vdash_L \varphi$ iff for any model $\mathfrak{M}$ based on a frame for L and any point x in $\mathfrak{M}$, $x \models \Gamma$ implies $x \models \varphi$.*

(ii) *For $L \in$ NExt$\mathbf{K}$, $\Gamma \vdash^*_L \varphi$ iff for any model $\mathfrak{M}$ based on a frame for L, $\mathfrak{M} \models \Gamma$ implies $\mathfrak{M} \models \varphi$.*

However, usually more specific completeness results are required. What is the "geometry" of frames for a given logic? Are Kripke or even finite frames enough to characterize it? Questions of this sort will be addressed in the next several sections.

1.3 Persistence

The structure of Kripke frames for many standard modal logics can be described by rather simple conditions on the accessibility relation which are expressed in the first order language with equality and a binary (accessibility) predicate R. (This observation was actually the starting point of investigations in *Correspondence Theory* studying the relation between modal and first (or higher) order languages; see Chapter 4 of this volume.) Moreover, in many cases it turns out that the universal frame $\mathfrak{F}_L(\omega)$ for such a logic L also satisfies the corresponding first order condition ϕ. Since ϕ says nothing about sets of possible values in $P_L(\omega)$, it follows immediately that the canonical (Kripke) frame $\boldsymbol{\kappa}\mathfrak{F}_L(\omega)$ also satisfies ϕ and so characterizes L. Thus we obtain a completeness theorem of the form:

$$\varphi \in L \text{ iff } \mathfrak{F} \models \varphi \text{ for every Kripke frame } \mathfrak{F} \text{ satisfying } \phi.$$

This method of establishing Kripke completeness, known as the *method of canonical models*, is based essentially upon two facts: first, that L is characterized by its universal frame $\mathfrak{F}_L(\omega)$ and second, that L is "persistent" under the transition from $\mathfrak{F}_L(\omega)$ to its underlying Kripke frame. Of course, instead of $\mathfrak{F}_L(\omega)$ we can take any other class of frames $\mathcal{C}$ with respect to which L is complete and try to show that L is *$\mathcal{C}$–persistent* in the sense that, for every $\mathfrak{F} = \langle W, R, P\rangle$ in $\mathcal{C}$, if $\mathfrak{F} \models L$ then $\boldsymbol{\kappa}\mathfrak{F} = \langle W, R\rangle$ validates L as well.

PROPOSITION 20. *If a logic is both $\mathcal{C}$–complete and $\mathcal{C}$–persistent, then it is complete with respect to the class $\{\boldsymbol{\kappa}\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$ of Kripke frames.*

It follows in particular that L is Kripke complete whenever it is $\mathcal{DF}$–, or $\mathcal{R}$–, or $\mathcal{D}$–persistent. Since every descriptive frame for L is a generated subframe of a suitable universal frame for L, L is $\mathcal{D}$–persistent iff it is persistent with respect to the class of its universal frames. It is an open problem, however, whether *canonicity*, i.e., $\mathfrak{F}_L(\omega)$–persistence, implies $\mathcal{D}$–persistence. Here are two simple examples.

THEOREM 21 (van Benthem 1983). *A logic is persistent with respect to the class of all general frames iff it is axiomatizable by a set of variable free formulas.*

It is easily checked that a Kripke frame validates $\mathbf{Alt}_n$ iff no point in it has more than n distinct successors (see [Segerberg 1971]).

THEOREM 22 (Bellissima 1988). *Every $L \in \mathrm{NExt}\mathbf{Alt}_n$ is $\mathcal{DF}$–persistent, for any $n < \omega$.*

Proof. The proof is based on the fact that, for any differentiated frame $\mathfrak{F} = \langle W, R, P\rangle$, any finite $X \subseteq W$, and any $y \in X$, there is $Y \in P$ such that $X \cap Y = \{y\}$. It follows that at most n distinct points are accessible from every point in a differentiated frame for L; in particular, $\mathbf{Alt}_n$ is $\mathcal{DF}$–persistent. Suppose now that a formula $\varphi \in L$ is refuted at a point x under a valuation $\mathfrak{V}$ in $\boldsymbol{\kappa}\mathfrak{F}$, $\mathfrak{F}$ a differentiated frame for L. Let X be the set of points accessible from x in $\leq md(\varphi)$ steps.[6] Since X is finite, there is a valuation $\mathfrak{U}$ in $\mathfrak{F}$ such that $\mathfrak{U}(p) \cap X = \mathfrak{V}(p)$, for every variable p. Consequently, φ is false in $\mathfrak{F}$ at x under $\mathfrak{U}$, which is a contradiction. ■

The proof of Fine's [1974c] Theorem that all logics of finite width, i.e., logics in $\mathrm{NExt}\mathbf{K4BW}_n$, for $n < \omega$, are Kripke complete (a sketch can be found in Section 18 of *Basic Modal Logic*) may also be regarded as a proof of persistence. Recall that a point x in a transitive frame $\mathfrak{F} = \langle W, R, P\rangle$ is called *non-eliminable* (relative to R) if there is $X \in P$ such that $x \in X$ but no proper successor of x is in X (in other words, x is *maximal* in X); in this case we write $x \in \max_R X$. Denote by W_r the set of all non-eliminable points in $\mathfrak{F}$ and put $\mathfrak{F}_r = \langle W_r, R_r, P_r\rangle$, where $R_r = R \restriction W_r$, $P_r = \{X \cap W_r : X \in P\}$. (Fine called the frame $\mathfrak{F}_r$ *reduced*.)

THEOREM 23 (Fine 1985). *Let $\mathfrak{F} = \langle W, R, P\rangle$ be a transitive descriptive frame and $x \in X \in P$. Then* (i) *there exists a point $y \in \max_R X \cap x\underline{\uparrow}$ and* (ii) *$\mathfrak{F}_r$ is a refined frame whose dual $\mathfrak{F}_r^+$ is isomorphic to $\mathfrak{F}^+$.*

Proof. (i) Suppose otherwise, i.e., there is no maximal point in $X \cap x\underline{\uparrow}$. Let Y be a maximal chain of points in $X \cap x\underline{\uparrow}$ (that it exists follows from Zorn's Lemma) and $\mathcal{X} = \{Z \in P : \exists y \in Y\ \underline{y}\uparrow \cap Y \subseteq Z\}$. Clearly, $\mathcal{X}$ is non-empty and has the finite intersection property (because $X \cap x\underline{\uparrow}$ has no maximal point). By compactness, we then have a point z in $\bigcap \mathcal{X}$ which, by tightness, is maximal in Y, contrary to $X \cap x\underline{\uparrow}$ having no maximal point. (ii) is a consequence of (i). ■

It follows that to establish the Kripke completeness of a logic $L \in \mathrm{NExt}\mathbf{K4}$ it is enough to show that it is persistent with respect to the class

$$\mathcal{RE} = \{\mathfrak{F}_r : \mathfrak{F} \text{ a finitely generated descriptive frame}\}.$$

That is what Fine [1974c] actually did for logics of finite width.

[6] Here $md(\varphi)$, the *modal degree* of φ, is the length of the longest chain of nested modal operators in φ.

THEOREM 24 (Fine 1974c). *All logics of finite width are $\mathcal{RE}$–persistent and so Kripke complete.*

Let us return, however, to the method of canonical models. Having tried it for a number of standard systems, Lemmon and Scott [1977] found a rather general sufficient condition for its applicability and put forward a conjecture concerning a further extension (which was proved by Goldblatt [1976b]). This direction of completeness (and correspondence) theory culminated in the theorem of Sahlqvist [1975] who proved an optimal (in a sense) generalization of the condition of [Lemmon and Scott 1977]. To formulate it we require the following definition. Say that a formula is *positive* (*negative*) if it is constructed from variables (negated variables) and the constants $\top$, $\bot$ using $\wedge$, $\vee$, $\Diamond$ and $\Box$.

THEOREM 25 (Sahlqvist 1975). *Suppose φ is a formula which is equivalent in $\mathbf{K}$ to a formula of the form $\Box^k(\psi \to \chi)$, where $k \geq 0$, χ is positive and ψ is constructed from variables and their negations, $\bot$ and $\top$ with the help of $\wedge$, $\vee$, $\Box$ and $\Diamond$ in such a way that no ψ's subformula of the form $\psi_1 \vee \psi_2$ or $\Diamond\psi_1$, containing an occurrence of a variable without $\neg$, is in the scope of some $\Box$. Then one can effectively construct a first order formula $\phi(x)$ in R and $=$ having x as its only free variable and such that, for every descriptive or Kripke frame $\mathfrak{F}$ and every point a in $\mathfrak{F}$,*

$$(\mathfrak{F}, a) \models \varphi \textit{ iff } \mathfrak{F} \models \phi(x)[a].$$

(Here $(\mathfrak{F}, a) \models \varphi$ means that φ is true at a in $\mathfrak{F}$ under any valuation.)

Proof. We present a sketch of the proof found by Sambin and Vaccaro [1989]. Given a formula $\varphi(p_1, \dots, p_n)$, a frame $\mathfrak{F} = \langle W, R, P\rangle$ and sets $X_1, \dots, X_n \in P$, denote by $\varphi(X_1, \dots, X_n)$ the set of points in $\mathfrak{F}$ at which φ is true under the valuation $\mathfrak{V}$ defined by $\mathfrak{V}(p_i) = X_i$, i.e., $\varphi(X_1, \dots, X_n) = \mathfrak{V}(\varphi)$. Using this notation, we can say that

$$(\mathfrak{F}, x) \models \varphi(p_1, \dots, p_n) \text{ iff } \forall X_1, \dots, X_n \in P\ x \in \varphi(X_1, \dots, X_n).$$

EXAMPLE 26. Let us consider the formula $\Box p \to p$ and try to extract a first order equivalent for it in the class of tight frames directly from the equivalence above and the condition of tightness. For every tight frame $\mathfrak{F} = \langle W, R, P\rangle$ we have:

$$\begin{array}{rll} (\mathfrak{F}, x) \models \Box p \to p & \text{iff} & \forall X \in P\ x \in (\Box X \to X) \\ & \text{iff} & \forall X \in P\ (x \in \Box X \to x \in X) \\ & \text{iff} & \forall X \in P\ (x{\uparrow} \subseteq X \to x \in X). \end{array}$$

To eliminate the variable X ranging over P, we can use two simple observations. The first one is purely set-theoretic:

(3) $\forall X \in P\ (Y \subseteq X \to x \in X)$ iff $x \in \bigcap\{X \in P : Y \subseteq X\}$.

And the second one is just a reformulation of the characteristic property of tight frames:

(4) $\bigcap\{X \in P : x{\uparrow} \subseteq X\} = x{\uparrow}$.

With the help of (3) and (4) we can continue the chain of equivalences above with two more lines:

$$
\begin{array}{lll}
(\mathfrak{F}, x) \models \Box p \to p & \text{iff} & \dots \\
& \text{iff} & x \in \bigcap\{X \in P : x{\uparrow} \subseteq X\} \\
& \text{iff} & x \in x{\uparrow}.
\end{array}
$$

Thus, $\mathfrak{F} \models \Box p \to p$ iff $\forall x\ x \in x{\uparrow}$ iff $\forall x\ xRx$.

The proof of Sahlqvist's Theorem is a (by no means trivial) generalization of this argument. Define by induction $x{\uparrow}^0 = \{x\}$, $x{\uparrow}^{n+1} = (x{\uparrow}^n){\uparrow}$, and notice that in (4) we can replace $x{\uparrow}$ by any term of the form $x_1{\uparrow}^{n_1} \cup \dots \cup x_k{\uparrow}^{n_k}$, thus obtaining the equality

(5) $\bigcap\{X \in P :\ x_1{\uparrow}^{n_1} \cup \dots \cup x_k{\uparrow}^{n_k} \subseteq X\} = x_1{\uparrow}^{n_1} \cup \dots \cup x_k{\uparrow}^{n_k}$

which holds for every descriptive frame $\mathfrak{F} = \langle W, R, P\rangle$, all $x_1, \dots, x_k \in W$ and all $n_1, \dots, n_k \geq 0$.

A frame-theoretic term $x_1{\uparrow}^{n_1} \cup \dots \cup x_k{\uparrow}^{n_k}$ with (not necessarily distinct) world variables $x_1, \dots, x_k$ will be called an *R-term*. It is not hard to see that for any R-term T, the relation $x \in T$ on $\mathfrak{F} = \langle W, R, P\rangle$ is first order expressible in R and $=$. Consequently, we obtain

LEMMA 27. *Suppose $\varphi(p_1, \dots, p_n)$ is a modal formula and $T_1, \dots, T_n$ are R-terms. Then the relation $x \in \varphi(T_1, \dots, T_n)$ is expressible by a first order formula (in R and $=$) having x as its only free variable.*

Syntactically, R-terms with a single world variable correspond to modal formulas of the form $\Box^{m_1} p_1 \wedge \dots \wedge \Box^{m_k} p_k$ with not necessarily distinct propositional variables $p_1, \dots, p_k$. Such formulas are called *strongly positive.* By induction on the construction of φ, one can prove the following

LEMMA 28. *Suppose $\varphi(p_1, \dots, p_n)$ is a strongly positive formula containing all the variables $p_1, \dots, p_n$ and $\mathfrak{F} = \langle W, R, P\rangle$ is a frame. Then one can effectively construct R-terms $T_1, \dots, T_n$ (with one variable x) such that for any $x \in W$ and any $X_1, \dots, X_n \in P$,*

$$x \in \varphi(X_1, \dots, X_n) \text{ iff } T_1 \subseteq X_1 \wedge \dots \wedge T_n \subseteq X_n.$$

Now, trying to extend the method of Example 26 to a wider class of formulas, we see that it still works if we replace the antecedent $\Box p$ in $\Box p \to p$

with an arbitrary strongly positive formula ψ. As to generalizations of the consequent, let us take first an arbitrary formula χ instead of p and see what properties it should satisfy to be handled by our method.

Thus, for a modal formula $(\psi \to \chi)(p_1, \dots, p_n)$ with strongly positive ψ and a descriptive frame $\mathfrak{F} = \langle W, R, P \rangle$, we have:

$$\begin{aligned}
(\mathfrak{F}, x) \models \psi \to \chi \text{ iff } & \forall X_1, \dots, X_n \in P \; (x \in \psi(X_1, \dots, X_n) \to \\
& \quad x \in \chi(X_1, \dots, X_n)) \\
\text{iff } & \forall X_1, \dots, X_n \in P \; (T_1 \subseteq X_1 \wedge \dots \wedge T_n \subseteq X_n \to \\
& \quad x \in \chi(X_1, \dots, X_n)) \\
\text{iff } & \forall X_1, \dots, X_{n-1} \in P \; (T_1 \subseteq X_1 \wedge \dots \wedge T_{n-1} \subseteq X_{n-1} \to \\
& \quad \forall X_n \in P \; (T_n \subseteq X_n \to x \in \chi(X_1, \dots, X_n))).
\end{aligned}$$

(3) does not help us here, but we can readily generalize it to

$$\begin{aligned}
& \forall X \in P \; (Y \subseteq X \to x \in \chi(\dots, X, \dots)) \text{ iff} \\
& \qquad x \in \bigcap \{\chi(\dots, X, \dots) : \; Y \subseteq X \in P\}. \qquad (6)
\end{aligned}$$

So

$$\begin{aligned}
(\mathfrak{F}, x) \models \psi \to \chi \text{ iff } & \forall X_1, \dots, X_{n-1} \in P \; (T_1 \subseteq X_1 \wedge \dots \wedge T_{n-1} \subseteq X_{n-1} \to \\
& \quad x \in \bigcap \{\chi(X_1, \dots, X_n) : \; T_n \subseteq X_n \in P\}).
\end{aligned}$$

But now (4) and (5) are useless. In fact, what we need is the equality

$$\begin{aligned}
& \bigcap \{\chi(\dots, X, \dots) : T \subseteq X \in P\} = \\
& \qquad \chi(\dots, \bigcap \{X \in P : T \subseteq X\}, \dots) \qquad (7)
\end{aligned}$$

which, with the help of (5), would give us

$$\bigcap \{\chi(\dots, X, \dots) : T \subseteq X \in P\} = \chi(\dots, T, \dots). \qquad (8)$$

Of course, (7) is too good to hold for an arbitrary χ, but suppose for a moment that our χ satisfies it. Then we can eliminate step by step all the variables $X_1, \dots, X_n$ like this:

$$\begin{aligned}
(\mathfrak{F}, x) \models \psi \to \chi \text{ iff } & \forall X_1, \dots, X_{n-1} \in P \; (T_1 \subseteq X_1 \wedge \dots \wedge T_{n-1} \subseteq X_{n-1} \to \\
& \quad x \in \chi(X_1, \dots, X_{n-1}, T_n)) \\
\text{iff } & \dots \text{ (by the same argument)} \\
\text{iff } & x \in \chi(T_1, \dots, T_n).
\end{aligned}$$

And the last relation can be effectively rewritten in the form of a first order formula $\phi(x)$ in R and $=$ having x as its only free variable. So, finally we shall have $\mathfrak{F} \models \psi \to \chi$ iff $\forall x \; \phi(x)$.

Now, to satisfy (7), χ should have the property that all its operators distribute over intersections. Clearly, $\to$ and $\neg$ are not suitable for this goal. But all the other operators turn out to be good enough at least in descriptive and Kripke frames. So we can take as χ any positive modal formula. The main property of a positive formula $\varphi(\dots,p,\dots)$ is its *monotonicity* in every variable p which means that, for all sets X, Y of worlds in a frame, $X \subseteq Y$ implies $\varphi(\dots,X,\dots) \subseteq \varphi(\dots,Y,\dots)$.

To prove that all positive formulas satisfy (7) in Kripke frames and descriptive frames, recall that $\Box$ distributes over arbitrary intersections in any frame. As to $\Diamond$, we have the following lemma in which a family $\mathcal{X}$ of non-empty subsets of some space W is called *downward directed* if for all $X, Y \in \mathcal{X}$ there is $Z \in \mathcal{X}$ such that $Z \subseteq X \cap Y$.

LEMMA 29 (Esakia 1974). *Suppose $\mathfrak{F} = \langle W, R, P\rangle$ is a descriptive frame. Then for every downward directed family $\mathcal{X} \subseteq P$,*

$$\Diamond \bigcap_{X\in\mathcal{X}} X = \bigcap_{X\in\mathcal{X}} \Diamond X.$$

Using Esakia's Lemma, by induction on the construction of φ one can prove

LEMMA 30. *Suppose that $\mathfrak{F} = \langle W, R, P\rangle$ is a Kripke or descriptive frame and $\varphi(p,\dots,q,\dots,r)$ is a positive formula. Then for every $Y \subseteq W$ and all $U,\dots,V \in P$,*

$$\bigcap\{\varphi(U,\dots,X,\dots,V) :\ Y \subseteq X \in P\} = \varphi(U,\dots,\bigcap\{X \in P :\ Y \subseteq X\},\dots,V). \tag{9}$$

It follows from this lemma and considerations above that Sahlqvist's Theorem holds for formulas $\varphi = \psi \to \chi$ with strongly positive ψ and positive χ. The remaining part of the proof is purely syntactic manipulations with modal and first order formulas.

Notice that using the monotonicity of positive formulas, equivalence (6) can be generalized to the following one: for every $\mathfrak{F} = \langle W, R, P\rangle$, every positive $\chi_i(\dots,p,\dots)$ and every $x_i \in W$,

$$\forall X \in P\ (Y \subseteq X \to \bigvee_{i\leq n} x_i \in \chi_i(\dots,X,\dots)) \text{ iff } \bigvee_{i\leq n} x_i \in \bigcap\{\chi_i(\dots,X,\dots) :\ Y \subseteq X \in P\}. \tag{10}$$

Say that a modal formula ψ is *untied* if it can be constructed from negative formulas and strongly positive ones using only $\wedge$ and $\Diamond$. If $\nu(p_1,\dots,p_n)$ is negative then $\neg\nu(p_1,\dots,p_n)$ is clearly equivalent in $\mathbf{K}$ to a positive formula; we denote it by $\nu^*(\neg p_1,\dots,\neg p_n)$.

LEMMA 31. *Let $\psi(p_1, \dots, p_n)$ be an untied formula and $\mathfrak{F} = \langle W, R, P\rangle$ a frame. Then for every $x \in W$ and all $X_1, \dots, X_n \in P$,*

$$x \in \psi(X_1, \dots, X_n) \text{ iff } \exists y_1, \dots, y_l (\vartheta \wedge \bigwedge_{i \le n} T_i \subseteq X_i \wedge \bigwedge_{j \le m} z_j \in \nu_j(X_1, \dots, X_n))$$

where the formula in the right-hand side, effectively constructed from ψ, has only one free individual variable x, ϑ is a conjunction of formulas of the form uRv, T_i are suitable R-terms and $\nu_j(p_1, \dots, p_n)$ are negative formulas.

We are ready now to prove Sahlqvist's Theorem. To construct a first order equivalent for $\Box^k(\psi \to \chi)$ supplied by the formulation of our theorem, we observe first that one can equivalently reduce ψ to a disjunction $\psi_1 \vee \dots \vee \psi_m$ of untied formulas, and hence $\Box^k(\psi \to \chi)$ is equivalent in $\mathbf{K}$ to the formula

$$\Box^k(\psi_1 \to \chi) \wedge \dots \wedge \Box^k(\psi_m \to \chi).$$

So all we need is to find a first order equivalent for an arbitrary formula $\Box^k(\psi \to \chi)$ with untied ψ and positive χ. Let $p_1, \dots p_n$ be all the variables in ψ and χ and $\mathfrak{F} = \langle W, R, P\rangle$ a descriptive or Kripke frame. Then, for any $x \in W$, we have:

$$\begin{aligned}
(\mathfrak{F}, x) \models \Box^k(\psi \to \chi) &\text{ iff } \forall X_1, \dots, X_n \in P\ x \in \Box^k(\psi \to \chi)(X_1, \dots, X_n)\\
\text{(by Lemma 31)} &\text{ iff } \forall X_1, \dots, X_n \in P\ \forall y\ (xR^k y \to (\exists y_1, \dots, y_l\ (\vartheta \wedge\\
&\quad \bigwedge_{i \le n} T_i \subseteq X_i \wedge \bigwedge_{j \le m} z_j \in \nu_j(X_1, \dots, X_n)) \to\\
&\quad y \in \chi(X_1, \dots, X_n)))\\
&\text{ iff } \forall X_1, \dots, X_n \in P\ \forall y, y_1, \dots, y_l\ (\vartheta' \wedge \bigwedge_{i \le n} T_i \subseteq X_i \wedge\\
&\quad \bigwedge_{j \le m} z_j \in \nu_j(X_1, \dots, X_n) \to y \in \chi(X_1, \dots, X_n))
\end{aligned}$$

where $\vartheta' = xR^k y \wedge \vartheta$. Let $\pi_j(p_1, \dots, p_n) = \nu_j^*(\neg p_1, \dots, \neg p_n)$. We continue this chain of equivalences as follows:

$$\begin{aligned}
&\text{iff}\quad \forall y, y_1, \dots, y_l\ (\vartheta' \to \forall X_1, \dots, X_n \in P\ (\bigwedge_{i \le n} T_i \subseteq X_i \to\\
&\quad \bigvee_{j \le m+1} z_j \in \pi_j(X_1, \dots, X_n)))
\end{aligned}$$

(where $\pi_{m+1}(p_1, \dots, p_n) = \chi(p_1, \dots, p_n)$ and $z_{m+1} = y$)

$$\text{iff}\quad \forall y, y_1, \dots, y_l\ (\vartheta' \to \bigvee_{j \le m+1} z_j \in \pi_j(T_1, \dots, T_n)),$$

as follows from (10), Lemma 30 and equality (5). It remains to use Lemma 27. ■

The formulas φ defined in the formulation of Theorem 25 are called *Sahlqvist formulas*. It follows from this theorem that if L is a $\mathcal{D}$–persistent logic and Γ a set of Sahlqvist formulas then $L \oplus \Gamma$ is also $\mathcal{D}$–persistent. Moreover, $L \oplus \Gamma$ is *elementary* (in the sense that the class of Kripke frames for it coincides with the class of all models for some set of first order formulas in R and $=$) whenever L is so.

Other proofs of Sahlqvist's Theorem were found by Kracht [1993] and Jónsson [1994] (the latter is based upon the algebraic technique developed in [Jónsson and Tarski 1951]). Venema [1991] extended Sahlqvist's Theorem to logics with non-standard inference rules, like Gabbay's [1981a] irreflexivity rule. In [Chagrov and Zakharyaschev 1995b] it is shown that there is a continuum of Sahlqvist logics above **S4** and that not all of them have the finite model property (above **T** such a logic was constructed by Hughes and Cresswell [1984]). As we shall see later in this chapter, there are even undecidable finitely axiomatizable Sahlqvist logics in NExt**K**. It would be of interest to find out whether such logics exist above **K4** or **S4**.

Kracht [1993] described syntactically the set of first order equivalents of Sahlqvist formulas. To formulate his criterion we require the fragment $\mathcal{S}$ of first order logic defined inductively as follows. Formulas of the form $xR^m y$ are in $\mathcal{S}$ for all variables x, y and every $m < \omega$; besides, if ϕ, ϕ' are in $\mathcal{S}$ then the formulas

$$\forall x \in y{\uparrow}^m\, \phi,\ \exists x \in y{\uparrow}^m\, \phi,\ \phi \wedge \phi',\ \text{and}\ \phi \vee \phi'$$

are also in $\mathcal{S}$. For simplicity we assume that all occurrences of quantifiers in a formula bind pairwise distinct variables. Call a variable y in a formula $\phi \in \mathcal{S}$ *inherently universal* if either all occurences of y are free in ϕ or ϕ contains a subformula $\forall y \in x{\uparrow}^m\, \phi'$ which is not in the scope of $\exists$.

THEOREM 32 (Kracht 1993). *For every first order formula $\phi(x)$ (in R and $=$) with one free variable x, the following conditions are equivalent:*

(i) *$\phi(x)$ is classically equivalent to a formula $\phi'(x) \in \mathcal{S}$ such that any subformula of the form $yR^m z$ of $\phi'(x)$ contains at least one inherently universal variable;*

(ii) *$\phi(x)$ corresponds to a Sahlqvist formula in the sense of Theorem 25.*

Condition (i) is satisfied, for example, by the formula

$$\forall u \in x{\uparrow}\, \forall v \in x{\uparrow}\, \exists z \in u{\uparrow}\, vRz$$

which corresponds to $\Diamond\Box p \to \Box\Diamond p$. On the other hand,

$$\phi(x) = \exists y \in x{\uparrow}\, \forall z \in y{\uparrow}\, zR^0 y$$

does not satisfy (i). In fact, even relative to **S4** the condition expressed by $\phi(x)$ does not correspond to any Sahlqvist formula. Notice, however, that

$\mathbf{S4} \oplus \Box\Diamond p \to \Diamond\Box p$ is a $\mathcal{D}$-persistent logic whose frames are precisely the transitive and reflexive frames validating $\forall x\phi(x)$.

We conclude this section by mentioning two more important results connecting persistence and elementarity (the idea of the proof was discussed in Section 22 of *Basic Modal Logic*.)

THEOREM 33. (i) (Fine 1975b, van Benthem 1980) *If a logic L is characterized by a first order definable class of Kripke frames then L is $\mathcal{D}$–persistent.*

(ii) (Fine 1975b) *If L is $\mathcal{R}$-persistent then the class of Kripke frames for L is first order definable.*

It is an open problem whether every $\mathcal{D}$–persistent logic is determined by a first order definable class of Kripke frames; for more information about this and related problems consult [Goldblatt 1995].

1.4 The degree of Kripke incompleteness

All known logics in NExt$\mathbf{K}$ of "natural origin" are complete with respect to Kripke semantics. On the other hand, there are many examples of "artificial" logics that cannot be characterized by any class of Kripke frames (see Sections 19, 20 of *Basic Modal Logic* or the examples below). To understand the phenomenon of Kripke incompleteness Fine [1974b] proposed to investigate how many logics may share the same Kripke frames with a given logic L. The number of them is called the *degree of Kripke incompleteness* of L. Of course, this number depends on the lattice of logics under consideration. The degree of Kripke incompleteness of logics in NExt$\mathbf{K}$ was comprehensively studied by Blok [1978]. In this section we present the main results of that paper following [Chagrov and Zakharyaschev 1997].

By Theorem 12, all Kripke complete union-splittings of NExt$\mathbf{K}$ have degree of incompleteness 1. And it turns out that no other union-splitting exists.

THEOREM 34 (Blok 1978). *Every union-splitting of* NExt$\mathbf{K}$ *has the finite model property.*

Proof. Let $\mathcal{F}$ be a class of finite rooted cycle free frames. We prove that $L = \mathbf{K}/\mathcal{F}$ has the finite model property using a variant of filtration, which is applied to an n-generated refined frame $\mathfrak{F} = \langle W, R, P\rangle$ for L refuting a formula $\varphi(p_1, \dots, p_n)$ under a valuation $\mathfrak{V}$.

Since $\mathfrak{F}$ is differentiated, for every $m \geq 1$ there are only finitely many points x in $\mathfrak{F}$ such that $x \models \Box^m\bot \wedge \neg\Box^{m-1}\bot$; we shall call them *points of type m*. Given $\Delta \subseteq \mathbf{Sub}\varphi$, $\mathbf{Sub}\varphi$ the set of all subformulas in φ, we put $m_\Delta = m$ if m is the minimal number such that a point in $\mathfrak{F}$ is of type $\leq m$ whenever $x \models \Delta$ and the formulas in $\mathbf{Sub}\varphi - \Delta$ are false at x (under $\mathfrak{V}$); if

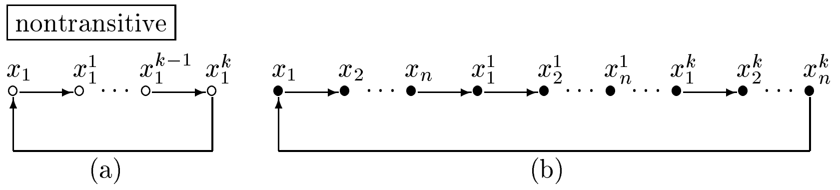


Figure 3.

no such m exists, we put $m_\Delta = 0$. Let

$$k = \max\{m_\Delta : \Delta \subseteq \mathbf{Sub}\varphi\}, \quad \Gamma = \mathbf{Sub}(\varphi \wedge \Box^k \bot).$$

Now we divide $\mathfrak{F}$ into two parts: W_1 consisting of points of type $\leq k$ and $W_2 = W - W_1$. For $x, y \in W$, put $x \sim y$ if either $x, y \in W_1$ and $x = y$ or $x, y \in W_2$ and exactly the same formulas in Γ are true at x and y. Let $\mathfrak{N} = \langle \mathfrak{G}, \mathfrak{U} \rangle$ be the smallest filtration (see Section 12 of *Basic Modal Logic*) of $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ through Γ with respect to $\sim$. Since W_1 is finite, $\mathfrak{G}$ is also finite and, by the Filtration Theorem, $(\mathfrak{M}, x) \models \psi$ iff $(\mathfrak{N}, [x]) \models \psi$, for every $\psi \in \Gamma$. So it remains to show that $\mathfrak{G} \models L$. Notice that $[x]$ in $\mathfrak{G}$ is of type $m \leq k$ iff x has type m in $\mathfrak{F}$. Moreover, there is no $[x]$ of type $l > k$. For otherwise $x \not\models \Box^k \bot$ and $m_\Delta = 0$ for $\Delta = \{\psi \in \mathbf{Sub}\varphi : x \models \psi\}$, which means that arbitrary long chains (of not necessarily distinct points) start from $[x]$, contrary to $[x]$ being of type l. Thus $\mathfrak{G}$ consists of two parts: points of type $\leq k$, which form the generated subframe $\langle W_1, R \restriction W_1 \rangle$ of $\mathfrak{F}$, and points involved in cycles. Since $\mathfrak{F} \models L$ and frames in $\mathcal{F}$ are cycle free, it follows from Lemma 13 and Theorem 17 that $\mathfrak{G} \models L$. ■

THEOREM 35 (Blok 1978). *If a logic L is inconsistent or a union-splitting of* NExt$\mathbf{K}$*, then L is strictly Kripke complete. Otherwise L has degree of Kripke incompleteness $2^{\aleph_0}$ in* NExt$\mathbf{K}$.

Proof. That **For** is strictly complete follows from Example 10 and Theorem 12. Suppose now that a consistent L is not a union-splitting and L' is the greatest union-splitting contained in L. Since L' has the finite model property, there is a finite rooted frame $\mathfrak{F} = \langle W, R \rangle$ for L' refuting some $\varphi \in L$ and such that every proper generated subframe of $\mathfrak{F}$ validates L. Clearly, $\mathfrak{F}$ is not cycle free. Let $x_1 R x_2 R \dots R x_n R x_1$ be the shortest cycle in $\mathfrak{F}$ and $k = md(\varphi) + 1$. We construct a new frame $\mathfrak{F}'$ by extending the cycle $x_1, \dots, x_n, x_1$ as is shown in Fig. 3 ((a) for $n = 1$ and (b) for $n > 1$). More precisely, we add to $\mathfrak{F}$ copies $x_i^1, \dots, x_i^k$ of x_i for each $i \in \{1, \dots, n\}$, organize them into the nontransitive cycle shown in Fig. 3 and draw an arrow from x_i^j to $y \in W - \{x_1, \dots, x_n\}$ iff $x_i R y$. Denote the resulting frame by $\mathfrak{F}' = \langle W', R' \rangle$ and let $x' = x_n^k$. By the construction, $\mathfrak{F}$ is a reduct of $\mathfrak{F}'$.

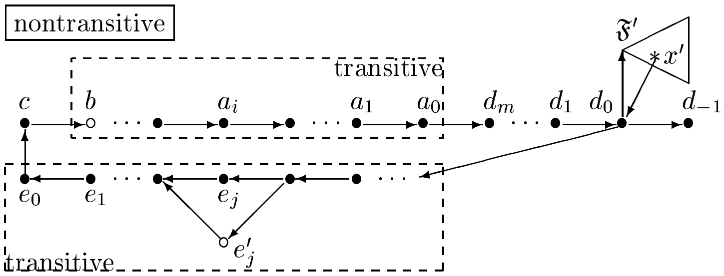


Figure 4.

Therefore, for every models $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and $\mathfrak{M}' = \langle \mathfrak{F}', \mathfrak{V}' \rangle$ such that

$$\mathfrak{V}'(p) = \mathfrak{V}(p) \cup \{x_i^j : x_i \in \mathfrak{V}(p),\ j < k\}$$

and for every $x \in W$, $\psi \in \mathbf{Sub}\varphi$, we have $(\mathfrak{M}, x) \models \psi$ iff $(\mathfrak{M}', x) \models \psi$. So we can hook some other model on x', and points in W will not feel its presence by means of φ's subformulas. The frame to be hooked on x' depends on whether $\bullet \models L$ or $\circ \models L$. We consider only the former alternative.

Fix some $m > |W'|$. For each $I \subseteq \omega - \{0\}$, let $\mathfrak{F}_I = \langle W_I, R_I, P_I \rangle$ be the frame whose diagram is shown in Fig. 4 (d_0 sees the root of $\mathfrak{F}'$, all points e_i and e'_j and is seen from x'; the subframes in dashed boxes are transitive, $e'_i \in W_I$ iff $i \in I$, and P_I consists of sets of the form $X \cup Y$ such that X is a finite or cofinite subset of $W_I - \{b, a_i : i < \omega\}$ and Y is either a finite subset of $\{a_i : i < \omega\}$ or is of the form $\{b\} \cup Y'$, where Y' is a cofinite subset of $\{a_i : i < \omega\}$. It is not hard to see that the points a_i, c, e_i and e'_i are characterized by the variable free formulas

$$\alpha_0 = \Diamond(\delta_m \wedge \Diamond(\delta_{m-1} \wedge \cdots \wedge \Diamond\delta_0)\dots) \wedge \neg\Diamond^2(\delta_m \wedge \Diamond(\delta_{m-1} \wedge \cdots \wedge \Diamond\delta_0)\dots),$$

$$\alpha_{i+1} = \Diamond\alpha_i \wedge \neg\Diamond^2\alpha_i,\ \gamma = \Diamond^2\alpha_0 \wedge \neg\Diamond\alpha_0,$$

$$\epsilon_0 = \Diamond\gamma,\ \epsilon_{i+1} = \Diamond\epsilon_i \wedge \neg\Diamond^2\epsilon_i,\ \epsilon'_{i+1} = \Diamond\epsilon_i \wedge \neg\Diamond^+\epsilon_{i+1},$$

(in the sense that $x \models \alpha_i$ iff $x = a_i$, etc.), where

$$\delta_0 = \Diamond\Box\bot,\ \delta_1 = \Diamond\delta_0 \wedge \neg\delta_0,\ \delta_2 = \Diamond\delta_1 \wedge \neg\delta_1 \wedge \neg\Diamond^+\delta_0,$$

$$\delta_{k+1} = \Diamond\delta_k \wedge \neg\delta_k \wedge \neg\Diamond^+\delta_{k-1} \wedge \cdots \wedge \neg\Diamond^+\delta_0.$$

Define L_I to be the logic determined by the class of frames for L and $\mathfrak{F}_I$, i.e., $L_I = L \cap \mathrm{Log}\mathfrak{F}_I$. Since $\neg(\epsilon'_i \wedge \Diamond^{m+6}\neg\varphi) \in L_J - L_I$ for $i \in I - J$ (φ is refuted at the root of $\mathfrak{F}'$), $|\{L_I : I \subseteq \omega - \{0\}\}| = 2^{\aleph_0}$.

Let us show now that L_I has the same Kripke frames as L. Since $L_I \subseteq L$, we must prove that every Kripke frame for L_I validates L. Suppose there is a rooted Kripke frame $\mathfrak{G}$ such that $\mathfrak{G} \models L_I$ but $\mathfrak{G} \not\models \psi$, for some $\psi \in L$. Since ψ is in L, it is valid in all frames for L, in particular, $\bullet \models \psi$. And since $\psi \notin L_I$, ψ is refuted in $\mathfrak{F}_I$. Moreover, by the construction of $\mathfrak{F}_I$, it is refuted at a point from which the root of $\mathfrak{F}'$ can be reached by a finite number of steps. Therefore, the following formulas are valid in $\mathfrak{F}_I$ and so belong to L_I and are valid in $\mathfrak{G}$:

$$(11)\quad \neg\psi \to \bigvee_{i=0}^{l} \Diamond^i \gamma,$$

$$(12)\quad \neg\psi \to \bigwedge_{i=0}^{l} \Box^i(\gamma \to \Box(\Box_0(\Box_0 p \to p) \to p)),$$

where p does not occur in ψ and l is a sufficiently big number so that any point in $\mathfrak{F}_I$ is accessible by $\leq l$ steps from every point in the selected cycle and every point at which ψ may be false, and $\Box_0\chi = \Box(\Diamond\alpha_0 \to \chi)$. According to (11), $\mathfrak{G}$ contains a point at which γ is true. By the construction of γ, this point has a successor y at which, by (12), $\Box_0(\Box_0 p \to p) \to p$ is true *under any valuation* in $\mathfrak{G}$ and $y \models \Diamond\alpha_0$. Define a valuation $\mathfrak{U}$ in $\mathfrak{G}$ by taking $\mathfrak{U}(p) = y\!\uparrow$. Then $y \models \Box_0(\Box_0 p \to p)$, from which $y \models p$ and so $y \in y\!\uparrow$. Now define another valuation $\mathfrak{U}'$ so that $\mathfrak{U}'(p) = y\!\uparrow - \{y\}$. Since y is reflexive, we again have $y \models \Box_0(\Box_0 p \to p)$, whence $y \models p$, which is a contradiction. ■

This construction can be used to obtain one more important result.

THEOREM 36 (Blok 1978). *Every union-splitting* $\mathbf{K}/\mathcal{F}$ *has* $\varkappa \leq \aleph_0$ *immediate predecessors in* NExt$\mathbf{K}$*, where* $\varkappa$ *is the number of frames in* $\mathcal{F}$ *which are not reducts of generated subframes of other frames in* $\mathcal{F}$*. Every consistent logic different from union-splittings has* $2^{\aleph_0}$ *immediate predecessors in* NExt$\mathbf{K}$*. (*$\mathbf{For}$ *has 2 immediate predecessors in* NExt$\mathbf{K}$*.)*

Proof. The former claim follows from Theorem 12. To establish the latter, we continue the proof of Theorem 35. One can show that L is finitely axiomatizable over L_I (the proof is rather technical, and we omit it here). Then, by Zorn's Lemma, NExtL_I contains an immediate predecessor L_I' of L. Besides, $L_I \oplus L_J = L$ whenever $I \neq J$. Indeed,

$$L_I \oplus L_J = (L \cap \mathrm{Log}\mathfrak{F}_I) \oplus (L \cap \mathrm{Log}\mathfrak{F}_J) = L \cap (\mathrm{Log}\mathfrak{F}_I \oplus \mathrm{Log}\mathfrak{F}_J)$$

and if $i \in I - J$ then, for every $\chi \in L$ and a sufficiently big l,

$$\neg \bigvee_{k=0}^{l} \Diamond^k \epsilon_i' \to \chi \in \mathrm{Log}\mathfrak{F}_I, \quad \neg\epsilon_i' \in \mathrm{Log}\mathfrak{F}_J,$$

from which $\chi \in \mathrm{Log}\mathfrak{F}_I \oplus \mathrm{Log}\mathfrak{F}_J$ and so $L \subseteq \mathrm{Log}\mathfrak{F}_I \oplus \mathrm{Log}\mathfrak{F}_J$. It follows that $L_I' \neq L_J'$ whenever $I \neq J$. ■

It is worth noting that tabular logics, proper extensions of **D** and extensions of **K4** are not union-splittings in NExt**K**. Similar results hold for the lattices NExt**D** and NExt**T**, where every consistent logic has degree of incompleteness $2^{\aleph_0}$ (see [Blok 1978, 1980b]). It would be of interest to describe the behavior of this function in NExt**K4**, NExt**GL**, NExt**S4**, NExt**Grz** (where Theorem 34 does not hold and where every tabular logic has finitely many immediate predecessors) and other lattices of logics to be considered later in this chapter.

1.5 Stronger forms of Kripke completeness

In the two preceding sections we were considering the problem of characterizing *logics* $L \in$ NExt**K** by classes of Kripke frames. The same problem arises in connection with the two consequence relations $\vdash_L$ and $\vdash^*_L$ as well. Theorem 19 shows a way of introducing the corresponding concepts of completeness.

With each Kripke frame $\mathfrak{F}$ let us associate a consequence relation $\models_{\mathfrak{F}}$ by putting, for any formula φ and any set Γ of formulas, $\Gamma \models_{\mathfrak{F}} \varphi$ iff $(\mathfrak{M}, x) \models \Gamma$ implies $(\mathfrak{M}, x) \models \varphi$ for every model $\mathfrak{M}$ based on $\mathfrak{F}$ and every point x in $\mathfrak{F}$. Clearly, a modal logic L is Kripke complete iff, for any *finite* set of formulas Γ and any formula φ, $\Gamma \not\vdash_L \varphi$ only if there is a Kripke frame $\mathfrak{F}$ for L such that $\Gamma \not\models_{\mathfrak{F}} \varphi$. Now, let us call L *strongly Kripke complete*[7] if this implication holds for arbitrary sets Γ. In other words, L is strongly complete if every L-consistent set of formulas holds at some point in a model based on a Kripke frame for L. Another reformulation: L is strongly complete iff L is Kripke complete and the relation $\bigcap\{\models_{\mathfrak{F}} : \mathfrak{F}$ is a Kripke frame for $L\}$ is finitary. It follows from the construction of the canonical models that every canonical (in particular, $\mathcal{D}$–persistent) logic is strongly complete, which provides us with many examples of such logics in NExt**K**.

By Theorem 33, all logics characterized by first order definable classes of Kripke frames are strongly complete. The converse does not hold: there exist strongly complete logics which are not canonical. The simplest is the bimodal logic of the frame $\langle \mathbb{R}, <, > \rangle$; see Example 144 below. By applying the Thomason simulation (to be introduced in Section 2.3) to this logic we obtain a logic in NExt**K** with the same properties; see Theorem 123. Moreover, in contrast to $\mathcal{D}$–persistence, strong Kripke completeness is not preserved under finite sums of logics (see [Wolter 1996b]). It is an open problem, however, whether such logics exist in NExt**K4**.

Perhaps the simplest examples of Kripke complete logics which are not strongly complete are **GL** and **Grz** (use Theorem 58 and the fact that these logics are not elementary; see *Correspondence Theory*). It is much more

[7]Fine [1974c] calls such logics *compact*, which does not agree with the use of this term by Thomason [1972].

difficult to prove that the McKinsey logic $\mathbf{K} \oplus \Box\Diamond p \to \Diamond\Box p$ is not strongly complete; the proof can be found in [Wang 1992]. For other examples of modal logics that are not strongly complete see Section 3.4. It is worth noting also that, as was shown in [Fine 1974c], every finite width logic in a *finite* language turns out to be strongly Kripke complete, though this is not the case for logics in an infinite language, witness

$$\mathbf{GL.3} = \mathbf{GL} \oplus \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p).$$

For the consequence relation $\vdash_L^*$, we should take the "global" version $\models_{\mathfrak{F}}^*$ of $\models_{\mathfrak{F}}$. Namely, we put $\Gamma \models_{\mathfrak{F}}^* \varphi$ if $\mathfrak{M} \models \Gamma$ implies $\mathfrak{M} \models \varphi$ for any model $\mathfrak{M}$ based on $\mathfrak{F}$. A modal logic L is called *globally Kripke complete* if for any finite set of formulas Γ and any formula φ, $\Gamma \nvdash_L^* \varphi$ only if there is a frame $\mathfrak{F}$ for L such that $\Gamma \not\models_{\mathfrak{F}}^* \varphi$. L is *strongly globally complete* if this holds for arbitrary (not only finite) Γ. We also say that L has the *global finite model property* if for every finite Γ and every φ, $\Gamma \nvdash_L^* \varphi$ only if there is a finite frame $\mathfrak{F}$ for L such that $\Gamma \not\models_{\mathfrak{F}}^* \varphi$.

The global finite model property (FMP, for short) of many standard logics can be proved by filtration. Say that a logic L *strongly admits filtration* if for every generated submodel $\mathfrak{M}$ of the canonical model $\mathfrak{M}_L$ and every finite set of formulas Σ closed under subformulas, there is a filtration of $\mathfrak{M}$ through Σ based on a frame for L.

PROPOSITION 37 (Goranko and Passy 1992). *If L strongly admits filtration then L has global FMP.*

Proof. Suppose that $\Gamma \nvdash_L^* \varphi$, Γ finite. Then $\Box^{<\omega} \bigwedge \Gamma \nvdash_L \varphi$ and so the set $\Delta = \Box^{<\omega} \bigwedge \Gamma \cup \{\neg\varphi\}$ is L-consistent. It remains to filtrate through $\mathbf{Sub}\Gamma \cup \mathbf{Sub}\varphi$ the submodel of $\mathfrak{M}_L$ generated by a maximal L-consistent set containing Δ. ■

It follows in particular that $\mathbf{K}$, $\mathbf{T}$, $\mathbf{D}$, $\mathbf{KB}$ have global FMP.

PROPOSITION 38. *Suppose L is globally complete (has global FMP) and Γ is a finite set of variable free formulas. Then $L \oplus \Gamma$ is globally complete (has global FMP) as well.*

Proof. Let $L' = L \oplus \Gamma$ and $\Delta \nvdash_{L'}^* \varphi$, Δ finite. Then $\Gamma \cup \Delta \nvdash_L^* \varphi$ and so there exists a (finite) Kripke frame $\mathfrak{F}$ for L such that $\Gamma \cup \Delta \not\models_{\mathfrak{G}}^* \varphi$. Since Γ contains no variables, $\mathfrak{F} \models L'$. ■

For n-transitive logics L the global consequence relation $\vdash_L^*$ is reducible to the "local" $\vdash_L$ and so L is Kripke complete (has FMP, is strongly complete) iff L is globally complete (has global FMP, is strongly globally complete). In general the global properties are stronger than the "local" ones. Although L is globally complete (has global FMP) only if L is complete (has FMP), the converse does not hold (see [Wolter 1994a] and [Kracht 1999]).

EXAMPLE 39. Let $L = \mathbf{Alt}_3 \oplus p \to \Box\Diamond p \oplus (\Box p \wedge \neg p) \to \neg(\Diamond q \wedge \Diamond\neg q)$. A Kripke frame $\mathfrak{F}$ validates L iff no point in $\mathfrak{F}$ has more than three successors, $\mathfrak{F}$ is symmetric, and irreflexive points in it have at most one successor. By Proposition 22, L is Kripke complete. The class of Kripke frames for L is closed under (not necessarily generated) subframes. So, by Proposition 59 to be proved below, L has FMP. We show now that it does not have global FMP. To this end we require the formulas:

$$\alpha_1 = q_1 \wedge \neg q_2 \wedge \neg q_3,\ \alpha_2 = \neg q_1 \wedge q_2 \wedge \neg q_3,\ \alpha_3 = \neg q_1 \wedge \neg q_2 \wedge q_3,$$

$$\varphi = \Box p \wedge \neg p \wedge \alpha_1,\ \psi = \bigwedge\{\alpha_i \to \Diamond\alpha_{i+1} : i = 1, 2\} \wedge \alpha_3 \to \Diamond\alpha_1.$$

Let $\mathfrak{F} = \langle W, R\rangle$, where $W = \omega$ and

$$R = \{\langle m, m\rangle : m > 0\} \cup \{\langle m, m+1\rangle : m < \omega\} \cup \{\langle m, m-1\rangle : m > 0\}.$$

We then have $\psi \not\models^*_{\mathfrak{F}} \neg\varphi$. In fact, φ is true at 0 and ψ is true everywhere under the valuation $\mathfrak{V}$ defined by $\mathfrak{V}(p) = W - \{0\}$ and $\mathfrak{V}(q_i) = \{3n + i : n < \omega\}$. Clearly, $\mathfrak{F} \models L$ and so $\psi \not\vdash^*_L \neg\varphi$. Suppose now that $(\mathfrak{N}, x_0) \models \varphi$ and $\mathfrak{N} \models \psi$, for a model $\mathfrak{N}$ based on a Kripke frame $\mathfrak{G} = \langle V, S\rangle$ for L. Then we can find a sequence x_j, $j < \omega$, such that $x_j S x_{j+1}$ and $x_{3j+i} \models \alpha_{i+1}$, for $j < \omega$ and $i = 1, 2, 3$. The reader can verify that all points x_j are distinct.

Let us consider now the algebraic meaning of the notions introduced above. A logic L is Kripke complete iff the variety $\mathrm{Alg}L$ of modal algebras for L is generated by the class $\mathrm{Kr}L = \{\mathfrak{F}^+ : \mathfrak{F} \text{ is a Kripke frame for } L\}$. By Birkhoff's Theorem (see e.g. [Mal'cev 1973]), this means that

$$\mathrm{Alg}L = \mathsf{HSP}\mathrm{Kr}L,$$

(i.e., $\mathrm{Alg}L$ is obtained by taking the closure of $\mathrm{Kr}L$ under direct products, then the closure of the result under (isomorphic copies of) subalgebras and finally under homomorphic images). Clearly, L is globally complete iff precisely the same quasi-identities hold in $\mathrm{Kr}L$ and $\mathrm{Alg}L$. And since the quasi-variety generated by a class of algebras $\mathcal{C}$ is $\mathsf{SPP_U}\mathcal{C}$ (where $\mathsf{P_U}$ denotes the closure under ultraproducts; see [Mal'cev 1973]), L is globally complete iff

$$\mathrm{Alg}L = \mathsf{SPP_U}\mathrm{Kr}L.$$

Goldblatt [1989] calls the variety $\mathrm{Alg}L$ *complex* if $\mathrm{Alg}L = \mathsf{S}\mathrm{Kr}L$, or, equivalently, if $\mathrm{Alg}L = \mathsf{SP}\mathrm{Kr}L$ (this follows from the fact that the dual of the disjoint union of a family of Kripke frames $\{\mathfrak{F}_i : i \in I\}$ is isomorphic to the product $\prod_{i\in I}\mathfrak{F}_i^+$). We say a logic L is *$\varkappa$-complex*, $\varkappa$ a cardinal, if every modal algebra for L with $\leq \varkappa$ generators is a subalgebra of $\mathfrak{F}^+$ for some Kripke frame $\mathfrak{F} \models L$. As was shown in [Wolter 1993], this notion turns out to be the algebraic counterpart of both strong completeness and strong global completeness of logics in *infinite languages* with $\varkappa$ variables.

THEOREM 40. *For every normal modal logic L in an infinite language with $\varkappa$ variables the following conditions are equivalent:*
(i) *L is strongly Kripke complete;*
(ii) *L is globally strongly complete;*
(iii) *L is $\varkappa$-complex.*

Proof. (i) $\Rightarrow$ (iii) Suppose the cardinality of $\mathfrak{A} \in \mathrm{Alg}L$ does not exceed $\varkappa$. Denote by $\mathfrak{L}$ the algebra of modal formulas over $\varkappa$ propositional variables and take some homomorphism h from $\mathfrak{L}$ onto $\mathfrak{A}$. For each ultrafilter ∇ in $\mathfrak{A}$, the set $h^{-1}(\nabla)$ is maximal L-consistent. Since L is strongly complete, there is a model $\mathfrak{M}_\nabla = \langle \mathfrak{F}_\nabla, \mathfrak{V}_\nabla \rangle$ with root x_∇ based on a Kripke frame $\mathfrak{F}_\nabla$ for L and such that $(\mathfrak{M}_\nabla, x_\nabla) \models h^{-1}(\nabla)$. Without loss of generality we may assume that the frames $\mathfrak{F}_\nabla$ for distinct ∇ are disjoint. Let $\mathfrak{F}$ be the disjoint union of all of them. Define a homomorphism $\mathfrak{V}$ from $\mathfrak{L}$ into $\mathfrak{F}^+$ by taking

$$\mathfrak{V}(p) = \bigcup \{\mathfrak{V}_\nabla(p) : \nabla \text{ is an ultrafilter in } \mathfrak{A}\}.$$

Then $\mathfrak{V}(\mathfrak{L})$ is a subalgebra of $\mathfrak{F}^+ \in \mathrm{Alg}L$ isomorphic to $\mathfrak{A}$.

The implication (iii) $\Rightarrow$ (ii) is trivial. To prove (ii) $\Rightarrow$ (i), consider an L-consistent set of formulas Γ of cardinality $\leq \varkappa$ and put

$$\Delta = \{p\} \cup \{\Box^n(p \to \varphi) : n < \omega, \varphi \in \Gamma\},$$

where the variable p does not occur in formulas from Γ. It is easily checked that all finite subsets of Δ are L-consistent, so Δ is L-consistent too. It follows that $\{p \to \varphi : \varphi \in \Gamma\} \not\vdash_L^* \neg p$. And since L is globally strongly complete, there exists a model $\mathfrak{M}$ based on a Kripke frame for L such that $\mathfrak{M} \models \{p \to \varphi : \varphi \in \Gamma\}$ and $(\mathfrak{M}, x) \models p$, for some x. But then $(\mathfrak{M}, x) \models \Gamma$. ■

1.6 Canonical formulas

The main problem of completeness theory in modal logic is not only to find a sufficiently simple class of frames with respect to which a given logic L is complete but also to characterize the constitution of frames for L (in this class). The first order approach to the characterization problem, discussed in Section 1.3 in connection with Sahlqvist's Theorem, comes across two obstacles. First, there are formulas whose Kripke frames cannot be described in the first order language with R and $=$. The best known example is probably the *Löb axiom*

$$\boldsymbol{la} = \Box(\Box p \to p) \to \Box p.$$

$\mathfrak{F} \models \boldsymbol{la}$ iff $\mathfrak{F}$ is transitive, irreflexive (i.e., a strict partial order) and *Noetherian* in the sense that it contains no infinite ascending chain of distinct points. And as is well known, the condition of Noetherianness is not a first order one. The second obstacle is that this approach deals only with logics that are Kripke complete; it does not take into account sets of possible values.

There is another, purely frame-theoretic method of characterizing the structure of frames. For instance, a frame $\mathfrak{G}$ validates $\mathbf{K}/\mathfrak{F}$ iff $\mathfrak{G}$ does not contain a generated subframe reducible to $\mathfrak{F}$. It was shown in [Zakharyaschev 1984, 1988, 1992] that in a similar manner one can describe *transitive* frames validating an arbitrary modal formula. It is not clear whether characterizations of this sort can be extended to the class of all frames (an important step in this direction would be a generalization to n-transitive frames). That is why all frames in this section are assumed to be transitive. First we illustrate this method by a simple example.

EXAMPLE 41. Suppose a frame $\mathfrak{F} = \langle W, R, P\rangle$ refutes $\boldsymbol{la}$ under some valuation. Then the set $V = \{x \in W : x \not\models \boldsymbol{la}\}$ is in P and $V \subseteq V\downarrow$. It follows from the former that $\mathfrak{G} = \langle V, R\restriction V, \{X \cap V : X \in P\}\rangle$ is a frame—we call it the *subframe of* $\mathfrak{F}$ *induced by* V. And the latter condition means that $\mathfrak{G}$ is reducible to the single reflexive point $\circ$ which is the simplest refutation frame for $\boldsymbol{la}$. Moreover, one can readily check that the converse also holds: if there is a subframe $\mathfrak{G}$ of $\mathfrak{F}$ reducible to $\circ$ then $\mathfrak{F} \not\models \boldsymbol{la}$.

This example motivates the following definitions. Given frames $\mathfrak{F} = \langle W, R, P\rangle$ and $\mathfrak{G} = \langle V, S, Q\rangle$, a partial (i.e., not completely defined, in general) map f from W onto V is called a *subreduction* of $\mathfrak{F}$ to $\mathfrak{G}$ if it satisfies the reduction conditions (R1)–(R3) for all x and y in the domain of f and all $X \in Q$. The domain of f will be denoted by $\mathrm{dom} f$. In other words, an f-subreduct of $\mathfrak{F}$ is a reduct of the subframe of $\mathfrak{F}$ induced by $\mathrm{dom} f$. A frame $\mathfrak{G} = \langle V, S, Q\rangle$ is a *subframe* of $\mathfrak{F} = \langle W, R, P\rangle$ if $V \subseteq W$ and the identity map on V is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$, i.e., if $S = R\restriction V$ and $Q \subseteq P$. Note that a generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ is not in general a subframe of $\mathfrak{F}$, since V may be not in P.

Thus, the result of Example 41 can be reformulated like this: $\mathfrak{F} \not\models \boldsymbol{la}$ iff $\mathfrak{F}$ is subreducible to $\circ$.

A subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$ is called *cofinal* if

$$\mathrm{dom} f\uparrow \subseteq \mathrm{dom} f\overline{\downarrow}.$$

This important notion can be motivated by the following observation: $\mathfrak{F}$ refutes $\Diamond\top$ iff $\mathfrak{F}$ is cofinally subreducible to $\bullet$ (a plain subreduction is not enough).

THEOREM 42. *Every refutation frame $\mathfrak{F} = \langle W, R, P\rangle$ for $\varphi(p_1, \ldots, p_n)$ is cofinally subreducible to a finite rooted refutation frame for φ containing at most $c_\varphi = 2^n \cdot (c_n(1) + \cdots + c_n(2^{|\mathbf{Sub}\varphi|}))$ points.*[8]

[8]The function $c_n(m)$ was defined in Section 1.2.

Proof. Suppose φ is refuted in $\mathfrak{F}$ under a valuation $\mathfrak{V}$. Without loss of generality we can assume $\mathfrak{F}$ to be generated by $\mathfrak{V}(p_1), \dots, \mathfrak{V}(p_n)$. Let $X_1, \dots, X_m$ be all distinct maximal 0-cyclic sets in $\mathfrak{F}$. Clearly, $m \leq c_n(1)$ but unlike Theorem 8, $\mathfrak{F}$ is not in general refined and so these sets are not necessarily clusters of depth 1. However, they can be easily reduced to such clusters. Define an equivalence relation $\sim$ on W by putting $x \sim y$ iff $x = y$ or $x, y \in X_i$, for some $i \in \{1, \dots, m\}$, and $x \sim_\Sigma y$ (as before $\Sigma = \{p_1, \dots, p_n\}$). Let $[x]$ be the equivalence class under $\sim$ generated by x and $[X] = \{[x] : x \in X\}$, for $X \in P$. By the definition of cyclic sets, xRy iff $[x] \subseteq [y]\!\downarrow$. So the map $x \mapsto [x]$ is a reduction of $\mathfrak{F}$ to the frame $\mathfrak{F}_1' = \langle W_1', R_1', P_1' \rangle$ which results from $\mathfrak{F}$ by "folding up" the 0-cyclic sets X_i into clusters of depth 1 and leaving the other points untouched: $W_1' = [W]$, $[x]R_1'[y]$ iff $[x] \subseteq [y]\!\downarrow$ and $P_1' = \{[X] : X \in P\}$. (Roughly, we refine that part of $\mathfrak{F}$ which gives points of depth 1.) Put $\mathfrak{V}_1'(p_i) = [\mathfrak{V}(p_i)]$. Then by the Reduction (or P-morphism) Theorem, we have $x \models \psi$ iff $[x] \models \psi$, for every $\psi \in \mathbf{Sub}\varphi$.

Let X be the set of all points in $\mathfrak{F}_1'$ of depth > 1 having $\mathbf{Sub}\varphi$-equivalent successors of depth 1. It is not hard to see that $X \in P_1'$. Denote by $\mathfrak{F}_1 = \langle W_1, R_1, P_1 \rangle$ the subframe of $\mathfrak{F}_1'$ induced by $W_1' - X$ and let $\mathfrak{V}_1$ be the restriction of $\mathfrak{V}_1'$ to $\mathfrak{F}_1$. By induction on the construction of $\psi \in \mathbf{Sub}\varphi$ one can readily show that ψ has the same truth-values at common points in $\mathfrak{F}_1'$ and $\mathfrak{F}_1$ (under $\mathfrak{V}_1'$ and $\mathfrak{V}_1$, respectively) and so $\mathfrak{F}_1 \not\models \varphi$. The partial map $x \mapsto [x]$, for $[x] \in W_1$, is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{F}_1$.

Then we take the maximal 1-cyclic sets in $\mathfrak{F}_1$, "fold" them up into clusters of depth 2 and remove those points of depth > 2 that have $\mathbf{Sub}\varphi$-equivalent successors of depth 2. The resulting frame $\mathfrak{F}_2$ will be a cofinal subreduct of $\mathfrak{F}_1$ and so of $\mathfrak{F}$ as well. After that we form clusters of depth 3, and so forth. In at most $2^{|\mathbf{Sub}\varphi|}$ steps of that sort we shall construct a cofinal subreduct of $\mathfrak{F}$ refuting φ and containing $\leq c_\varphi$ points. It remains to select in it a suitable rooted generated subframe. ∎

For the majority of standard modal axioms the converse also holds. However, not for all. The simplest counterexample is the density axiom $\boldsymbol{den} = \Box\Box p \to \Box p$. It is refuted by the chain $\mathfrak{H}$ of two irreflexive points but becomes valid if we insert between them a reflexive one. In fact, $\mathfrak{F} \not\models \boldsymbol{den}$ iff there is a subreduction f of $\mathfrak{F}$ to $\mathfrak{H}$ such that $f(x\!\uparrow) = \{a\}$, for no point x in $\mathrm{dom} f\!\uparrow - \mathrm{dom} f$, where a is the final point in $\mathfrak{H}$.

Loosely, every refutation frame for formulas like $\boldsymbol{la}$ can be constructed by adding new points to a frame $\mathfrak{G}$ that is reducible to some finite refutation frame of fixed size. For formulas like $\Diamond\top$ we have to take into account the cofinality condition and do not put new points "above" $\mathfrak{G}$. And formulas like $\boldsymbol{den}$ impose another restriction: some places inside $\mathfrak{G}$ may be "closed" for inserting new points. These "closed domains" can be singled out in the following way.

Suppose $\mathfrak{N} = \langle \mathfrak{H}, \mathfrak{U} \rangle$ is a model and $\mathfrak{a}$ an antichain in $\mathfrak{H}$. Say that $\mathfrak{a}$ is an *open domain* in $\mathfrak{N}$ relative to a formula φ if there is a pair $t_{\mathfrak{a}} = (\Gamma_{\mathfrak{a}}, \Delta_{\mathfrak{a}})$ such that $\Gamma_{\mathfrak{a}} \cup \Delta_{\mathfrak{a}} = \mathbf{Sub}\varphi$, $\bigwedge \Gamma_{\mathfrak{a}} \to \bigvee \Delta_{\mathfrak{a}} \notin \mathbf{K4}$ and

- $\Box\psi \in \Gamma_{\mathfrak{a}}$ implies $\psi \in \Gamma_{\mathfrak{a}}$,
- $\Box\psi \in \Gamma_{\mathfrak{a}}$ iff $a \models \Box^{+}\psi$ for all $a \in \mathfrak{a}$.

Otherwise $\mathfrak{a}$ is called a *closed domain* in $\mathfrak{N}$ relative to φ. A reflexive singleton $\mathfrak{a} = \{a\}$ is always open: just take

$$t_{\mathfrak{a}} = (\{\psi \in \mathbf{Sub}\varphi : a \models \psi\}, \{\psi \in \mathbf{Sub}\varphi : a \not\models \psi\}).$$

It is easy to see also that antichains consisting of points from the same clusters are open or closed simultaneously; we shall not distinguish between such antichains.

For a frame $\mathfrak{H}$ and a (possibly empty) set $\mathfrak{D}$ of antichains in $\mathfrak{H}$, we say a subreduction f of $\mathfrak{F}$ to $\mathfrak{H}$ satisfies the *closed domain condition* for $\mathfrak{D}$ if

(CDC) $\neg\exists x \in \mathrm{dom} f{\uparrow} - \mathrm{dom} f \; \exists \mathfrak{d} \in \mathfrak{D} \; f(x{\uparrow}) = \mathfrak{d}\underline{\uparrow}$.

Notice that the cofinal subreduction f of $\mathfrak{F}$ to the resulting finite rooted frame $\mathfrak{H}$ in the proof of Theorem 42 satisfies (CDC) for the set $\mathfrak{D}$ of closed domains in the corresponding model $\mathfrak{N}$ on $\mathfrak{H}$ refuting φ. Indeed, every $x \in \mathrm{dom} f{\uparrow} - \mathrm{dom} f$ has a $\mathbf{Sub}\varphi$-equivalent successor $y \in \mathrm{dom} f$, and so an antichain $\mathfrak{d}$ such that $f(x{\uparrow}) = \mathfrak{d}\underline{\uparrow}$ is open, since we can take

$$t_{\mathfrak{d}} = (\{\psi \in \mathbf{Sub}\varphi : y \models \psi\}, \{\psi \in \mathbf{Sub}\varphi : y \not\models \psi\}).$$

On the other hand, we have

PROPOSITION 43. *Suppose $\mathfrak{N} = \langle \mathfrak{H}, \mathfrak{U} \rangle$ is a finite countermodel for φ and $\mathfrak{D}$ the set of all closed domains in $\mathfrak{N}$ relative to φ. Then $\mathfrak{F} \not\models \varphi$ whenever there is a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{D}$. Moreover, if φ is negation free (i.e., contains no $\bot$, $\neg$, $\Diamond$) then a plain subreduction satisfying (CDC) for $\mathfrak{D}$ is enough.*

Proof. If f is cofinal and $\mathfrak{F} = \langle W, R, P \rangle$ then we can assume $\mathrm{dom} f\underline{\uparrow} = W$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ as follows. If $x \in \mathrm{dom} f$ then we take $x \models p$ iff $f(x) \models p$, for every variable p in φ. If $x \notin \mathrm{dom} f$ then $f(x{\uparrow}) \neq \emptyset$, since f is cofinal. Let $\mathfrak{a}$ be an antichain in $\mathfrak{H}$ such that $\mathfrak{a}\underline{\uparrow} = f(x{\uparrow})$. By (CDC), $\mathfrak{a}$ is an open domain in $\mathfrak{N}$, and we put $y \models p$ iff $p \in \Gamma_{\mathfrak{a}}$, for every $y \notin \mathrm{dom} f$ such that $f(y{\uparrow}) = f(x{\uparrow})$. One can show that $\mathfrak{V}$ is really a valuation in $\mathfrak{F}$ and, for every $\psi \in \mathbf{Sub}\varphi$, $x \models \psi$ iff $f(x) \models \psi$ in the case $x \in \mathrm{dom} f$, and $x \models \psi$ iff $\psi \in \Gamma_{\mathfrak{a}}$, where $\mathfrak{a}$ is the open domain in $\mathfrak{N}$ associated with x, in the case $x \notin \mathrm{dom} f$.

If φ is negation free and f is a plain subreduction then $f(x{\uparrow})$ may be empty. In such a case we just put $x \models p$, for all variables p. ■

Now let us summarize what we have got. Given an arbitrary formula φ, we can effectively construct a finite collection of finite rooted frames $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ (underlying all possible rooted countermodels for φ with $\leq c_\varphi$ points) and select in them sets $\mathfrak{D}_1, \ldots, \mathfrak{D}_n$ of antichains (open domains in those countermodels) such that, for any frame $\mathfrak{F}$, $\mathfrak{F} \not\models \varphi$ iff there is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{F}_i$, for some i, satisfying (CDC) for $\mathfrak{D}_i$. If φ is negation free then a plain subreduction satisfying (CDC) is enough.

This general characterization of the constitution of refutation transitive frames can be presented in a more convenient form if with every finite rooted frame $\mathfrak{F} = \langle W, R\rangle$ and a set $\mathfrak{D}$ of antichains in $\mathfrak{F}$ we associate formulas $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha(\mathfrak{F}, \mathfrak{D})$ such that $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ ($\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D})$) iff there is a cofinal (respectively, plain) subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. For instance, one can take

$$\alpha(\mathfrak{F}, \mathfrak{D}, \bot) = \bigwedge_{a_i R a_j} \varphi_{ij} \wedge \bigwedge_{i=0}^{n} \varphi_i \wedge \bigwedge_{\mathfrak{d} \in \mathfrak{D}} \varphi_{\mathfrak{d}} \wedge \varphi_\bot \to p_0$$

where $a_0, \ldots, a_n$ are all points in $\mathfrak{F}$ and a_0 is its root,

$$\begin{aligned}
\varphi_{ij} &= \Box^+(\Box p_j \to p_i), \\
\varphi_i &= \Box^+\Big(\Big(\bigwedge_{\neg a_i R a_k} \Box p_k \wedge \bigwedge_{j=0, j\neq i}^{n} p_j \to p_i\Big) \to p_i, \\
\varphi_{\mathfrak{d}} &= \Box^+\Big(\bigwedge_{a_i \in W - \mathfrak{d}\underline{\uparrow}} \Box p_j \wedge \bigwedge_{i=0}^{n} p_i \to \bigvee_{a_j \in \mathfrak{d}} \Box p_j\Big), \\
\varphi_\bot &= \Box^+\Big(\bigwedge_{i=0}^{n} \Box^+ p_i \to \bot\Big).
\end{aligned}$$

$\alpha(\mathfrak{F}, \mathfrak{D})$ results from $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ by deleting the conjunct $\varphi_\bot$. $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha(\mathfrak{F}, \mathfrak{D})$ are called the *canonical* and *negation free canonical formulas* for $\mathfrak{F}$ and $\mathfrak{D}$, respectively. It is not hard to check that if $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is refuted in $\mathfrak{G} = \langle V, S, Q\rangle$ under some valuation then the partial map defined by $x \mapsto a_i$ if the premise of $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is true at x and p_i false is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$; and conversely, if f is such a subreduction then the valuation $\mathfrak{U}$ defined by $\mathfrak{U}(p_i) = V - f^{-1}(a_i)$ refutes $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ at any point in $f^{-1}(a_0)$.

THEOREM 44. *There is an algorithm which, given a formula φ, returns canonical formulas $\alpha(\mathfrak{F}_1, \mathfrak{D}_1, \bot), \ldots, \alpha(\mathfrak{F}_n, \mathfrak{D}_n, \bot)$ such that*

$$\mathbf{K4} \oplus \varphi = \mathbf{K4} \oplus \alpha(\mathfrak{F}_1, \mathfrak{D}_1, \bot) \oplus \cdots \oplus \alpha(\mathfrak{F}_n, \mathfrak{D}_n, \bot).$$

So the set of canonical formulas is complete for the class NExt$\mathbf{K4}$. *If φ is negation free then one can use negation free canonical formulas.*

It is not hard to see that $\mathbf{K4} \oplus \varphi$ is a splitting of NExt$\mathbf{K4}$ iff φ is deductively equivalent in NExt$\mathbf{K4}$ to a formula of the form $\alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$, where $\mathfrak{D}^\sharp$ is the set of all antichains in $\mathfrak{F}$ (in this case $\mathbf{K4}/\mathfrak{F} = \mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$). Such formulas are known as *Jankov formulas* (Jankov [1963] introduced them for intuitionistic logic), or *frame formulas* (cf. [Fine 1974a]), or *Jankov–Fine formulas*. Since $\mathbf{GL}$ is not a union-splitting of NExt$\mathbf{K4}$, this class of logics has no axiomatic basis.

We conclude this section by showing in Table 2 canonical axiomatizations of some standard modal logics in the field of $\mathbf{K4}$. For brevity we write $\alpha(\mathfrak{F}, \bot)$ instead of $\alpha(\mathfrak{F}, \emptyset, \bot)$ and $\alpha^\sharp(\mathfrak{F}, \bot)$ instead of $\alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$. Each $*$ in the table is to be replaced by both $\circ$ and $\bullet$.

For more information about the canonical formulas the reader is referred to [Zakharyaschev 1992, 1997b].

1.7 Decidability via the finite model property

Although, for cardinality reason, there are "much more" undecidable logics than decidable ones, almost all "natural" propositional systems close to those we deal with in this chapter turn out to be decidable. Relevant and linear logics are probably the best known among very few exceptions (see [Urquhart 1984], [Lincoln *et al.* 1992]).

The majority of decidability results in modal logic was obtained by means of establishing the finite model property. FMP by itself does not ensure yet decidability (there is a continuum of logics with FMP); some additional conditions are required to be satisfied. For instance, to prove the decidability of $\mathbf{S4}$ McKinsey [1941] used two such conditions: that the logic under consideration is characterized by an effective class of finite frames (or algebras, matrices, models, etc.) and that there is an effective (exponential in the case of $\mathbf{S4}$) upper bound for the size of minimal refutation frames. Under these conditions, a formula belongs to the logic iff it is validated by (finite) frames in a finite family which can be effectively constructed. Another sufficient condition of decidability is provided by the following well known

THEOREM 45 (Harrop 1958). *Every finitely axiomatizable logic with FMP is decidable.*

Here we need not to know a priori anything about the structure of frames for a given logic. This information is replaced by checking the validity of its axioms in finite frames, and the restriction of the size of refutation frames is replaced by constructing all possible derivations: in a finite number of steps we either separate a tested formula from the logic or derive it. Note that unlike the previous case now we cannot estimate the time required to complete this algorithm.

The condition of finite axiomatizability in Harrop's Theorem cannot be weakened to that of recursive axiomatizability. For there is a logic of depth

Table 2. Canonical axioms of standard modal logics

D4	$=$	$\mathbf{K4} \oplus \alpha(\bullet, \bot)$
S4	$=$	$\mathbf{K4} \oplus \alpha(\bullet)$
GL	$=$	$\mathbf{K4} \oplus \alpha(\circ)$
Grz	$=$	$\mathbf{K4} \oplus \alpha(\bullet) \oplus \alpha(\boxed{\circ\ \circ})$
K4.1	$=$	$\mathbf{K4} \oplus \alpha(\bullet, \bot) \oplus \alpha(\boxed{\circ\ \circ}, \bot)$
Triv	$=$	$\mathbf{K4} \oplus \alpha(\bullet) \oplus \alpha(\boxed{\circ\ \circ}) \oplus \alpha(\begin{array}{c}\circ\\ \uparrow\\ \circ\end{array})$
Verum	$=$	$\mathbf{K4} \oplus \alpha(\circ) \oplus \alpha(\begin{array}{c}\bullet\\ \uparrow\\ \bullet\end{array})$
S5	$=$	$\mathbf{S4} \oplus \alpha(\begin{array}{c}\circ\\ \uparrow\\ \circ\end{array})$
K4B	$=$	$\mathbf{K4} \oplus \alpha(\begin{array}{c}*\\ \uparrow\\ *\end{array})$ (4 axioms)
A*	$=$	$\mathbf{GL} \oplus \alpha(\begin{array}{ccc}1\,\bullet & & \bullet\,2\\ & \nwarrow\ \nearrow & \\ & \bullet & \end{array}, \{\{1\},\{1,2\}\})$
K4.2	$=$	$\mathbf{K4} \oplus \alpha(\begin{array}{c}\bullet\\ \uparrow\\ \bullet\\ \uparrow\\ \bullet\end{array}, \bot) \oplus \alpha(\begin{array}{c}\bullet\\ \uparrow\\ \circ\end{array}, \bot) \oplus \alpha(\begin{array}{ccc}* & & *\\ & \nwarrow\ \nearrow & \\ & * & \end{array}, \bot)$ (8 axioms)
K4.3	$=$	$\mathbf{K4} \oplus \alpha(\begin{array}{ccc}* & & *\\ & \nwarrow\ \nearrow & \\ & * & \end{array})$ (6 axioms)
Dum	$=$	$\mathbf{S4} \oplus \alpha(\begin{array}{ccc}\boxed{\circ\ \circ} & & \begin{array}{c}\circ\\ \circ\end{array}\\ & \nwarrow\ \nearrow & \\ & \circ & \end{array}) \oplus \alpha(\begin{array}{c}\circ\\ \uparrow\\ \boxed{\circ\ \circ}\end{array})$
K4BW$_n$	$=$	$\mathbf{K4} \oplus \alpha(\begin{array}{c}\overbrace{* \ \cdots \ *}^{n+1}\\ \nwarrow\ \nearrow\\ *\end{array})$ $(2n+4$ axioms)
K4BD$_n$	$=$	$\mathbf{K4} \oplus \alpha(\begin{array}{l}*\ n\\ \vdots\\ *\ 1\\ \uparrow\\ *\ 0\end{array})$ $(2^{n+1}$ axioms)
K4$_{n,m}$	$=$	$\mathbf{K4} \oplus \alpha(\begin{array}{l}\bullet\ m\\ \vdots\\ \bullet\ 1\\ \uparrow\\ \bullet\ 0\end{array}, \mathfrak{D}^\sharp)$

3 in NExt**K4** (i.e., a logic in NExt$\mathbf{K4BD}_3$) with an infinite set of independent axioms; so the logic of depth 3 axiomatizable by some recursively enumerable but not recursive sequence of formulas in this set is undecidable and has FMP. On the other hand there are examples of undecidable logics characterized by decidable classes of finite frames (see e.g. [Chagrov and Zakharyaschev 1997]). Yet one can generalize Harrop's Theorem in the following way. A logic is decidable iff it is recursively enumerable and characterized by a recursive class of recursive algebras. However, this criterion is absolutely useless in its generality. In this connection we note two open problems posed by Kuznetsov [1979]. Is every finitely axiomatizable logic characterized by recursive algebras? Is every finitely axiomatizable logic, characterized by recursive algebras, decidable? (That *finite* axiomatizability is essential here is explained by the following fact: if a lattice of logics contains a logic with a continuum of immediate predecessors then there is no countable sequence of algebras such that every logic in the lattice is characterized by one of its subsequences. For details see [Chagrov and Zakharyaschev 1997].)

FMP of almost all standard systems was proved using various forms of filtration (consult Section 12 *Basic Modal Logic* and [Gabbay 1976]). However, the method of filtration is rather capricious; one needs a special craft to apply it in each particular case (for instance, to find a suitable "filter"). In this and two subsequent sections we discuss other methods of proving FMP which are applicable to families of logics and provide in fact sufficient conditions of FMP. (It is to be noted that the families of Kripke complete logics considered in Section 1.3 contain logics without FMP.) A pair of such conditions was already presented in *Basic Modal Logic*:

THEOREM 46 (Segerberg 1971). *Each logic in* NExt**K4** *characterized by a frame of finite depth (or, which is equivalent, containing* $\mathbf{K4BD}_n$*, for some* $n < \omega$*) has FMP.*

THEOREM 47 (Bull 1966b, Fine 1971). *Each logic in* NExt**S4.3** *has FMP and is finitely axiomatizable (and so decidable).*

The former result, covering a continuum of logics, follows immediately from the description of finitely generated refined frames for **K4** in Section 1.2 and the latter is a consequence of Theorem 52 and Example 54 below. It is worth noting also that since $\mathfrak{F}_L(n)$ is finite for every logic $L \in$ NExt**K4** of finite depth and every $n < \omega$, there are only finitely many pairwise non-equivalent in L formulas with n variables. Logics with this property are called *locally tabular* (or *locally finite*). Moreover, as was observed by Maksimova [1975a], the converse is also true: if $L \in$ NExt**K4** has frames of any depth $< \omega$ then the formulas in the sequence $\varphi_1 = p$, $\varphi_{n+1} = p \vee \Box(p \to \Box\varphi_n)$ are not equivalent in L. Thus, a logic in NExt**K4** is locally tabular iff it is of finite depth. For $L \in$ NExt**S4** this criterion can be reformulated in the following way: L is not locally tabular iff $L \subseteq$ **Grz.3**, where $\mathbf{Grz.3} = \mathbf{S4.3} \oplus \mathbf{Grz}$. Likewise, $L \in \mathrm{NExt}\mathbf{GL}$ is not locally tabular iff $L \subseteq \mathbf{GL.3}$. Nagle and Thomason [1985] showed that all normal extensions of $\mathbf{K5}$ are locally tabular.

Uniform logics Fine [1975a] used a modal analog of the full disjunctive normal form for constructing finite models and proving FMP of a family of logics in $\mathrm{NExt}\mathbf{D}$ (containing in particular the McKinsey system $\mathbf{K} \oplus \Box\Diamond p \to \Diamond\Box p$ which had resisted all attempts to prove its completeness by the method of canonical models and filtration). Let us notice first that every formula $\varphi(p_1, \dots, p_m)$ is equivalent in $\mathbf{K}$ either to $\bot$ or to a disjunction of normal forms (in the variables $p_1, \dots, p_m$) of degree $md(\varphi)$, which are defined inductively in the following way. $\mathbf{NF}_0$, the set of *normal forms of degree* 0, contains all formulas of the form $\neg_1 p_1 \wedge \dots \wedge \neg_m p_m$, where each $\neg_i$ is either blank or $\neg$. $\mathbf{NF}_{n+1}$, the set of *normal forms of degree* $n+1$, consists of formulas of the form

$$\theta \wedge \neg_1 \Diamond\theta_1 \wedge \dots \wedge \neg_k \Diamond\theta_k,$$

where $\theta \in \mathbf{NF}_0$ and $\theta_1, \dots, \theta_k$ are all distinct normal forms in $\mathbf{NF}_n$. Put $\mathbf{NF} = \bigcup_{n<\omega} \mathbf{NF}_n$. Using the fact that $\bigvee\{\Diamond\theta : \theta \in \mathbf{NF}_n\} \in \mathbf{D}$ it is not hard to see also that in $\mathbf{D}$ every formula φ with $md(\varphi) \leq n$ is equivalent either to $\bot$ or to a disjunction of normal forms of degree n such that at least one of $\neg_1, \dots, \neg_k$ in the inductive step of the definition above is blank. Such normal forms are called $\mathbf{D}$-*suitable*.

It should be clear that, for any distinct $\theta', \theta'' \in \mathbf{NF}_n$, $\neg(\theta' \wedge \theta'') \in \mathbf{K}$. Consequently, for every $\theta \in \mathbf{NF}_n$ and every $\varphi(p_1, \dots, p_m)$ with $md(\varphi) \leq n$, we have either $\theta \to \varphi \in \mathbf{K}$ or $\theta \to \neg\varphi \in \mathbf{K}$.

With each $\mathbf{D}$-suitable normal form θ we associate a model $\mathfrak{M}_\theta = \langle \mathfrak{F}_\theta, \mathfrak{V}_\theta \rangle$ on a frame $\mathfrak{F}_\theta = \langle W_\theta, R_\theta \rangle$ by taking

$$W_\theta = \{\top\} \cup \{\theta' \in \mathbf{NF} : \theta' <^n \theta, \text{ for some } n \geq 0\},$$

$$\theta' < \theta'' \text{ iff } \Diamond\theta' \text{ is a conjunct of } \theta'',$$

$$\theta' R_\theta \theta'' \text{ iff either } \theta' > \theta'' \text{ or } md(\theta') = 0 \text{ and } \theta'' = \top,$$

$$\mathfrak{V}_\theta(p) = \{\theta' \in W_\theta : p \text{ is a conjunct of } \theta'\}.$$

According to the definition, $\top$ is the reflexive last point in $\mathfrak{F}_\theta$ and so $\mathfrak{F}_\theta$ is serial. By a straightforward induction on the degree of $\theta' \in W_\theta$ one can readily show that $(\mathfrak{M}_\theta, \theta') \models \theta'$. It follows immediately that $\mathbf{D}$ has FMP. Indeed, given $\varphi \notin \mathbf{D}$, we reduce $\neg\varphi$ to a disjunction of $\mathbf{D}$-suitable normal forms with at least one disjunct θ, and then $(\mathfrak{M}_\theta, \theta) \models \theta$.

It turns out that in the same way we can prove FMP of all logics in $\mathrm{NExt}\mathbf{D}$ axiomatizable by *uniform formulas*, which are defined as follows. Every φ without modal operators is a *uniform formula of degree* 0; and if

$\varphi = \psi(\bigcirc_1\chi_1, \ldots, \bigcirc_m\chi_m)$, where $\bigcirc_i \in \{\Box, \Diamond\}$, $md(\psi(p_1, \ldots, p_m)) = 0$ and $\chi_1, \ldots, \chi_m$ are uniform formulas of degree n, then φ is a *uniform formula of degree* $n+1$. A remarkable property of uniform formulas is the following:

PROPOSITION 48. *Suppose φ is a uniform formula of degree n and $\mathfrak{M}$, $\mathfrak{N}$ are models based upon the same frame and such that, for some point x, $(\mathfrak{M}, y) \models p$ iff $(\mathfrak{N}, y) \models p$ for every $y \in x{\uparrow}^n$ and every variable p in φ. Then $(\mathfrak{M}, x) \models \varphi$ iff $(\mathfrak{N}, x) \models \varphi$.*

Given a logic L, we call a normal form θ *L-suitable* if $\mathfrak{F}_\theta \models L$.

THEOREM 49 (Fine 1975a). *Every logic $L \in$ NExt**D** axiomatizable by uniform formulas has FMP.*

Proof. It suffices to prove that each formula φ with $md(\varphi) \leq n$ is equivalent in L either to $\bot$ or to a disjunction of L-suitable normal forms of degree n. And this fact will be established if we show that every **D**-suitable normal form θ such that $\theta \to \bot \notin L$ is L-suitable. Suppose otherwise. Let θ be an L-consistent and **D**-suitable normal form of the least possible degree under which it is not L-suitable. Then there are a uniform formula $\psi \in L$ of some degree m and a model $\mathfrak{M} = \langle \mathfrak{F}_\theta, \mathfrak{V} \rangle$ such that $(\mathfrak{M}, \theta) \not\models \psi$.

For every variable p in ψ, let $\Gamma_p = \{\theta' \in \theta{\uparrow}^m\colon (\mathfrak{M}, \theta') \models p\}$ and let $\delta_p = \bigvee \Gamma_p$ (if $\Gamma_p = \emptyset$ then $\delta_p = \bot$). Observe that for every $\theta' \in \theta{\uparrow}^m$ we have $(\mathfrak{M}_\theta, \theta') \models \delta_p$ iff $\theta' \in \Gamma_p$ iff $(\mathfrak{M}, \theta') \models p$. Therefore, by Proposition 48, the formula ψ' which results from ψ by replacing each p with δ_p is false at θ in $\mathfrak{M}_\theta$. Now, if $md(\psi') > n$ then $m > n$ and so $\delta_p = \bot$ for every p in ψ, i.e., ψ' is variable free. But then ψ' is equivalent in **D** to $\top$ or $\bot$, contrary to $\mathfrak{F}_\theta \not\models \psi'$ and L being consistent. And if $md(\psi') \leq n$ then either $\theta \to \psi' \in$ **K**, which is impossible, since $(\mathfrak{M}_\theta, \theta) \not\models \theta \to \psi'$, or $\theta \to \neg\psi' \in$ **K**, from which $\psi' \to \neg\theta \in$ **K** and so $\neg\theta \in L$, contrary to θ being L-consistent. ∎

Logics with $\Box\Diamond$-axioms Another result, connecting FMP of logics with the distribution of $\Box$ and $\Diamond$ over their axioms, is based on the following

LEMMA 50. *For any φ and ψ, $\Diamond\varphi \leftrightarrow \Diamond\psi \in$ **S5** iff $\Box\Diamond\varphi \leftrightarrow \Box\Diamond\psi \in$ **K4**.*

Proof. Suppose $\Box\Diamond\varphi \to \Box\Diamond\psi \notin$ **K4**. Then there is a finite model $\mathfrak{M}$, based on a transitive frame, and a point x in it such that $x \models \Box\Diamond\varphi$ and $x \not\models \Box\Diamond\psi$. It follows from the former that every final cluster accessible from x, if any, is non-degenerate and contains a point where φ is true. The latter means that x sees a final cluster C at all points of which ψ is false. Now, taking the generated submodel of $\mathfrak{M}$ based on C, we obtain a model for **S5** refuting $\Diamond\varphi \to \Diamond\psi$. The rest is obvious, since $\Diamond p \leftrightarrow \Diamond\Box p$ is in **S5** and **K4** $\subseteq$ **S5**. ∎

Formulas in which every occurrence of a variable is in the scope of a modality $\Box\Diamond$ will be called *$\Box\Diamond$-formulas*.

THEOREM 51 (Rybakov 1978). *If a logic $L \in$ NExt**K4** is decidable (or has FMP) and ψ is a $\Box\Diamond$-formula then $L \oplus \psi$ is also decidable (has FMP).*

Proof. Let $\psi = \psi'(\Box\Diamond\chi_1, \dots, \Box\Diamond\chi_n)$, for some formula $\psi'(q_1, \dots, q_n)$. If $\varphi(p_1, \dots, p_m) \in L \oplus \psi$ then there exists a derivation of φ in $L \oplus \psi$ in which substitution instances of ψ contain no variables different from $p_1, \dots, p_m$. Each of these instances has the form $\psi'(\Box\Diamond\chi_1', \dots, \Box\Diamond\chi_n')$, where every χ_i' is some substitution instance of χ_i containing only $p_1, \dots, p_m$. By Lemma 50 and in view of the local tabularity of **S5** (it is of depth 1), there are finitely many pairwise non-equivalent in **K4** substitution instances of $\Box\Diamond\chi_i$ of that sort (the reader can easily estimate the number of them). So there exist only finitely many pairwise non-equivalent in **K4** substitution instances of ψ containing $p_1, \dots, p_m$, say $\psi_1, \dots, \psi_k$, and we can effectively construct them. Then, by the Deduction Theorem,

$$\varphi \in L \oplus \psi \text{ iff } \psi_1, \dots, \psi_k \vdash_L^* \varphi \text{ iff } \Box^+(\psi_1 \wedge \dots \wedge \psi_k) \to \varphi \in L$$

and so $L \oplus \psi$ is decidable (or has FMP) whenever L is decidable (has FMP). ■

It should be noted that by adding to L with FMP infinitely many $\Box\Diamond$-formulas we can construct an incomplete logic. For a concrete example see [Rybakov 1977]. By adding a variable free formula to a logic in NExt**K** with FMP one can get a logic without FMP. However, $\mathbf{K} \oplus \varphi$, φ variable free, has FMP, as can be easily shown by the standard filtration through the set $\mathbf{Sub}\varphi \cup \mathbf{Sub}\psi$, where $\psi \notin \mathbf{K} \oplus \varphi$. Infinitely many variable free formulas can axiomatize a normal extension of **K4** without FMP (for a concrete example see [Chagrov and Zakharyaschev 1997]).

1.8 Subframe and cofinal subframe logics

A very useful source of information for investigating various properties of logics in NExt**K4** is their canonical axioms. Notice, for instance, that the canonical axioms of all logics in Table 2, save $\mathbf{A}^*$ and $\mathbf{K4}_{n,m}$, contain no closed domains. Canonical and negation free canonical formulas of the form $\alpha(\mathfrak{F})$ and $\alpha(\mathfrak{F}, \bot)$ are called *subframe* and *cofinal subframe formulas*, respectively, and logics in NExt**K4** axiomatizable by them are called *subframe* and *cofinal subframe logics*. The classes of such logics will be denoted by $\mathcal{SF}$ and $\mathcal{CSF}$. Subframe and cofinal subframe logics in NExt**K4** were studied by Fine [1985] and Zakharyaschev [1984, 1988, 1996].

THEOREM 52. *All logics in $\mathcal{SF}$ and $\mathcal{CSF}$ have FMP.*

Proof. Suppose $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \bot) : i \in I\}$ and $\varphi \notin L$. By Theorem 44, without loss of generality we may assume that φ is a canonical formula, say, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Now consider two cases. (1) For no $i \in I$, $\mathfrak{F}$ is cofinally subreducible to $\mathfrak{F}_i$. Then $\mathfrak{F} \models L$, $\mathfrak{F} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, and we are done. (2) $\mathfrak{F}$ is cofinally subreducible to $\alpha(\mathfrak{F}_i, \bot)$, for some $i \in I$. In this case we have $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \mathbf{K4} \oplus \alpha(\mathfrak{F}_i, \bot) \subseteq L$, which is a contradiction. Indeed, suppose $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Then there is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$. And since the composition of (cofinal) subreductions is again a (cofinal) subreduction, $\mathfrak{G}$ is cofinally subreducible to $\mathfrak{F}_i$, which means that $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i, \bot)$. Subframe logics are treated analogously. ■

The names "subframe logic" and "cofinal subframe logic" are explained by the following frame-theoretic characterization of these logics. A subframe $\mathfrak{G} = \langle V, S, Q \rangle$ of a frame $\mathfrak{F}$ is called *cofinal* if $V\!\uparrow \subseteq \overline{V\!\downarrow}$ in $\mathfrak{F}$. Say that a class $\mathcal{C}$ of frames is *closed under (cofinal) subframes* if every (cofinal) subframe of $\mathfrak{F}$ is in $\mathcal{C}$ whenever $\mathfrak{F} \in \mathcal{C}$.

THEOREM 53. *$L \in$ NExt$\mathbf{K4}$ is a (cofinal) subframe logic iff it is characterized by a class of frames that is closed under (cofinal) subframes.*

Proof. Suppose $L \in \mathcal{CSF}$. We show that the class of all frames for L is closed under cofinal subframes. Let $\mathfrak{G} \models L$ and $\mathfrak{H}$ be a cofinal subframe of $\mathfrak{G}$. If $\mathfrak{H} \not\models \alpha(\mathfrak{F}, \bot)$, for some $\alpha(\mathfrak{F}, \bot) \in L$, then (since $\mathfrak{G}$ is cofinally subreducible to $\mathfrak{H}$) $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$, which is a contradiction. So $\mathfrak{H} \models L$.

Now suppose that L is characterized by some class of frames $\mathcal{C}$ closed under cofinal subframes. We show that $L = L'$, where

$$L' = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}, \bot) : \mathfrak{F} \not\models L\}.$$

If $\mathfrak{F}$ is a finite rooted frame and $\mathfrak{F} \not\models L$ then $\alpha(\mathfrak{F}, \bot) \in L$, for otherwise $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$ for some $\mathfrak{G} \in \mathcal{C}$, and hence there is a cofinal subframe $\mathfrak{H}$ of $\mathfrak{G}$ which is reducible to $\mathfrak{F}$; but $\mathfrak{H} \in \mathcal{C}$ and so, by the Reduction Theorem, $\mathfrak{F}$ is a frame for L, which is a contradiction. Thus, $L' \subseteq L$. To prove the converse, suppose $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$. Then $\mathfrak{F} \not\models L$, and hence $\alpha(\mathfrak{F}, \bot) \in L'$, from which $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L'$.

Subframe logics are considered in the same way. ■

It follows in particular that $\mathcal{SF} \subset \mathcal{CSF}$ ($\mathbf{K4.1}$ and $\mathbf{K4.2}$ are cofinal subframe logics but not subframe ones). One can easily show also that $\mathcal{CSF}$ is a complete sublattice of NExt$\mathbf{K4}$ and $\mathcal{SF}$ a complete sublattice of $\mathcal{CSF}$.

EXAMPLE 54. Every normal extension of $\mathbf{S4.3}$ is axiomatizable by canonical formulas which are based on chains of non-degenerate clusters and so have no closed domains. Therefore, NExt$\mathbf{S4.3} \subset \mathcal{CSF}$.

The classes $\mathcal{SF}$ and $\mathcal{CSF} - \mathcal{SF}$ contain a continuum of logics. And yet, unlike NExt$\mathbf{K}$ or NExt$\mathbf{K4}$, their structure and their logics are not so complex. For instance, it is not hard to see that every logic in $\mathcal{CSF}$ is uniquely axiomatizable by an independent set of cofinal subframe formulas and so these formulas form an axiomatic basis for $\mathcal{CSF}$.

The concept of subframe logic was extended in [Wolter 1993] to the class NExt**K** by taking the frame-theoretic characterization of Theorem 53 as the definition. Namely, we say that $L \in \mathrm{NExt}\mathbf{K}$ is a *subframe logic* if the class of frames for L is closed under subframes. In other words, subframe logics are precisely those logics whose axioms "do not force the existence of points". For example, **K**, **KB**, **K5**, **T**, and $\mathbf{Alt}_n$ are subframe logics. To give a syntactic characterization of subframe logics we require the following formulas.

For a formula φ and a variable p not occurring in φ, define a formula φ^p inductively by taking

$$\begin{array}{lcl} q^p & = & q \wedge p,\ q \text{ an atom}, \\ (\psi \odot \chi)^p & = & \psi^p \odot \chi^p, \quad \text{for } \odot \in \{\wedge, \vee, \rightarrow\}, \\ (\Box\psi)^p & = & \Box(p \rightarrow \psi^p) \wedge p \end{array}$$

and put $\varphi^{sf} = p \rightarrow \varphi^p$.

LEMMA 55. *For any frame $\mathfrak{F}$, $\mathfrak{F} \models \varphi^{sf}$ iff φ is valid in all subframes of $\mathfrak{F}$.*

Proof. It suffices to notice that if $\mathfrak{M}$ is a model based on $\mathfrak{F}$, $\mathfrak{M}'$ a model based on the subframe of $\mathfrak{F}$ induced by $\{y : (\mathfrak{M}, y) \models p\}$ and $(\mathfrak{M}, x) \models q$ iff $(\mathfrak{M}', x) \models q$, for all variables q, then $(\mathfrak{M}, x) \models \varphi^p$ iff $(\mathfrak{M}', x) \models \varphi$. ■

PROPOSITION 56. *The following conditions are equivalent for any modal logic L:*

(i) *L is a subframe logic;*

(ii) *$L = \mathbf{K} \oplus \{\varphi^{sf} : \varphi \in \Gamma\}$, for some set of formulas Γ;*

(iii) *L is characterized by a class of frames closed under subframes.*

Proof. The implication (i) $\Rightarrow$ (iii) is trivial; (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are consequences of Lemma 55. ■

It follows that the class of subframe logics forms a complete sublattice of NExt**K**. However, not all of them have FMP and even are Kripke complete.

EXAMPLE 57. Let L be the logic of the frame $\mathfrak{F}$ constructed in Example 7. Since every rooted subframe $\mathfrak{G}$ of $\mathfrak{F}$ is isomorphic to a generated subframe of $\mathfrak{F}$, L is a subframe logic. We show that L has the same Kripke frames as **GL.3**. Suppose $\mathfrak{G}$ is a rooted Kripke frame for **GL.3** refuting $\varphi \in L$. Then clearly $\mathfrak{G}$ contains a finite subframe $\mathfrak{H}$ refuting φ. Since $\mathfrak{H}$ is a finite chain of irreflexive points, it is isomorphic to a generated subframe of $\mathfrak{F}$, contrary to $\mathfrak{F} \not\models \varphi$. Thus $\mathfrak{G} \models L$. Conversely, suppose $\mathfrak{G}$ is a Kripke frame for L. Then $\mathfrak{G}$ is irreflexive. For otherwise $\mathfrak{G}$ refutes the formula $\varphi = \Box^2(\Box p \rightarrow p) \wedge \Box(\Box p \rightarrow p) \rightarrow \Box p$, which is valid in $\mathfrak{F}$. Let us show now that $\mathfrak{G}$ is transitive. Suppose otherwise. Then $\mathfrak{G}$ refutes the formula $\Box p \to \Box(\Box p \vee (\Box q \to q))$, which is valid in $\mathfrak{F}$ because ω is a reflexive point. Finally, since $\mathfrak{G} \models \varphi$, $\mathfrak{G}$ is Noetherian and since $\mathfrak{F}$ is of width 1, we may conclude that $\mathfrak{G} \models \mathbf{GL.3}$. It follows that the subframe logic L is Kripke incomplete. Indeed, it shares the same class of Kripke frames with $\mathbf{GL.3}$ but $\Box p \to \Box\Box p \in \mathbf{GL.3} - L$.

The following theorem provides a frame-theoretic characterization of those complete subframe logics in NExt$\mathbf{K}$ that are elementary, $\mathcal{D}$–persistent and strongly complete. Say that a logic L has the *finite embedding property* if a Kripke frame $\mathfrak{F}$ validates L whenever all finite subframes of $\mathfrak{F}$ are frames for L.

THEOREM 58 (Fine 1985). *For each Kripke complete subframe logic L the following conditions are equivalent:*

(i) *L is universal;*[9]
(ii) *L is elementary;*
(iii) *L is $\mathcal{D}$–persistent;*
(iv) *L is strongly Kripke complete;*
(v) *L has the finite embedding property.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial; (ii) $\Rightarrow$ (iii) follows from Fine's [1975b] Theorem formulated in Section 1.3 and (v) $\Rightarrow$ (i) from [Tarski 1954]. Thus it remains to show that (iv) $\Rightarrow$ (v). Suppose $\mathfrak{F}$ is a Kripke frame with root r such that $\mathfrak{F} \not\models L$ but all finite subframes of $\mathfrak{F}$ validate L. Then it is readily checked that all finite subsets of $\Gamma = \{p_r\} \cup \Box^{<\omega}\Delta_{\mathfrak{F}}$ are L-consistent. Hence the whole set Γ is L-consistent. On the other hand, similarly to the proof of Lemma 13 one can show that Γ is satisfiable in a Kripke frame iff the frame is subreducible to $\mathfrak{F}$. So Γ cannot be satisfied in a Kripke frame for L and L is not strongly complete. ■

A similar criterion for the cofinal subframe logics in NExt$\mathbf{K4}$ can be found in [Zakharyaschev 1996]. Note, however, that they are not in general universal and certainly do not have the finite embedding property, but (ii), (iii) and (iv) are still equivalent.

PROPOSITION 59. *Every subframe logic $L \in$ NExt$\mathbf{Alt}_n$ has FMP.*

Proof. Suppose $\varphi \notin L$. By Theorem 22, there is a Kripke frame $\mathfrak{F}$ for L refuting φ at a point x. Denote by X the set of points in $\mathfrak{F}$ accessible from x by $\leq md(\varphi)$ steps. Clearly, X is finite and the subframe of $\mathfrak{F}$ induced by X validates L and refutes φ. ■

[9] I.e., universal is the class of Kripke frames for L considered as models of the first order language with R and $=$.

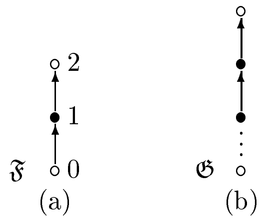


Figure 5.

To understand the place of incomplete logics in the lattice of subframe logics we call a subframe logic L *strictly sf-complete* if it is Kripke complete and no other subframe logic has the same Kripke frames as L. Example 57 shows that $\mathbf{GL.3}$ is not strictly sf-complete. However, the logics $\mathbf{T}$, $\mathbf{S4}$ and $\mathbf{Grz}$ turn out to be strictly sf-complete. The following result clarifies the situation. It is proved by applying the splitting technique to lattices of subframe logics.

THEOREM 60. *A subframe logic L containing $\mathbf{K4}$ is strictly sf-complete iff $L \not\subseteq \mathbf{GL.3}$. All subframe logics in $\mathrm{NExt}\mathbf{Alt}_n$ are strictly sf-complete. A subframe logic is tabular iff there are only finitely many subframe logics containing it.*

1.9 More sufficient conditions of FMP

As follows from Theorem 52, a logic in $\mathrm{NExt}\mathbf{K4}$ does not have FMP only if at least one of its canonical axioms contains closed domains. We illustrate their role by a simple example.

EXAMPLE 61. Consider the logic $L = \mathbf{K4.3} \oplus \alpha^\sharp(\mathfrak{F}, \bot)$ and the formula $\alpha(\mathfrak{F}, \bot)$, where $\mathfrak{F}$ is the frame depicted in Fig. 5 (a). The frame $\mathfrak{G}$ in Fig. 5 (b) separates $\alpha(\mathfrak{F}, \bot)$ from L. Indeed, $\mathfrak{F}$ is a cofinal subframe of $\mathfrak{G}$ and so $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$. To show that $\mathfrak{G} \models \alpha^\sharp(\mathfrak{F}, \bot)$, suppose f is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$. Then $f^{-1}(1)$ contains only one point, say x; $f^{-1}(0)$ also contains only one point, namely the root of $\mathfrak{G}$. So the infinite set of points between x and the root is outside $\mathrm{dom} f$, which means that f does not satisfy (CDC) for $\{\{1\}\}$. On the other hand, if $\mathfrak{H}$ is a finite refutation frame of width 1 for $\alpha(\mathfrak{F}, \bot)$ then $\mathfrak{H}$ contains a generated subframe reducible to $\mathfrak{F}$, from which $\mathfrak{H} \not\models L$. Thus, L fails to have FMP. In the same manner the reader can prove that $\mathbf{A}^*$ in Table 2 does not have FMP either.

We show now two methods developed in [Zakharyaschev 1997a] for establishing FMP of logics whose canonical axioms contain closed domains. One of them uses the following lemma, which is an immediate consequence of the refutability criterion for the canonical formulas.

LEMMA 62. *Suppose* $\alpha(\mathfrak{F}, \mathfrak{D})$ *and* $\alpha(\mathfrak{G}, \mathfrak{E})$ *(*$\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ *and* $\alpha(\mathfrak{G}, \mathfrak{E}, \bot)$*) are canonical formulas such that there is a (cofinal) subreduction* f *of* $\mathfrak{G}$ *to* $\mathfrak{F}$ *satisfying (CDC) for* $\mathfrak{D}$ *and an antichain* $\mathfrak{e} \subseteq \mathrm{dom} f\uparrow$ *is in* $\mathfrak{E}$ *whenever* $f(\mathfrak{e}\underline{\uparrow}) = \mathfrak{d}\underline{\uparrow}$ *for some* $\mathfrak{d} \in \mathfrak{D}$*. Then* $\alpha(\mathfrak{G}, \mathfrak{E}) \in \mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{D})$ *(respectively,* $\alpha(\mathfrak{G}, \mathfrak{E}, \bot) \in \mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$*).*

THEOREM 63. $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\} \oplus \{\alpha(\mathfrak{F}_j, \mathfrak{D}_j) : j \in J\}$ *has FMP provided that either all frames* $\mathfrak{F}_i$*, for* $i \in I \cup J$*, are irreflexive or all of them are reflexive.*

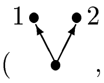
Proof. Suppose all $\mathfrak{F}_i$ are irreflexive and $\alpha(\mathfrak{G}, \mathfrak{E}, \bot)$ is an arbitrary canonical formula. We construct from $\mathfrak{G}$ a new finite frame $\mathfrak{H}$ by inserting into it new *reflexive* points. Namely, suppose $\mathfrak{e}$ is an antichain in $\mathfrak{G}$ such that $\mathfrak{e} \notin \mathfrak{E}$. Suppose also that $C_1, \ldots, C_n$ are all clusters in $\mathfrak{G}$ such that $\mathfrak{e} \subseteq C_i\uparrow$ and $\mathfrak{e} \cap C_i = \emptyset$, for $i = 1, \ldots, n$, but no successor of C_i possesses this property. Then we insert in $\mathfrak{G}$ new reflexive points $x_1, \ldots, x_n$ so that each x_i could see only the points in $\mathfrak{e}$ and their successors and could be seen only from the points in C_i and their predecessors. The same we simultaneously do for all antichains $\mathfrak{e}$ in $\mathfrak{G}$ of that sort. The resulting frame is denoted by $\mathfrak{H}$. Since no new point was inserted just below an antichain in $\mathfrak{E}$, $\mathfrak{H} \not\models \alpha(\mathfrak{G}, \mathfrak{E}, \bot)$.

Suppose now that $\alpha(\mathfrak{G}, \mathfrak{E}, \bot) \notin L$ and show that $\mathfrak{H} \models L$. If this is not so then either $\mathfrak{H} \not\models \alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$, for some $i \in I$, or $\mathfrak{H} \not\models \alpha(\mathfrak{F}_j, \mathfrak{D}_j)$, for some $j \in J$. We consider only the former case, since the latter one is treated similarly. Thus, we have a cofinal subreduction f of $\mathfrak{H}$ to $\mathfrak{F}_i$ satisfying (CDC) for $\mathfrak{D}_i$. Since $\mathfrak{F}_i$ is irreflexive, no point that was added to $\mathfrak{G}$ is in $\mathrm{dom} f$. So f may be regarded as a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}_i$ satisfying (CDC) for $\mathfrak{D}_i$. We clearly may assume also that the subframe of $\mathfrak{G}$ generated by $\mathrm{dom} f$ is rooted. Let $\mathfrak{e}$ be an antichain in $\mathfrak{G}$ belonging to $\mathrm{dom} f\uparrow$ and such that $f(\mathfrak{e}\underline{\uparrow}) = \mathfrak{d}\underline{\uparrow}$ for some $\mathfrak{d} \in \mathfrak{D}_i$. If $\mathfrak{e} \notin \mathfrak{E}$ then there is a reflexive point x in $\mathfrak{H}$ such that $x \in \mathrm{dom} f\uparrow$ and x sees only $\mathfrak{e}\underline{\uparrow}$ and, of course, itself. But then $f(x\uparrow) = f(\mathfrak{e}\underline{\uparrow}) = \mathfrak{d}\underline{\uparrow}$ and so, by (CDC), $x \in \mathrm{dom} f$, which is impossible. Therefore, $\mathfrak{e} \in \mathfrak{E}$ and so, by Lemma 62, $\alpha(\mathfrak{G}, \mathfrak{E}, \bot) \in L$, contrary to our assumption.

In the case of reflexive frames *irreflexive* points are inserted. ■

EXAMPLE 64. According to Theorem 63, the logic

$$L = \mathbf{K4} \oplus \alpha($$



$$, \{\{1\}, \{1,2\}\})$$

has FMP. However, Artemov's logic $\mathbf{A}^* = L \oplus \mathbf{GL}$ does not enjoy this property. So FMP is not in general preserved under sums of logics.

The scope of the method of inserting points is not bounded only by canonical axioms associated with homogeneous (irreflexive or reflexive) frames. It can be applied, for instance, to normal extensions of **K4** with modal reduction principles, i.e., formulas of the form $\boldsymbol{M}p \to \boldsymbol{N}p$, where $\boldsymbol{M}$ and $\boldsymbol{N}$ are strings of $\Box$ and $\Diamond$ (for first order equivalents of modal reduction principles see [van Benthem 1976]). One can show that each such logic is either of finite depth, or can be axiomatized by $\Box\Diamond$-formulas and canonical formulas based upon almost homogeneous frames (containing at most one reflexive point), for which the method works as well. So we have

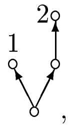
THEOREM 65. *All logics in* NExt**K4** *axiomatizable by modal reduction principles have FMP and are decidable.*

One of the most interesting open problems in completeness theory of modal logic is to prove an analogous theorem for logics in NExt**K** or to construct a counter-example. It is unknown, in particular, whether the logics of the form $\mathbf{K} \oplus \Box^m p \to \Box^n p$ have FMP; the same concerns the logics $\mathbf{K} \oplus \boldsymbol{tra}_n$.

The second method of proving FMP uses the more conventional technique of removing points. Suppose that $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot) : i \in I\}$ and $\alpha = \alpha(\mathfrak{H}, \mathfrak{C}, \bot) \notin L$. Then there exists a frame $\mathfrak{F}$ for L such that $\mathfrak{F} \not\models \alpha$, i.e., there is a cofinal subreduction h of $\mathfrak{F}$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{C}$. Construct the countermodel $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ for α as it was done in Section 1.6. Without loss of generality we may assume that $\mathrm{dom} h\underline{\uparrow} = \mathrm{dom} h\overline{\downarrow} = \mathfrak{F}$ and that $\mathfrak{F}$ is generated by the sets $\mathfrak{V}(p_i)$, p_i a variable in α.

Actually, the step-wise refinement procedure with deleting points having **Sub**α-equivalent successors, used in the proof of Theorem 42, establishes FMP of L when all $\mathfrak{D}_i$ are empty, i.e., L is a cofinal subframe logic. To tune it for L with non-empty $\mathfrak{D}_i$, we should follow a subtler strategy of deleting points, preserving those that are "responsible" for validating the axioms of L. Suppose we have already constructed a model $\mathfrak{M}'_n = \langle \mathfrak{F}'_n, \mathfrak{V}'_n \rangle$ by "folding up" $n-1$-cyclic sets into clusters of depth n (we use the same notations as in the proof of Theorem 42). Now we throw away points of two sorts.

First, for every proper cluster C of depth n such that some $x \in C$ has a **Sub**α-equivalent successor of depth $< n$, we remove from C all points except x. Second, call a point x of depth $> n$ *redundant* in $\mathfrak{M}'_n$ if it has a **Sub**α-equivalent successor of depth $\leq n$ and, for every $i \in I$ and every cofinal subreduction g of $(\mathfrak{F}'_n)^{\leq n}$ to the subframe of $\mathfrak{G}_i$ generated by some $\mathfrak{d} \in \mathfrak{D}_i$ such that $\mathfrak{d} \subseteq g(x\uparrow)$ and g satisfies (CDC) for $\mathfrak{D}_i$, there is a point $y \in x\uparrow$ of depth $\leq n$ such that $g(y\uparrow) = \mathfrak{d}\underline{\uparrow}$. Let X be the maximal set of redundant points in $\mathfrak{M}'_n$ which is upward closed in $(W'_n)^{>n}$. We define $\mathfrak{M}_{n+1} = \langle \mathfrak{F}_{n+1}, \mathfrak{V}_{n+1} \rangle$ as the submodel of $\mathfrak{M}'_n$ resulting from it by removing all points in X as well. Since all deleted points have **Sub**α-equivalent successors, $\mathfrak{M}_{n+1} \not\models \alpha$. And since we keep in $\mathfrak{F}_{n+1}$ points which violate (CDC) for $\mathfrak{D}_i$ of possible cofinal subreductions to $\mathfrak{G}_i$, $\mathfrak{F}_{n+1} \models L$. So FMP of L will be established if we manage to prove that this process eventually terminates.

EXAMPLE 66. Let $L = \mathbf{S4} \oplus \alpha(\mathfrak{G}, \{\{1,2\}\}, \bot)$, where $\mathfrak{G}$ is 

, and assume that our "algorithm", when being applied to $\mathfrak{F}$, α and L, works infinitely long. Then the frame $\mathfrak{F}_\omega = \langle W_\omega, R_\omega \rangle$, where

$$W_\omega = \bigcup_{0<i<\omega} W_i^{\leq i},\ R_\omega = \bigcup_{0<i<\omega} R_i^{\leq i},\ \mathfrak{F}_i = \langle W_i, R_i, P_i \rangle\,,$$

is of infinite depth. By König's Lemma, there is an infinite descending chain $\dots x_i R_\omega x_{i-1} \dots R_\omega x_2 R_\omega x_1$ in $\mathfrak{F}_\omega$ such that x_i is of depth i. Since there are only finitely many pairwise non-$\mathbf{Sub}\alpha$-equivalent points, there must be some $n > 0$ such that, for every $k \geq n$, each point in $C(x_k)$ has a $\mathbf{Sub}\alpha$-equivalent successor in $\mathfrak{F}_k^{<k}$. And since $\mathfrak{F}_1^{\leq 1}$ is finite, there is $m \geq n$ starting from which all x_i see the same points of depth 1. Let us consider now $\mathfrak{F}_m$ and ask why points in the m-cyclic set X, folded at step $m+1$ into $C(x_{m+1})$, were not removed at step m. X is upward closed in $W_m^{>m}$ and every point in it has a $\mathbf{Sub}\alpha$-equivalent successor in $\mathfrak{F}_m^{\leq m}$. So the only reason for keeping some $x \in X$ is that $\mathfrak{F}_m^{\leq m}$ is cofinally subreducible to $\mathfrak{G}^{\leq 1}$, x sees inverse images of both points in $\mathfrak{G}^{\leq 1}$ but none of its successors in $\mathfrak{F}_m^{\leq m}$ does. By the cofinality condition, these inverse images can be taken from $\mathfrak{F}_1^{\leq 1}$. But then they are also seen from x_m, which is a contradiction. Thus sooner or later our algorithm will construct a finite frame separating L from α, which proves that L has FMP.

The reason why we succeeded in this example is that inverse images of points in the closed domain $\{1,2\}$ can be found at a fixed finite depth in $\mathfrak{F}_\omega$, and so points violating (CDC) for it can also be found at finite depth (that was not the case in Example 61). The following definitions describe a big family of frames and closed domains of that sort.

A point x in a frame $\mathfrak{G}$ is called a *focus* of an antichain $\mathfrak{a}$ in $\mathfrak{G}$ if $x \notin \mathfrak{a}$ and $x{\uparrow} = \{x\} \cup \mathfrak{a}{\uparrow}$. Suppose $\mathfrak{G}$ is a finite frame and $\mathfrak{D}$ a set of antichains in $\mathfrak{G}$. Define by induction on n notions of *n-stable point* in $\mathfrak{G}$ (relative to $\mathfrak{D}$) and *n-stable antichain* in $\mathfrak{D}$. A point x is 1-*stable* in $\mathfrak{G}$ iff either x is of depth 1 in $\mathfrak{G}$ or the cluster $C(x)$ is proper. A point x is $n+1$-*stable* in $\mathfrak{G}$ (relative to $\mathfrak{D}$) iff it is not m-stable, for any $m \leq n$, and either there is an n-stable point in $\mathfrak{G}$ (relative to $\mathfrak{D}$) which is not seen from x or x is a focus of an antichain in $\mathfrak{D}$ containing an $n-1$-stable point and no n-stable point. And we say an antichain $\mathfrak{d}$ in $\mathfrak{D}$ is n-stable iff it contains an n-stable point in the subframe $\mathfrak{G}'$ of $\mathfrak{G}$ generated by $\mathfrak{d}$ (relative to $\mathfrak{D}$) and no m-stable point in $\mathfrak{G}'$ (relative to $\mathfrak{D}$), for $m > n$. A point or an antichain is *stable* if it is n-stable for some n. It should be clear that if a point in an antichain is stable then the rest points in the antichain are also stable.

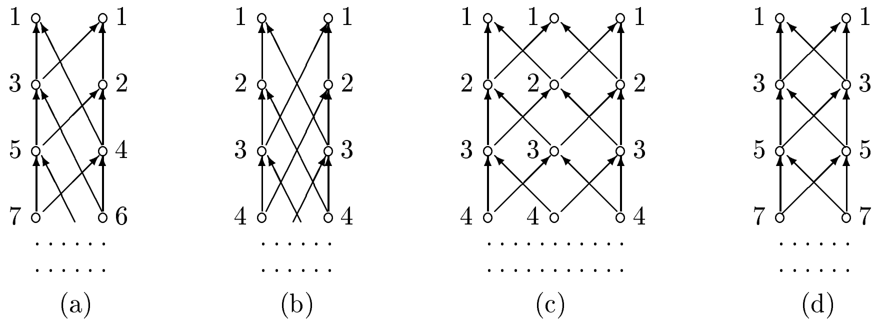


Figure 6.

EXAMPLE 67. (1) Suppose $\mathfrak{G}$ is a finite rooted generated subframe of one of the frames shown in Fig. 6 (a)–(c). Then, regardless of $\mathfrak{D}$, each point in $\mathfrak{G}$ different from its root is n-stable, where n is the number located near the point. Every antichain $\mathfrak{d}$ in $\mathfrak{G}$, containing at least two points, is also n-stable, with n being the maximal degree of stability of points in $\mathfrak{d}$.

(2) If $\mathfrak{G}$ is a rooted generated subframe of the frame depicted in Fig. 6 (d) and $\mathfrak{D}$ is the set of all two-point antichains in $\mathfrak{G}$ then every point in $\mathfrak{G}$ is n-stable (relative to $\mathfrak{D}$), where n stays near the point. However, for $\mathfrak{D} = \emptyset$ no point in $\mathfrak{G}$, save those of depth 1, is stable.

(3) If $\mathfrak{G}$ is a finite tree of clusters then every antichain in $\mathfrak{G}$, different from a non-final singleton, is either 1- or 2-stable in $\mathfrak{G}$ regardless of $\mathfrak{D}$. Every antichain containing a point x with proper $C(x)$ is 1- or 2-stable as well, whatever $\mathfrak{G}$ and $\mathfrak{D}$ are.

(4) Every antichain is stable in every irreflexive frame $\mathfrak{G}$ relative to the set $\mathfrak{D}^\sharp$ of all antichains in $\mathfrak{G}$. However, this is not so if $\mathfrak{G}$ contains reflexive points (for reflexive singletons are open domains and do not belong to $\mathfrak{D}^\sharp$).

The sufficient condition of FMP below is proved by arguments that are similar to those we used in Example 66.

THEOREM 68. *If $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{G}_i, \mathfrak{D}_i, \bot) : i \in I\}$ and there is $d > 0$ such that, for any $i \in I$, every closed domain $\mathfrak{d} \in \mathfrak{D}_i$ is n-stable in $\mathfrak{G}_i$ (relative to $\mathfrak{D}_i$), for some $n \leq d$, then L has FMP.*

Example 67 shows many applications of this condition. Moreover, using it one can prove the following

THEOREM 69. *Every normal extension of* **S4** *with a formula in one variable has FMP and is decidable.*

Note that, as was shown by Shehtman [1980], a formula with two variables or an infinite set of one-variable formulas can axiomatize logics in NExt**S4** without FMP (and even Kripke incomplete).

1.10 The reduction method

That a logic does not have FMP (or is Kripke incomplete) is not yet an evidence of its undecidability: it is enough to recall that the majority of decidability results for classical theories was proved without using any analogues of the finite model property (see e.g. [Rabin 1977], [Ershov 1980]). The first example of a decidable finitely axiomatizable modal logic without FMP was constructed by Gabbay [1971].

It seems unlikely that the methods of classical model theory can be applied directly for proving the decidability of propositional modal logics. However, sometimes it is possible to *reduce* the decision problem for a given modal logic L to that for a knowingly decidable first or higher order theory whose language is expressive enough for describing the structure of frames characterizing L. The most popular tools used for this purpose are Büchi's [1962] Theorem on the decidability of the weak monadic second order theory of the successor function on natural numbers and Rabin's [1969] Tree Theorem. Below we illustrate the use of Rabin's Theorem following [Gabbay 1975] and [Cresswell 1984].

Let ω^* be the set of all finite sequences of natural numbers and $\prec$ the lexicographic order on it. For $x \in \omega^*$ and $i < \omega$, put $r_i(x) = x * i$, where $*$ denotes the usual concatenation operation. Besides, define the following predicates $<_i$ on ω^*, for $0 \leq i \leq 2$,

$$x <_i y \text{ iff } y = x * (3n + i) \text{ for some } n < \omega.$$

It follows from [Rabin 1969] that the monadic second order theory SωS of the model $\langle \omega^*, \{r_i : i < \omega\}, \{<_i : 0 \leq i \leq 2\}, \prec, \emptyset \rangle$ ($\emptyset$ denotes the empty sequence) is decidable.

The theory SωS has a very strong expressive power which makes it possible to effectively describe semantical definitions of many modal (as well as some other) logics and thereby prove their decidability. In this way Gabbay [1975] established the decidability of, for instance,

$$\mathbf{K} \oplus \Box^m \Diamond p \to \Diamond p, \quad \mathbf{K} \oplus \Diamond^m \Box p \to \Box p,$$

$$\mathbf{K} \oplus \Box^m p \to \Diamond^n p, \quad \mathbf{K} \oplus \Diamond^m p \to \Box^n p.$$

By Sahlqvist's Theorem, all these logics are Kripke complete; however, we do not know whether they have FMP. General frames can also be described by means of SωS.

EXAMPLE 70. The frame $\mathfrak{F} = \langle W, R, P \rangle$ constructed in Example 7 can be represented in the language of SωS as follows. Let us encode each $n < \omega$ by the sequence $\langle 3n \rangle$, while ω and $\omega + 1$ by $r_1(\emptyset)$ and $r_2(\emptyset)$, respectively.

Then we have

$$
\begin{aligned}
x \in W \quad &\text{iff} \quad \emptyset <_0 x \vee x = r_1(\emptyset) \vee x = r_2(\emptyset),\\
xRy \quad &\text{iff} \quad (\emptyset <_0 x \wedge \emptyset <_0 y \wedge y \prec x \wedge x \neq y) \vee\\
&\qquad (x = r_1(\emptyset) \wedge \emptyset <_0 y) \vee x = y = r_1(\emptyset) \vee\\
&\qquad (x = r_2(\emptyset) \wedge y = r_1(\emptyset)),\\
X \in P \quad &\text{iff} \quad \forall x\ (x \in X \to x \in W) \wedge ((Fin(X) \wedge r_1(\emptyset) \notin X) \vee\\
&\qquad \forall Y\ (\forall y\ (y \in Y \leftrightarrow (y \in W \wedge y \notin X)) \to Fin(Y) \wedge r_1(\emptyset) \notin Y)),
\end{aligned}
$$

where $x = y$ means $x \prec y \wedge y \prec x$ and

$$Fin(X) = \exists x \forall y\ (y \in X \to y \prec x).$$

It follows that the logic $\mathrm{Log}\mathfrak{F}$ is decidable. Indeed, for every formula $\varphi(p_1, \dots, p_n)$, we have $\varphi \in \mathrm{Log}\mathfrak{F}$ iff the second order formula

$$\forall x \forall X_1, \dots, X_n\ (X_1 \in P \wedge \cdots \wedge X_n \in P \wedge x \in W \to ST(\varphi(X_1, \dots, X_n)))$$

belongs to SωS. Here $ST(\varphi(X_1, \dots, X_n))$, the *standard translation* of φ, is defined inductively in the following way (see also *Correspondence Theory*):

$$ST(X) = x \in X, \quad ST(\bot) = \bot,$$

$$ST(X \odot Y) = ST(X) \odot ST(Y), \text{ for } \odot \in \{\wedge, \vee, \to\},$$

$$ST(\Box X) = \forall y\ (xRy \to ST(X)\{y/x\}).$$

Recall that, as was shown in Example 57, $\mathrm{Log}\mathfrak{F}$ is Kripke incomplete.

Also, it is not hard to find examples of applications of this technique for proving the decidability of finitely axiomatizable quasi-normal unimodal and normal polymodal (in particular, tense) logics which do not have Kripke frames at all; perhaps, the simplest one is Solovay's logic **S**.

Sobolev [1977a] found another way of proving decidability by applying methods of automata theory on infinite sequences. Using the results of [Büchi and Siefkes 1973] he showed that all finitely axiomatizable superintuitionistic logics of finite width (see Section 3.4) containing the formula

$$(((p \to q) \to p) \to p) \vee (((q \to p) \to q) \to q).$$

are decidable. By the preservation theorem of Section 3.3, this result can be transferred to the corresponding extensions of **S4**.

If a logic is known to be complete with respect to a suitable class of frames, the methods discussed above are usually applicable to it in a rather straightforward manner. A relative disadvantage of this approach is that the resulting decision algorithms inherit the extremely high complexity of the decision algorithms for SωS or other "rich theories" used to prove decidability. On the other hand, the logic **S**, for instance, turns out to be decidable by an algorithm of the same complexity as that for **GL** (see Example 75), in particular, the derivability problem in **S** is ***PSPACE***-complete. The logic of the frame $\mathfrak{F}$ in Example 7 is "almost trivial"—it is polynomially equivalent to classical propositional logic, which follows from the fact that every formula φ refutable by $\mathfrak{F}$ can be also refuted in $\mathfrak{F}$ under a valuation giving the same truth-value to all variables in φ at all points i such that $|\mathbf{Sub}\varphi| < i < \omega$ (see Section 4.6). Actually, this sort of decidability proofs (ignoring "inessential" parts of infinite frames) was used already by Kuznetsov and Gerchiu [1970] for studying some superintuitionistic logics.

Recently more general semantical methods of obtaining decidability results without turning to "rich theories" have been developed. We demonstrate them in the next section by establishing the decidability of all finitely axiomatizable logics in NExt**K4.3**, which according to Example 61 do not in general have FMP. We show, however, that those logics are complete with respect to recursively enumerable classes of recursive frames in which the validity of formulas can be effectively checked—it was this rather than the finiteness of frames that we used in the proof of Harrop's Theorem. In Section 2.5 this result will be extended to linear tense logics which in general are not even Kripke complete. Our presentation follows [Zakharyaschev and Alekseev 1995].

1.11 Logics containing **K4.3**

Each logic in $L \in$ NExt**K4.3** is represented in the form

$$L = \mathbf{K4.3} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\},$$

where all $\mathfrak{F}_i$ are chains of clusters. So our decidability problem reduces to finding an algorithm which, given such a representation with finite I and a canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ built on a chain of clusters $\mathfrak{F}$, could decide whether $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$. Recall also that, by Fine's [1974c] Theorem, logics of width 1 are characterized by Kripke frames having the form of Noetherian chains of clusters.

LEMMA 71. *For any Noetherian chain of clusters $\mathfrak{G}$ and any canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ iff there is an injective*[10] *cofinal subreduction g of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.*

Proof. If $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ then there is a cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Clearly, $f^{-1}(x)$ is a singleton if x is irreflexive. Suppose now that x is a reflexive point in $\mathfrak{F}$. Since $\mathfrak{G}$ contains no infinite ascending chains, $f^{-1}(x)$ has a finite cover and so there is a reflexive point

[10] That is $g(x) \neq g(y)$, for every distinct $x, y \in \mathrm{dom} g$.

$u_x \in f^{-1}(x)$ such that $f^{-1}(x) \subseteq u_x{\downarrow}$. Fix such a u_x for each reflexive x and define a partial map g by taking

$$g(y) = \begin{cases} f(y) & \text{if either } f(y) \text{ is irreflexive or} \\ & f(y) \text{ is reflexive and } y = u_{f(y)} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

One can readily check that g is the injective cofinal subreduction we need. The converse is trivial. ■

Roughly, every Noetherian chain of clusters refuting $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ results from $\mathfrak{F}$ by inserting some Noetherian chains of clusters just below clusters $C(x)$ in $\mathfrak{F}$ such that $\{x\} \notin \mathfrak{D}$. We show now that if $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is not in $L \in \mathrm{NExt}\mathbf{K4.3}$ then it can be separated from L by a frame constructed from $\mathfrak{F}$ by inserting in open domains between its adjacent clusters either finite descending chains of irreflexive points possibly ending with a reflexive one or infinite descending chains of irreflexive points.

Let $C(x_0), \dots, C(x_n)$ be all distinct clusters in $\mathfrak{F}$ ordered in such a way that $C(x_0) \subset C(x_1)\overline{\downarrow} \subset \dots \subset C(x_n)\overline{\downarrow}$. Say that an n-tuple $t = \langle \xi_1, \dots, \xi_n \rangle$ is a *type* for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ if either $\xi_i = m$ or $\xi_i = m+$, for some $m < \omega$, or $\xi_i = \omega$, with $\xi_i = 0$ if $\{x_i\} \in \mathfrak{D}$. Given a type $t = \langle \xi_1, \dots, \xi_n \rangle$ for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, we define the *$t$-extension* of $\mathfrak{F}$ to be the frame $\mathfrak{G}$ that is obtained from $\mathfrak{F}$ by inserting between each pair $C(x_{i-1})$, $C(x_i)$ either a descending chain of m irreflexive points, if $\xi_i = m < \omega$, or a descending chain of $m+1$ points of which only the last (lowest) one is reflexive, if $\xi_i = m+$, or an infinite descending chain of irreflexive points, if $\xi_i = \omega$. It should be clear that $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$.

LEMMA 72. *If $L \in \mathrm{NExt}\mathbf{K4.3}$ and $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$ then $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is separated from L by the t-extension of $\mathfrak{F}$, for some type t for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$.*

Proof. By Lemma 71, we have a Noetherian chain of clusters $\mathfrak{G}$ for L and an injective cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. By the Generation Theorem, we may assume that f maps the root of $\mathfrak{G}$ to the root of $\mathfrak{F}$. Let $\mathfrak{G}_0$ be the subframe of $\mathfrak{G}$ obtained by removing from $\mathfrak{G}$ all those points that are not in $\mathrm{dom} f$ but belong to clusters containing some points in $\mathrm{dom} f$. The very same map f is an injective cofinal subreduction of $\mathfrak{G}_0$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$, and so $\mathfrak{G}_0 \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Since $\mathfrak{G}_0$ is a reduct of $\mathfrak{G}$, $\mathfrak{G}_0 \models L$.

Let $C(x_0), \dots, C(x_n)$ be all distinct clusters in $\mathfrak{G}_0$ such that

$$\mathrm{dom} f = \bigcup_{i=0}^{n} C(x_i),\ C(x_0) \subset C(x_1)\overline{\downarrow} \subset \dots \subset C(x_n)\overline{\downarrow}.$$

By induction on i we define a sequence of frames $\mathfrak{G}_0 \supseteq \dots \supseteq \mathfrak{G}_n$ such that (a) f is an injective cofinal subreduction of $\mathfrak{G}_i$ to $\mathfrak{F}$ satisfying (CDC) for

$\mathfrak{D}$, (b) between $C(x_{i-1})$ and $C(x_i)$ the frame $\mathfrak{G}_i$ contains either a finite descending chain of irreflexive points possibly ending with a reflexive one or an infinite descending chain of irreflexive points, and (c) $\mathfrak{G}_i \models L$.

Suppose $\mathfrak{G}_{i-1}$ has been already constructed and $\mathfrak{C}_i$ is the chain of clusters located between $C(x_{i-1})$ and $C(x_i)$. Three cases are possible. (1) $\mathfrak{C}_i$ is a finite chain of irreflexive points. Then we put $\mathfrak{G}_i = \mathfrak{G}_{i-1}$. (2) $\mathfrak{C}_i$ contains a non-degenerate cluster $C(x)$ having finitely many distinct successors in $\mathfrak{C}_i$ and all of them are irreflexive. Then $\mathfrak{G}_i$ results from $\mathfrak{G}_{i-1}$ by removing from $\mathfrak{C}_i$ all points save x and those successors. $\mathfrak{G}_i$ is a reduct of $\mathfrak{G}_{i-1}$ and so conditions (a)–(c) are satisfied. (3) Suppose (1) and (2) do not hold. Then $\mathfrak{C}_i$ contains an infinite descending chain Y of irreflexive points accessible from all other points in $\mathfrak{C}_i$. In this case $\mathfrak{G}_i$ is obtained from $\mathfrak{G}_{i-1}$ by removing all points in $\mathfrak{C}_i$ save those in Y. Clearly, $\mathfrak{G}_i$ satisfies (a) and (b). To prove (c) suppose $\mathfrak{G}_i \not\models \alpha(\mathfrak{H}, \mathfrak{E}, \bot)$ for some $\alpha(\mathfrak{H}, \mathfrak{E}, \bot) \in L$. Then there is an injective cofinal subreduction g of $\mathfrak{G}_i$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{E}$. Consider g as a cofinal subreduction of $\mathfrak{G}_{i-1}$ to $\mathfrak{H}$ and show that it also satisfies (CDC) for $\mathfrak{E}$. Indeed, (CDC) could be violated only by a point in $z \in \mathfrak{C}_i - Y$ such that $g(z\uparrow) = \underline{w\uparrow}$, for some $\{w\} \in \mathfrak{E}$. Since $g^{-1}(w)$ is a singleton and $Y \subseteq z\uparrow$, there is $y \in Y$ such that $g(y\uparrow) = \underline{w\uparrow}$ and $y \notin \mathrm{dom} g$, contrary to g satisfying (CDC) for $\mathfrak{E}$ as a subreduction of $\mathfrak{G}_i$ to $\mathfrak{H}$. ■

Thus, a frame separating $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$ from $L \in \mathrm{NExt}\mathbf{K4.3}$ can be found in the recursively enumerable class of t-extensions of $\mathfrak{F}$, t being a type for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Moreover, given a formula $\alpha(\mathfrak{H}, \mathfrak{E}, \bot)$ and a type t for $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, one can effectively check whether $\alpha(\mathfrak{H}, \mathfrak{E}, \bot)$ is valid in the t-extension of $\mathfrak{F}$. Indeed, let k be the number of irreflexive points in $\mathfrak{H}$, $t = \langle \xi_1, \dots, \xi_n \rangle$, and $\mathfrak{G}$ the t-extension of $\mathfrak{F}$. Construct a cofinal subframe $\mathfrak{G}_k$ of $\mathfrak{G}$ by "cutting off" the infinite descending chains inserted in $\mathfrak{F}$ (if any) just below their $k+1$th points, and let X be the set of all these $k+1$th points. Clearly, $\mathfrak{G}_k$ is finite. It is now an easy exercise to prove the following

LEMMA 73. *$\mathfrak{G} \not\models \alpha(\mathfrak{H}, \mathfrak{E}, \bot)$ iff there is an injective cofinal subreduction f of $\mathfrak{G}_k$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{E}$ and such that $X \cap \mathrm{dom} f = \emptyset$.*

As a consequence we obtain

THEOREM 74. *All finitely axiomatizable normal extensions of* **K4.3** *are decidable.*

1.12 *Quasi-normal modal logics*

All logics we have considered so far were *normal*, i.e., closed under the rule of necessitation $\varphi/\Box\varphi$. McKinsey and Tarski [1948] noticed, however, that by adding to **S4** the McKinsey axiom $\boldsymbol{ma} = \Box\Diamond p \to \Diamond\Box p$ and taking the closure under modus ponens and substitution we obtain a logic—let us denote it by **S4.1′**—which is not normal in that sense. To understand why

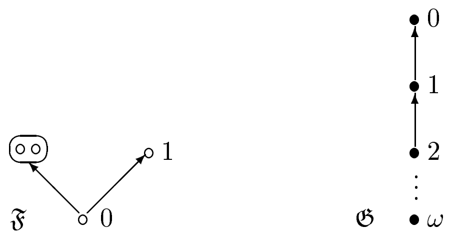


Figure 7.

this is so, consider the frame $\mathfrak{F}$ shown in Fig. 7. One can easily construct a model on $\mathfrak{F}$ such that $0 \not\models \Box \boldsymbol{ma}$ (0 sees a final proper cluster). On the other hand, $\boldsymbol{ma}$ and all its substitution instances are true at 0 (0 sees a final simple cluster), from which $\mathbf{S4.1}' \subseteq \{\varphi : 0 \models \varphi\}$ and so $\Box \boldsymbol{ma} \notin \mathbf{S4.1}'$.

A set of modal formulas containing $\mathbf{K}$ and closed under modus ponens and substitution was called by Segerberg [1971] a *quasi-normal logic*. The minimal quasi-normal extension of a logic L with formulas φ_i, $i \in I$, will be denoted by $L + \{\varphi_i : i \in I\}$ (i.e., the operation $+$ presupposes taking the closure under modus ponens and substitution only). Ext$L$ is the class of all quasi-normal logics above L. It is easy to see that a quasi-normal logic is normal iff it is closed under the congruence rule $p \leftrightarrow q / \Box p \leftrightarrow \Box q$.

Quasi-normal logics, introduced originally as some abstract (though natural) generalization of normal ones, attracted modal logicians' attention after Solovay [1976] constructed his provability logics $\mathbf{GL}$ and $\mathbf{S}$. The former one treats $\Box$ as "it is provable in Peano Arithmetic" and describes those properties of Gödel's provability predicate that are provable in PA; it is normal. The latter characterizes the properties of the provability predicate that are true in the standard arithmetic model, and in view of Gödel's Incompleteness Theorem it cannot be normal. (For a detailed discussion of provability logic consult *Modal Logic and Self-reference*.) Solovay showed in fact that

$$\mathbf{S} = \mathbf{GL} + \Box p \to p.$$

At first sight $\mathbf{S}$ may appear to be inconsistent: Löb's axiom requires frames to be irreflexive, while $\Box p \to p$ is refuted in them. And indeed, no Kripke frame validates both these axioms (in particular no consistent extension of $\mathbf{S}$ is normal).

Having the algebraic semantics for normal modal logics, it is fairly easy to construct an adequate algebraic semantics for a consistent $L \in \mathrm{Ext}\mathbf{K}$. Let M be a normal logic contained in L (for instance the greatest one, which is called the *kernel* of L) and $\mathfrak{A}_M$ its Tarski–Lindenbaum algebra (in Section 11 of *Basic Modal Logic* it was called the canonical modal algebra for M). The set

$$\nabla = \{[\varphi]_M : \varphi \in L\}$$

is clearly a filter in $\mathfrak{A}_M$. By the well known properties of the Tarski–Lindenbaum algebras, we then obtain the following completeness result: $\varphi \in L$ iff under every valuation in $\mathfrak{A}_M$ the value of φ belongs to ∇. Structures of the form $\langle \mathfrak{A}, \nabla \rangle$, where $\mathfrak{A}$ is a modal algebra and ∇ a filter in $\mathfrak{A}$, are known as *modal matrices*. Thus, every quasi-normal logic is characterized by a suitable class of modal matrices. It is not hard to see that L is normal iff it is characterized by a class of modal matrices with unit filters.

Now, going over to the dual (Stone–Jónsson–Tarski representation) $\mathfrak{A}_+$ of $\mathfrak{A}$ in a modal matrix $\langle \mathfrak{A}, \nabla \rangle$ and taking ∇_+ to be the set of ultrafilters in $\mathfrak{A}$ containing ∇, we arrive at the general frame $\mathfrak{A}_+$ with the set of *distinguished points* (or *actual worlds*) ∇_+. A formula φ is regarded to be valid in $\langle \mathfrak{A}_+, \nabla_+ \rangle$ iff under any valuation in $\mathfrak{A}_+$, φ is true at all points in ∇_+.

Taking into account the Generation Theorem, we can conclude that every quasi-normal modal logic is characterized by a suitable class of rooted general frames in which the root is regarded to be the only actual world. It follows in particular that, as was first observed by McKinsey and Tarski [1948],

$$\mathbf{K4} + \{\Box\varphi_i : i \in I\} = \mathbf{K4} \oplus \{\Box\varphi_i : i \in I\}.$$

However, one cannot replace here **K4** by **K** or **T**. Note also that as was shown by Segerberg [1971], **K**, **T** and some other standard normal logics are not finitely axiomatizable with modus ponens and substitution as the only postulated inference rules. Duality theory between modal matrices and frames with distinguished points can be developed along with duality theory for normal logics (for details see [Chagrov and Zakharyaschev 1997]). Kripke frames with distinguished points were used for studying quasi-normal logics by Segerberg [1971]. Modal matrices were considered by Blok and Köhler [1983] (under the name of filtered algebras), Chagrov [1985b], and Shum [1985].

EXAMPLE 75. Consider the (transitive) frame $\mathfrak{G} = \langle V, S, Q \rangle$ whose underlying Kripke frame is shown in Fig. 7 and Q consists of $\emptyset$, V, all finite sets of natural numbers and the complements to them in the space V (so $\omega \in X \in Q$ iff there is $n < \omega$ such that $m \in X$ for all $m \geq n$). Since $\mathfrak{G}$ is irreflexive and Noetherian, it validates **GL**. Moreover, we have $\langle \mathfrak{G}, \omega \rangle \models \Box p \to p$; for if under some valuation $\omega \models \Box p$ then p must be true at every point. It follows that $\mathfrak{G}$ with actual world ω validates **S**. (The reader can check that by making ω reflexive we again obtain a frame for **S**.)

By inserting the "tail" $\mathfrak{G}$ as in Fig. 7 into finite rooted frames for **GL** below their roots and using the fact that **GL** has FMP, one can readily show that, for every formula φ,

$$\varphi \in \mathbf{S} \text{ iff } \bigwedge_{\Box\psi \in \mathbf{Sub}\varphi} (\Box\psi \to \psi) \to \varphi \in \mathbf{GL}.$$

It follows in particular that **S** is decidable.

This example shows that the concepts of Kripke completeness and FMP do not play so important role in the quasi-normal case: even simple logics require infinite general frames. One possible way to cope with them at least in the transitive case is to extend the frame-theoretic language of the canonical formulas to the class Ext$\mathbf{K4}$.

Notice first that the canonical formulas, introduced in Section 1.6, cannot axiomatize all logics in Ext$\mathbf{K4}$. Indeed, $\langle \mathfrak{G}, w\rangle \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ iff there is a cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ and the following *actual world condition* as well:

(AWC) $f(w)$ is the root of $\mathfrak{F}$.

Now, consider the frame $\langle \mathfrak{G}, \omega\rangle$ constructed in Example 75. Since each set $X \in Q$ containing ω is infinite and has a dead end, it is impossible to reduce X to $\circ$ or $\bullet$, and so $\langle \mathfrak{G}, \omega\rangle$ validates all normal canonical formulas. On the other hand, we clearly have $\langle \mathfrak{G}, \omega\rangle \not\models B_n$ for every $n \geq 1$. So the logics $\mathbf{K4BD}_n$ cannot be axiomatized by normal canonical formulas without the postulated necessitation.

To get over this obstacle we have to modify the definition of subreduction so that such sets as X above may be "reduced" at least to irreflexive roots of frames. Given a frame $\mathfrak{G} = \langle V, S, Q\rangle$ with an *irreflexive* root u and a frame $\mathfrak{F} = \langle W, R, P\rangle$, we say a partial map f from W onto V is a *quasi-subreduction* of $\mathfrak{F}$ to $\mathfrak{G}$ if it satisfies (R1) for all $x, y \in \mathrm{dom} f$ such that $f(x) \neq u$ or $f(y) \neq u$, (R2) and (R3).[11] Thus, we may map all points in the frame $\mathfrak{G}$ in Fig. 7 to $\bullet$, and this map will be a quasi-reduction of $\mathfrak{G}$ to $\bullet$ satisfying (AWC). Actually, every frame is quasi-reducible to $\bullet$.

Now, given a finite frame $\mathfrak{F}$ with an irreflexive root a_0 and a set $\mathfrak{D}$ of antichains in $\mathfrak{F}$, we define the *quasi-normal canonical formula* $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot)$ as the result of deleting $\Box p_0$ from φ_0 in $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ (which says that a_0 is not self-accessible); the *quasi-normal negation free canonical formula* $\alpha^\bullet(\mathfrak{F}, \mathfrak{D})$ is defined in exactly the same way, starting from $\alpha(\mathfrak{F}, \mathfrak{D})$. It is not hard to see that $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot)$ (or $\alpha^\bullet(\mathfrak{F}, \mathfrak{D})$) is refuted in a frame $\langle \mathfrak{G}, w\rangle$ iff there is a cofinal (respectively, plain) quasi-subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$ and (AWC). The following result is obtained by an obvious generalization of the proof of Theorem 44 to frames with distinguished points (for details see [Zakharyaschev 1992]).

THEOREM 76. *There is an algorithm which, given a modal (negation free) formula φ, constructs a finite set Δ of normal and quasi-normal (negation free) canonical formulas such that $\mathbf{K4} + \varphi = \mathbf{K4} + \Delta$.*

For example, $\mathbf{S} = \mathbf{K4} + \alpha(\circ) + \alpha(\bullet)$. Since frames for $\mathbf{S4}$ are reflexive, we have

[11] Another possibility is to allow "reductions" of X to reflexive points by relaxing (R2); cf. Section 2.6.

COROLLARY 77. *There is an algorithm which, given a modal formula φ, constructs a finite set Δ of normal canonical formulas built on reflexive frames such that* $\mathbf{S4} + \varphi = \mathbf{S4} + \Delta$.

As a consequence we obtain

THEOREM 78 (Segerberg 1975). Ext$\mathbf{S4.3}$ = NExt$\mathbf{S4.3}$.

Proof. We must show that every logic $L \in \mathrm{Ext}\mathbf{S4.3}$ is normal, i.e., $\varphi \in L$ only if $\Box\varphi \in L$, for every φ. Suppose otherwise. Then by Corollary 77, there exists $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ such that $\Box\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \notin L$. Let $\langle \mathfrak{G}, w\rangle$ be a frame validating L and refuting $\Box\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Since $\mathfrak{G} \models \mathbf{S4.3}$, $\mathfrak{G}$ is a chain of non-degenerate clusters. And since it refutes $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ there is a cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$. It follows, in particular, that $\mathfrak{F}$ is also a chain of non-degenerate clusters and so $\mathfrak{D} = \emptyset$. Let a be the root of $\mathfrak{F}$. Define a map g by taking

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathrm{dom} f \\ a & \text{if } x \in f^{-1}(a){\downarrow} - \mathrm{dom} f \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It should be clear that g cofinally subreduces $\mathfrak{G}$ to $\mathfrak{F}$ and $g(w) = a$. Consequently, $\langle \mathfrak{G}, w\rangle \not\models \alpha(\mathfrak{F}, \bot)$, which is a contradiction. ■

Let us now briefly consider quasi-normal analogues of subframe and cofinal subframe logics in NExt$\mathbf{K4}$. Those logics that can be represented in the form

$$(\mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i) : i \in I\}) + \{\alpha(\mathfrak{F}_j) : j \in J\} + \{\alpha^\bullet(\mathfrak{F}_k) : k \in K\}$$

are called (*quasi-normal*) *subframe logics* and those of the form

$$(\mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \bot) : i \in I\}) + \{\alpha(\mathfrak{F}_j, \bot) : j \in J\} + \{\alpha^\bullet(\mathfrak{F}_k, \bot) : k \in K\}$$

are called (*quasi-normal*) *cofinal subframe logics*. The classes of quasi-normal subframe and cofinal subframe logics are denoted by $\mathcal{QSF}$ and $\mathcal{QCSF}$, respectively. The example of $\mathbf{S}$ shows that Theorem 52 cannot be extended to $\mathcal{QSF}$ and $\mathcal{QCSF}$. Yet one can show that all finitely axiomatizable logics in $\mathcal{QSF}$ and $\mathcal{QCSF}$ are decidable. We omit almost all proofs and confine ourselves mainly to formulations of relevant results. For details the reader is referred to [Zakharyaschev 1996].

We use the following notation. For a frame $\mathfrak{F} = \langle W, R\rangle$ with irreflexive root u and $0 < \xi < \omega$, $\mathfrak{F}^{ir}_\xi$ and $\mathfrak{F}^{r}_\xi$ denote the frames obtained from $\mathfrak{F}$ by replacing u with the descending chains $0, \dots, \xi - 1$ of irreflexive and reflexive points, respectively; $\mathfrak{F}^{ir}_{(\omega+1)^*} = \left\langle W_{(\omega+1)^*}, R^{ir}_{(\omega+1)^*}, P_{(\omega+1)^*}\right\rangle$ is the frame that results from $\mathfrak{F}$ by replacing u with the infinite descending chain

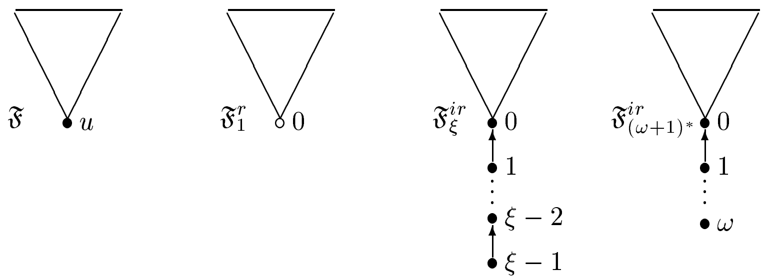


Figure 8.

$0, 1, \dots$ of irreflexive points and then adding irreflexive root ω, with $P_{(\omega+1)^*}$ containing all subsets of $W - \{u\}$, all finite subsets of natural numbers $\{0, 1, \dots\}$, all (finite) unions of these sets and all complements to them in the space $W_{(\omega+1)^*}$ (see Fig. 8). Note that $\mathfrak{F}$ is a quasi-reduct of every frame of the form $\mathfrak{F}^{ir}_{\xi}$, $\mathfrak{F}^{r}_{\xi}$ or $\mathfrak{F}^{ir}_{(\omega+1)^*}$.

The following theorem characterizes the canonical formulas belonging to logics in $\mathcal{QSF}$ and $\mathcal{QCSF}$.

THEOREM 79. *Suppose L is a subframe or cofinal subframe quasi-normal logic. Then*

(i) *for every finite frame $\mathfrak{F}$ with root u, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ iff $\langle \mathfrak{F}, u\rangle \not\models L$;*

(ii) *for every finite frame $\mathfrak{F}$ with irreflexive root u, $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ iff $\langle \mathfrak{F}, u\rangle \not\models L$, $\langle \mathfrak{F}^r_1, 0\rangle \not\models L$ and $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^*}, \omega \right\rangle \not\models L$.*

Proof. We prove only ($\Leftarrow$) of (ii). Let $\mathfrak{G} = \langle V, S, Q\rangle$ refute $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot)$ at its root w and show that $\langle \mathfrak{G}, w\rangle \not\models L$. We have a cofinal quasi-subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ such that $f(w) = u$. Consider the set $U = f^{-1}(u) \in Q$. Without loss of generality we may assume that $U = U\overline{\downarrow}$. There are three possible cases.

Case 1. The point w is irreflexive and $\{w\} \in Q$. Then the restriction of f to $\mathrm{dom} f - (U - \{w\})$ is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (AWC) and so $\langle \mathfrak{G}, w\rangle \not\models L$.

Case 2. There is $X \subseteq U$ such that $w \in X \in Q$ and, for every $x \in X$, there exists $y \in X \cap x\!\uparrow$. Then the restriction of f to $\mathrm{dom} f - (U - X)$ is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}^r_1$ satisfying (AWC) and so again $\langle \mathfrak{G}, w\rangle \not\models L$.

Case 3. If neither of the preceding cases holds then, for every $X \subseteq U$ such that $w \in X \in Q$, the set $D_X = X - X\!\downarrow$ of dead ends in X is a cover for X, i.e., $X \subseteq D_X\overline{\downarrow}$, and $w \in X - D_X \in Q$. Put

$$X_0 = D_U, \dots, X_{n+1} = D_{U-(X_0 \cup \dots \cup X_n)}, \dots, X_\omega = U - \bigcup_{\xi < \omega} X_\xi.$$

Each of these sets, save possibly X_ω, is an antichain of irreflexive points and belongs to Q. Besides, $X_\zeta \subset X_n{\downarrow} = \bigcup_{n<\xi\leq\omega} X_\xi$ for every $n < \zeta \leq \omega$. Therefore, the map g defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in V - U \\ \xi & \text{if } x \in X_\xi,\ 0 \leq \xi \leq \omega \end{cases}$$

is a cofinal quasi-subreduction of $\mathfrak{G}$ to $\mathfrak{F}^{ir}_{(\omega+1)^*}$ satisfying (AWC).

Now using the fact that $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^*}, \omega \right\rangle \not\models L$ and that the composition of (cofinal) (quasi-) subreductions is again a (cofinal) (quasi-) subreduction, it is not hard to see that $\langle \mathfrak{G}, w\rangle \not\models L$. ■

COROLLARY 80. *All subframe and cofinal subframe quasi-normal logics above* **S4** *have FMP.*

EXAMPLE 81. As an illustration let us use Theorem 79 to characterize those normal and quasi-normal canonical formulas that belong to **S**. Clearly, either $\alpha(\circ)$ or $\alpha(\bullet)$ is refuted at the root of every rooted Kripke frame. So all normal canonical formulas are in **S**. Every quasi-normal formula $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot)$ associated with $\mathfrak{F}$ containing a reflexive point is also in **S**, since $\Box\alpha(\circ)$ is refuted at the roots of $\mathfrak{F}$, $\mathfrak{F}^r_1$ and $\mathfrak{F}^{ir}_{(\omega+1)^*}$. But no quasi-normal formula $\alpha^\bullet(\mathfrak{F}, \mathfrak{D}, \bot)$ built on irreflexive $\mathfrak{F}$ belongs to **S**, because $\mathfrak{F}^{ir}_{(\omega+1)^*} \models \alpha(\circ)$ and $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^*}, \omega \right\rangle \models \alpha(\bullet)$, since $\{\omega\} \notin P_{(\omega+1)^*}$. Notice that incidentally we have proved the following completeness theorem for **S**.

THEOREM 82. **S** *is characterized by the class*

$$\{\left\langle \mathfrak{F}^{ir}_{(\omega+1)^*}, \omega \right\rangle : \mathfrak{F} \text{ is a finite rooted irreflexive frame}\}.$$

Theorem 79 reduces the decision problem for a logic L in $\mathcal{QSF}$ or $\mathcal{QCSF}$ to the problem of verifying, given a finite frame $\mathfrak{F}$ with root u, whether $\langle \mathfrak{F}, u\rangle$, $\langle \mathfrak{F}^r_1, 0\rangle$ and $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^*}, \omega \right\rangle$ refute an axiom of L. The two former frames present no difficulties: they are finite. As to the latter, it is not hard to see that, for instance, $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^*}, \omega \right\rangle \not\models \alpha^\bullet(\mathfrak{G}, \bot)$ iff $\left\langle \mathfrak{F}^{ir}_{\xi}, \xi - 1 \right\rangle$, for some $\xi \leq |\mathfrak{G}|$, is cofinally quasi-subreducible to $\mathfrak{G}$. Thus we obtain

THEOREM 83. *All finitely axiomatizable subframe and cofinal subframe quasi-normal logics are decidable.*

One can also give a frame-theoretic characterization of the classes $\mathcal{QSF}$ and $\mathcal{QCSF}$ similar to Theorem 53. Let us say that a frame $\mathfrak{F}$ with actual world u is a (*cofinal*) *subframe* of a frame $\mathfrak{G}$ with actual world w if $\mathfrak{F}$ is a (cofinal) subframe of $\mathfrak{G}$ and $u = w$.

THEOREM 84. *L is a (cofinal) subframe quasi-normal logic iff L is characterized by a class of frames with actual worlds that is closed under (cofinal) subframes.*

1.13 Tabular logics

Every logic L having the finite model property can be represented as the intersection of some *tabular logics*, that is logics characterized by finite frames (or models, algebras, matrices, etc.):

$$L = \bigcap \{\mathrm{Log}\mathfrak{F} : \mathfrak{F} \text{ is a finite frame for } L\}.$$

(It follows in particular that every fragment of L containing only those formulas whose length does not exceed some fixed $n < \omega$ is determined by a finite frame; for that reason logics with FMP are also called *finitely approximable*.) In many respects tabular logics are very easy to deal with. For instance, the key problem of recognizing whether a formula φ belongs to a tabular L is trivially decided by the direct inspection of all possible valuations of φ's variables in the finite frame characterizing L. That is why the question "is it tabular?" is one of the first items in the standard "questionnaire" for every new logical system.

First results concerning the tabularity of modal logics were obtained by Gödel [1932] and Dugundji [1940] who showed that intuitionistic propositional logic and all Lewis' modal systems **S1**–**S5** are not tabular. (Note that using the same method Drabbé [1967] proved that the three non-normal Lewis' systems **S1**–**S3** cannot be characterized by a matrix with a finite number of distinguished elements). For arbitrary logics in Ext**K** one can easily prove the following syntactical criterion of tabularity, which uses the formulas

$$\alpha_n = \neg(\varphi_1 \wedge \Diamond(\varphi_2 \wedge \Diamond(\varphi_3 \wedge \cdots \wedge \Diamond\varphi_n)\ldots)),$$

$$\beta_n = \bigwedge_{m=0}^{n-1} \neg\Diamond^m(\Diamond\varphi_1 \wedge \cdots \wedge \Diamond\varphi_n),$$

$$\boldsymbol{tab}_n = \alpha_n \wedge \beta_n,$$

where $\varphi_i = p_1 \wedge \cdots \wedge p_{i-1} \wedge \neg p_i \wedge p_{i+1} \wedge \cdots \wedge p_n$.

THEOREM 85. *$L \in \mathrm{Ext}\mathbf{K}$ is tabular iff $\boldsymbol{tab}_n \in L$, for some $n < \omega$.*

Proof. A frame $\mathfrak{F} = \langle W, R\rangle$ refutes α_n at a point x_1 iff a chain of length n starts from x_1, and $\mathfrak{F}$ refutes β_n at x_1 iff there is a chain $x_1Rx_2R\ldots Rx_m$ of length $m < n$ such that x_m is of branching n, i.e., $x_mRy_1, \ldots, x_mRy_n$ for some distinct $y_1, \ldots, y_n$. It follows that every rooted generated (by an actual world) subframe of the canonical frame for L containing $\boldsymbol{tab}_n$ has at most $1 + (n-1) + \cdots + (n-1)^{n-2}$ points. ■

As a consequence we immediately obtain

COROLLARY 86. *Every tabular modal logic has finitely many extensions and all of them are also tabular.*

The next theorem follows from general algebraic results of [Blok and Köhler 1983]; equally easy it can be proved using the characterization above.

THEOREM 87. *Every tabular logic $L \in$ Ext$\mathbf{K}$ is finitely axiomatizable.*

Proof. According to Theorem 85, L is an extension of $\mathbf{K} + \boldsymbol{tab}_n$, for some $n < \omega$. By Corollary 86, we have a chain

$$\mathbf{K} + \boldsymbol{tab}_n = L_1 \subset L_2 \subset \cdots \subset L_{k-1} \subset L_k = L$$

of quasi-normal logics such that $\{L' \in \mathrm{Ext}\mathbf{K} : \ L_i \subset L' \subset L_{i+1}\} = \emptyset$, for every $i = 1, \dots, k-1$. It remains to notice that if L' is finitely axiomatizable, $L' \subset L''$ and there is no logic located properly between L' and L'' then L'' is also finitely axiomatizable (e.g. $L'' = L' + \varphi$, for any $\varphi \in L'' - L'$). ■

Theorem 12 provides us in fact with an algorithm to decide, given a tabular logic $L \in$ NExt$\mathbf{K4}$ and an arbitrary formula φ, whether $\mathbf{K4} \oplus \varphi = L$. Indeed, notice first that we have

THEOREM 88. *Each finitely axiomatizable logic $L \in$ NExt$\mathbf{K4}$ of finite depth is a finite union-splitting, i.e., can be represented in the form*

$$L = \mathbf{K4} \oplus \{\alpha^\sharp(\mathfrak{F}_i, \bot) : i \in I\}$$

with finite I.

Proof. Let $L = \mathbf{K4} \oplus \varphi$ be a logic of depth n and let m be the number of variables in φ. We show that L coincides with the logic

$$L' = \mathbf{K4} \oplus \{\alpha^\sharp(\mathfrak{G}, \bot) : \ |\mathfrak{G}| \leq \sum_{i=1}^{n+1} 2^m c_m(i), \ \mathfrak{G} \not\models \varphi\}$$

($c_m(i)$ was defined in Section 1.2). The inclusion $L \supseteq L'$ is obvious. Suppose $\varphi \notin L'$. Then there is a rooted refined m-generated frame $\mathfrak{F}$ for L' refuting φ. Clearly, $\mathfrak{F}$ is of depth $\leq n$, since otherwise $\alpha^\sharp(\mathfrak{G}, \bot)$ is an axiom of L' for every rooted generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ of depth $n+1$ and so $\mathfrak{F} \not\models L'$, which is a contradiction. But then $\alpha^\sharp(\mathfrak{F}, \bot)$ is an axiom of L', contrary to our assumption. ■

Thus, all tabular logics in NExt$\mathbf{K4}$ are finite union-splittings and so, by Theorem 12, we obtain the following

THEOREM 89. *Let L be a tabular logic in* NExt$\mathbf{K4}$.

(i) (Blok 1980c) *L has finitely many immediate predecessors and they are also tabular.*

(ii) *The axiomatizability problem for L above* $\mathbf{K4}$ *is decidable.*

For logics in NExt$\mathbf{K}$ this is not the case, witness Theorems 36 and 205.

The tabularity criterion of Theorem 85 is not effective. Moreover, as we shall see in Section 4.4, no effective tabularity criterion exists in general. However, if we restrict attention to sufficiently strong logics, e.g. to the class NExt$\mathbf{S4}$, the tabularity problem turns out to be decidable. The key idea, proposed by Kuznetsov [1971], is to consider the so called pretabular logics.

A logic $L \in$ (N)ExtL_0 is said to be *pretabular* in the lattice (N)ExtL_0, if L is not tabular but every proper extension of L in (N)ExtL_0 is tabular. In other words, a pretabular logic in (N)ExtL_0 is a maximal non-tabular logic in (N)ExtL_0.

THEOREM 90. *In the lattices* Ext$\mathbf{K}$ *and* NExt$\mathbf{K}$ *every non-tabular logic is contained in a pretabular one.*

Proof. By Theorem 85, a logic is non-tabular iff it does not contain the formula $\boldsymbol{tab}_n$, for any $n < \omega$. It follows that the union of an ascending chain of non-tabular logics is a non-tabular logic as well. The standard use of Zorn's Lemma completes the proof. ■

If there is a simple description of all pretabular logics in a lattice, we obtain an effective (modulo the description) tabularity criterion for the lattice. Indeed, take for definiteness the lattice NExt$\mathbf{K4}$. How to determine, given a formula φ, whether $\mathbf{K4} \oplus \varphi$ is tabular? We may launch two parallel processes: one of them generates all derivations in $\mathbf{K4} \oplus \varphi$ and stops after finding a derivation of $\boldsymbol{tab}_n$, for some $n < \omega$; another process checks if φ belongs to a pretabular logic in NExt$\mathbf{K4}$ and stops if this is the case. The termination of the first process means that $\mathbf{K4} \oplus \varphi$ is tabular, while that of the second one shows that it is not tabular.

Unfortunately, it is impossible to describe in an effective way all pretabular logics in (N)Ext$\mathbf{K}$ and even (N)Ext$\mathbf{K4}$: Blok [1980c] and Chagrov [1989] constructed a continuum of them. However, for smaller lattices like NExt$\mathbf{S4}$ or NExt$\mathbf{GL}$ such descriptions were found by Maksimova [1975b], Esakia and Meskhi [1977] and Blok [1980c]. The five pretabular logics in NExt$\mathbf{S4}$ were presented in Section 17 of *Basic Modal Logic.* In NExt$\mathbf{GL}$ the picture is much more complicated.

THEOREM 91 (Blok 1980c, Chagrov 1989). *The set of pretabular logics in* NExt$\mathbf{GL}$ *is denumerable. It consists of the logics* $\mathbf{GL.3} = \mathrm{Log}\mathfrak{G}^{\omega}$ *and* $\mathrm{Log}\mathfrak{G}^{\omega}_{m,n}$*, for* $m \geq 0$*,* $n \geq 1$*, where* $\mathfrak{G}^{\omega}$ *and* $\mathfrak{G}^{\omega}_{m,n}$ *are the frames depicted in Fig. 9. If* $\langle m,n\rangle \neq \langle k,l\rangle$ *then* $\mathrm{Log}\mathfrak{G}^{\omega}_{m,n} \neq \mathrm{Log}\mathfrak{G}^{\omega}_{k,l}$.

Using this semantic description of pretabular logics in NExt$\mathbf{GL}$, it is not hard to find finite sets of formulas axiomatizing them. Moreover, all of them turn out to be decidable. For we have

THEOREM 92. *Every non-tabular logic* $L \in$ NExt$\mathbf{K4}$ *has a non-tabular extension with FMP, and so every pretabular logic in* NExt$\mathbf{K4}$ *has FMP.*

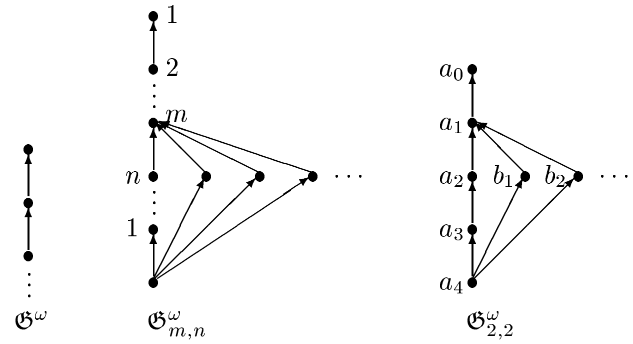


Figure 9.

Proof. Since L is non-tabular and characterized by the class of its rooted finitely generated refined frames, we have either a sequence $\mathfrak{F}_i$, $i = 1, 2, \ldots$, of rooted finite frames for L of depth i, or a sequence $\mathfrak{F}_i$ of rooted finite frames for L of width $\geq i$. In both cases the logic $\mathrm{Log}\{\mathfrak{F}_i : i < \omega\} \supseteq L$ is non-tabular and has FMP. ■

So we obtain the following result on the decidability of tabularity.

THEOREM 93. *The property of tabularity is decidable in* NExt**S4***,* Ext**S4***,* NExt**GL***,* Ext**GL**.

Since a logic in Ext**K4** is locally tabular iff it is determined by a frame of finite depth, the property of local tabularity is decidable in the lattices mentioned in Theorem 93 as well. However, this is not the case for Ext**K4** itself.

1.14 Interpolation

One of the fundamental properties of logics is their capability to provide explicit definitions of implicitly definable terms, which is known as the Beth property (Beth [1953] proved it for classical logic). In the modal case we say a logic L has the *Beth property* if, for any formula $\varphi(p_1, \ldots, p_n, p_{n+1})$ and variables p and q different from $p_1, \ldots, p_n$,

$$\varphi(p_1, \ldots, p_n, p) \wedge \varphi(p_1, \ldots p_n, q) \to (p \leftrightarrow q) \in L$$

only if there is a formula $\psi(p_1, \ldots p_n)$ such that

$$\varphi(p_1, \ldots, p_n, p) \to (p \leftrightarrow \psi(p_1, \ldots p_n)) \in L.$$

The Beth property turns out to be closely related to the interpolation property which was introduced by Craig [1957] for classical logic. Namely, we say that a logic L has the *interpolation property* if, for every implication $\alpha \to \beta \in L$, there exists a formula γ, called an *interpolant* for $\alpha \to \beta$ in L, such that $\alpha \to \gamma \in L$, $\gamma \to \beta \in L$ and every variable in γ, if any, occurs in both α and β. While in abstract model theory interpolation is weaker than Beth definability, for modal logics we have

THEOREM 94 (Maksimova 1992). *A normal modal logic has interpolation iff it has the Beth property.*

Say also that a normal modal logic L has the *interpolation property for the consequence relation* $\vdash_L^*$, $\vdash^*$*-interpolation* for short, if every time when $\alpha \vdash_L^* \beta$, there is a formula γ such that $\alpha \vdash_L^* \gamma$, $\gamma \vdash_L^* \beta$ and $\mathbf{Var}\gamma \subseteq \mathbf{Var}\alpha \cap \mathbf{Var}\beta$. (Here $\mathbf{Var}\varphi$ is the set of all variables in φ.) It should be clear that interpolation implies $\vdash^*$-interpolation.

By the end of the 1970s interpolation had been established for a good many standard modal systems. The semantical proofs, sometimes rather sophisticated, resemble the Henkin construction of the canonical models. Here are two examples of such proofs (which are due to Maksimova [1982b] and Smoryński [1978]).

THEOREM 95 (Gabbay 1972). *The logics* **K**, **K4**, **T**, **S4** *have the interpolation property.*

Proof. We consider only **S4**; for the other logics the proofs are similar. Suppose $\alpha \to \gamma \notin \mathbf{S4}$ and $\gamma \to \beta \notin \mathbf{S4}$ for any γ whose variables occur in both α and β, and show that in this case $\alpha \to \beta \notin \mathbf{S4}$.

Let $t = (\Gamma, \Delta)$ be a pair of sets of formulas such that $\mathbf{Var}\varphi \subseteq \mathbf{Var}\alpha$ if $\varphi \in \Gamma$ and $\mathbf{Var}\varphi \subseteq \mathbf{Var}\beta$ if $\varphi \in \Delta$. Say that t is *inseparable* if there are no formulas $\varphi_i \in \Gamma$, $\psi_j \in \Delta$ and γ with $\mathbf{Var}\gamma \subseteq \mathbf{Var}\alpha \cap \mathbf{Var}\beta$ such that $\bigwedge_{i=1}^n \varphi_i \to \gamma \in \mathbf{S4}$, $\gamma \to \bigvee_{i=1}^m \psi_i \in \mathbf{S4}$. The pair t is called *complete* if for every φ and ψ with $\mathbf{Var}\varphi \subseteq \mathbf{Var}\alpha$ and $\mathbf{Var}\psi \subseteq \mathbf{Var}\beta$, one of the formulas φ and $\neg\varphi$ is in Γ and one of ψ and $\neg\psi$ is in Δ.

LEMMA 96. *Every inseparable pair* $t_0 = (\Gamma_0, \Delta_0)$ *can be extended to a complete inseparable pair.*

Proof. Let $\varphi_1, \varphi_2, \dots$ and $\psi_1, \psi_2, \dots$ be enumerations of all formulas whose variables occur in α and β, respectively. Define pairs $t'_n = (\Gamma'_n, \Delta'_n)$ and $t_{n+1} = (\Gamma_{n+1}, \Delta_{n+1})$ inductively by taking

$$t'_n = \begin{cases} (\Gamma_n \cup \{\varphi_n\}, \Delta_n) & \text{if this pair is inseparable} \\ (\Gamma_n \cup \{\neg\varphi_n\}, \Delta_n) & \text{otherwise,} \end{cases}$$

$$t_{n+1} = \begin{cases} (\Gamma'_n, \Delta'_n \cup \{\psi_n\}) & \text{if this pair is inseparable} \\ (\Gamma'_n, \Delta'_n \cup \{\neg\psi_n\}) & \text{otherwise} \end{cases}$$

and put $t^* = (\Gamma^*, \Delta^*)$, where $\Gamma^* = \bigcup_{n<\omega} \Gamma_n$, $\Delta^* = \bigcup_{n<\omega} \Delta_n$. Clearly t^* is complete. Suppose it is separable, i.e., for some $\varphi_1, \dots, \varphi_n \in \Gamma^*$,

$\psi_1, \dots, \psi_m \in \Delta^*$ and some γ containing only those variables that occur in both α and β, we have $\bigwedge_{i=1}^n \varphi_i \to \gamma \in \mathbf{S4}$ and $\gamma \to \bigvee_{i=1}^m \psi_i \in \mathbf{S4}$. Then there is $k < \omega$ such that $\varphi_1, \dots, \varphi_n \in \Gamma_k$ and $\psi_1, \dots, \psi_m \in \Delta_k$, which means that t_k is separable. So it remains to show that if $t = (\Gamma, \Delta)$ is inseparable, $\mathbf{Var}\varphi \subseteq \mathbf{Var}\alpha$ and $\mathbf{Var}\psi \subseteq \mathbf{Var}\beta$ then

- one of the pairs $(\Gamma \cup \{\varphi\}, \Delta)$ or $(\Gamma \cup \{\neg\varphi\}, \Delta)$ is inseparable and
- one of the pairs $(\Gamma, \Delta \cup \{\psi\})$ or $(\Gamma, \Delta \cup \{\neg\psi\})$ is inseparable.

We prove only the former claim. Suppose, on the contrary, that both pairs are separable, i.e., there are formulas γ_1, γ_2 in variables occurring in both α and β such that, for some $\varphi_1, \dots, \varphi_n \in \Gamma$, $\psi_1, \dots, \psi_m \in \Delta$, we have

$$\varphi_1 \wedge \dots \wedge \varphi_n \wedge \varphi \to \gamma_1 \in \mathbf{S4},\ \gamma_1 \to \psi_1 \vee \dots \vee \psi_m \in \mathbf{S4},$$

$$\varphi_1 \wedge \dots \wedge \varphi_n \wedge \neg\varphi \to \gamma_2 \in \mathbf{S4},\ \gamma_2 \to \psi_1 \vee \dots \vee \psi_m \in \mathbf{S4}.$$

Then we obtain $(\varphi_1 \wedge \dots \wedge \varphi_n \wedge \varphi) \vee (\varphi_1 \wedge \dots \wedge \varphi_n \wedge \neg\varphi) \to \gamma_1 \vee \gamma_2 \in \mathbf{S4}$, $\gamma_1 \vee \gamma_2 \to \psi_1 \vee \dots \vee \psi_m \in \mathbf{S4}$, from which

$$\varphi_1 \wedge \dots \wedge \varphi_n \to \gamma_1 \vee \gamma_2 \in \mathbf{S4},\ \gamma_1 \vee \gamma_2 \to \psi_1 \vee \dots \vee \psi_m \in \mathbf{S4},$$

contrary to t being inseparable. ■

Now we define a frame $\mathfrak{F} = \langle W, R \rangle$ by taking W to be the set of all complete and inseparable pairs and, for $t_1 = (\Gamma_1, \Delta_1)$, $t_2 = (\Gamma_2, \Delta_2)$ in W, $t_1 R t_2$iff $\Box\varphi \in \Gamma_1$ implies $\varphi \in \Gamma_2$. Using the axioms $\Box p \to p$ and $\Box p \to \Box\Box p$ of $\mathbf{S4}$, one can readily check that R is a quasi-order on W, i.e., $\mathfrak{F} \models \mathbf{S4}$.

Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking for every variable $p \in \mathbf{Var}(\alpha \to \beta)$, $\mathfrak{V}(p) = \{(\Gamma, \Delta) \in W :$ either $p \in \Gamma$ or $p \in \mathbf{Var}\beta$ and $p \notin \Delta\}$. Put $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$. By induction on the construction of formulas φ and ψ with $\mathbf{Var}\varphi \subseteq \mathbf{Var}\alpha$, $\mathbf{Var}\psi \subseteq \mathbf{Var}\beta$ one can show that for every $t = (\Gamma, \Delta)$ in $\mathfrak{F}$

$$(\mathfrak{M}, t) \models \varphi \text{ iff } \varphi \in \Gamma,\ (\mathfrak{M}, t) \not\models \psi \text{ iff } \psi \in \Delta.$$

Indeed, the basis of induction follows from the definition of $\mathfrak{V}$ and the completeness and inseparability of t. The cases of the Boolean connectives present no difficulty. So suppose $\varphi = \Box\varphi_1$. If $t \models \Box\varphi_1$ then, for every $t' = (\Gamma', \Delta') \in t\uparrow$, we have $t' \models \varphi_1$ and so $\varphi_1 \in \Gamma'$. Suppose $\Box\varphi_1 \notin \Gamma$. Then $\neg\Box\varphi_1 \in \Gamma$. Consider the pair $t_0 = (\Gamma_0, \Delta_0)$, where

$$\Gamma_0 = \{\neg\varphi_1\} \cup \{\chi :\ \Box\chi \in \Gamma\},\ \Delta_0 = \{\neg\chi :\ \neg\Box\chi \in \Delta\},$$

and show that it is inseparable. Assume otherwise. Then there is γ with $\mathbf{Var}\gamma \subseteq \mathbf{Var}\alpha \cap \mathbf{Var}\beta$ such that, for some formulas $\Box\chi_1, \dots, \Box\chi_n \in \Gamma$, $\neg\Box\chi_{n+1}, \dots, \neg\Box\chi_m \in \Delta$,

$$\neg\varphi_1 \wedge \chi_1 \wedge \dots \wedge \chi_n \to \gamma \in \mathbf{S4},\ \gamma \to \neg\chi_{n+1} \vee \dots \vee \neg\chi_m \in \mathbf{S4}.$$

It follows that

$$\neg\Box\varphi_1 \wedge \Box\chi_1 \wedge \cdots \wedge \Box\chi_n \to \Diamond\gamma \in \mathbf{S4},$$

$$\Diamond\gamma \to \neg\Box\chi_{n+1} \vee \cdots \vee \neg\Box\chi_m \in \mathbf{S4},$$

contrary to t being inseparable. Let $t' = (\Gamma', \Delta')$ be a complete inseparable extension of t_0. By the definition of t_0, we have tRt' and so $\varphi_1 \in \Gamma'$, contrary to $\neg\varphi_1 \in \Gamma_0 \subseteq \Gamma'$ and t' being inseparable.

Suppose now that $\Box\varphi_1 \in \Gamma$. Then for every $t' = (\Gamma', \Delta')$ such that tRt', we have $\varphi_1 \in \Gamma$ and so $t' \models \varphi_1$. Consequently, $t \models \Box\varphi_1$. The formula ψ is treated in the dual way.

To complete the proof it remains to observe that $\mathfrak{M} \not\models \alpha \to \beta$. ■

This proof does not always go through for different kinds of logics. However, sometimes suitable modifications are possible.

THEOREM 97. **GL** *has the interpolation property.*

Proof. Suppose $\alpha \to \beta$ has no interpolant in **GL**. Our goal is to construct a finite irreflexive transitive frame refuting $\alpha \to \beta$.

This time we consider finite pairs $t = (\Gamma, \Delta)$ such that all formulas in Γ and Δ are constructed from variables and their negations using $\wedge$, $\vee$, $\Box$, $\Diamond$. Without loss of generality we will assume α and β to be formulas of that sort. Say that t is *separable* if there is a formula γ with $\mathbf{Var}\gamma \subseteq \mathbf{Var}\alpha \cap \mathbf{Var}\beta$ such that $\bigwedge\Gamma \to \gamma \in \mathbf{GL}$ and $\gamma \to \bigvee\Delta \in \mathbf{GL}$. It should be clear that if $t = (\Gamma, \Delta)$ is a finite inseparable pair then in the same way as in the proof of Theorem 95 but taking only subformulas of α and β we can obtain a finite inseparable pair $t^\star = (\Gamma^\star, \Delta^\star)$ satisfying the conditions: for every $\varphi \in \mathbf{Sub}\alpha$ and $\psi \in \mathbf{Sub}\beta$, one of the formulas φ and $\neg\varphi$ (an equivalent formula of the form under consideration, to be more precise) is in $\Gamma^\star$ and one of ψ and $\neg\psi$ is in $\Delta^\star$.

Now we construct by induction a finite rooted model for **GL** refuting $\alpha \to \beta$. As its root we take $(\{\alpha\}^\star, \{\beta\}^\star)$. If we have already put in our model a pair $t = (\Gamma, \Delta)$ and it has not been considered yet, then for every $\Diamond\varphi \in \Gamma$ and every $\Box\psi \in \Delta$, we add to the model the pairs

$$t_1 = (\{\chi, \Box\chi, \Box\neg\varphi, \varphi : \Box\chi \in \Gamma\}^\star, \{\chi, \Diamond\chi : \Diamond\chi \in \Delta\}^\star),$$

$$t_2 = \{\chi, \Box\chi : \Box\chi \in \Gamma\}^\star, \{\chi, \Diamond\chi, \Diamond\neg\psi, \psi : \Diamond\chi \in \Delta\}^\star).$$

One can readily show that if t is inseparable then t_1 and t_2 are also inseparable. Put $tR't_1$ and $tR't_2$. The process of adding new pairs must eventually terminate, since each step reduces the number of formulas of the form $\Diamond\varphi$ and $\Box\psi$ in the left and right parts of pairs. Let W be the set of all pairs constructed in this way and R the transitive closure of R'. Clearly, the resulting frame $\mathfrak{F} = \langle W, R\rangle$ validates **GL**. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ by taking, for each variable p,

$$\mathfrak{V}(p) = \{(\Gamma, \Delta) \in W : \ p \in \Gamma\}.$$

As in the proof of Theorem 95, it is easily shown that $\alpha \to \beta$ is refuted in $\mathfrak{F}$ under $\mathfrak{V}$. ■

To clarify the algebraic meaning of interpolation we require the following well known proposition.

PROPOSITION 98. *If ∇ is a normal filter*[12] *in a modal algebra $\mathfrak{A}$ then the relation $\sim_\nabla$, defined by $a \sim_\nabla b$ iff $a \leftrightarrow b \in \nabla$, is a congruence relation. The map $\nabla \mapsto \sim_\nabla$ is an isomorphism from the lattice of normal filters in $\mathfrak{A}$ onto the lattice of congruences in $\mathfrak{A}$.*

Denote by $\mathfrak{A}/\nabla$ the quotient algebra $\mathfrak{A}/\sim_\nabla$ and let $\|a\|_\nabla = \{b : a \sim_\nabla b\}$.

Say that a class $\mathcal{C}$ of algebras is *amalgamable* if for all algebras $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ in $\mathcal{C}$ such that $\mathfrak{A}_0$ is embedded in $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by isomorphisms f_1 and f_2, respectively, there exist $\mathfrak{A} \in \mathcal{C}$ and isomorphisms g_1 and g_2 of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ into $\mathfrak{A}$ with $g_1(f_1(x)) = g_2(f_2(x))$, for any x in $\mathfrak{A}_0$. If in addition we have

$$g_i(x) \leq g_j(y) \text{ implies } \exists z \in A_0 \ (x \leq_i f_i(z) \text{ and } f_j(z) \leq_j y)$$

for all $x \in A_i$, $y \in A_j$ such that $\{i, j\} = \{1, 2\}$, then $\mathcal{C}$ is called *superamalgamable*. Here A_i is the universe of $\mathfrak{A}_i$ and $\leq_i$ its lattice order.

THEOREM 99 (Maksimova 1979). *L has the interpolation property iff the variety* Alg$L$ *of modal algebras for L is superamalgamable. L has the $\vdash^*$-interpolation property iff* Alg$L$ *is amalgamable.*

Proof. We prove only the former claim. ($\Rightarrow$) Suppose L has the interpolation property and $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ are modal algebras for L such that $\mathfrak{A}_0$ is a subalgebra of both $\mathfrak{A}_1$ and $\mathfrak{A}_2$. With each element $a \in A_i$, $i = 0, 1, 2$, we associate a variable p_a^i in such a way that, for $a \in A_0$, $p_a^0 = p_a^1 = p_a^2$. Denote by $\mathcal{L}_i$ the language with the variables p_a^i, for $a \in A_i$, $i = 0, 1, 2$, and let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$. We will assume that $\mathcal{L}$ is the language of L.

Fix the valuation $\mathfrak{V}_i$ of $\mathcal{L}_i$ in $\mathfrak{A}_i$, defined by $\mathfrak{V}_i(p_a^i) = a$, and put

$$\Sigma_i = \{\varphi \in \mathbf{For}\mathcal{L}_i : \mathfrak{V}_i(\varphi) = \top\}.$$

Let Σ be the closure of $\Sigma_1 \cup \Sigma_2 \cup L$ under modus ponens. We show that, for every $\varphi \in \mathbf{For}\mathcal{L}_i$, $\psi \in \mathbf{For}\mathcal{L}_j$ such that $\{i, j\} = \{1, 2\}$,

(13) $\varphi \to \psi \in \Sigma$ iff $\exists \chi \in \mathbf{For}\mathcal{L}_0$ $(\varphi \to \chi \in \Sigma_i$ and $\chi \to \psi \in \Sigma_j)$.

[12]A filter ∇ is *normal* (or open, as in Section 10 of *Basic Modal Logic*) if $\Box a \in \nabla$ whenever $a \in \nabla$.

Suppose $\varphi \to \psi \in \Sigma$. Then there exist finite sets $\Gamma_i \subseteq \Sigma_i$ and $\Gamma_j \subseteq \Sigma_j$ such that

$$\bigwedge \Gamma_i \wedge \varphi \to (\bigwedge \Gamma_j \to \psi) \in L.$$

Since L has interpolation, there is a formula $\chi \in \mathbf{For}\mathcal{L}_0$ such that

$$\bigwedge \Gamma_i \wedge \varphi \to \chi \in L, \ \bigwedge \Gamma_j \to (\chi \to \psi) \in L,$$

from which $\varphi \to \chi \in \Sigma_i$ and $\chi \to \psi \in \Sigma_j$. The converse implication is obvious.

Now construct an algebra $\mathfrak{A}$ by taking the set $\{\|\varphi\| : \varphi \in \Sigma\}$ as its universe, where $\|\varphi\| = \{\psi : \varphi \leftrightarrow \psi \in \Sigma\}$, $\|\varphi\| \wedge \|\psi\| = \|\varphi \wedge \psi\|$ and $\odot\|\varphi\| = \|\odot\varphi\|$, for $\odot \in \{\neg, \Box\}$. One can readily prove that $\mathfrak{A} \in \mathrm{Alg}L$. Define maps g_i from $\mathfrak{A}_i$ into $\mathfrak{A}$ by taking $g_i(a) = \|p_a^i\|$. It is not difficult to show that g_i is an embedding of $\mathfrak{A}_i$ in $\mathfrak{A}$. And for $a \in A_0$, we have

$$g_1(a) = \|p_a^0\| = g_2(a).$$

It remains to check the condition for superamalgamability: Suppose $a \in A_i$, $b \in A_j$, $\{i, j\} = \{1, 2\}$, and $g_i(a) \leq g_j(b)$. Then $g_i(a) \to g_j(b) = \top$ and so $\|p_a^i \to p_b^j\| = \top$, i.e., $p_a^i \to p_b^j \in \Sigma$. By (13), we have $\chi \in \mathbf{For}\mathcal{L}_0$ with $\mathfrak{V}(\chi) = c$ such that $a \leq_i c \leq_j b$.

($\Leftarrow$) Assuming $\mathrm{Alg}L$ to be superamalgamable, we show that L has the interpolation property. To this end we require

LEMMA 100. *Suppose $\mathfrak{A}_0$ is a subalgebra of modal algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, $a \in A_1$, $b \in A_2$ and there is no $c \in A_0$ such that $a \leq_1 c \leq_2 b$. Then there are ultrafilters ∇_1 in $\mathfrak{A}_1$ and ∇_2 in $\mathfrak{A}_2$ such that $a \in \nabla_1$, $b \notin \nabla_2$ and $\nabla_1 \cap A_0 = \nabla_2 \cap A_0$.*

Suppose $\varphi(p_1, \dots, p_m, q_1, \dots, q_n)$ and $\psi(q_1, \dots, q_n, r_1, \dots, r_l)$ are formulas for which there is no $\chi(q_1, \dots, q_n)$ such that $\varphi \to \chi \in L$ and $\chi \to \psi \in L$. We show that in this case there exists an algebra $\mathfrak{A} \in \mathrm{Var}L$ refuting $\varphi \to \psi$.

Let $\mathfrak{A}_0'$, $\mathfrak{A}_1'$ and $\mathfrak{A}_2'$ be the free algebras in $\mathrm{Alg}L$ generated by the sets $\{c_1, \dots, c_n\}$, $\{a_1, \dots, a_m, c_1, \dots, c_n\}$ and $\{c_1, \dots, c_n, b_1, \dots, b_l\}$, respectively. According to this definition, $\mathfrak{A}_0'$ is a subalgebra of both $\mathfrak{A}_1'$ and $\mathfrak{A}_2'$. By Lemma 100, there are ultrafilters ∇_1 in $\mathfrak{A}_1'$ and ∇_2 in $\mathfrak{A}_2'$ such that we have $\varphi(a_1, \dots, a_m, c_1, \dots, c_n) \in \nabla_1$ and $\psi(c_1, \dots, c_n, b_1, \dots, b_l) \notin \nabla_2$. Define normal filters

$$\nabla_i^* = \{a \in A_i' : \forall m < \omega \ \Box^m a \in \nabla_i\}$$

and put $\mathfrak{A}_1 = \mathfrak{A}_1'/\nabla_1^*$, $\mathfrak{A}_2 = \mathfrak{A}_2'/\nabla_2^*$. Construct an algebra $\mathfrak{A}_0$ by taking $A_0 = \{\|a\|_{\nabla_1^*} : a \in A_0'\}$. By the definition, $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$, i.e., is embedded in $\mathfrak{A}_1$ by the map $f_1(x) = x$. One can show that $\mathfrak{A}_0$ is embedded in $\mathfrak{A}_2$ by the map $f_2(\|x\|_{\nabla_1}) = \|x\|_{\nabla_2^*}$. Then there are an algebra $\mathfrak{A}$ for L and isomorphisms g_1 and g_2 of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ into $\mathfrak{A}$ satisfying the conditions of superamalgamability. Define a valuation $\mathfrak{V}$ in $\mathfrak{A}$ by taking $\mathfrak{V}(p_i) =$

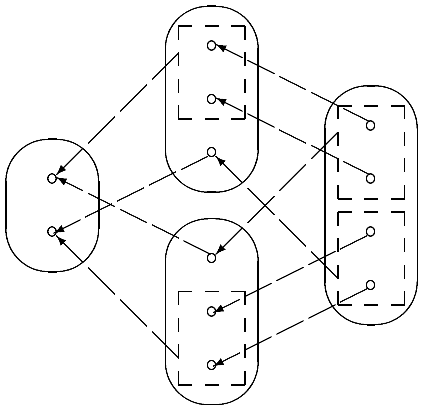


Figure 10.

$g_1(\|a_i\|_{\nabla_1})$, $\mathfrak{V}(q_j) = g_1(\|c_j\|_{\nabla_1}) = g_2(\|c_j\|_{\nabla_2})$ and $\mathfrak{V}(r_k) = g_2(\|b_k\|_{\nabla_2})$. Then $\mathfrak{V}(\varphi) \not\leq \mathfrak{V}(\psi)$ because otherwise there would exist $\{i,j\} = \{1,2\}$ and $z \in A_0$ such that $\mathfrak{V}(\varphi) \leq_i f_i(z)$ and $f_j(z) \leq_j \mathfrak{V}(\psi)$. Thus, $\mathfrak{A} \not\models \varphi \to \psi$ and so $\varphi \to \psi \notin L$. ■

Using this theorem Maksimova [1979] discovered a surprising fact: there are only finitely many logics in NExt**S4** with the interpolation property (not more than 38, to be more exact) and all of them turned out to be union-splittings. By Theorem 12, we obtain then

THEOREM 101 (Maksimova 1979). *There is an algorithm which, given a modal formula φ, decides whether* $\mathbf{S4} \oplus \varphi$ *has interpolation.*

We illustrate this result by considering a much simpler class of logics.

THEOREM 102. *Only four logics in* NExt**S5** *have the interpolation property:* **S5** *itself, the logic of the two-point cluster,* **Triv** *and* **For**.

Proof. We have already demonstrated how to prove that a logic has interpolation. So now we show only that no logic L in NExt**S5** different from those mentioned in the formulation has the interpolation property. Suppose on the contrary that L has interpolation. We use the amalgamability of the variety of modal algebras for L to show that an arbitrary big finite cluster is a frame for L, from which it will follow that $L = \mathbf{S5}$.

Figure 10 demonstrates two ways of reducing the three-point cluster to the two-point one. By the amalgamation property, there must exist a cluster reducible to the two depicted copies of the two-point cluster, with the reductions satisfying the amalgamation condition. It should be clear from

Fig. 10 that such a cluster contains at least four points. By the same scheme one can prove now that every n-point cluster validates L. ■

It would be naive to expect that such a simple picture can be extended to classes like NExt**K4** or NExt**K**. Even in NExt**GL** the situation is quite different from that in NExt**S4**: Maksimova [1989] discovered that there is a continuum of logics in NExt**GL** having the interpolation property. This result is based upon the following observation. For $L \in$ NExt**K4**, we call a formula $\alpha(p)$ *conservative* in NExt$L$ if

$$\Box^+(\alpha(\bot) \wedge \alpha(p) \wedge \alpha(q)) \to \alpha(p \to q) \wedge \alpha(\Box p) \in L.$$

For example, in NExt**S4** conservative are $\Box\Diamond p \to \Diamond\Box p$, $\Box\Diamond p \leftrightarrow \Diamond\Box p$, and $\Box p \leftrightarrow \Diamond p$.

THEOREM 103 (Maksimova 1987). *If $L \in$ NExt**K4** has the interpolation property and formulas α_i, for $i \in I$, are conservative in NExt$L$, then the logic $L \oplus \{\alpha_i : i \in I\}$ also has the interpolation property.*

Proof. Suppose $\varphi \to \psi \in L \oplus \{\alpha_i : i \in I\}$. Then there is a finite $J \subseteq I$, say $J = \{1, \dots, l\}$, such that $\varphi \to \psi \in L \oplus \{\alpha_i : i \in J\}$ and so, as follows from the definition of conservative formulas and the Deduction Theorem for **K4**,

$$\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(\bot) \wedge \alpha_j(p_1) \wedge \cdots \wedge \alpha_j(p_n)) \to (\varphi \to \psi) \in L,$$

where $p_1, \dots, p_m, p_{m+1}, \dots, p_k$ and $p_{m+1}, \dots, p_k, p_{k+1}, \dots, p_n$ are all the variables in φ and ψ, respectively. Consequently

$$\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(\bot) \wedge \alpha_j(p_1) \wedge \cdots \wedge \alpha_j(p_k)) \wedge \varphi \to$$
$$(\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(p_{m+1}) \wedge \cdots \wedge \alpha_j(p_n)) \to \psi) \in L.$$

Since L has the interpolation property, there is $\chi(p_{m+1}, \dots, p_k)$ such that

$$\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(\bot) \wedge \alpha_j(p_1) \wedge \cdots \wedge \alpha_j(p_k)) \wedge \varphi \to \chi \in L,$$

$$\Box^+ \bigwedge_{j=1}^{l} (\alpha_j(p_{m+1}) \wedge \cdots \wedge \alpha_j(p_n)) \to (\chi \to \psi) \in L.$$

Then we obtain $\varphi \to \chi \in L \oplus \{\alpha_i : \ i \in I\}$ and $\chi \to \psi \in L \oplus \{\alpha_i : \ i \in I\}$, i.e., χ is an interpolant for $\varphi \to \psi$ in $L \oplus \{\alpha_i : i \in I\}$. ■

Using the formulas

$$\alpha_i = \Box^+(\Diamond^{i+1}\top \wedge \Box^{i+2}\bot \to \Box^{i+1}p \vee \Box^{i+1}\neg p)$$

which are conservative in NExt**GL**, one can readily construct a continuum of logics in this class with the interpolation property. The set of logics in NExt**GL** without interpolation is also continual.

In general, an interpolant γ for an implication $\alpha \to \beta \in L$ depends on both α and β. Say that a logic L has *uniform interpolation* if, for any finite set of variables Ξ and any formula α, there exists a formula γ such that $\mathbf{Var}\gamma \subseteq \Xi$ and $\alpha \to \gamma \in L$, $\gamma \to \beta \in L$ whenever $\mathbf{Var}\alpha \cap \mathbf{Var}\beta \subseteq \Xi$ and $\alpha \to \beta \in L$. In this case γ is called a *post-interpolant* for α and Ξ. Roughly speaking, a logic has uniform interpolation if we can choose an interpolant for $\alpha \to \beta \in L$ independly from the actual shape of β. Uniform interpolation was first investigated by Pitts [1992] who proved that intuitionistic logic enjoys it. It is fairly easy to find multiple examples of modal logics with uniform interpolation by observing that any locally tabular logic with interpolation has uniform interpolation as well. Indeed, for every formula α and every set of variables Ξ, we can define a post-interpolant γ as the conjunction of a maximal set of pairwise non-equivalent in L formulas γ' such that $\mathbf{Var}\gamma' \subseteq \Xi$ and $\alpha \to \gamma' \in L$ (which is finite in view of the local tabularity of L). It follows, for instance, that **S5** has uniform interpolation. In general, however, interpolation does not imply uniform interpolation: [Ghilardi and Zawadowski 1995] showed that **S4** does not enjoy the latter, witness the following formula without a post-interpolant for $\{r\}$ in **S4**

$$p \wedge \Box(p \to \Diamond q) \wedge \Box(q \to \Diamond p) \wedge \Box(p \to r) \wedge \Box(q \to \neg r).$$

Only a few positive results on the uniform interpolation of modal logics are known: Shavrukov [1993] proved it for **GL**, Ghilardi [1995] for **K**, and Visser [1996] for **Grz**.

A property closely related to interpolation is so called Halldén completeness. A logic L is said to be *Halldén complete* if $\varphi \vee \psi \in L$ and $\mathbf{Var}\varphi \cap \mathbf{Var}\psi = \emptyset$ imply $\varphi \in L$ or $\psi \in L$. Since every variable free formula is equivalent in **D** either to $\top$ or to $\bot$, $L \in \mathrm{Ext}\mathbf{D}$ is Halldén complete whenever it has interpolation. **K**, **K4**, **GL** are examples of Halldén incomplete logics with interpolation: each of them contains $\Diamond\top \vee \neg\Diamond\top$ but not $\Diamond\top$ and $\neg\Diamond\top$. On the other hand, **S4.3** is a Halldén complete logic (see [van Benthem and Humberstone 1983]) without interpolation (see [Maksimova 1982a]). Actually, there is a continuum of Halldén complete logics in NExt**S4** (see [Chagrov and Zakharyaschev 1993]).

Halldén completeness has an interesting lattice-theoretic characterization.

THEOREM 104 (Lemmon 1966c). *A logic $L \in \mathrm{Ext}\mathbf{K}$ is Halldén complete iff it is $\bigcap$-irreducible in* $\mathrm{Ext}L$.

Since the lattice $\mathrm{Ext}\mathbf{S5}$ is linearly ordered by inclusion, all logics above **S5** are Halldén complete. There are various semantic criteria for Halldén completeness (see e.g. [Maksimova 1995]). Here we note only the following generalization of the result of [van Benthem and Humberstone 1983].

THEOREM 105. *Suppose a logic $L \in \mathrm{Ext}\mathbf{K}$ is characterized by a class $\mathcal{C}$ of descriptive rooted frames with distinguished roots. Then L is Halldén complete iff, for all frames $\langle \mathfrak{F}_1, d_1\rangle$ and $\langle \mathfrak{F}_2, d_2\rangle$ in $\mathcal{C}$, there is a frame $\langle \mathfrak{F}, d\rangle$ for L reducible*[13] *to both $\langle \mathfrak{F}_1, d_1\rangle$ and $\langle \mathfrak{F}_2, d_2\rangle$.*

For more results and references on Halldén completeness consult [Chagrov and Zakharyaschev 1991].

2 POLYMODAL LOGICS

So far we have confined ourselves to considering modal logics with only one necessity operator. From a theoretical point of view this restriction is not such a great loss as it may seem at first sight. In fact, really important concepts of modal logic do not depend on the number of boxes and can be introduced and investigated on the basis of just one. We shall give a precise meaning to this claim in Section 2.3 below where it is shown that polymodal logic is reduced in a natural way to unimodal logic. However, there are at least two reasons for a detailed discussion of polymodal logic in this chapter.

First, a number of interesting phenomena are easily missed in unimodal logic and actually appear in a representative form only in the polymodal case. For example, with the exception of $\mathrm{NExt}\mathbf{K4.3}$ and $\mathcal{QCSF}$ all known general decidability results in unimodal logic have been obtained by proving the finite model property. In fact, nearly all natural classes of logics in $\mathrm{NExt}\mathbf{K}$ turned out to be describable by their finite frames. The situation drastically changes with the addition of just one more box. Even in the case of linear tense logics or bimodal provability logics one has to start with a thorough investigation of their infinite frames: FMP becomes a rather rare guest. While the result on $\mathrm{NExt}\mathbf{K4.3}$ indicated the need for general methods of establishing decidability without FMP, this need becomes of vital importance only in the context of polymodal logic.

The second reason is that various applications of modal logic require polymodal languages. For example, in tense logic we have two necessity-like operators $\Box_1$ and $\Box_2$. One of them, say the former, is interpreted as "it will always be true" and the other as "it was always true". Kripke frames for tense logics are structures $\langle W, R_1, R_2\rangle$ with two binary relations R_1 and R_2

[13] By reductions that map d to d_i.

such that R_2 coincides with the converse R_1^{-1} of R_1 (which reflects the fact that a moment x is earlier than y iff y is later than x). The characteristic axioms connecting the two tense operators are

$$p \to \Box_1 \Diamond_2 p \text{ and } p \to \Box_2 \Diamond_1 p.$$

For more information about tense systems consult *Basic Tense Logic*.

Another example is basic temporal logic in which we have two necessity-like operators: one of them—usually called Next—is interpreted by the successor relation in ω and the other by its transitive and reflexive closure. Details can be found in [Segerberg 1989]. Propositional dynamic logic **PDL** and its extensions, like deterministic **PDL**, can also be regarded as polymodal logics (see *Dynamic Logic*).

A number of provability logics use two or more modal operators; see e.g. Boolos [1993]. In **GLB**, for instance, we have one operator $\Box_1$ understood as provability in PA and another operator $\Box_2$ interpreted as ω-provability in PA. The unimodal fragments of **GLB** coincide with **GL**. The axioms connecting $\Box_1$ and $\Box_2$ are

$$\Box_1 p \to \Box_2 p \text{ and } \Diamond_1 p \to \Box_2 \Diamond_1 p.$$

In epistemic logics we need an operator $\Box_i$ for each agent i; $\Box_i \varphi$ is interpreted as "agent i believes (or knows) φ". One possible way to axiomatize the logic of knowledge with m agents is to take the axioms of **S5** for each agent without any principles connecting different $\Box_i$ and $\Box_j$. We denote the resultant logic by $\bigotimes_{i=1}^m \mathbf{S5}$. Often $\bigotimes_{i=1}^m \mathbf{S5}$ is extended by the common knowledge operator C with the intended meaning

$$\mathsf{C}\varphi = \mathsf{E}\varphi \wedge \mathsf{E}^2\varphi \wedge \cdots \wedge \mathsf{E}^n\varphi \wedge \ldots, \text{ where } \mathsf{E}\varphi = \textstyle\bigwedge_{i=1}^m \Box_i \varphi$$

(see e.g. [Halpern and Moses 1992] and [Meyer and van der Hoek 1995]).

The reader will find more items for this list in other chapters of the *Handbook*.

From the semantical point of view, many standard polymodal logics can be obtained by applying Boolean or various natural closure operators to the accessibility relations of Kripke frames. For instance, in frames $\langle W, R_1, \ldots, R_n \rangle$ for epistemic logic the common knowledge operator is interpreted by the transitive closure of $R_1 \cup \cdots \cup R_n$. Tense frames result from usual $\langle W, R \rangle$ by adding the converse of R. Humberstone [1983] and Goranko [1990a] study the bimodal logic of *inaccessible worlds* determined by frames of the form $\langle W, R, W^2 - R \rangle$. This list of examples can be continued; for a general approach and related topics consult [Goranko 1990b], [Gargov *et al.* 1987], [Gargov and Passy 1990].

Let us see now how polymodal logics in general fit into the theory developed so far. We begin by demonstrating how the concepts introduced in the unimodal case transfer to polymodal logic and showing that a few general results—like Sahlqvist's and Blok's Theorems—have natural analogues in polymodal logic. We hope to convince the reader that up to this point no new difficulties arise when one switches from the unimodal language to the polymodal one. After that, in Section 2.2, we start considering subtler features of polymodal logics.

2.1 From unimodal to polymodal

Let $\mathcal{L}_I$ be the propositional language with a finite number of necessity operators $\Box_i$, $i \in I$. A *normal polymodal logic* in $\mathcal{L}_I$ is a set of $\mathcal{L}_I$-formulas containing all classical tautologies, the axioms $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$ for all $i \in I$, and closed under substitution, modus ponens and the rule of necessitation $\varphi/\Box_i\varphi$ for every $i \in I$. If the language is clear from the context, we call these logics just (*normal*) *modal logics* and denote by $\mathrm{NExt}L$ the family of all normal extensions of L (in the language $\mathcal{L}_I$). The smallest normal modal logic with n necessity operators is denoted by $\mathbf{K}_n$ ($\mathbf{K} = \mathbf{K}_1$, of course).

Given a logic L_0 in $\mathcal{L}_I$ and a set of $\mathcal{L}_I$-formulas Γ, we again denote by $L_0 \oplus \Gamma$ the smallest normal logic (in $\mathcal{L}_I$) containing $L_0 \cup \Gamma$. A number of other notions and results also transfer in a rather straightforward way, e.g. Theorems 4 and 6, Proposition 5 and all concepts involved in their formulations. More care has to be taken to generalize Theorems 1, 2 and 3. Denote by $\boldsymbol{M}_I^*$ the set of non-empty strings (words) over $\{\Box_i : i \in I\}$ which do not contain any $\Box_i$ twice and put

$$\Box_I\varphi = \bigwedge\{\boldsymbol{M}\varphi : \boldsymbol{M} \in \boldsymbol{M}_I^*\},\ \Box_I^{\leq m}\varphi = \bigwedge\{\Box_I^n\varphi : n \leq m\}.$$

In the language $\mathcal{L}_I$ the operator $\Box_I$ serves as a sort of surrogate for $\Box$ in $\mathbf{K}$. For example, the following polymodal version of Theorem 1 holds.

THEOREM 106 (Deduction). *For every modal logic L in $\mathcal{L}_I$, every set of $\mathcal{L}_I$-formulas Γ, and all $\mathcal{L}_I$-formulas φ and ψ,*

$$\Gamma, \psi \vdash_L^* \varphi \text{ iff } \exists m \geq 0\ \Gamma \vdash_L^* \Box_I^{\leq m}\psi \to \varphi.$$

Theorems 2 and 3 can be reformulated analogously by replacing $\Box$ with $\Box_I$ (a logic L in $\mathcal{L}_I$ is *n-transitive* if it contains $\Box_I^{\leq n}p \to \Box_I^{n+1}p$).

Basic semantic concepts are lifted to the polymodal case in a straightforward manner. The algebraic counterpart of $L \in \mathrm{NExt}\mathbf{K}_n$ is the variety of Boolean algebras with n unary operators validating L. A structure $\mathfrak{F} = \langle W, \langle R_i : i \in I\rangle, P\rangle$ is called a (*general polymodal*) *frame* whenever every $\langle W, R_i, P\rangle$, for $i \in I$, is a unimodal frame. We then put

$$\Box_i X = \{x \in W : \forall y\ (xR_iy \to y \in X)\}.$$

Differentiated, *refined* and *descriptive frames* and the truth-preserving operations can also be defined in the same component-wise way. For instance, a frame $\mathfrak{F} = \langle W, \langle R_i : i \in I\rangle, P\rangle$ is differentiated if all the unimodal frames $\langle W, R_i, P\rangle$, for $i \in I$, are differentiated. $\mathfrak{F} = \langle W, \langle R_i : i \in I\rangle, P\rangle$ is a (*generated*) *subframe* of $\mathfrak{G} = \langle V, \langle S_i : i \in I\rangle, Q\rangle$ if all $\langle W, R_i, P\rangle$ are (generated) subframes of $\langle V, S_i, Q\rangle$, and f is a *reduction* of $\mathfrak{F}$ to $\mathfrak{G}$ if f is a reduction of $\langle W, R_i, P\rangle$ to $\langle V, S_i, Q\rangle$, for every $i \in I$.

There are some exceptions to this rule. A point r is called a root of $\mathfrak{F}$ if it is a root of the unimodal frame $\langle W, \bigcup_{i\in I} R_i\rangle$. This does not mean that r is a root of all unimodal reducts of $\mathfrak{F}$. Another important exception: as before, a polymodal frame is *$\varkappa$-generated* if the algebra $\mathfrak{F}^+$ is $\varkappa$-generated; however, this does not mean that the unimodal reducts of $\mathfrak{F}$ are $\varkappa$-generated.

Splittings and the degree of Kripke incompleteness The semantic criterion of splittings by finite frames given in Theorem 15 transfers to polymodal logics by replacing $\Box$ with $\Box_I$. Again, all finite rooted frames split $\mathrm{NExt}L_0$, if L_0 is an n-transitive logic in $\mathcal{L}_I$. Notice, however, that n-transitivity is a rather strong condition in the polymodal case. For example, it is easily checked that the fusion $\mathbf{S5} \otimes \mathbf{S5}$ as well as the minimal tense logic $\mathbf{K4}.t$ containing $\mathbf{K4}$ are not n-transitive, for any $n < \omega$ (see Sections 2.2 and 2.4 for precise definitions). In fact, only $\circ$ splits the lattice $\mathrm{NExt}(\mathbf{S5} \otimes \mathbf{S5})$ and only $\bullet$ splits $\mathrm{NExt}\mathbf{K4}.t$ (see [Wolter 1993] and [Kracht 1992], respectively).

Call a frame $\langle W, \langle R_i : i \in I\rangle\rangle$ *cycle free* if the unimodal frame $\langle W, \bigcup_{i\in I} R_i\rangle$ is cycle free. Kracht [1990] showed that precisely the finite cycle free frames split $\mathrm{NExt}\mathbf{K}_n$.

It is not difficult now to extend Blok's result on the degree of Kripke incompleteness to the polymodal case. Note, however, that the degree of incompleteness of **For** in $\mathrm{NExt}\mathbf{K}_n$ is $2^{\aleph_0}$ whenever $n \geq 2$. So, we do not have a polymodal analog of Makinson's Theorem. (An example of an incomplete maximal consistent logic in $\mathrm{NExt}\mathbf{K}_2$ is the logic determined by the tense frame $\mathfrak{C}(0, \circ)$ introduced in Section 2.5).

THEOREM 107. *Let $n > 1$. If L is a union-splitting of* $\mathrm{NExt}\mathbf{K}_n$*, then L is strictly Kripke complete. Otherwise L has degree of Kripke incompleteness* $2^{\aleph_0}$ *in* $\mathrm{NExt}\mathbf{K}_n$.

Sahlqvist's Theorem and persistence The proof of the following polymodal version of Sahlqvist's Theorem is a straightforward extension of the proof in the unimodal case. Say that φ is a *Sahlqvist formula* (in $\mathcal{L}_I$) if the result of replacing all $\Box_i$ and $\Diamond_i$, $i \in I$, in φ with $\Box$ and $\Diamond$, respectively, is a unimodal Sahlqvist formula.

THEOREM 108. *Suppose that φ is equivalent in* $\mathrm{NExt}\mathbf{K}_n$ *to a Sahlqvist formula. Then $\mathbf{K}_n \oplus \varphi$ is $\mathcal{D}$-persistent, and one can effectively construct a* *first order formula $\phi(x)$ in $R_1, \dots, R_n$ and $=$ such that, for every descriptive or Kripke frame $\mathfrak{F}$ and every point a in $\mathfrak{F}$, $(\mathfrak{F}, a) \models \varphi$ iff $\mathfrak{F} \models \phi(x)[a]$.*

Bellissima's result on the $\mathcal{DF}$-persistence of all logics in $\mathrm{NExt}\mathbf{Alt}_n$ has a polymodal analog as well. Denote by $\bigotimes_{i \in I} \mathbf{Alt}_n$ the smallest polymodal logic in $\mathcal{L}_I$ containing $\mathbf{Alt}_n$ in all its unimodal fragments. It is easy to see that every $L \in \mathrm{NExt} \bigotimes_{i \in I} \mathbf{Alt}_n$ is $\mathcal{DF}$-persistent and so Kripke complete. However, in contrast to the lattice $\mathrm{NExt}\mathbf{Alt}_1$—which is countable and all logics in which have FMP (see [Segerberg 1986] and [Bellissima 1988])—the lattice $\mathrm{NExt}(\mathbf{Alt}_1 \otimes \mathbf{Alt}_1)$ is rather complex: as was shown by Grefe [1994], it contains logics without FMP (even without finite frames at all) and uncountably many maximal consistent logics.

Some FMP results Fine's Theorem on uniform logics can be extended to a suitable class of polymodal logics in $\mathcal{L}_I$, namely those logics that contain $\Diamond_i \top$, for all $i \in I$, and are axiomatizable by formulas φ in which all maximal sequences of nested modal operators coincide with respect to the distribution of the indices i of $\Box_i$ and $\Diamond_i$, $i \in I$.

Now consider a result of Lewis [1974] which we have not proved in its unimodal formulation. Call a normal polymodal logic *non-iterative* if it is axiomatizable by formulas without nested modalities. Examples of non-iterative logics are $\mathbf{T} = \mathbf{K} \oplus \Box p \to p$, $\mathbf{Alt}_m \otimes \mathbf{Alt}_n$ and $\mathbf{K}_2 \oplus \Box_2 p \to \Box_1 p$.

THEOREM 109 (Lewis 1974). *All non-iterative normal logics have FMP.*

Proof. Suppose the axioms of $L = \mathbf{K}_n \oplus \Gamma$ have no nested modal operators and $\varphi \notin L$. By a *φ-description* we mean any set of subformulas of φ together with the negations of the remaining formulas in $\mathbf{Sub}\varphi$. For each L-consistent φ-description Θ select a maximal L-consistent set Δ_Θ containing Θ. Denote by W the (finite) set of the selected Δ_Θ and define $\mathfrak{F} = \langle W, \langle R_i : i \in I \rangle \rangle$ and $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ by taking

$$\Delta_\Theta R_i \Delta_\Psi \text{ iff } \Diamond_i \bigwedge \Psi \in \Delta_\Theta$$

and $\mathfrak{V}(p) = \{\Delta_\Theta \in W : p \in \Delta_\Theta\}$. It is easily proved that $(\mathfrak{M}, \Delta_\Theta) \models \psi$ iff $\psi \in \Delta_\Theta$, for all subformulas ψ of φ and $\Delta_\Theta \in W$. Hence $\mathfrak{F} \not\models \varphi$. It is also easy to see that for all truth-functional compounds ψ of subformulas in φ,

(14) $(\mathfrak{M}, \Delta_\Theta) \models \Diamond_i \psi$ iff $\Diamond_i \psi \in \Delta_\Theta$.

Consider now a model $\mathfrak{M}' = \langle \mathfrak{F}, \mathfrak{V}' \rangle$ and $\chi \in \Gamma$. For each variable p put

$$\psi_p = \bigvee \left\{ \bigwedge \Theta : \Delta_\Theta \in \mathfrak{V}(p) \right\}$$

and denote by χ' the result of substituting ψ_p for p, for each p in χ. Then $\mathfrak{M}' \models \chi$ iff $\mathfrak{M} \models \chi'$. In view of (14), we have $\mathfrak{M} \models \chi'$ because χ' has no nested modalities. Therefore, $\mathfrak{F} \models \chi$ and so $\mathfrak{F} \models L$. ■

Tabular Logics Needless to say that all polymodal tabular logics are finitely axiomatizable and have only finitely many extensions. (The proof is the same as in the unimodal case.) A more interesting observation concerns the complexity of polymodal logics whose unimodal fragments are tabular or pretabular. In fact, it is not difficult to construct two tabular unimodal logics L_1 and L_2 such that their fusion $L_1 \otimes L_2$ has uncountably many normal extensions (see e.g. [Grefe 1994]). However, those logics are $\mathcal{DF}$-persistent and so Kripke complete. Wolter [1994b] showed that the lattice NExt$\mathbf{T}$ can be embedded into the lattice NExt(Log  $\otimes$ $\mathbf{S5}$) in such a way that properties like FMP, decidability and Kripke completeness are reflected under this embedding. It follows that almost all "negative" phenomena of modal logic are exhibited by bimodal logics one unimodal fragment of which is tabular and the other pretabular.

2.2 Fusions

The simplest way of constructing polymodal logics from unimodal ones is to form the *fusions* (alias *independent joins*) of them. Namely, given two unimodal logics L_1 and L_2 in languages with the same set of variables and distinct modal operators $\Box_1$ and $\Box_2$, respectively, the *fusion* $L_1 \otimes L_2$ of L_1 and L_2 is the smallest bimodal logic to contain $L_1 \cup L_2$. If Γ_1 and Γ_2 axiomatize L_1 and L_2, then $L_1 \otimes L_2$ is axiomatized by $\Gamma_1 \cup \Gamma_2$, i.e., $L_1 \otimes L_2 = \mathbf{K}_2 \oplus \Gamma_1 \oplus \Gamma_2$. So the fusions are precisely those bimodal logics that are axiomatizable by sets of formulas each of which contains only one of $\Box_1, \Box_2$. From the model-theoretic point of view this means that a frame $\langle W, R_1, R_2, P\rangle$ validates $L_1 \otimes L_2$ iff $\langle W, R_i, P\rangle \models L_i$ for $i = 1, 2$.

PROPOSITION 110 (Thomason 1980). *If logics L_1 and L_2 are consistent, then $L_1 \otimes L_2$ is a conservative extension of both L_1 and L_2.*

Proof. Suppose for definiteness that $\varphi \notin L_1$, for some formula φ in the language of L_1, and consider the Tarski–Lindenbaum algebras

$$\mathfrak{A}_{L_1}(\omega) = \left\langle A, \wedge^A, \neg^A, \Box_1 \right\rangle \text{ and } \mathfrak{A}_{L_2}(\omega) = \left\langle B, \wedge^B, \neg^B, \Box_2 \right\rangle.$$

The Boolean reducts of them are countably infinite atomless Boolean algebras which are known to be isomorphic (see e.g. [Koppelberg 1988]). So we may assume that $A = B$, $\wedge^A = \wedge^B$, $\neg^A = \neg^B$. Since the algebra $\mathfrak{A}_{L_1}(\omega)$ refutes φ, $\left\langle A, \wedge^A, \neg^A, \Box_1, \Box_2 \right\rangle$ is then an algebra for $L_1 \otimes L_2$ refuting φ. ■

Having constructed the fusion of logics, it is natural to ask which of their properties it inherits. For example, the first order theory of a single equivalence relation has the finite model property and is decidable, but the theory of two equivalence relations is undecidable and so does not have the finite model property (see [Janiczak 1953]). So neither decidability nor the finite model property is preserved under joins of first order theories. On the other hand, as was shown by Pigozzi [1974], decidability is preserved under fusions of equational theories in languages with mutually disjoint sets of operation symbols.

For modal logics we have:

THEOREM 111. *Suppose L_1 and L_2 are normal unimodal consistent logics and $\mathcal{P}$ is one of the following properties: FMP, (strong) Kripke completeness, decidability, Halldén completeness, interpolation, uniform interpolation. Then $L = L_1 \otimes L_2$ has $\mathcal{P}$ iff both L_1 and L_2 have $\mathcal{P}$.*

Proof. We outline proofs of some claims in this theorem; the reader can consult [Fine and Schurz 1996], [Kracht and Wolter 1991], and [Wolter 1997b] for more details.

The implication ($\Rightarrow$) presents no difficulties. So let us concentrate on ($\Leftarrow$). With each formula φ of the form $\Box_i\psi$ we associate a new variable q_φ which will be called the *surrogate* of φ. For a formula φ containing no surrogate variables, denote by $\varphi^{\mathbf{1}}$ the formula that results from φ by replacing all occurrences of formulas $\Box_2\psi$, which are not within the scope of another $\Box_2$, with their surrogate variables $q_{\Box_2\psi}$. So $\varphi^{\mathbf{1}}$ is a unimodal formula containing only $\Box_1$. Denote by $\Theta^{\mathbf{1}}(\varphi)$ the set of variables in φ together with all subformulas of $\Box_2\psi \in \mathbf{Sub}\varphi$. The formula $\varphi^{\mathbf{2}}$ and the set $\Theta^{\mathbf{2}}(\varphi)$ are defined symmetrically.

Suppose now that both L_1 and L_2 are Kripke complete and $\varphi \notin L$. To prove the completeness of L we construct a Kripke frame for L refuting φ. Since we know only how to build refutation frames for the unimodal fragments of L, the frame is constructed by steps alternating between $\Box_1$ and $\Box_2$. First, since L_1 is complete, there is a unimodal model $\mathfrak{M}$ based on a Kripke frame for L_1 and refuting $\varphi^{\mathbf{1}}$ at its root r. Our aim now is to ensure that the formulas of the form $\Box_2\psi$ have the same truth-values as their surrogates $q_{\Box_2\psi}$. To do this, with each point x in $\mathfrak{M}$ we can associate the formula

$$\varphi_x = \bigwedge\{\psi \in \Theta^{\mathbf{1}}(\varphi) : (\mathfrak{M}, x) \models \psi^{\mathbf{1}}\} \wedge \bigwedge\{\neg\psi : \psi \in \Theta^{\mathbf{1}}(\varphi),\ (\mathfrak{M}, x) \not\models \psi^{\mathbf{1}}\},$$

construct a model $\mathfrak{M}_x$ based on a frame for L_2 and satisfying $\varphi_x^{\mathbf{2}}$ at its root y, and then hook $\mathfrak{M}_x$ to $\mathfrak{M}$ by identifying x and y. After that we can switch to $\Box_1$ and in the same manner ensure that formulas $\Box_1\psi$ have the same truth-values as $q_{\Box_1\psi}$ at all points in every $\mathfrak{M}_x$. And so forth.

However, to realize this quite obvious scheme we must be sure that φ_x is really satisfiable in a frame for L_2, which may impose some restrictions on the models we choose. First, one can show that in the construction above it is enough to deal with points x accessible from r by at most $m = md(\varphi)$ steps. Let X be the set of all such points. Now, a sufficient and necessary condition for φ_x to be L- (and so L_2-) consistent can be formulated as follows. Call a $\Theta^{\mathbf{1}}(\varphi)$-*description* the conjunction of formulas in any maximal L-consistent subset of $\Theta^{\mathbf{1}}(\varphi) \cup \{\neg\psi : \psi \in \Theta^{\mathbf{1}}(\varphi)\}$. It should be clear that φ_x is L-consistent iff it is a $\Theta^{\mathbf{1}}(\varphi)$-description. Denote by $\Sigma_1(\varphi)$ the set of all $\Theta^{\mathbf{1}}(\varphi)$-descriptions. It follows that all φ_x, for $x \in X$, are L-consistent iff $(\mathfrak{M}, r) \models \Box_1^{\leq m}(\bigvee \Sigma_1(\varphi))^{\mathbf{1}}$. In other words, we should start with a model $\mathfrak{M}$ satisfying $\varphi^{\mathbf{1}} \wedge \Box_1^{\leq m}(\bigvee \Sigma_1(\varphi))^{\mathbf{1}}$ at its root r. Of course, the subsequent models $\mathfrak{M}_x$, for $x \in X$, must satisfy $\varphi_x^{\mathbf{2}} \wedge \Box_2^{\leq m}(\bigvee \Sigma_2(\varphi_x))^{\mathbf{2}}$, where $\Sigma_2(\varphi_x)$ is the set of all $\Theta^{\mathbf{2}}(\varphi_x)$-descriptions, etc.

In this way we can prove that Kripke completeness is preserved under fusions. The preservation of strong completeness and FMP can be established in a similar manner. The following lemma plays the key role in the proof of the preservation of the four remaining properties.

LEMMA 112. *The following conditions are equivalent for every φ:*

(i) $\varphi \in L_1 \otimes L_2$;

(ii) $\Box_1^{\leq m}(\bigvee \Sigma_1(\varphi))^{\mathbf{1}} \to \varphi^{\mathbf{1}} \in L_1$, *where* $m = md(\varphi)$;

(iii) $\Box_2^{\leq m}(\bigvee \Sigma_2(\varphi))^{\mathbf{2}} \to \varphi^{\mathbf{2}} \in L_2$.

For Kripke complete L_1 and L_2 this lemma was first proved by Fine and Schurz [1996] and Kracht and Wolter [1991]; actually, it is an immediate consequence of the consideration above. The proof for the arbitrary case is also based upon a similar construction combined with the algebraic proof of Proposition 110; for details see [Wolter 1997b].

Now we show how one can use this lemma to prove the preservation of the remaining properties. Define $a^1(\varphi)$ to be the length of the longest sequence $\Box_2, \Box_1, \Box_2, \dots$ of boxes starting with $\Box_2$ such that a subformula of the form $\Box_2(\dots\Box_1(\dots\Box_2(\dots\dots)))$ occurs in φ. The function $a^2(\varphi)$ is defined analogously by exchanging $\Box_1$ and $\Box_2$, and $a(\varphi) = a^1(\varphi) + a^2(\varphi)$. It is easy to see that

$$a(\varphi) > a(\bigvee \Sigma_1(\varphi)) \ \text{ or } \ a(\varphi) > a(\bigvee \Sigma_2(\varphi)).$$

The preservation of decidability, Halldén completeness, interpolation, and uniform interpolation can be proved by induction on $a(\varphi)$ with the help of Lemma 112. We illustrate the method only for Halldén completeness. Notice first that, modulo the Boolean equivalence, we have

$$\bigvee \Sigma_1(\varphi \vee \psi) = \bigvee \Sigma_1(\varphi) \wedge \bigvee \Sigma_1(\psi) \wedge \bigwedge \Delta(\varphi, \psi),$$

where

$$\Delta(\varphi, \psi) = \{\chi_1 \to \neg\chi_2 : \chi_1 \in \Sigma_1(\varphi), \chi_2 \in \Sigma_1(\psi), \chi_1 \to \neg\chi_2 \in L\}.$$

Suppose both L_1 and L_2 are Halldén complete. By induction on $n = a(\varphi \vee \psi)$ we prove that $\varphi \vee \psi \in L$ implies $\varphi \in L$ or $\psi \in L$ whenever φ and ψ have no common variables. The basis of induction is trivial. So suppose $a(\varphi \vee \psi) = n > 0$ and $\varphi \vee \psi \in L$. We may also assume that $a(\varphi \vee \psi) > a(\bigvee \Sigma_1(\varphi \vee \psi))$. By the induction hypothesis, it follows that $\Delta(\varphi, \psi) = \emptyset$. Hence, up to the Boolean equivalence, $\bigvee \Sigma_1(\varphi \vee \psi) = \bigvee \Sigma_1(\varphi) \wedge \bigvee \Sigma_1(\psi)$ and, by Lemma 112,

$$\Box_1^{\leq m}(\bigvee \Sigma_1(\varphi))^{\mathbf{1}} \wedge \Box_1^{\leq m}(\bigvee \Sigma_1(\psi))^{\mathbf{1}} \to (\varphi \vee \psi)^{\mathbf{1}} \in L_1,$$

for $m = md(\varphi \vee \psi)$. Then

$$(\Box_1^{\leq m}(\bigvee \Sigma_1(\varphi))^{\mathbf{1}} \to \varphi^{\mathbf{1}}) \vee (\Box_1^{\leq m}(\bigvee \Sigma_1(\psi))^{\mathbf{1}} \to \psi^{\mathbf{1}}) \in L_1$$

and, by the Halldén completeness of L_1, one of the disjuncts in this formula belongs to L_1. By Lemma 112, this means that $\varphi \in L$ or $\psi \in L$. ■

Remark. This theorem can be generalized to fusions of polymodal logics with polyadic modalities.

Note that in languages with finitely many variables both **GL.3** and **K** are strongly complete but **GL.3** $\otimes$ **K** is not strongly complete even in the language with one variable (see [Kracht and Wolter 1991]).

It is natural now to ask whether there exist interesting axioms φ containing both $\Box_1$ and $\Box_2$ and such that $(L_1 \otimes L_2) \oplus \varphi$ inherits basic properties of $L_1, L_2 \in \mathrm{Ext}\mathbf{K}$. Let us start with the observation that even such a simple axiom as $\Box_1 p \leftrightarrow \Box_2 p$ destroys almost all “good” properties because (i) we can identify the logic $(L_1 \otimes L_2) \oplus \Box_1 p \leftrightarrow \Box_2 p$ with the sum of the translation of L_1 and L_2 into a common unimodal language and (ii) such properties as FMP, decidability, and Kripke completeness are not preserved under sums of unimodal logics (see Example 64 and [Chagrov and Zakharyaschev 1997]). Even for the simpler formula $\Box_2 p \to \Box_1 p$ no general results are available. To demonstrate this we consider the following way of constructing a bimodal logic L_u for a given $L \in \mathrm{NExt}\mathbf{K}$:

$$L_u = (L \otimes \mathbf{S5}) \oplus \Box_2 p \to \Box_1 p.$$

The modal operator $\Box_2$ in L_u is called the *universal modality.* Its meaning is explained by the following lemma:

LEMMA 113 (Goranko and Passy 1992). *For every normal unimodal logic L and all unimodal formulas φ and ψ,*

$$\varphi \vdash_L^* \psi \textit{ iff } \vdash_{L_u} \Box_2 \varphi \to \psi.$$

Proof. Follows immediately from Theorem 19 (ii), since

$$\langle W, R, P \rangle \models L \text{ iff } \langle W, R, W \times W, P \rangle \models L_u,$$

for every frame $\langle W, R, P \rangle$ and every unimodal logic L. ■

The universal modality is used to express those properties of frames $\mathfrak{F} = \langle W, R, W \times W \rangle$ that cannot be expressed in the unimodal language. For example, $\mathfrak{F}$ validates $\Box_2(p \to \Diamond_1 p) \to \neg p$ iff it contains no infinite R-chains. Recall that there is no corresponding unimodal axiom, since $\mathbf{K}$ is determined by the class of frames without infinite R-chains. We refer the reader to [Goranko and Passy 1992] for more information on this matter.

THEOREM 114 (Goranko and Passy 1992). *For any $L \in \mathrm{NExt}\mathbf{K}$,*
(i) *L is globally Kripke complete iff L_u is Kripke complete;*
(ii) *L has global FMP iff L_u has FMP.*

Proof. We prove only (i). Suppose that L_u is Kripke complete and $\varphi \not\vdash_L^* \psi$. Then by Lemma 113, $\Box_2\varphi \to \psi \notin L_u$ and so $\Box_2\varphi \to \psi$ is refuted in a Kripke frame $\mathfrak{F} = \langle W, R_1, R_2 \rangle$ for L_u. We may assume that $R_2 = W \times W$. But then $\varphi \vdash_L^* \psi$ is refuted in $\langle W, R_1 \rangle$. Conversely, suppose that L is globally Kripke complete and $\varphi \notin L_u$, for a (possibly bimodal) formula φ. Using the properties of $\mathbf{S5}$ it is readily checked that φ is (effectively) equivalent in $\mathbf{K}_u$ to a formula φ' which is a conjunction of formulas ψ of the form

$$\psi = \chi_0 \vee \Diamond_2\chi_1 \vee \Box_2\chi_2 \vee \Box_2\chi_3 \vee \cdots \vee \Box_2\chi_n$$

such that $\chi_0, \ldots, \chi_n$ are unimodal formulas in the language with $\Box_1$. Let ψ be a conjunct of φ' such that $\psi \notin L_u$. Then $\neg\chi_1 \not\vdash_L^* \chi_i$, for every $i \in \{0, 2, 3, \ldots, n\}$. Since L is globally complete, we have Kripke frames $\langle W_i, R_i \rangle$ for L refuting $\neg\chi_1 \vdash_L^* \chi_i$, for $i \in \{0, 2, \ldots, n\}$. Denote by $\langle W, R \rangle$ the disjoint union of those frames. Then $\langle W, R, W \times W \rangle$ is a Kripke frame for L_u refuting φ. ■

We have seen in Section 1.5 that there are Kripke complete logics (logics with FMP) which do not enjoy the corresponding global property. In view of Theorem 114, we conclude that neither FMP nor Kripke completeness is preserved under the map $L \mapsto L_u$.

Another interesting way of adding to fusions new axioms mixing the necessity operators is to use the so called *inductive* (or *Segerberg's*) *axioms*. First, we extend the language $\mathcal{L}_I$ with m necessity operators by introducing the operators E and C and then let

$$\boldsymbol{ind} = \{\mathsf{E}p \leftrightarrow \bigwedge_{i \in I} \Box_i p,\ \mathsf{C}p \to \mathsf{EC}p;\ \mathsf{C}(p \to \mathsf{E}p) \to (p \to \mathsf{C}p)\}.$$

Now, given $L \in \mathrm{NExt}\mathbf{K}_m$, we put

$$L\mathbf{EC}_m = (L \otimes \mathbf{K}_{\mathsf{E}} \otimes \mathbf{S4}_{\mathsf{C}}) \oplus \boldsymbol{ind},$$

where $\mathbf{K}_{\mathsf{E}}$ and $\mathbf{S4}_{\mathsf{C}}$ are just $\mathbf{K}$ and $\mathbf{S4}$ in the languages with E and C, respectively. The following proposition explains the meaning of the inductive axioms.

PROPOSITION 115. *A frame* $\langle W, R_1, \ldots, R_m, R_E, R_C\rangle$ *validates* $L\mathbf{EC}_m$ *iff* $\langle W, R_1, \ldots, R_m\rangle \models L$, $R_E = R_1 \cup \cdots \cup R_m$ *and* R_C *is the transitive reflexive closure of* R_E.

EXAMPLE 116. The logic $(\mathbf{Alt}_1 \oplus \mathbf{D})\mathbf{EC}_1$ is determined by the frame $\langle \omega, S, \leq\rangle$ in which S is the successor relation in ω. (Here we omit writing R_E because $R_E = S$.) For details consult [Segerberg 1989].[14]

No general results are known about the preservation properties of the map $L \mapsto L\mathbf{EC}_m$. In fact, it is easy to extend the counter-examples for the map $L \mapsto L_u$ to the present case (see [Hemaspaandra 1996]). However, at least in some cases—especially those that are of importance for epistemic logic—the logic $L\mathbf{EC}_m$ enjoys a number of desirable properties.

THEOREM 117 (Halpern and Moses 1992). *For every* $m \geq 1$, *the logics* $(\bigotimes_{i=1}^m \mathbf{K})\mathbf{EC}_m$, $(\bigotimes_{i=1}^m \mathbf{S4})\mathbf{EC}_m$ *and* $(\bigotimes_{i=1}^m \mathbf{S5})\mathbf{EC}_m$ *have FMP.*

Proof. We consider only $L = (\bigotimes_{i=1}^m \mathbf{S5})\mathbf{EC}_m$. The proof is by filtration and so the main difficulty is to find a suitable "filter". Suppose that $\varphi \notin L$ and let $\mathfrak{M} = \langle\langle W, R_1, \ldots, R_m, R_E, R_C\rangle, \mathfrak{U}\rangle$ be the canonical model for L. Denote by $\Gamma^\neg$ the closure of a set of formulas Γ under negations and define a filter $\Phi = \Phi_1^\neg \cup \Phi_2^\neg \cup \Phi_3^\neg$, where $\Phi_1 = \mathbf{Sub}\varphi$, $\Phi_2 = \{\Box_i\psi : \mathsf{E}\psi \in \Phi_1^\neg\}$ and $\Phi_3 = \{\mathsf{EC}\psi, \Box_i\mathsf{C}\psi : \mathsf{C}\psi \in \Phi_1^\neg\}$. Certainly, Φ is finite and closed under subformulas. Now, we filter $\mathfrak{M}$ through Φ, i.e., put $W^* = \{[x] : x \in W\}$, where $[x]$ consists of all points that validate the same formulas in Φ as x, and

$$[x]R_i[y] \text{ iff } \forall \Box_i\psi \in \Phi\ ((\mathfrak{M}, x) \models \Box_i\psi \rightarrow (\mathfrak{M}, y) \models \Box_i\psi),$$

$$R_E^* = R_1^* \cup \cdots \cup R_m^*,$$

and R_C^* is the transitive and reflexive closure of R_E^*. A rather tedious inductive proof shows that $\langle W^*, R_1^*, \ldots, R_m^*, R_E^*, R_C^*\rangle$ refutes φ under the valuation $\mathfrak{U}^*(p) = \{[x] : x \models p\}$, p a variable in φ. For details we refer the reader to [Halpern and Moses 1992] and [Meyer and van der Hoek 1995]. ■

It would be of interest to look for big classes of logics L for which $L\mathbf{EC}_m$ inherits basic properties of L.

2.3 Simulation

In the preceding section we saw how results concerning logics in NExt$\mathbf{K}$ can be extended to a certain class of polymodal logics. More generally, we may ask whether—at least theoretically—polymodal logics are reducible to unimodal ones. The first to attack this problem was Thomason [1974b, 1975c]

[14]Krister Segerberg kindly informed us that this result was independently obtained by D. Scott, H. Kamp, K. Fine and himself.

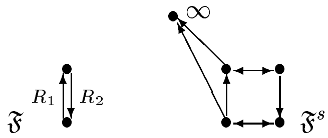


Figure 11.

who proved that each polymodal logic L can be embedded into a unimodal logic L^s in such a way that L inherits almost all interesting properties of L^s. Using this result one can construct unimodal logics with various "negative" properties by presenting first polymodal logics with the corresponding properties, which is often much easier. It was in this way that Thomason [1975c] constructed Kripke incomplete and undecidable unimodal calculi. Kracht [1996] strengthened Thomason's result by showing that his embedding not only reflects but also (i) preserves almost all important properties and (ii) induces an isomorphism from the lattice $\mathrm{NExt}\mathbf{K}_2$ onto the interval $[\mathbf{Sim}, \mathbf{K} \oplus \Box\bot]$, for some normal unimodal logic $\mathbf{Sim}$. Thus indeed, in many respects polymodal logics turn out to be reducible to unimodal ones.

Below we outline Thomason's construction following [Kracht 1999] and [Kracht and Wolter 1999]. To define the unimodal "simulation" L^s of a bimodal logic L, let us first transform each bimodal frame into a unimodal one.

So suppose $\mathfrak{F} = \langle W, R_1, R_2, P\rangle$ is a bimodal frame. Construct a unimodal frame $\mathfrak{F}^s = \langle W^s, R^s, P^s\rangle$—the *simulation* of $\mathfrak{F}$—by taking

$$\begin{array}{rcl} W^s & = & W \times \{1,2\} \cup \{\infty\}, \\ R^s & = & \{\langle\langle x,1\rangle, \langle x,2\rangle\rangle : x \in W\} \cup \\ & & \{\langle\langle x,2\rangle, \langle x,1\rangle\rangle : x \in W\} \cup \\ & & \{\langle\langle x,1\rangle, \infty\rangle : x \in W\} \cup \\ & & \{\langle\langle x,1\rangle, \langle y,1\rangle\rangle : x,y \in W, xR_1y\} \cup \\ & & \{\langle\langle x,2\rangle, \langle y,2\rangle\rangle : x,y \in W, xR_2y\}, \\ P^s & = & \{(X \times \{2\}) \cup (Y \times \{1\}) \cup Z : X, Y \in P, Z \subseteq \{\infty\}\}. \end{array}$$

This construction is illustrated by Fig. 11. One can easily prove that $\mathfrak{F}^s$ is a Kripke (differentiated, refined, descriptive) frame whenever $\mathfrak{F}$ is so. Notice also that if $W = \emptyset$ then $\mathfrak{F}^s \cong \bullet$. Now, given a bimodal logic L, define the *simulation* L^s of L to be the unimodal logic

$$\mathrm{Log}\{\mathfrak{F}^s : \mathfrak{F} \models L\}.$$

To formulate the translation which embeds L into L^s we require the following formulas and notations:

$$\begin{array}{rclcrcl}\gamma & = & \Box\bot & \qquad & \Box_\gamma\varphi & = & \Box(\gamma\to\varphi)\\ \alpha & = & \Diamond\Box\bot & & \Box_\alpha\varphi & = & \Box(\alpha\to\varphi)\\ \beta & = & \neg\gamma\wedge\neg\Diamond\gamma & & \Box_\beta\varphi & = & \Box(\beta\to\varphi).\end{array}$$

$\Diamond_\gamma$, $\Diamond_\alpha$ and $\Diamond_\beta$ are defined dually. Observe that the formula γ is true in $\mathfrak{F}^s$ only at ∞, α is true precisely at the points in the set $\{\langle x,1\rangle : x\in W\}$, and β is true at the points $\{\langle x,2\rangle : x\in W\}$ and only at them. Put

$$\begin{array}{lcl}p^s & = & p,\\ (\neg\varphi)^s & = & \alpha\wedge\neg\varphi^s,\\ (\varphi\wedge\psi)^s & = & \varphi^s\wedge\psi^s,\\ (\Box_1\varphi)^s & = & \Box_\alpha\varphi^s,\\ (\Box_2\varphi)^s & = & \Box_\beta\Box_\beta\Box_\alpha\varphi^s.\end{array}$$

By an easy induction on the construction of φ one can prove

LEMMA 118. *Let $\mathfrak{M}=\langle\mathfrak{F},\mathfrak{V}\rangle$ be a bimodal model, $X=\{x : x\models\alpha\}$ and let $\mathfrak{M}^s=\langle\mathfrak{F}^s,\mathfrak{V}^s\rangle$ be a model such that $\mathfrak{V}^s(p)\cap X=\mathfrak{V}(p)\times\{1\}$, for all variables p. Then for every bimodal formula φ,*

$$\begin{array}{rcl}(\mathfrak{M},x)\models\varphi & \textit{iff} & (\mathfrak{M}^s,\langle x,1\rangle)\models\varphi^s,\\ \mathfrak{M}\models\varphi & \textit{iff} & \mathfrak{M}^s\models\alpha\to\varphi^s,\\ \mathfrak{F}\models\varphi & \textit{iff} & \mathfrak{F}^s\models\alpha\to\varphi^s.\end{array}$$

Using this lemma, both consequence relations $\vdash_L$ and $\vdash_L^*$ can be reduced to the corresponding consequence relations for L^s.

PROPOSITION 119. *Let L be a bimodal logic, Δ a set of bimodal formulas and φ a bimodal formula. Then*

$$\begin{array}{rcl}\Delta\vdash_L\varphi & \textit{iff} & \alpha\to\Delta^s\vdash_{L^s}\alpha\to\varphi^s,\\ \Delta\vdash_L^*\varphi & \textit{iff} & \alpha\to\Delta^s\vdash_{L^s}^*\alpha\to\varphi^s,\end{array}$$

where $\alpha\to\Delta^s=\{\alpha\to\delta : \delta\in\Delta^s\}$.

To axiomatize L^s, given an axiomatization of L, we require the following formulas:

(a) $\alpha\to(\Diamond_\gamma p\leftrightarrow\Box_\gamma p),\ \alpha\wedge\Diamond_\gamma p\to\Box_\alpha\Diamond_\gamma p,$

(b) $\alpha\to(\Diamond_\beta p\leftrightarrow\Box_\beta p),$

(c) $\beta\to(\Diamond_\alpha p\leftrightarrow\Box_\alpha p),$

(d) $\alpha\wedge p\to\Box_\beta\Box_\alpha p,\ \beta\wedge p\to\Box_\alpha\Box_\beta p,$

(e) $\alpha\wedge\Diamond_\gamma p\to\Box_\beta\Box_\beta\Box_\alpha\Diamond_\gamma p.$

Let $\mathbf{Sim}=\mathbf{K}\oplus\{(a),\dots,(e)\}$. Obviously, $\mathfrak{F}^s$ is a frame for $\mathbf{Sim}$ whenever $\mathfrak{F}$ is a bimodal frame. Consider now a differentiated frame $\mathfrak{F}=\langle W,R,P\rangle$

for **Sim** which contains only one point where γ is true. (Actually, every rooted differentiated frame for **Sim** satisfies this condition.) Construct a bimodal frame $\mathfrak{F}_s = \langle V, R_1, R_2, Q\rangle$, called the *unsimulation* of $\mathfrak{F}$, in the following way. Put $V = \{x \in W : x \models \alpha\}$, $V^\bullet = \{x \in W : x \models \beta\}$ and $U = \{x \in W : x \models \gamma\}$. Since $\gamma \vee \alpha \vee \beta \in \mathbf{K}$, we have $W = V \cup V^\bullet \cup U$. It is not hard to verify using (b) and (c) (and the differentiatedness of $\mathfrak{F}$) that for every $x \in V$ there exists a unique $x^\bullet \in V^\bullet$ such that $xRx^\bullet$, and for every $y \in V^\bullet$ there exists $y^\circ \in V$ such that yRy°. By (d), $x = x^{\bullet\circ}$. Finally, we put $R_1 = R \cap V^2$, $R_2 = \{\langle x, y\rangle \in V^2 : x^\bullet R y^\bullet\}$ and $Q = \{X \cap V : X \in P\}$. It is easily proved that $\mathfrak{F}_s$ is a bimodal frame. The name *unsimulation* is justified by the following lemma.

LEMMA 120. *For every differentiated bimodal frame $\mathfrak{F}$, $(\mathfrak{F}^s)_s \cong \mathfrak{F}$.*

Now we have:

THEOREM 121. *For every bimodal logic $L = \mathbf{K}_2 \oplus \Delta$,*

$$L^s = \mathbf{Sim} \oplus \alpha \to \Delta^s.$$

Proof. Clearly, $\mathbf{Sim} \oplus \alpha \to \Delta^s \subseteq L^s$. Assume that the converse inclusion does not hold. Then there exists a rooted differentiated $\mathfrak{F}$ such that $\mathfrak{F} \not\models L^s$ but $\mathfrak{F} \models \mathbf{Sim} \oplus \alpha \to \Delta^s$. By Lemma 120, $(\mathfrak{F}_s)^s \not\models L^s$. By the definition of L^s, we then conclude that $\mathfrak{F}_s \not\models L$. And by Proposition 119, we have $(\mathfrak{F}_s)^s \not\models \alpha \to \Delta^s$, from which $\mathfrak{F} \not\models \alpha \to \Delta^s$. ■

Given $L \in [\mathbf{Sim}, \mathbf{K} \oplus \Box\bot]$, the logic $L_s = \{\varphi : \alpha \to \varphi^s \in L\}$ is called the *unsimulation* of L.

LEMMA 122. *If L is determined by a class $\mathcal{C}$ of frames in which γ is true only at one point then $L_s = \mathrm{Log}\{\mathfrak{F}_s : \mathfrak{F} \in \mathcal{C}\}$.*

We are in a position now to formulate the main result of this section.

THEOREM 123 (Kracht 1999). *The map $L \mapsto L^s$ is an isomorphism from the lattice* $\mathrm{NExt}\mathbf{K}_2$ *onto the interval* $[\mathbf{Sim}, \mathbf{K}_1 \oplus \Box\bot]$. *The inverse map is $L \mapsto L_s$. Both these maps preserve tabularity, (global) FMP, (global) Kripke completeness, decidability, interpolation, strong completeness, $\mathcal{R}$- and $\mathcal{D}$-persistence, elementarity.*

Proof. To prove the first claim it suffices to show that $(L_s)^s = L$ for every $L \in [\mathbf{Sim}, \mathbf{K} \oplus \Box\bot]$. That $L \subseteq (L_s)^s$ is clear. Consider the set $\mathcal{C}$ of all differentiated frames $\mathfrak{F}_s$ such that $\mathfrak{F} \models L$ and γ is true only at one point in $\mathfrak{F}$. By Lemma 122, $\mathcal{C}$ characterizes L_s. It is not difficult to show now that the class $\{\mathfrak{F}_s^+ : \mathfrak{F} \in \mathcal{C}\}$ is closed under subalgebras, homomorphic images and direct products; so it is a variety. Consequently, $\mathcal{C}$ is (up to isomorphic copies) the class of all differentiated frames for L_s.

Take a differentiated frame $\mathfrak{F}$ for $(L_s)^s$. Then $\mathfrak{F}_s \models L_s$. So there exists $\mathfrak{G}_s \in \mathcal{C}$ which is isomorphic to $\mathfrak{F}_s$. Hence $(\mathfrak{F}_s)^s \cong (\mathfrak{G}_s)^s$ and $\mathfrak{F} \models L$, since $\mathfrak{G} \models L$. It follows that L^s is determined by $\{\mathfrak{F}^s : \mathfrak{F} \in \mathcal{C}\}$ whenever L is determined by $\mathcal{C}$.

The preservation of tabularity, (global) FMP, (global) Kripke completeness, and strong completeness under both maps is proved with the help of Lemma 122 and the observation above. It is also clear that L is decidable whenever L^s is decidable. For the remaining (rather technical) part of the proof the reader is referred to [Kracht 1999] and [Kracht and Wolter 1999]. ∎

Besides its theoretical significance, this theorem can be used to transfer rather subtle counter-examples from polymodal logic to unimodal logic. For instance, Kracht [1996] constructs a polymodal logic which has FMP and is globally Kripke incomplete. By Theorem 123, we obtain a unimodal logic with the same properties.

2.4 Minimal tense extensions

Now let us turn to *tense logics* which may be regarded as normal bimodal logics containing the axioms $p \to \Box_1\Diamond_2 p$ and $p \to \Box_2\Diamond_1 p$. Usually studies in Tense Logic concern some special systems representing various models of time, like cyclic time, discrete or dense linear time, branching time, relativistic time, etc. Such systems are discussed in *Basic Tense Logic* (see also [Gabbay *et al.* 1994], [Goldblatt 1987] and [van Benthem 1991]). However, as before our concern is general methods which make it possible to obtain results not only for this or that particular system but for wide classes of logics. This direction of studies in Tense Logic is quite new and actually not so many general results are available. In this and the next section we consider two natural families of tense logics—the minimal tense extensions of unimodal logics and tense logics of linear frames. Our aim is to find out to what extent the theory developed for unimodal logics in NExt**K** and especially NExt**K4** can be "lifted" to these families.

The smallest tense logic $\mathbf{K}.t$ is determined by the class of bimodal Kripke frames $\langle W, R, R^{-1}\rangle$ in which R is the accessibility relation for $\Box_1$ and R^{-1} for $\Box_2$. Frames of this type are known as *tense Kripke frames*; general frames of the form $\langle W, R, R^{-1}, P\rangle$ will be called just *tense frames*. Notice that not all unimodal general frames $\langle W, R, P\rangle$ can be converted into tense frames $\langle W, R, R^{-1}, P\rangle$ because P is not necessarily closed under the operation

$$\Diamond_2 X = \{x \in W : \exists y \in X \; xR^{-1}y\}.$$

For instance, in the frame $\mathfrak{F}$ of Example 7 we have $\Diamond_2\{\omega + 1\} = \{\omega\} \notin P$.

Each normal unimodal logic $L = \mathbf{K} \oplus \Gamma$ in the language with $\Box_1$ gives rise to its *minimal tense extension* $L.t = \mathbf{K}.t \oplus \Gamma$. From the semantical point of view $L.t$ is the logic determined by the class of tense frames $\langle W, R, R^{-1}, P\rangle$ such that $\langle W, R, P\rangle \models L$. The formation of the minimal tense extensions is the simplest way of constructing tense logics from unimodal ones. Of "natural" tense logics, minimal tense extensions are, for instance, the logics of (converse) transitive trees, (converse) well-founded frames, (converse) transitive directed frames, etc. The main aim of this section is to describe conditions under which various properties of L are inherited by $L.t$.

Notice first that unlike fusions, $L.t$ is not in general a conservative extension of L, witness $L = \mathrm{Log}\mathfrak{F}$ where $\mathfrak{F}$ is again the frame constructed in Example 7: one can easily check that $\mathbf{K4}.t \subseteq L.t$. However, if L is Kripke complete then $L.t$ is a conservative extension of L and so $L'.t = L.t$ implies $L' \subseteq L$. This example may appear to be accidental (as the first examples of Kripke incomplete logics in NExt**K**). However, we can repeat (with a slight modification) Blok's construction of Theorem 35 and prove the following

THEOREM 124. *If L is a union-splitting of* NExt**K** *or $L = \mathbf{For}$, then $L'.t = L.t$ implies $L' = L$. Otherwise there is a continuum of logics in* NExt**K** *having the same minimal tense extension as L.*

It is not known whether there exists $L \in \mathrm{NExt}\mathbf{K4}$ such that $L.t$ is not a conservative extension of L.

Theorem 124 leaves us little hope to obtain general positive results for the whole family of minimal tense extensions. As in the case of unimodal logics we can try our luck by considering logics with transitive frames. So in the rest of this section it is assumed that the unimodal and tense logics we deal with contain **K4** and $\mathbf{K4}.t$, respectively, and that frames are transitive. But even in this case we do not have general preservation results: Wolter [1996b] constructed a logic $L \in \mathrm{NExt}\mathbf{K4}$ having FMP and such that $L.t$ is not Kripke complete. However, the situation turns out to be not so hopeless if we restrict attention to the well-behaved classes of logics in NExt**K4**, namely logics of finite width, finite depth and cofinal subframe logics. First, we have the following results of [Wolter 1997a].

THEOREM 125. *If $L \in$* NExt**K4** *is a logic of finite depth then $L.t$ has FMP. If $L \in$* NExt**K4** *is a logic of finite width then $L.t$ is Kripke complete.*

It is to be noted that tense logics of finite depth are much more complex than their unimodal counterparts. For example, there exists an undecidable finitely axiomatizable logic containing $\mathbf{K4}.t \oplus \Box_1\Box_1\bot$ (for details see [Kracht and Wolter 1999]).

The minimal tense extensions of cofinal subframe logics were investigated in [Wolter 1995, 1997a].

THEOREM 126. *If $L \in$* NExt**K4** *is a cofinal subframe logic then*

(i) *$L.t$ is Kripke complete;*

(ii) *$L.t$ has FMP iff L is canonical;*

(iii) *$L.t$ is decidable whenever L is finitely axiomatizable.*

Before outlining the idea of the proof we note some immediate consequences for a few standard tense logics.

EXAMPLE 127. (i) The logic of the converse well-founded tense frames is $\mathbf{GL}.t$; it does not have FMP but is decidable. (ii) The logic of the converse transitive trees is $\mathbf{K4.3}.t$; it has FMP and is decidable. (iii) The logic of the converse well-founded directed tense frames is $\mathbf{GL}.t \oplus \mathbf{K4.2}.t$; it does not have FMP and is decidable.

Proof. The proof of the negative part, i.e., that $L.t$ does not have FMP if L is not canonical, is rather technical; it is based on the characterization of the canonical cofinal subframe logics of [Zakharyaschev 1996]. The reader can get some intuition from the following example: neither $\mathbf{Grz}.t$ nor $\mathbf{GL}.t$ has FMP. Indeed, the Grzegorczyk axiom

$$\Box_2(\Box_2(p \to \Box_2 p) \to p) \to p$$

is refuted in $\langle \omega, \geq, \leq \rangle$ and so does not belong to $\mathbf{Grz}.t$; however, it is valid in all finite partial orders. The argument for $\mathbf{GL}.t$ is similar: take the Löb axiom in $\Box_2$ and the frame $\langle \omega, >, < \rangle$.

We sketch now the proof of the positive part. For a tense Kripke frame $\mathfrak{F} = \langle W, R, R^{-1} \rangle$, let rp be a partial function associating with some clusters in $\mathfrak{F}$ one of the frames

$$\langle \omega, >, < \rangle \text{ or } \langle \omega, \geq, \leq \rangle.$$

We call it a *replacement function* for $\mathfrak{F}$ and define $\mathfrak{F}^{rp}$ to be the result of replacing in $\mathfrak{F}$ all clusters C in the domain of rp by (disjoint copies of) $\mathsf{rp}C$. Our first observation is that for each cofinal subframe logic L, $L.t$ is determined by a set of frames of the form $\mathfrak{F}^{rp}$ such that $\mathfrak{F}$ is of finite depth. Indeed, suppose $\varphi \notin L.t$ and consider a countermodel $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ for φ based on a descriptive finitely generated tense frame $\mathfrak{F} = \langle W, R, R^{-1}, P \rangle$ for $L.t$. Say that a point $x \in W$ is *non-eliminable* (relative to φ) if there are a subformula ψ of φ and $S \in \{R, R^{-1}\}$ such that $x \in \max_S\{y \in W : y \models \psi\}$ or $x \in \max_S\{y \in W : y \models \neg\psi\}$. Denote by W_e the set of non-eliminable points in W and construct a new model $\mathfrak{M}_e$ on the frame $\mathfrak{F}_e = \langle W_e, R{\restriction}W_e, R^{-1}{\restriction}W_e \rangle$ by taking $\mathfrak{V}_e(p) = \mathfrak{V}(p) \cap W_e$ for all variables p in φ. Clearly, the Kripke frame $\mathfrak{F}_e$ is of finite depth ($d(\mathfrak{F}_e) \leq 2l(\varphi)$, to be more precise). Besides, using Theorem 23 one can easily show that $(\mathfrak{M}_e, y) \models \psi$ iff $(\mathfrak{M}, y) \models \psi$, for all $\psi \in \mathbf{Sub}\varphi$ and $y \in W_e$. (Note that Theorem 23 is applicable in this case, since $\langle W, R, P \rangle$ is descriptive whenever $\langle W, R, R^{-1}, P \rangle$ is descriptive.) Moreover, the R-reduct $\langle W_e, R{\restriction}W_e \rangle$ of $\mathfrak{F}_e$ is a cofinal subframe of the R-reduct $\langle W, R \rangle$ of the underlying Kripke frame of $\mathfrak{F}$. So $\mathfrak{F}_e$ is a frame for $L.t$ whenever L is canonical (= $\mathcal{D}$-persistent). However, this is not so if L is not canonical.

EXAMPLE 128. Consider the frame $\mathfrak{F} = \langle W, R, R^{-1}, P\rangle$, where $\langle W, R\rangle$ is the reflexive point ∞ followed by the chain $\langle \omega, >\rangle$ and P consists of all cofinite sets containing ∞ and their complements. Then $\mathfrak{F} \models \mathbf{GL}.t$ but (for an arbitrary φ) $\mathfrak{F}_e$ contains ∞ and so $\mathfrak{F}_e \not\models \mathbf{GL}.t$.

A rather tedious proof (see [Wolter 1997a]) shows, however, that there exists a replacement function rp for $\mathfrak{F}_e$ such that $\mathfrak{F}_e^{rp}$ validates $L.t$ and all points in clusters from domrp are eliminable relative to R in $\mathfrak{F}$. (In the example above we put $\mathsf{rp}\{\infty\} = \langle \omega, >, <\rangle$ and ∞ is eliminable relative to R.) So let us assume that such rp is given and that its domain is empty if L is canonical. Define a model $\mathfrak{M}_e^{rp} = (\mathfrak{F}_e^{rp}, \mathfrak{V}^{rp})$ as follows. First we put $y \in \mathfrak{V}^{rp}(p)$ whenever $y \in \mathfrak{V}_e(p)$ and $y \notin$ domrp. Consider now a cluster $C = \{a_0, \dots, a_{m-1}\}$ in domrp. $\mathfrak{V}^{rp}$ is defined in $\mathsf{rp}C$ by unravelling C into the chain $\mathsf{rp}C$; more precisely, we put

$$\mathfrak{V}^{rp}(p) \cap \mathsf{rp}C = \{mj + i : j < \omega,\ a_i \in \mathfrak{V}(p)\}.$$

Using the fact that domrp contains only R-eliminable points, one can show by induction that, for every $\psi \in \mathbf{Sub}\varphi$, $(\mathfrak{M}_e, y) \models \psi$ iff $(\mathfrak{M}_e^{rp}, y) \models \psi$, if $C(y)$ does not belong to domrp, and

$$\{n \in \mathsf{rp}C : (\mathfrak{M}_e^{rp}, n) \models \psi\} = \{mj + i : j < \omega,\ (\mathfrak{M}_e, a_i) \models \psi\},$$

if a cluster $C = \{a_0, \dots, a_{m-1}\}$ is in domrp. Thus $\mathfrak{F}_e^{rp}$ refutes φ, which proves that $L.t$ is Kripke complete.

To show that all canonical logics $L.t$ do have FMP we reduce $\mathfrak{F}_e^{rp}$ once again. Define an equivalence relation $\sim$ on W_e by induction on the R-depth $d_R(x)$ of a point x in $\mathfrak{F}_e$. Suppose that $d_R(x) = d_R(y)$ and $\sim$ is already defined for all points of R-depth $< d_R(x)$ and put $x \sim y$ if the following conditions are satisfied: (a) $x \models \psi$ iff $y \models \psi$, for all $\psi \in \mathbf{Sub}\varphi$ ($x \sim_\varphi y$, for short), (b) if z is an R-successor of y and $C(z) \neq C(y)$ then there exists an R-successor z' of x with $C(z') \neq C(x)$ such that $z \sim z'$ and vice versa, (c) the cluster $C(x)$ is degenerate iff $C(y)$ is degenerate, (d) $\mathsf{rp}C(x) = \mathsf{rp}C(y)$, (e) for each $z \in C(x)$ there exists $z' \in C(y)$ such that $z \sim_\varphi z'$ and vice versa.

Let $[x]$ denote the equivalence class generated by x. Define a frame $\mathfrak{G} = \langle V, S, S^{-1}\rangle$ by taking $V = \{[x] : x \in W_e\}$, and $[x]S[y]$ iff there are $x' \in [x]$ and $y' \in [y]$ such that $x'Ry'$. Since $\mathfrak{F}_e$ is of finite depth, V is finite. Moreover, the map $x \mapsto [x]$ is a reduction of the unimodal frame $\langle W_e, R \upharpoonright W_e\rangle$ to $\langle V, S\rangle$. It follows that $\mathfrak{G}$ is a frame for $L.t$ whenever L is canonical. Define a valuation in $\mathfrak{G}$ by putting $[x] \models p$ iff $x \models p$, for all $x \in W_e$ and all variables p in φ. Then one can show that $[x] \models \psi$ iff $x \models \psi$, for all $\psi \in \mathbf{Sub}\varphi$. So $\mathfrak{G} \not\models \varphi$, as required, which means that $L.t$ has FMP.

To prove the decidability of a finitely axiomatizable $L.t$ we first show its completeness with respect to a rather simple class of frames.

Define a replacement function rf for $\mathfrak{G}$ as follows. For each cluster C in $\mathfrak{F}_e$ the set $[C] = \{[x] : x \in C\}$ is a cluster in $\mathfrak{G}$, and moreover, every cluster in $\mathfrak{G}$ can be presented in this way. So we put $\mathsf{rf}[C] = \mathsf{rp}C$, for all clusters $[C]$ in $\mathfrak{G}$. Notice that by (d), rf is well-defined. It is easily shown now that the R-reduct of $\mathfrak{F}_e^{rp}$ is reducible to the R-reduct of $\mathfrak{G}^{rf}$ and that $\mathfrak{G}^{rf}$ refutes φ. Thus we obtain

LEMMA 129. *For each cofinal subframe logic L,*

$$L.t = \mathrm{Log}\{\mathfrak{G}^{rp} : \mathfrak{G}^{rp} \models L.t,\ \mathfrak{G} \textit{ finite, } \mathsf{rp} \textit{ a replacement function for } \mathfrak{G}\}.$$

So, to establish the decidability of a finitely axiomatizable $L.t$ it is enough now to present an algorithm which is capable of deciding, given an rp for a finite $\mathfrak{G}$ and φ, whether $\mathfrak{G}^{rp} \models \varphi$. To this end we require the notion of a *cluster assignment* $\mathbf{t} = \langle \mathbf{t}_1, \mathbf{t}_2 \rangle$ in a tense frame $\mathfrak{G}$, which is any function from the set of clusters in $\mathfrak{G}$ into the set $\{\mathsf{m}, \mathsf{j}\} \times \{\mathsf{m}, \mathsf{j}\}$ such that $\mathbf{t}C = (\mathsf{m}, \mathsf{m})$ if C is degenerate (here m and j are just two symbols; m stands for "maximal" and j for "joker"). A valuation $\mathfrak{V}$ in $\mathfrak{G}$ is called *φ-good for* $(\mathfrak{G}, \mathbf{t})$ if the following conditions hold:

- if $\mathbf{t}_1 C = \mathsf{j}$ then $C \cap \max_R(\mathfrak{V}(\psi)) = \emptyset$, for all $\psi \in \mathbf{Sub}\varphi$;
- if $\mathbf{t}_2 C = \mathsf{j}$ then $C \cap \max_{R^{-1}}(\mathfrak{V}(\psi)) = \emptyset$, for all $\psi \in \mathbf{Sub}\varphi$.

EXAMPLE 130. Let $\mathfrak{F}$ be the frame constructed in Example 128 and suppose that $\mathbf{t}\{\infty\} = (\mathsf{j}, \mathsf{m})$. Then each valuation $\mathfrak{V}$ in $\mathfrak{F}$ is φ-good for $(\mathfrak{G}, \mathbf{t})$ no matter what φ is, because ∞ is eliminable relative to R. The point ∞ is not R^{-1}-eliminable, since $\infty \in \max_{R^{-1}}(\top)$.

Given a formula φ, a finite frame $\mathfrak{F}$ and a replacement function rp for $\mathfrak{F}$, we construct a finite frame $\mathfrak{G} = \langle V, S, S^{-1} \rangle$ with a cluster assignment $\mathbf{t}$ as follows. Let k be the number of variables in φ. Then $\mathfrak{G}$ is obtained from $\mathfrak{F}^{rp}$ by replacing every $\mathsf{rp}C = \langle \omega, >, < \rangle$ with a non-degenerate cluster C' of cardinality 2^k, S-followed by a chain of $2l(\varphi)$ irreflexive points, and by replacing every $\mathsf{rp}C = \langle \omega, \geq, \leq \rangle$ with a non-degenerate cluster C' of cardinality 2^k, S-followed by a chain of $2l(\varphi)$ reflexive points. The cluster assignment $\mathbf{t}$ in $\mathfrak{G}$ is defined by putting $\mathbf{t}C' = (\mathsf{j}, \mathsf{m})$, for all new clusters C' of cardinality 2^k, and $\mathbf{t}C' = (\mathsf{m}, \mathsf{m})$, for all the other clusters. It is not difficult now to prove that $\mathfrak{F}^{rp} \models \varphi$ iff $(\mathfrak{G}, \mathfrak{U}) \models \varphi$, for all φ-good for $(\mathfrak{G}, \mathbf{t})$ valuations $\mathfrak{U}$ in $\mathfrak{G}$. This equivalence provides an effective procedure for deciding whether $\mathfrak{F}^{rp} \models \varphi$. ■

Note that a similar technique can be used to prove completeness and decidability of various tense logics that are not minimal tense extensions. For instance, all logics of the form $L.t \oplus \Diamond_2\Box_2 p \to \Box_2\Diamond_2 p$, where L is a cofinal subframe logic, are complete and decidable if finitely axiomatizable.

2.5 Tense logics of linear frames

One of the most important types of tense logics are logics characterized by linear tense frames, i.e., transitive frames $\langle W, R, R^{-1}, P\rangle$ such that, for all $x, y \in W$, xRy or $xR^{-1}y$ or $x = y$. For example, Bull [1968] and Segerberg [1970] axiomatized the logics of the frames, $\langle \mathbb{Z}, <, >\rangle$, $\langle \mathbb{Q}, <, >\rangle$ and $\langle \mathbb{R}, <, >\rangle$ ($\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are the sets of integer, rational and real numbers, respectively).

Linear tense logics form the lattice NExt**Lin**, where

$$\mathbf{Lin} = \mathbf{K4}.t \oplus \Diamond_1\Diamond_2 p \vee \Diamond_2\Diamond_1 p \rightarrow p \vee \Diamond_1 p \vee \Diamond_2 p$$

is the tense logic determined by the class of all linearly ordered Kripke frames $\langle W, R, R^{-1}\rangle$. As we saw in Section 1.11, even unimodal logics of linear orders are rather non-trivial (for instance, they do not always enjoy FMP). Yet they can be characterized by Kripke frames with a transparent structure, which yields a decision algorithm for those of them that are finitely axiomatizable. Tense logics of linear frames turn out to be even more complicated. In fact, one can find almost all kinds of "monsters" among them: uncountably many logics without Kripke frames, strongly complete logics that are not canonical, canonical logics that are not $\mathcal{R}$-persistent, incomplete subframe logics, etc. Nevertheless, in this section we show that these logics are quite manageable. Our exposition follows [Wolter 1996b, c], where the reader can find the omitted details. All frames in this section are assumed to be linear.

Given a finite sequence $\overline{\mathfrak{F}} = \langle \mathfrak{F}_i = \langle W_i, R_i, P_i\rangle : 1 \leq i \leq n\rangle$ of disjoint frames, we denote by $[\overline{\mathfrak{F}}] = \mathfrak{F}_1 \lhd \cdots \lhd \mathfrak{F}_n$ the ordered sum of them, i.e., the frame $\langle W, R, R^{-1}, P\rangle$ in which

$$W = \bigcup_{i=1}^{n} W_i,\ R = \bigcup_{i=1}^{n} R_i \cup \bigcup_{1\leq i<j\leq n} (W_i \times W_j)$$

and $P = \{X_1 \cup \cdots \cup X_n : X_i \in P_i\}$. Each finite frame can be represented then as the ordered sum $C_1 \lhd \cdots \lhd C_n$ of its clusters.

We begin our study by developing a language of "canonical formulas" for axiomatizing logics in NExt**Lin** and characterizing the constitution of their frames. It will play the same role as the language of canonical formulas for **K4**. With every finite frame $\mathfrak{F} = \langle W, R, R^{-1}\rangle = C_1 \lhd \cdots \lhd C_n$ and a cluster assignment $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ in it we associate the formula

$$\alpha(\mathfrak{F}, \mathbf{t}) = \delta(\mathfrak{F}, \mathbf{t}) \wedge \Box_1\delta(\mathfrak{F}, \mathbf{t}) \wedge \Box_2\delta(\mathfrak{F}, \mathbf{t}) \rightarrow \neg p_r,$$

where r is an arbitrary fixed point in $\mathfrak{F}$ and

$$\begin{aligned}
\delta(\mathfrak{F},\mathbf{t}) \quad = \quad & \bigwedge\{p_x \to \Diamond_1 p_y : xRy, \neg(yRx)\} \wedge \\
& \bigwedge\{p_x \to \Diamond_2 p_y : xR^{-1}y, \neg(xRy)\} \wedge \\
& \bigwedge\{p_x \to \neg p_y : x \neq y\} \wedge \bigwedge\{p_x \to \neg\Diamond_2 p_y : \neg(xRy)\} \wedge \\
& \bigwedge\{p_x \to \Diamond_1 p_y : \exists i \leq n\ (\mathbf{t}_1 C_i = \mathsf{m} \wedge x, y \in C_i \wedge xRy)\} \wedge \\
& \bigwedge\{p_x \to \Diamond_2 p_y : \exists i \leq n\ (\mathbf{t}_2 C_i = \mathsf{m} \wedge x, y \in C_i \wedge xR^{-1}y)\} \wedge \\
& \bigvee\{p_y : y \in W\}.
\end{aligned}$$

To explain the semantical meaning of these formulas, notice first that if $\mathbf{t}C = (\mathsf{m},\mathsf{m})$ for all clusters C then $\mathfrak{G} \not\models \alpha(\mathfrak{F},\mathbf{t})$ iff $\mathfrak{G}$ is reducible to $\mathfrak{F}$; so $\mathbf{Lin} \oplus \alpha(\mathfrak{F},\mathbf{t})$ is a splitting of NExt$\mathbf{Lin}$. Suppose now that $\mathbf{t}_i C = \mathsf{j}$ for some $i \in \{1,2\}$ and some cluster C in $\mathfrak{F}$. In this case $\mathfrak{G} \not\models \alpha(\mathfrak{F},\mathbf{t})$ iff there exist frames $\mathfrak{G}_i$, for $1 \leq i \leq n$, such that $\mathfrak{G} = \mathfrak{G}_1 \lhd \cdots \lhd \mathfrak{G}_n$ and $\mathfrak{G}_i \not\models \alpha(C_i, \mathbf{t}\restriction C_i)$ for all $1 \leq i \leq n$. So it suffices to examine the situation when $\mathfrak{G} \not\models \alpha(C,\mathbf{t})$ for a cluster C. Assume for simplicity that $\mathfrak{G}$ is a Kripke frame. *Case 1:* $\mathbf{t}C = (\mathsf{j},\mathsf{j})$. Then $\mathfrak{G} \not\models \alpha(C,\mathbf{t})$ iff $|\mathfrak{G}| \geq |C|$. *Case 2:* $\mathbf{t}C = (\mathsf{m},\mathsf{j})$. Then C is non-degenerate and $\mathfrak{G} \not\models \alpha(C,\mathbf{t})$ iff either $\mathfrak{G}$ contains an R-final cluster of cardinality $\geq |C|$ or it has no R-final point at all. *Case 3:* $\mathbf{t}C = (\mathsf{j},\mathsf{m})$. This is the mirror image of Case 2. *Case 4:* $\mathbf{t}C = (\mathsf{m},\mathsf{m})$. If C is an irreflexive point then $\mathfrak{G}$ is an irreflexive point as well whenever $\mathfrak{G} \not\models \alpha(C,\mathbf{t})$. If C is non-degenerate and $\mathfrak{G} \not\models \alpha(C,\mathbf{t})$ then $\mathfrak{G}$ satisfies the conditions of Cases 2 and 3.

EXAMPLE 131. Let $\alpha = \alpha(\overset{a}{\circ}\!\!\rightarrow\!\!\overset{b}{\circ}, \mathbf{t})$ where $\mathbf{t}a = (\mathsf{m},\mathsf{j})$ and $\mathbf{t}b = (\mathsf{j},\mathsf{m})$. Then $\mathfrak{F} \not\models \alpha$ iff there exists a non-empty upward closed set $X \in P$ such that $\forall x \in X \exists y \in X\ yRx$, $W - X \neq \emptyset$ and $\forall x \in W - X \exists y \in W - X\ xRy$. Hence $\langle \mathbb{Q}, <, > \rangle \not\models \alpha$ (take $X = \{y \in \mathbb{Q} : \sqrt{2} < y\}$) but $\langle \mathbb{R}, <, > \rangle \models \alpha$, since the real line contains no gaps.

THEOREM 132. *There is an algorithm which, given a formula φ, returns formulas $\alpha(\mathfrak{F}_1,\mathbf{t}_1), \ldots, \alpha(\mathfrak{F}_n,\mathbf{t}_n)$ such that*

$$\mathbf{Lin} \oplus \varphi = \mathbf{Lin} \oplus \alpha(\mathfrak{F}_1,\mathbf{t}_1) \oplus \cdots \oplus \alpha(\mathfrak{F}_n,\mathbf{t}_n).$$

Proof. Let $(\mathfrak{F}_i,\mathbf{t}_i)$, $1 \leq i \leq n$, be the collection of all finite frames with type assignments such that, for each i, (a) there is a countermodel $\mathfrak{M}_i = \langle \mathfrak{F}_i, \mathfrak{V}_i \rangle$ for φ in which $\mathfrak{V}_i$ is φ-good for $(\mathfrak{F}_i,\mathbf{t}_i)$, (b) the depth of $\mathfrak{F}_i$ does not exceed $4l(\varphi) + 1$, and (c) no cluster in $\mathfrak{F}_i$ contains more than $2^{v(\varphi)}$ points, where $v(\varphi)$ is the number of variables in φ.

Let $\mathfrak{F}$ refute $\alpha(\mathfrak{G}_i,\mathbf{t}_i)$ under a valuation $\mathfrak{U}$. By the definition of $(\mathfrak{F}_i,\mathbf{t}_i)$, the model $\mathfrak{M}_i$ refutes φ. Define a valuation $\mathfrak{U}'$ in $\mathfrak{F}$ by taking, for all variables

p in φ,

$$\mathfrak{U}'(p) = \bigcup\{\mathfrak{U}(p_x) : x \in \mathfrak{V}_i(p)\}.$$

It is not hard to show by induction that $\mathfrak{U}'(\psi) = \bigcup\{\mathfrak{U}(p_x) : x \in \mathfrak{V}_i(\psi)\}$ for all $\psi \in \mathbf{Sub}\varphi$, and so $\mathfrak{F}$ refutes φ under $\mathfrak{U}'$. Thus $\mathfrak{F} \models \varphi$ implies $\mathfrak{F} \models \alpha(\mathfrak{F}_i, \mathbf{t}_i)$ for every i. The converse direction is rather technical; we refer the reader to [Wolter 1996c]. ■

"Canonical" axiomatizations of some standard linear tense logics are shown in Table 3, where we use the following abbreviations. Given a finite frame $\mathfrak{F} = C_1 \lhd \cdots \lhd C_n$, we write $\alpha((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n))$ instead of $\alpha(\mathfrak{F}, \mathbf{t})$ and $\alpha(-, (C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n))$ instead of

$$\alpha((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n)) \oplus \alpha((\circ, (\mathsf{j}, \mathsf{j})) \lhd (C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n)).$$

$\alpha((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n), -)$ is defined analogously.

Now we exploit the formulas $\alpha(\mathfrak{F}, \mathbf{t})$ to characterize the $\bigcap$-irreducible logics in NExt**Lin**. Recall that every logic $L \in \text{NExt}L_0$ is represented as

$$L = \bigcap\{L' \supseteq L : L' \text{ is } \bigcap\text{-irreducible}\}.$$

So such a characterization can open the door to a better understanding of the structure of the lattice NExt**Lin**. The $\bigcap$-irreducible logics will be described semantically as the logics determined by certain descriptive frames.

DEFINITION 133. *(1) Denote by* $ⓚ$ *the non-degenerate cluster with* $k > 0$ *points.*

(2) Let $\omega^<(0)$ *be the strictly ascending chain* $\langle \omega, <, > \rangle$ *of natural numbers,* $\omega^<(1)$ *the chain* $\langle \omega, \leq, \geq \rangle$*,* $\omega^<(2)$ *the ascending chain of natural numbers in which precisely the even points are reflexive,* $\omega^<(3)$ *the chain in which precisely the multiples of* 3 *are reflexive, and so on;* $\omega^>(n)$ *is the mirror image of* $\omega^<(n)$*.*

(3) $\mathfrak{C}(0, ①)$ *is the mirror image of the frame introduced in Example 128, i.e.,* $\mathfrak{C}(0, ①) = \langle \omega^<(0) \lhd ①, P \rangle$*, where* P *consists of all cofinite sets containing* ① *and their complements. We generalize this construction to chains* $\omega^<(n)$ *and clusters* ⓚ*. Namely, for* $n < \omega$*,* $k > 1$ *and* $ⓚ = \{a_0, \ldots, a_{k-1}\}$*, we put*

$$\mathfrak{C}(n, ⓚ) = \langle \omega^<(n) \lhd ⓚ, P \rangle,$$

where P *is the set of possible values generated by* $\{X_i : 0 \leq i \leq k-1\}$*, for* $X_i = \{a_i\} \cup \{kj + i : j \in \omega\}$*,* $0 \leq i \leq k-1$*.* $\mathfrak{C}(ⓚ, n)$ *denotes the mirror image of* $\mathfrak{C}(n, ⓚ)$*.*

(4) $\mathfrak{C}(0, ①, 0) = \langle \omega^<(0) \lhd ① \lhd \omega^>(0), P \rangle$*, where* P *consists of all cofinite sets containing* ① *and their complements.*

Table 3. Axiomatizations of standard tense logics

$\mathbf{Ord}_t$	$=$	$\mathrm{Log}\{\langle \xi, <, > \rangle : \xi \text{ an ordinal}\} =$
		$\mathbf{Lin} \oplus \alpha(-, (\circ, (\mathsf{j}, \mathsf{m})))$
$\mathbf{E}_t$	$=$	$\mathbf{Lin} \oplus \Diamond_1 \top \oplus \Diamond_2 \top =$
		$\mathbf{Lin} \oplus \alpha(-, (\bullet, (\mathsf{m}, \mathsf{m}))) \oplus \alpha((\bullet, (\mathsf{m}, \mathsf{m})), -)$
$\mathbf{O}_n$	$=$	$\mathrm{Log}\langle \omega n, <, > \rangle =$
		$\mathbf{Ord}_t \oplus \alpha(\underbrace{(\circ, (\mathsf{m}, \mathsf{j})) \lhd \cdots \lhd (\circ, (\mathsf{m}, \mathsf{j})))}_{n+1} \oplus \alpha(-, (\bullet, (\mathsf{m}, \mathsf{m})))$
$\mathbf{RD}$	$=$	$\mathrm{Log}\{\mathfrak{G} : \forall x(\neg xRx \to \exists y(xRy \wedge \{z : xRzRy\} = \emptyset))\} =$
		$\mathbf{Lin} \oplus \alpha(-, (\bullet, (\mathsf{m}, \mathsf{m}))) \oplus \alpha(-, (\bullet, (\mathsf{m}, \mathsf{m})) \lhd (\circ, (\mathsf{m}, \mathsf{j})))$
$\mathbf{LD}$	$=$	the mirror image of $\mathbf{RD}$
$\mathbf{Z}_t$	$=$	$\mathrm{Log}\langle \mathbb{Z}, <, > \rangle =$
		$\mathbf{RD} \oplus \mathbf{LD} \oplus \alpha((\circ, (\mathsf{j}, \mathsf{j})) \lhd (\circ, (\mathsf{j}, \mathsf{m}))) \oplus$
		$\alpha((\circ, (\mathsf{m}, \mathsf{j})) \lhd (\circ, (\mathsf{j}, \mathsf{j})))$
$\mathbf{Ds}_n$	$=$	$\mathbf{Lin} \oplus \Box_1^{n+1} p \to \Box_1^n p =$
		$\mathbf{Lin} \oplus \alpha(-, \underbrace{(\bullet, (\mathsf{m}, \mathsf{m}) \lhd \cdots \lhd (\bullet, (\mathsf{m}, \mathsf{m}))}_{n+1}, -)$
$\mathbf{Q}_t$	$=$	$\mathrm{Log}\langle \mathbb{Q}, <, > \rangle =$
		$\mathbf{Ds}_1 \oplus \mathbf{E}_t$
$\mathbf{R}_t$	$=$	$\mathrm{Log}\langle \mathbb{R}, <, > \rangle =$
		$\mathbf{Q}_t \oplus \alpha((\circ, (\mathsf{m}, \mathsf{j})) \lhd (\circ, (\mathsf{j}, \mathsf{m})))$
$\mathbf{Rd}_t$	$=$	$\mathrm{Log}\{\langle \xi, \leq, \geq \rangle : \xi \text{ an ordinal}\} =$
		$\mathbf{Lin} \oplus \alpha(-, (②, (\mathsf{j}, \mathsf{m})))$

It is easy to check that the frames defined in (3) and (4) are descriptive and a singleton $\{x\}$ is in P iff $x \notin ⓚ$.

For a class of frames $\mathcal{C}$, we denote by $\mathcal{C}^*$ the class of finite sequences of frames from $\mathcal{C}$ and let $[\mathcal{C}^*] = \{[\overline{\mathfrak{F}}] : \overline{\mathfrak{F}} \in \mathcal{C}^*\}$. The class of finite clusters and the frames of the form (3) in Definition 133 is denoted by $\mathcal{B}_0$; put also $\mathcal{B} = \{\mathfrak{C}(0, ①, 0)\} \cup \mathcal{B}_0$.

THEOREM 134. *Each logic $L \in$ NExt**Lin** is determined by a set $\mathcal{C} \subseteq [\mathcal{B}^*]$. If L is finitely axiomatizable then $L = \mathrm{Log}\mathcal{C}$ for some set $\mathcal{C} \subseteq [\mathcal{B}_0^*]$.*

Proof. We explain the idea of the proof of the first claim. Suppose that $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is a countermodel for $\alpha = \alpha((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n))$ based on a descriptive frame $\mathfrak{F} = \langle W, R, R^{-1}, P \rangle$. We must show that there exists $\mathfrak{G} \in [\mathcal{B}^*]$ refuting α and such that $\mathrm{Log}\mathfrak{G} \supseteq \mathrm{Log}\mathfrak{F}$. Consider the sets

$$W_i = \{y \in W : (\mathfrak{M}, y) \models \bigvee \{p_x : x \in C_i\}\}.$$

One can easily show that W_i are intervals in $\mathfrak{F}$ and $\mathfrak{F} = \mathfrak{F}_1 \lhd \cdots \lhd \mathfrak{F}_n$, for the subframes $\mathfrak{F}_i$ of $\mathfrak{F}$ induced by W_i. Moreover, $\mathfrak{G} = [\overline{\mathfrak{G}}]$ is as required if $\overline{\mathfrak{G}} = \langle \mathfrak{G}_1, \ldots, \mathfrak{G}_n \rangle$ is a sequence in $\mathcal{B}^*$ such that $\mathrm{Log}\mathfrak{G}_i \supseteq \mathrm{Log}\mathfrak{F}_i$, and $\mathfrak{G}_i \not\models \alpha(C_i, \mathbf{t}C_i)$, for $1 \leq i \leq n$. Frames $\mathfrak{G}_i$ with those properties are constructed in [Wolter96d]. ■

EXAMPLE 135. The logic $\mathbf{Q}_t$ is determined by the frames $\mathfrak{F} \in [\mathcal{B}^*]$ which contain no pair of adjacent irreflexive points, and $\mathbf{R}_t$ is determined by the frames $\mathfrak{F} \in [\mathcal{B}^*]$ which contain neither a pair of adjacent irreflexive points nor a pair of adjacent non-degenerate clusters.

It is not difficult to show now that the logics $\mathrm{Log}\mathfrak{F}$, for $\mathfrak{F} \in [\mathcal{B}^*]$, coincide with the $\bigcap$-irreducible logics in NExt**Lin**. Our first aim is achieved, and in the remaining part of this section we shall draw consequences of this result. Using the same sort of arguments as in the proof of Theorem 126 and Kruskal's [1960] Tree Theorem one can prove

COROLLARY 136. (i) *All finitely axiomatizable logics in* NExt**Lin** *are decidable.*

(ii) *A logic L is finitely axiomatizable whenever there exists $n < \omega$ such that $L \in$ NExt$\mathbf{Ds}_n$.*

It follows in particular that all logics in NExt$\mathbf{Q}_t$ and all logics of reflexive frames are finitely axiomatizable and decidable.

Now we formulate two corollaries concerning the Kripke completeness of linear tense logics. First, it is not hard to see that every logic in NExt**Lin** characterized by an infinite frame in $[\mathcal{B}^*]$ is Kripke incomplete. Using this observation one can prove

COROLLARY 137. *Suppose $L \in$ NExt**Lin** and there is a Kripke frame of infinite depth for L. Then there exists a Kripke incomplete logic in* NExt$L$.

This result means in particular that in Tense Logic we do not have analogues of the unimodal completeness results of Bull [1966b] and Fine [1974c]. However, if a logic is complete then it is determined by a simple class of frames. Let $\mathcal{K}$ be the class frames containing finite clusters and frames of the form (2) in Definition 133.

THEOREM 138. *Each Kripke complete logic in* NExt**Lin** *is determined by a subset of* $[\mathcal{K}^*]$.

One of the main types of logics considered in conventional Tense Logic are logics determined by strict linear orders, known also as *time-lines.* We call them *t-line logics.* All logics in Table 3, save $\mathbf{Rd}_t$, are t-line logics. T-line logics were defined semantically, and now we are going to determine a necessary syntactic condition for a linear tense logic to be a t-line logic.

Given a frame $\mathfrak{F}$, we denote by $\mathfrak{F}^\circ$ the frame that results from $\mathfrak{F}$ by replacing its proper clusters with reflexive points. Call $L \in$ NExt**Lin** a *t-axiom logic* if L is axiomatizable by a set of formulas of the form $\alpha(\mathfrak{F}, \mathbf{t})$ in which $\mathfrak{F}$ contains no proper clusters.

PROPOSITION 139. *The following conditions are equivalent for all logics* $L \in$ NExt**Lin***:*

(i) *L is a t-axiom logic;*

(ii) *$\mathfrak{F}^\circ \models L$ implies $\mathfrak{F} \models L$, for every $\mathfrak{F} \in [\mathcal{B}^*]$.*

(iii) *$\alpha(\mathfrak{G}, \mathbf{t}) \in L$ implies $\alpha(\mathfrak{G}^\circ, \mathbf{t}) \in L$,*[15] *for every finite $\mathfrak{G}$.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are clear. To prove that (ii) $\Rightarrow$ (iii), suppose $\alpha(\mathfrak{G}^\circ, \mathbf{t}) \notin L$. Then there exists a frame $\mathfrak{F} \in [\mathcal{B}^*]$ for L refuting $\alpha(\mathfrak{G}^\circ, \mathbf{t})$. Without loss of generality we may assume that $\mathfrak{F}$ contains no proper clusters. By enlarging some clusters in $\mathfrak{F}$ we can construct a frame $\mathfrak{H} \in [\mathcal{B}^*]$ such that $\mathfrak{H}^\circ = \mathfrak{F}$ and $\mathfrak{H} \not\models \alpha(\mathfrak{G}, \mathbf{t})$. In view of (ii), $\mathfrak{H} \models L$ and so $\alpha(\mathfrak{G}, \mathbf{t}) \notin L$. ■

It follows that the t-axiom logics form a complete sublattice of the lattice NExt**Lin**.

THEOREM 140. (i) *All finitely axiomatizable t-axiom logics are Kripke complete.*

(ii) *All t-line logics are t-axiom logics.*

Proof. (i) Suppose that $L = \mathbf{Lin} \oplus \{\alpha(\mathfrak{G}_i^\circ, \mathbf{t}_i) : i \in I\}$, for some finite set I. By Theorem 134, L is determined by a subset of $[\mathcal{B}_0^*]$. For $\mathfrak{F} \in [\mathcal{B}_0^*]$, let $k\mathfrak{F}$ be the Kripke frame that results from $\mathfrak{F}$ by replacing all $\mathfrak{C}(n, \textcircled{k})$ and $\mathfrak{C}(\textcircled{k}, n)$ with $\omega^<(n)$ and $\omega^>(n)$, respectively. Then we clearly have Log$k\mathfrak{F} \subseteq$ Log$\mathfrak{F}$, and $\mathfrak{F} \models \alpha(\mathfrak{G}^\circ, \mathbf{t})$ iff $k\mathfrak{F} \models \alpha(\mathfrak{G}^\circ, \mathbf{t})$. It follows that L is Kripke complete. (ii) Suppose that L is a t-line logic. By Proposition 139 (3), it suffices to

[15] We assume that $\mathbf{t}C = \mathbf{t}\circ$ whenever $\circ$ replaces C in $\mathfrak{G}$.

observe that $\mathfrak{F} \models \alpha(\mathfrak{G}^\circ, \mathbf{t})$ iff $\mathfrak{F} \models \alpha(\mathfrak{G}, \mathbf{t})$, for all time-lines $\mathfrak{F}$ and all finite $\mathfrak{G}$. ■

So the fact that in Table 3 all t-line logics are axiomatized by canonical formulas of the form $\alpha(\mathfrak{G}^\circ, \mathbf{t})$ is no accident. Finding and verifying axiomatizations of t-line logics becomes almost trivial now.

EXAMPLE 141. Let us check the axiomatization of $\mathbf{Z}_t$ in Table 3. Put

$$L = \mathbf{RD} \oplus \mathbf{LD} \oplus \alpha((\circ, (\mathsf{j}, \mathsf{j})) \lhd (\circ, (\mathsf{j}, \mathsf{m}))) \oplus \alpha((\circ, (\mathsf{m}, \mathsf{j})) \lhd (\circ, (\mathsf{j}, \mathsf{j}))).$$

By Theorem 140, L is complete. By Theorem 138, L is then determined by a subset of $[\mathcal{K}^*]$. Clearly this set contains $\langle \mathbb{Z}, <, > \rangle$, possibly ⓚ for $k > 0$, and nothing else. But the logic of ⓚ contains $\mathbf{Z}_t$, for all $k > 0$.

We conclude this section by discussing the decidability of properties of logics in NExt**Lin**. In Section 4.4 it will be shown that almost all interesting properties of calculi are undecidable in NExt**K** and even in NExt**S4**. In NExt**Lin** the situation is different, as was proved in [Wolter 1996c, 1997c].

THEOREM 142. (i) *There are algorithms which, given a formula φ, decide whether* $\mathbf{Lin} \oplus \varphi$ *has FMP, interpolation, whether it is Kripke complete, strongly complete, canonical, $\mathcal{R}$-persistent.*

(ii) *A linear tense logic is canonical iff it is $\mathcal{D}$-persistent iff it is complete and its frames are first order definable.*

(iii) *If a logic in* NExt**Lin** *has a frame of infinite depth then it does not have interpolation.*

So NExt**Lin** provides an interesting example of a rather complex lattice of modal logics for which almost all important properties of calculi are decidable. We shall not go into details of the proof here but discuss quite natural criteria for canonicity and strong completeness of logics in NExt**Lin** required to prove this theorem. Denote by $\mathcal{B}_+$ the class of frames containing $\mathcal{B}$ together with frames $\mathfrak{C}(n_1, ⓚ, n_2)$ defined as follows. Suppose $k > 1$, $n_1, n_2 < \omega$ are such that $n_1 + n_2 > 0$ and $ⓚ = \{a_0, \dots, a_{k-1}\}$. Then

$$\mathfrak{C}(n_1, ⓚ, n_2) = \langle \omega^{<}(n_1) \lhd ⓚ \lhd \omega^{>}(n_2), P \rangle,$$

where P is the set of possible values generated by $\{X_i : 0 \le i \le k-1\}$, for

$$X_i = \{a_i\} \cup \{kj + i : j \in \omega\} \cup \{k^* j^* + i^* : j \in \omega\}$$

and $\{0^*, 1^*, \dots, n^*, \dots\}$ being the points in $\omega^{>}(n_2)$.

Let $\mathcal{F}$ be the class of frames of the form

$$\langle \{0, \dots, n_1\}, <, > \rangle \lhd ① \lhd \langle \{0, \dots, n_2\}, <, > \rangle \text{ or } \langle \{0, \dots, n\}, <, > \rangle .$$

THEOREM 143. (i) *A logic* $L \in$ NExt**Lin** *is canonical iff the underlying Kripke frame of each frame $\mathfrak{F} \in [\mathcal{B}_+^*]$ for L validates L as well.*

(ii) *A logic* $L \in \mathrm{NExt}\mathbf{Lin}$ *is strongly complete iff for each frame* $\mathfrak{F} \in [\mathcal{B}_+^*]$ *validating* L*, there exists a Kripke frame* $\mathfrak{G}$ *for* L *which results from* $\mathfrak{F}$ *by replacing*

- *every* $\mathfrak{C}(n, (k))$ *with* $\omega^<(n)$ *or* $\omega^<(n) \lhd \mathfrak{H} \lhd (k)$*, for some* $\mathfrak{H} \in \mathcal{F}$*, and*
- *every* $\mathfrak{C}((k), n)$ *with* $\omega^>(n)$ *or* $(k) \lhd \mathfrak{H} \lhd \omega^>(n)$*, for some* $\mathfrak{H} \in \mathcal{F}$*, and*
- *every* $\mathfrak{C}(n_1, (k), n_2)$ *with* $\omega^<(n_1) \lhd \mathfrak{H} \lhd \omega^>(n_2)$*, for some* $\mathfrak{H} \in \mathcal{F}$*.*

EXAMPLE 144. The logic $\mathbf{R}_t$ is not canonical because $\mathfrak{C}(2, (2)) \models \mathbf{R}_t$ but $\omega^<(2) \lhd (2) \not\models \mathbf{R}_t$. However, $\mathbf{R}_t$ is strongly complete, since $\mathfrak{F} \models \mathbf{R}_t$ whenever $\mathfrak{G} \in [\mathcal{B}_+^*]$ validates $\mathbf{R}_t$ and $\mathfrak{F}$ is obtained from $\mathfrak{G}$ as in the formulation of Theorem 143 with $\mathfrak{H} = \bullet \in \mathcal{F}$.

One can also use Theorem 143 to construct two strongly complete logics $L_1, L_2 \in \mathrm{NExt}\mathbf{Lin}$ whose sum $L_1 \oplus L_2$ is not strongly complete (see [Wolter 1996b]).

2.6 Bimodal provability logics

Bimodal provability logics emerge when combinations of two different provability predicates are investigated, for example, if $\Box_1$ is understood as "it is provable in PA" and $\Box_2$ as "it is provable in ZF". In contrast to the situation in unimodal provability logic, where almost all provability predicates behave like the necessity operator $\Box$ in $\mathbf{GL}$, there exist quite a lot of different types of bimodal provability logics. Various completeness results extending Solovay's completeness theorem for $\mathbf{GL}$ to the bimodal case were established by Smoryński [1985], Montagna [1987], Beklemishev [1994, 1996] and Visser [1995]. Here we will not deal with the interpretation of modal operators as provability predicates but sketch some results on modal logics containing the bimodal provability logic

$$\mathbf{CSM}_0 = (\mathbf{GL} \otimes \mathbf{GL}) \oplus \Box_1 p \to \Box_2 p \oplus \Box_2 p \to \Box_1 \Box_2 p$$

(named so by Visser [1995] after Carlson, Smoryński and Montagna). A number of provability logics is included in this class, witness the list below. (As in unimodal provability logic we have quasi-normal logics among them, i.e., sets of formulas containing $\mathbf{K}_2$ and closed under modus ponens and substitutions (but not necessarily under $\varphi/\Box_i\varphi$). Recall that we denote by $L + \Gamma$ the smallest quasi-normal logic containing L and Γ.)

- $\mathbf{CSM}_1 = \mathbf{CSM}_0 \oplus \Box_2(\Box_1 p \to p)$. (This is $\mathbf{PRL}_{ZF}$ in [Smoryński 1985] and $\mathbf{F}$ in [Montagna 1987].)
- $\mathbf{NB}_1 = \mathbf{CSM}_0 \oplus (\neg\Box_1 p \wedge \Box_2 p) \to \Box_2(\Box_1 q \to q)$.

- $\mathbf{CSM}_2 = \mathbf{CSM}_1 + \Box_1 p \to p$. (This is $\mathbf{PRL}_{ZF}$ + Reflection$_{\Box_1}$ in [Smoryński 1985] and $\mathbf{F}_1$ in [Montagna 1987].)
- $\mathbf{CSM}_3 = \mathbf{CSM}_2 + \Box_2 p \to p$. (This is $\mathbf{PRL}_{ZF}$ + Reflection$_{\Box_2}$ in [Smoryński 1985].)
- $\mathbf{NB}_2 = \mathbf{NB}_1 + \Box_2 p \to p + \Box_2 p \to \Box_1 p$.

A remarkable feature of $\mathbf{CSM}_0$ is that—like in **GL**—we have uniquely determined definable fixed points.

THEOREM 145 (Smoryński 1985). *Let $\varphi(p)$ be a formula in which every occurrence of p lies within the scope of some $\Box_1$ or some $\Box_2$. Then*

(i) *there exists a formula ψ containing only the propositional variables of $\varphi(p)$ different from p such that $\psi \leftrightarrow \varphi(\psi) \in \mathbf{CSM}_0$;*

(ii) $\Box_1((p \leftrightarrow \varphi(p)) \wedge (q \leftrightarrow \varphi(q))) \to (p \leftrightarrow q) \in \mathbf{CSM}_0$.

In the remaining part of this section we are concerned with subframe logics containing $\mathbf{CSM}_0$, the main result stating that those of them that are finitely axiomatizable are decidable. All the provability logics introduced above turn out to be subframe logics, so we obtain a uniform proof of their decidability. An interesting trait of subframe logics in Ext$\mathbf{CSM}_0$ is that (as a rule) they are Kripke incomplete; in the list above such are $\mathbf{CSM}_i$, $i = 1, 2, 3$, and $\mathbf{NB}_i$, $i = 1, 2$. The proof extends the techniques introduced by Visser [1995]; for details we refer the reader to [Wolter 1998].

First we develop—as was done for NExt**K4** and NExt**Lin**—a frame theoretic language for axiomatizing subframe logics in the lattice Ext$\mathbf{CSM}_0$. A finite frame $\mathfrak{G} = \langle W, R_1, R_2\rangle$ validates $\mathbf{CSM}_0$ iff both R_1 and R_2 are transitive, irreflexive, $R_2 \subseteq R_1$ and

$$\forall x, y, z\ (xR_1y \wedge yR_2z \to xR_2z).$$

In this section all (not only finite) frames are assumed to satisfy these conditions, *save irreflexivity.*

A finite frame $\mathfrak{F}$ is called a *surrogate frame* if it has precisely one root r and all points different from r are R_2-irreflexive. Surrogate frames will provide the language to axiomatize subframe logics in Ext$\mathbf{CSM}_0$. A *normal surrogate frame* $\langle W, R_1, R_2\rangle$ is a surrogate frame in which the root r is R_1-irreflexive. We write $xR_i^p y$ iff xR_iy and $\neg yR_ix$. Given a frame $\mathfrak{G} = \langle V, S_1, S_2, Q\rangle$ for $\mathbf{CSM}_0$ and a surrogate frame $\mathfrak{F} = \langle W, R_1, R_2\rangle$, a map h from V onto W is called a *weak reduction* of $\mathfrak{G}$ to $\mathfrak{F}$ if for $i \in \{1, 2\}$ and all $x, y \in V$,

- xS_iy implies $f(x)R_if(y)$,
- $f(x)R_i^p f(y)$ implies $\exists z \in V\ (xS_iz \wedge f(z) = f(y))$,
- $f^{-1}(X) \in Q$ for all $X \subseteq W$.

(The standard definition of reduction is relaxed here in the second condition.) Each weak reduction to a $\mathbf{CSM}_0$-frames is a usual reduction, since in this case $R_i^p = R_i$. A frame $\mathfrak{G}$ is said to be *weakly subreducible* to a surrogate frame $\mathfrak{F}$ if a subframe of $\mathfrak{G}$ is weakly reducible to $\mathfrak{F}$. To describe weak subreducibility syntactically, with each surrogate frame $\mathfrak{F} = \langle W, R_1, R_2 \rangle$ we associate the formula

$$\alpha(\mathfrak{F}) = \delta(\mathfrak{F}) \wedge \Box_1 \delta(\mathfrak{F}) \to \neg p_r,$$

where r is the root of $\mathfrak{F}$ and

$$\begin{aligned}\delta(\mathfrak{F}) \quad = \quad & \bigwedge\{p_x \to \Diamond_1 p_y : xR_1^p y,\ x,y \in W\} \wedge \\ & \bigwedge\{p_x \to \Diamond_2 p_y : xR_2^p y,\ x,y \in W\} \wedge \\ & \bigwedge\{p_x \to \neg p_y : x \neq y,\ x,y \in W\} \wedge \\ & \bigwedge\{p_x \to \neg\Diamond_1 p_y : \neg(xR_1 y),\ x,y \in W\} \wedge \\ & \bigwedge\{p_x \to \neg\Diamond_2 p_y : \neg(xR_2 y),\ x,y \in W\}.\end{aligned}$$

LEMMA 146. *For every surrogate frame $\mathfrak{F}$ and every $\mathbf{CSM}_0$-frame $\mathfrak{G}$, we have $\mathfrak{G} \not\models \alpha(\mathfrak{F})$ iff $\mathfrak{G}$ is weakly subreducible to $\mathfrak{F}$.*

It follows immediately that $\mathbf{CSM}_0 \oplus \alpha(\mathfrak{F})$ and $\mathbf{CSM}_0 + \alpha(\mathfrak{F})$ are subframe logics. Conversely, we have the following completeness result.

THEOREM 147. (i) *There is an algorithm which, given a formula φ such that $\mathbf{CSM}_0 + \varphi$ is a subframe logic, returns surrogate frames $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ for which*

$$\mathbf{CSM}_0 + \varphi = \mathbf{CSM}_0 + \alpha(\mathfrak{F}_1) + \cdots + \alpha(\mathfrak{F}_n).$$

(ii) *There is an algorithm which, given a formula φ such that $\mathbf{CSM}_0 \oplus \varphi$ is a subframe logic, returns normal surrogate frames $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ such that*

$$\mathbf{CSM}_0 \oplus \varphi = \mathbf{CSM}_0 \oplus \alpha(\mathfrak{F}_1) \oplus \cdots \oplus \alpha(\mathfrak{F}_n).$$

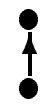
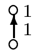
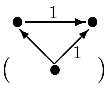
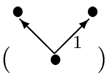
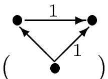
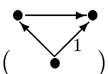
Table 4 shows axiomatizations of the logics introduced above by means of formulas of the form $\alpha(\mathfrak{F})$. In this section we adopt the convention that in figures we place the number 1 nearby an arrow from x to y if xR_1y and $\neg xR_2y$. An arrow without a number means that xR_2y (and therefore xR_1y as well).

The proof of decidability is based on the completeness of subframe logics in Ext$\mathbf{CSM}_0$ with respect to rather simple descriptive frames. With every surrogate frame $\mathfrak{F}$ we associate a finite set of frames

$$\mathrm{E}(\mathfrak{F}) = \{\mathfrak{F}_{\overline{A}} : \overline{A} \in \mathrm{Seq}\mathfrak{F}\}.$$

Loosely, it is defined as follows. Let us first assume that the root r of $\mathfrak{F}$ is R_2-irreflexive. Then the frames in $\mathrm{E}(\mathfrak{F})$ are the results of inserting an

Table 4. Axiomatizations of provability logics

$\mathbf{CSM}_1$	$=$	$\mathbf{CSM}_0 \oplus \alpha($  $)$
$\mathbf{CSM}_0 + \Box_1 p \to p$	$=$	$\mathbf{CSM}_0 + \alpha(\bullet)$
$\mathbf{CSM}_0 + \Box_2 p \to p$	$=$	$\mathbf{CSM}_0 + \alpha($  $)$
$\mathbf{CSM}_0 + \Box_2 p \to \Box_1 p$	$=$	$\mathbf{CSM}_0 + \alpha($  $)$
$\mathbf{NB}_1$	$=$	$\mathbf{CSM}_0 \oplus \alpha($  $) \oplus \alpha($  $)\oplus$ $\alpha($  $) \oplus \alpha($  $)$

infinite strictly descending R_1-chain, denoted by $C(\omega)$, between each non-degenerate R_1-cluster C and its R_1-successors. This defines R_1 uniquely. However, R_2 may be defined in different ways, since a point R_2-seeing a point in C need not (but may) R_2-see certain points in the chain $C(\omega)$.

To be more precise, the set $\mathrm{Seq}\mathfrak{F}$ consists of all sequences $\overline{A}$ of the form

$$\overline{A} = \langle A_x : xR_1x,\ x \in W\rangle.$$

where A_x is a subset of $\{y \in W - C : yR_2x\}$ such that for all y and z, $y \in A_x$ and zR_1y imply $z \in A_x$. For each non-degenerate R_1-cluster C, denote by $C(\omega)$ the set $\{(n, C) : n \in \omega\}$. Finally, given $\overline{A} \in \mathrm{Seq}\mathfrak{F}$, we construct $\mathfrak{F}_{\overline{A}} = \langle V, S_0, S_1\rangle$ as the frame satisfying the following conditions:

- $V = W \cup \bigcup\{C(\omega) : C$ a non-degenerate R_1-cluster in $\mathfrak{F}\}$;
- $R_i = S_i \cap (W \times W)$, for $i \in \{1, 2\}$;
- S_1 is defined so that $C(\omega)$ becomes an infinite descending chain between C and its immediate successors;
- for every non-degenerate R_1-cluster C,
  - $((C(\omega) \cup C) \times (C(\omega) \cup C)) \cap S_2 = \emptyset$,
  - for all $y \in W - C$ and $x \in C(\omega)$, xS_2y iff CR_2y,
  - for all $y \in W - C$, $C = \{j : 0 \le j \le m - 1\}$ and $x \in C(\omega)$, yS_2x iff $\exists i \in \omega \exists j \le m - 1$ $(x = (im + j, C) \wedge y \in A_j)$,

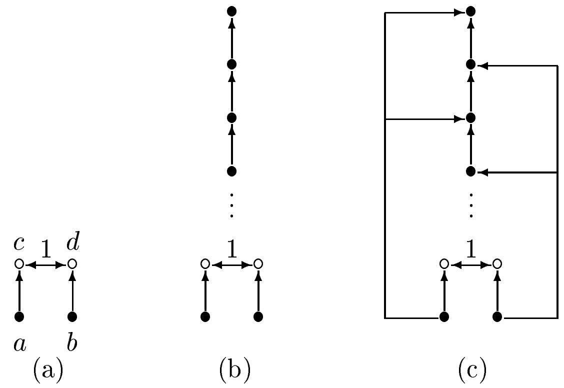


Figure 12.

– for all $x \in C(\omega)$ and $y \in V - C$, xS_2y iff CS_2y.

We illustrate this technical definition by a simple example.

EXAMPLE 148. Construct $\mathrm{E}(\mathfrak{F})$ for the frame $\mathfrak{F}$ in Fig. 12 (a). In this case we have two R_1-reflexive points, namely c and d. So, $\mathrm{Seq}\mathfrak{F}$ consists of pairs $\langle A_c, A_d \rangle$. There are four different pairs and so we have four frames in $\mathrm{E}(\mathfrak{F})$: the frame in Fig. 12 (b) is $\mathfrak{F}_{\langle \emptyset, \emptyset \rangle}$ and that in (c) is $\mathfrak{F}_{\langle \{a\}, \{b\} \rangle}$. $\mathfrak{F}_{\langle \emptyset, \{b\} \rangle}$ is obtained from $\mathfrak{F}_{\langle \{a\}, \{b\} \rangle}$ by omitting the R_2-arrows starting from a, save the arrow to c, and $\mathfrak{F}_{\langle \{a\}, \emptyset \rangle}$ is obtained from $\mathfrak{F}_{\langle \{a\}, \{b\} \rangle}$ by omitting the R_2-arrows starting from b, save the arrow to d.

Suppose now that the root r of $\mathfrak{F} = \langle W, R_1, R_2 \rangle$ is R_2-reflexive. We define $\mathfrak{F}_{\overline{A}}$ as in the previous case, but this time we also insert an infinite strictly descending R_2-chain $C(\omega)$ between r and its R_1-successors.

We have defined the relational component of our frames and now turn to their sets of possible values. Given $\mathfrak{F}_{\overline{A}} = \langle V, S_1, S_2 \rangle$ and a non-degenerate R_1-cluster $C = \{j : 0 \le j \le m-1\}$ in $\mathfrak{F}$, let

$$P_C = \{\{j\} \cup \{(im + j, C) : i \in \omega\} : j = 0, \dots, m-1\}$$

and denote by P the closure of

$$\{\{x\} : x \in V, \neg x S_1 x\} \cup \{P_C : C \text{ is a non-degenerate } R_1\text{-cluster in } \mathfrak{F}\}$$

under intersections and complements in V. The resultant general frame is denoted by $\mathfrak{G}(\mathfrak{F}_{\overline{A}}) = \langle V, S_1, S_2, P \rangle$. One can check that it is a descriptive frame for $\mathbf{CSM}_0$. The following completeness result is proved similarly to that in Section 2.4.

THEOREM 149. (i) *Each subframe logic in* $\mathrm{NExt}\mathbf{CSM}_0$ *is determined by a set of frames of the form* $\mathfrak{G}(\mathfrak{F}_{\overline{A}})$*, in which* $\mathfrak{F}$ *is a normal surrogate frame and* $\overline{A} \in \mathrm{Seq}\mathfrak{F}$.

(ii) *Each subframe logic in* Ext$\mathbf{CSM}_0$ *is determined by a set of frames with distinguished worlds of the form* $\langle\mathfrak{G}(\mathfrak{F}_{\overline{A}}), r\rangle$ *in which* $\mathfrak{F}$ *is a surrogate frame with root* r *and* $\overline{A} \in \mathrm{Seq}\mathfrak{F}$.

As a consequence of Theorem 149 and the fact that, for each surrogate frame $\mathfrak{F}$ with root r and each $\overline{A} \in \mathrm{Seq}\mathfrak{F}$, both the logics of $\mathfrak{G}(\mathfrak{F}_{\overline{A}})$ and $\langle\mathfrak{G}(\mathfrak{F}_{\overline{A}}), r\rangle$ are decidable, we obtain

THEOREM 150. *All finitely axiomatizable subframe logics in* Ext$\mathbf{CSM}_0$ *are decidable.*

We conjecture that the method above can be extended to logics without the **GL**-axioms, i.e., all finitely axiomatizable subframe logics containing

$$(\mathbf{K4} \otimes \mathbf{K4}) \oplus \Box_1 p \to \Box_2 p \oplus \Box_2 p \to \Box_1 \Box_2 p$$

are decidable.

2.7 Cartesian products of modal logics

Polymodal logics can be used for talking about multi-dimensional relational structures such as Cartesian products of Kripke frames. The formation of products is probably the most natural way of introducing a concept of dimension in modal logic in order to reflect interactions between modal operators representing time, space, knowledge, actions, etc. Products of modal logics (i.e., the sets of polymodal formulas that are valid in Cartesian products of Kripke frames for those logics) have been studied in both pure modal logic (see e.g. [Segerberg 1973], [Shehtman 1978a], [Marx and Venema 1997], [Gabbay and Shehtman 1998], [Kurucz 2000], [Wolter 2000]) and applications in computer science and artificial intelligence (see e.g. [Reif and Sistla 1985], [Fagin *et al.* 1995], [Baader and Ohlbach 1995], [Finger and Reynolds 1999], [Wolter and Zakharyaschev 1998, 1999b, 1999c, 2000]) since the 1970s. (Products of modal logics are also relevant to finite variable fragments of modal and intermediate predicate logics; see [Gabbay and Shehtman 1993].)

The (*Cartesian*) *product* of two frames $\mathfrak{F}_1 = \langle W_1, R_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2\rangle$ is the bimodal frame of the form

$$\mathfrak{F}_1 \times \mathfrak{F}_2 = \langle W_1 \times W_2, R_h, R_v\rangle$$

in which, for all $u_1, u_2 \in W_1$ and $v_1, v_2 \in W_2$,

$$\begin{aligned} \langle u_1, v_1\rangle\, R_h\, \langle u_2, v_2\rangle &\text{ iff } u_1 R_1 u_2 \text{ and } v_1 = v_2, \\ \langle u_1, v_1\rangle\, R_v\, \langle u_2, v_2\rangle &\text{ iff } u_1 = u_2 \text{ and } v_1 R_2 v_2. \end{aligned}$$

The subscripts h and v appeal to the geometrical intuition of considering R_h as the "horizontal" accessibility relation in $\mathfrak{F}_1 \times \mathfrak{F}_2$ and R_v as the "vertical" one.

Let $\mathcal{L}_2$ be the bimodal language with boxes $\boxminus$ and $\boxbar$ (and their duals $\diamondminus$ and $\diamondbar$). Frames for this language will be denoted by $\mathfrak{F} = \langle W, R_h, R_v \rangle$, so that $\boxminus$ and $\boxbar$ are interpreted by the relations R_h and R_v, respectively. Products are just special frames of this form.

Every product $\mathfrak{F} = \mathfrak{F}_1 \times \mathfrak{F}_2$ satisfies the following two important properties: $R_v \circ R_h = R_h \circ R_v$ and $R_v^{-1} \circ R_h \subseteq R_h \circ R_v^{-1}$, known as *commutativity* and the *Church–Rosser property*, respectively. $\mathfrak{F}$ is commutative iff it validates the formula

$$\boldsymbol{com} = \diamondbar \diamondminus p \leftrightarrow \diamondminus \diamondbar p,$$

and $\mathfrak{F}$ is Church–Rosser iff it validates

$$\boldsymbol{chr} = \diamondminus \boxbar p \to \boxbar \diamondminus p.$$

It is to be noted, however, that these properties are not characteristic for products: there are bimodal commutative and Church–Rosser frames that are not (isomorphic to) products of any two frames.

Given two classes of Kripke frames $\mathcal{C}_1$ and $\mathcal{C}_2$, we define their (*Cartesian*) *product* $\mathcal{C}_1 \times \mathcal{C}_2$ by taking

$$\mathcal{C}_1 \times \mathcal{C}_2 = \{\mathfrak{F}_1 \times \mathfrak{F}_2 : \mathfrak{F}_1 \in \mathcal{C}_1, \mathfrak{F}_2 \in \mathcal{C}_2\}.$$

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the classes of all Kripke frames for complete unimodal logics L_1 and L_2. Their (*Cartesian*) *product* $L_1 \times L_2$ is defined as the bimodal logic $\mathrm{Log}(\mathcal{C}_1 \times \mathcal{C}_2)$ in the language $\mathcal{L}_2$. It is easy to see that $L_1 \times L_2$ is a conservative extension of both L_1 and L_2, and that

$$L_1 \times L_2 \supseteq (L_1 \otimes L_2) \oplus \boldsymbol{com} \oplus \boldsymbol{chr}.$$

In some important cases the converse inclusion also holds:

THEOREM 151 (Gabbay and Shehtman 1998). *Suppose both L_1 and L_2 are axiomatized by variable free formulas and formulas of the form*

$$\Diamond^n \Box p \to \Box^m p.$$

Then

$$L_1 \times L_2 = (L_1 \otimes L_2) \oplus \boldsymbol{com} \oplus \boldsymbol{chr}.$$

This theorem yields, for instance, axiomatizations for products of **K**, **D**, **K4**, **T**, **S4**, **S5**. However there are many products of standard logics which cannot be axiomatized in this canonical way, for instance, **Grz** $\times$ **Grz** (see [Gabbay and Shehtman 1998]). Moreover, products of logics of linear frames may be even not recursively enumerable, e.g. **GL.3** $\times$ **GL.3** (see [Reynolds and Zakharyaschev 2000]). (On the other hand, as was observed in [Gabbay and Shehtman 1998], if classes $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are elementary and recursive then $\mathrm{Log}(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n)$ is recursively axiomatizable.)

In contrast to the unimodal case, usually it is rather difficult to prove positive results about products of even simple standard logics. Here we illustrate one of the methods of establishing FMP and decidability developed in [Wolter and Zakharyaschev 1998, 1999b] by applying it to $\mathbf{S5}\times\mathbf{S5}$. Other techniques—filtration, finite depth method, and mosaic—can be found in [Gabbay and Shehtman 1998] and [Marx and Venema 1997].

$\mathbf{S5}\times\mathbf{S5}$ is clearly determined by the class of products of universal frames which will be called **S5**-*rectangles.* Suppose we are given a formula φ and want to find out whether it is satisfiable in some **S5**-rectangle.

Let us call a *type* for φ any subset $\boldsymbol{t}$ of $\mathbf{Sub}\varphi$ such that

- $\psi \wedge \chi \in \boldsymbol{t}$ iff $\psi \in \boldsymbol{t}$ and $\chi \in \boldsymbol{t}$, for every $\psi \wedge \chi \in \mathbf{Sub}\varphi$,
- $\neg\psi \in \boldsymbol{t}$ iff $\psi \notin \boldsymbol{t}$, for every $\neg\psi \in \mathbf{Sub}\varphi$.

A *type-cluster* for φ is a set $\boldsymbol{T}$ of distinct types for φ such that

$$\forall \boldsymbol{t} \in \boldsymbol{T} \forall \Diamond\psi \in \mathbf{Sub}\varphi \ (\Diamond\psi \in \boldsymbol{t} \leftrightarrow \exists \boldsymbol{t}' \in \boldsymbol{T}\ \psi \in \boldsymbol{t}').$$

Let $\mathfrak{Q}$ be a non-empty set elements a in which are labelled by type-clusters $\boldsymbol{T}$ for φ. In other words, we can think of $\mathfrak{Q}$ as consisting of pairs of the form $\boldsymbol{T}_a$ with pairwise distinct a.

A *run through* $\mathfrak{Q}$ (or a $\mathfrak{Q}$-*run*, for short) is a function $\boldsymbol{r}$ from $\mathfrak{Q}$ to the set of types for φ such that

- $\boldsymbol{r}(\boldsymbol{T}_a) \in \boldsymbol{T}$ for every $\boldsymbol{T}_a \in \mathfrak{Q}$, and
- $\forall \boldsymbol{T}_a \in \mathfrak{Q} \forall \diamondsuit\psi \in \mathbf{Sub}\varphi \ (\diamondsuit\psi \in \boldsymbol{r}(\boldsymbol{T}_a) \leftrightarrow \exists \boldsymbol{sc}_b \in \mathfrak{Q}\ \psi \in \boldsymbol{r}(\boldsymbol{sc}_b))$.

Say that $\mathfrak{Q}$ is a *quasimodel* for φ if, for every $\boldsymbol{T}_a \in \mathfrak{Q}$ and every $\boldsymbol{t} \in \boldsymbol{T}$, there is a $\mathfrak{Q}$-run $\boldsymbol{r}$ coming through $\boldsymbol{t}$, i.e., $\boldsymbol{r}(\boldsymbol{T}_a) = \boldsymbol{t}$. $\mathfrak{Q}$ *satisfies* φ if φ belongs to a type occurring in a type-cluster in $\mathfrak{Q}$.

One can readily show that a formula φ is satisfied in an **S5**-rectangle iff φ is satisfied in some quasimodel for φ.

We prove now that every satisfiable formula φ is satisfied in a quasimodel of some bounded size. Let $\mathfrak{Q}$ be a quasimodel satisfying φ. Construct a subquasimodel $\mathfrak{Q}'$ of $\mathfrak{Q}$ in the following way. To begin with, we put in $\mathfrak{Q}'$ an element a_0 from $\mathfrak{Q}$ labelled by a type-cluster $\boldsymbol{T}$ containing a type with φ. Then, for every $\boldsymbol{t} \in \boldsymbol{T}$ we take a run $\boldsymbol{r}$ coming through $\boldsymbol{t}$, and for each $\diamondsuit\psi \in \boldsymbol{t}$ select $\boldsymbol{r}(a)$ containing ψ and put in $\mathfrak{Q}'$ the element a together with its copy a' (labelled by the same type-cluster as a). Thus the resulting $\mathfrak{Q}'$ contains at most $2^{|\mathbf{Sub}\varphi|} \cdot 2\,|\mathbf{Sub}\varphi|$ elements. It is now easy to see that $\mathfrak{Q}'$ is a quasimodel satisfying φ. For suppose we have an element $a \in \mathfrak{Q}'$ labelled by $\boldsymbol{T}$ and a $\boldsymbol{t} \in \boldsymbol{T}$. If $a = a_0$ then, by the construction, we have a $\mathfrak{Q}'$-run coming through $\boldsymbol{t}$. Assume now that $a \neq a_0$. We know that there is a $\mathfrak{Q}$-run

$\boldsymbol{r}'$ through $\boldsymbol{t}$. Let $\boldsymbol{r}'(a_0) = \boldsymbol{t}'$. By the construction we have a $\mathfrak{Q}'$-run $\boldsymbol{r}''$ through $\boldsymbol{t}'$. But then the function $\boldsymbol{r}$ defined by

$$\boldsymbol{r}(b) = \begin{cases} \boldsymbol{r}'(b) & \text{if } b = a \\ \boldsymbol{r}''(b) & \text{otherwise,} \end{cases}$$

for $b \in \mathfrak{Q}'$, is a run in $\mathfrak{Q}'$ coming through $\boldsymbol{t}$. Note that this gives us $2^{2|\mathbf{Sub}\varphi|} \cdot 2\,|\mathbf{Sub}\varphi|$ runs in $\mathfrak{Q}'$ coming through all its types. As a consequence we obtain:

THEOREM 152. *Every formula satisfiable in an* $\mathbf{S5}$*-rectangle is satisfied in an* $\mathbf{S5}$*-rectangle containing at most* $2^{3|\mathbf{Sub}\varphi|} \cdot 4\,|\mathbf{Sub}\varphi|^2$ *points. Thus* $\mathbf{S5} \times \mathbf{S5}$ *has FMP and is decidable.*

Unfortunately, there is no general transfer theorem that could guarantee the preservation of such properties of logics as decidability, axiomatizability, or interpolation under the formation of products. If we consider only products of standard modal logics, then the results obtained so far can roughly be described as follows (for more details consult [Spaan 1993], [Marx and Venema 1997], [Gabbay and Shehtman 1998], [Marx and Areces 1998], [Marx 1999], [Wolter 2000], [Reynolds and Zakharyaschev 2000]):

- logics of the form $\mathbf{K} \times L$ and $\mathbf{S5} \times L$ are usually decidable (in particular, for $L \in \{\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{K4.3}, \mathbf{S4}, \mathbf{S5}\}$);
- products of logics determined by infinite linear orders are undecidable,
- the computational complexity of a decidable product is usually substantially higher than the complexity of its components; for example, the satisfiability problem for $\mathbf{S5} \times \mathbf{S5}$ is $\boldsymbol{NEXPTIME}$-complete.

The decidability and FMP of logics like $\mathbf{K4} \times \mathbf{K4}$, $\mathbf{S4} \times \mathbf{S4}$, $\mathbf{S4} \times \mathbf{S4.3}$ remain challenging open problems of the field.

In higher dimensions—for $n \geq 3$—the first results related to products of modal logics were obtained in the framework of algebraic logic: as follows from [Maddux 1980] and [Johnson 1969], $\mathbf{S5}^n$ is undecidable and not finitely axiomatizable. However, as we mentioned above, products like $\mathbf{S5}^n$ and $\mathbf{K}^n$ are recursively enumerable. It is worth noting that although $\mathbf{K}^n$ have FMP for all $n < \omega$ [Gabbay and Shehtman 1998], this could imply the decidabiliy of $\mathbf{K}^n$ only if the class of finite frames for $\mathbf{K}^n$ were recursive. We could have such a test if $\mathbf{K}^n$ were finitely axiomatizable; however this is not the case [Kurucz 2000]. To prove the decidability of $\mathbf{K}^n$, it would also be enough to show that it has the product FMP, i.e., it is characterized by the class of products of n-many finite frames. But this approach does not work either: as has been shown by Hirsch *et al.* [2000], all logics L such that $\mathbf{K}^n \subseteq L \subseteq \mathbf{S5}^n$ are undecidable, non finitely axiomatizable, do not have the product FMP, and it is undecidable whether a finite frame is a frame for

L. (The only known example of a decidable higher dimensional product of non-tabular logics is $\mathbf{Alt}^n$ [Gabbay and Shehtman 1998].)

3 SUPERINTUITIONISTIC LOGICS

Although C.I. Lewis constructed his first modal calculus **S3** in 1918, it was Gödel's [1933] two page note that attracted serious attention of mathematical logicians to modal systems. While Lewis [1918] used an abstract necessity operator to avoid paradoxes of material implication, Gödel [1933] and earlier Orlov [1928][16] treated $\Box$ as "it is provable" to give a classical interpretation of intuitionistic propositional logic **Int** by means of embedding it into a modal "provability" system which turned out to be equivalent to Lewis' **S4**.

Approximately at the same time Gödel [1932] observed that there are infinitely many logics located between **Int** and classical logic **Cl**, which—together with the creation of constructive (proper) extensions of **Int** by Kleene [1945] and Rose [1953] (realizability logic), Medvedev [1962] (logic of finite problems), Kreisel and Putnam [1957]—gave an impetus to studying the class of logics intermediate between **Int** and **Cl**, started by Umezawa [1955, 1959]. Gödel's embedding of **Int** into **S4**, presented in an algebraic form by McKinsey and Tarski [1948] and extended to all intermediate logics by Dummett and Lemmon [1959], made it possible to develop the theories of modal and intermediate logics in parallel ways. And the structural results of Blok [1976] and Esakia [1979a,b], establishing an isomorphism between the lattices Ext**Int** and NExt**Grz**, along with preservation results of Maksimova and Rybakov [1974] and Zakharyaschev [1991], transferring various properties from modal to intermediate logics and back, showed that in many respects the theory of intermediate logics is reducible to the theory of logics in NExt**S4**.

To demonstrate this as well as some features of intermediate logics is the main aim of this part. We will use the same system of notations as in the modal case. In particular, Ext**Int** is the lattice of all logics of the form **Int** + Γ (where Γ is an arbitrary set of formulas in the language of **Int** and + as before means taking the closure under modus ponens and substitution); we call them *superintuitionistic logics* or *si-logics* for short. Basic facts about the syntax and semantics of **Int** and relevant references can be found in *Intuitionistic Logic*. A list of some "standard" si-logics is given in Table 5.

[16] Orlov's paper remained unnoticed till the end of the 1980s. It is remarkable also for constructing the first system of relevant logic.

Table 5. A list of standard superintuitionistic logics

For	=	$\mathbf{Int} + p$
Cl	=	$\mathbf{Int} + p \vee \neg p$
SmL	=	$\mathbf{Int} + (\neg q \to p) \to (((p \to q) \to p) \to p)$
KC	=	$\mathbf{Int} + \neg p \vee \neg\neg p$
LC	=	$\mathbf{Int} + (p \to q) \vee (q \to p)$
SL	=	$\mathbf{Int} + ((\neg\neg p \to p) \to \neg p \vee p) \to \neg p \vee \neg\neg p$
KP	=	$\mathbf{Int} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$
$\mathbf{BD}_n$	=	$\mathbf{Int} + \boldsymbol{bd}_n$, where $\boldsymbol{bd}_1 = p_1 \vee \neg p_1,\ \boldsymbol{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \to \boldsymbol{bd}_n)$
$\mathbf{BW}_n$	=	$\mathbf{Int} + \bigvee_{i=0}^{n}(p_i \to \bigvee_{j \neq i} p_j)$
$\mathbf{BTW}_n$	=	$\mathbf{Int} + \bigwedge_{0 \leq i < j \leq n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n}(\neg p_i \to \bigvee_{j \neq i} \neg p_j)$
$\mathbf{T}_n$	=	$\mathbf{Int} + \bigwedge_{i=0}^{n}((p_i \to \bigvee_{i \neq j} p_j) \to \bigvee_{i \neq j} p_j) \to \bigvee_{i=0}^{n} p_i$
$\mathbf{B}_n$	=	$\mathbf{Int} + \bigwedge_{i=0}^{n}(\neg p_i \leftrightarrow \bigvee_{i \neq j} p_j) \to \bigvee_{i=0}^{n} p_i$
$\mathbf{NL}_n$	=	$\mathbf{Int} + \boldsymbol{nf}_n$, where $\boldsymbol{nf}_0 = \bot,\ \boldsymbol{nf}_1 = p,\ \boldsymbol{nf}_2 = \neg p,\ \boldsymbol{nf}_\omega = \top$ $\boldsymbol{nf}_{2m+3} = \boldsymbol{nf}_{2m+1} \vee \boldsymbol{nf}_{2m+2}$, $\boldsymbol{nf}_{2m+4} = \boldsymbol{nf}_{2m+3} \to \boldsymbol{nf}_{2m+1}$

3.1 Intuitionistic frames

As in the case of modal logics, the adequate relational semantics for si-logics can be constructed on the base of the Stone representation of the algebraic "models" for **Int**, known as *Heyting* (or *pseudo-Boolean*) *algebras*. It is hard to trace now who was the first to introduce intuitionistic general frames—the earliest references we know are [Esakia 1974] and [Rautenberg 1979]—but in any case, having at hand [Jónsson and Tarski 1951] and [Goldblatt 1976a], the construction must have been clear.

An *intuitionistic* (*general*) *frame* is a triple $\mathfrak{F} = \langle W, R, P \rangle$ in which R is a partial order on $W \neq \emptyset$ and P, the *set of possible values* in $\mathfrak{F}$, is a collection of upward closed subsets (cones) in W containing $\emptyset$ and closed under the Boolean $\cap$, $\cup$, and the operation $\supset$ (for $\to$) defined by

$$X \supset Y = \{x \in W : \forall y \in x\uparrow\ (y \in X \to y \in Y)\}.$$

If P contains all upward closed subsets in W then we call $\mathfrak{F}$ a *Kripke frame* and denote it by $\mathfrak{F} = \langle W, R \rangle$. An important feature of intuitionistic models $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ ($\mathfrak{V}$, a *valuation* in $\mathfrak{F}$, maps propositional variables to sets in P)

is that $\mathfrak{V}(\varphi)$, the *truth-value* of a formula φ, is always upward closed.

Every intuitionistic frame $\mathfrak{F} = \langle W, R, P\rangle$ gives rise to the Heyting algebra $\mathfrak{F}^+ = \langle P, \cap, \cup, \supset, \emptyset\rangle$ called the *dual* of $\mathfrak{F}$. Conversely, given a Heyting algebra $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, \bot\rangle$, we construct its relational representation $\mathfrak{A}_+ = \langle W, R\rangle$ by taking W to be the set of all prime filters in $\mathfrak{A}$ (a filter ∇ is *prime* if it is proper and $a \vee b \in \nabla$ implies $a \in \nabla$ or $b \in \nabla$), R to be the set-theoretic inclusion $\subseteq$ and

$$P = \{\{\nabla \in W : a \in \nabla\} : a \in A\}.$$

It is readily checked that $\mathfrak{A}_+$, the *dual* of $\mathfrak{A}$, is an intuitionistic frame, $\mathfrak{A} \cong (\mathfrak{A}_+)^+$ and $\mathfrak{A}_+$ is differentiated, *tight* in the sense that

$$xRy \text{ iff } \forall X \in P\ (x \in X \rightarrow y \in X),$$

and *compact*, i.e., for any families $\mathcal{X} \subseteq P$ and $\mathcal{Y} \subseteq \{W - X :\ X \in P\}$,

$$\bigcap(\mathcal{X} \cup \mathcal{Y}) = \{x \in W : \forall X \in \mathcal{X} \forall Y \in \mathcal{Y}\ (x \in X \wedge x \in Y)\} \neq \emptyset$$

whenever $\bigcap(\mathcal{X}' \cup \mathcal{Y}') \neq \emptyset$ for every finite subfamilies $\mathcal{X}' \subseteq \mathcal{X}$, $\mathcal{Y}' \subseteq \mathcal{Y}$. Frames with these three properties (actually differentiatedness follows from tightness) are called *descriptive*. In the same way as in the modal case one can prove that $\mathfrak{F}$ is descriptive iff $\mathfrak{F} \cong (\mathfrak{F}^+)_+$. Duality between the basic truth-preserving operations on algebras and descriptive frames (the definitions of generated subframes, reductions and disjoint unions do not change) is also established by the same technique.

Since every consistent si-logic L is characterized by its Tarski–Lindenbaum algebra $\mathfrak{A}_L$, we conclude that L is characterized also by a class of intuitionistic frames, say by the dual of $\mathfrak{A}_L$.

Refined finitely generated frames for **Int** look similarly to those for **K4**: the only difference is that now all clusters are simple and the truth-sets must be upward closed. Fig. 13 showing (a) the free 1-generated Heyting algebra $\mathfrak{A}_{\mathbf{Int}}(1)$ and (b) its dual $\mathfrak{F}_{\mathbf{Int}}(1)$ will help the reader to restore the details. $\mathfrak{A}_{\mathbf{Int}}(1)$ was first constructed by Rieger [1949] and Nishimura [1960]; it is called the *Rieger–Nishimura lattice*. The formulas $\boldsymbol{nf}_n$ defined in Table 5 and used for the construction are known as *Nishimura formulas* (see also Section 3 of *Intuitionistic Logic*).

At the algebraic level the connection between **Int** and **S4** discovered by Gödel is reflected by the fact, established in [Mckinsey and Tarski 1946], that the algebra of open elements (i.e., elements a such that $\Box a = a$) of every modal algebra for **S4** (known as a *topological Boolean algebra*; see [Rasiowa and Sikorski 1963]) is a Heyting algebra and conversely, every Heyting algebra is isomorphic to the algebra of open elements of a suitable algebra for **S4**. We explain this result in the frame-theoretic language.

Given a frame $\mathfrak{F} = \langle W, R, P\rangle$ for **S4** (which means that R is a quasi-order on W), we denote by $\boldsymbol{\rho} W$ the set of clusters in $\mathfrak{F}$—more generally,

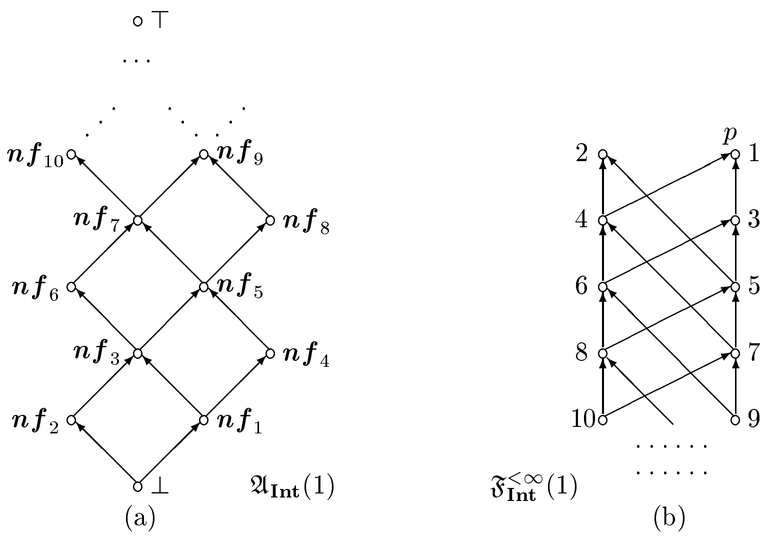


Figure 13.

$\boldsymbol{\rho} X = \{C(x) : x \in X\}$—and put $C(x)\boldsymbol{\rho}C(y)$ iff xRy,

$$\boldsymbol{\rho} P = \{\boldsymbol{\rho} X : \; X \in P \wedge X = \Box X\} = \{\boldsymbol{\rho} X : \; X \in P \wedge X = X{\uparrow}\}.$$

It is readily checked that the structure $\boldsymbol{\rho}\mathfrak{F} = \langle \boldsymbol{\rho} W, \boldsymbol{\rho} R, \boldsymbol{\rho} P\rangle$ is an intuitionistic frame (for instance, $\boldsymbol{\rho}(X) \supset \boldsymbol{\rho}(Y) = \boldsymbol{\rho}(\Box(-X \cup Y))$); we call it the *skeleton* of $\mathfrak{F}$. The *skeleton* of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ for **S4** is the intuitionistic model $\boldsymbol{\rho}\mathfrak{M} = \langle \boldsymbol{\rho}\mathfrak{F}, \boldsymbol{\rho}\mathfrak{V}\rangle$, where $\boldsymbol{\rho}\mathfrak{V}(p) = \mathfrak{V}(\Box p)$.

Denote by T the *Gödel translation* prefixing $\Box$ to all subformulas of a given intuitionistic formula.[17] By induction on the construction of φ one can easily prove the following

LEMMA 153 (Skeleton). *For every model $\mathfrak{M}$ for* **S4**, *every intuitionistic formula φ and every point x in $\mathfrak{M}$,*

$$(\boldsymbol{\rho}\mathfrak{M}, C(x)) \models \varphi \textit{ iff } (\mathfrak{M}, x) \models T(\varphi).$$

It follows that $\varphi \in \mathbf{Int}$ implies $T(\varphi) \in \mathbf{S4}$. To prove the converse we should be able to convert intuitionistic frames $\mathfrak{F}$ into modal ones with the skeleton (isomorphic to) $\mathfrak{F}$. This is trivial if $\mathfrak{F}$ is a Kripke frame—we can just regard it to be a frame for **S4**, which in view of the Kripke completeness

[17]The translation defined in [Gödel 1933] does not prefix $\Box$ to conjunctions and disjunctions. However this difference is of no importance as far as embeddings into logics in NExt**S4** are concerned.

of both **Int** and **S4**, shows that T really embeds the former into the latter, i.e.,

$$\varphi \in \mathbf{Int} \text{ iff } T(\varphi) \in \mathbf{S4}.$$

In general, the most obvious way of constructing a modal frame from an intuitionistic frame $\mathfrak{F} = \langle W, R, P\rangle$ is to take the closure $\boldsymbol{\sigma} P$ of P under the Boolean operations $\cap$, $\cup$ and $\to$. It is well known in the theory of Boolean algebras (see [Rasiowa and Sikorski 1963]) that for every $X \subseteq W$, X is in $\boldsymbol{\sigma} P$ iff

$$X = (-X_1 \cup Y_1) \cap \cdots \cap (-X_n \cup Y_n)$$

for some $X_1, Y_1, \dots, X_n, Y_n \in P$ and $n \geq 1$. It follows that if $X \in \boldsymbol{\sigma} P$ then

$$\Box X = (X_1 \supset Y_1) \cap \cdots \cap (X_n \supset Y_n) \in P \subseteq \boldsymbol{\sigma} P,$$

and so $\boldsymbol{\sigma} P$ is closed under $\Box$ in $\langle W, R\rangle$ and P coincides with the set of upward closed sets in $\boldsymbol{\sigma} P$. Thus, $\langle W, R, \boldsymbol{\sigma} P\rangle$ is a partially ordered modal frame; we shall denote it by $\boldsymbol{\sigma}\mathfrak{F}$. Moreover, we clearly have $\mathfrak{F} \cong \boldsymbol{\rho\sigma}\mathfrak{F}$. If $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ is an intuitionistic model then $\boldsymbol{\sigma}\mathfrak{M} = \langle \boldsymbol{\sigma}\mathfrak{F}, \mathfrak{V}\rangle$ is a modal model having $\mathfrak{M}$ as its skeleton. So by the Skeleton Lemma,

$$(\mathfrak{M}, x) \models \varphi \text{ iff } (\boldsymbol{\sigma}\mathfrak{M}, x) \models T(\varphi),$$

for every intuitionistic formula φ and every point x in $\mathfrak{F}$.

It is worth noting that if $\mathfrak{F} = \langle W, R\rangle$ is a finite intuitionistic Kripke frame then $\boldsymbol{\sigma}\mathfrak{F}$ is also a Kripke frame. However, for an infinite $\mathfrak{F}$, $\boldsymbol{\sigma}\mathfrak{F}$ is not in general a Kripke frame, witness $\langle \omega, \leq\rangle$.

The operator $\boldsymbol{\sigma}$ is not the only one which, given an intuitionistic frame $\mathfrak{F}$, returns a modal frame whose skeleton is isomorphic to $\mathfrak{F}$. As an example, we define now an infinite class of such operators. For Kripke frames $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{G} = \langle V, S\rangle$, denote by $\mathfrak{F} \times \mathfrak{G}$ the *direct product* of $\mathfrak{F}$ and $\mathfrak{G}$, i.e., the frame $\langle W \times V, R \times S\rangle$ in which the relation $R \times S$ is defined component-wise:

$$\langle x_1, y_1\rangle \, (R \times S) \, \langle x_2, y_2\rangle \text{ iff } x_1 R x_2 \text{ and } y_1 S y_2.$$

Let $0 < k \leq \omega$. We will regard k to be the set $\{0, \dots, k-1\}$ if $k < \omega$ and $\{0, 1, \dots\}$ if $k = \omega$. Denote by $\boldsymbol{\tau}_k$ an operator which, given an intuitionistic frame $\mathfrak{F} = \langle W, R, P\rangle$, returns a modal frame $\boldsymbol{\tau}_k\mathfrak{F} = \langle kW, kR, kP\rangle$ such that

(i) $\langle kW, kR\rangle$ is the direct product of the k-point cluster $\langle k, k^2\rangle$ and $\langle W, R\rangle$ (in other words, $\langle kW, kR\rangle$ is obtained from $\langle W, R\rangle$ by replacing its every point with a k-point cluster);

(ii) $\boldsymbol{\rho\tau}_k\mathfrak{F} \cong \mathfrak{F}$;

(iii) $I \times X \in kP$, for every $I \subseteq k$ and $X \in \boldsymbol{\sigma} P$.

For instance, we can take kP to be the Boolean closure of the set

$$\{I \times X : \ I \subseteq k, \ X \in \boldsymbol{\sigma} P\}.$$

For a Kripke frame $\mathfrak{F} = \langle W, R, \mathrm{Up}W\rangle$ we can, of course, take $kP = 2^{kW}$ and then $\boldsymbol{\tau}_k\mathfrak{F} = \langle kW, kR, 2^{kW}\rangle$.

3.2 Canonical formulas

The language of canonical formulas, axiomatizing all si-logics and characterizing the structure of their frames, can be easily developed following the scheme of constructing the canonical formulas for **K4** outlined in Section 1.6 and using the connection between modal and intuitionistic frames established above. We confine ourselves here only to pointing out the differences from the modal case and some interesting peculiarities; details can be found in [Zakharyaschev 1983, 1989] and [Chagrov and Zakharyaschev 1997].

Actually, there are two important differences. First, in the definition of subreduction of $\mathfrak{F} = \langle W, R, P\rangle$ to $\mathfrak{G}$ the condition (R3) does not correspond to the fact that all sets in P are upward closed. We replace it by the following condition

$$\text{(R3}'\text{)} \qquad \forall X \in \overline{Q}\ f^{-1}(X)\!\downarrow \in \overline{P},$$

where $\overline{Q} = \{V - X : X \in Q\}$ and $\overline{P} = \{W - X : X \in P\}$. For a completely defined f satisfying (R1) and (R2) the condition (R3′) is clearly equivalent to (R3) and so every reduction is also a subreduction. If $\mathfrak{G}$ is a finite Kripke frame then (R3′) is equivalent to $\forall z \in V\ f^{-1}(z)\!\downarrow \in \overline{P}$. $\mathfrak{G}$ is a *subframe* of $\mathfrak{F}$ if $\boldsymbol{\kappa}\mathfrak{G}$ is a subframe of $\boldsymbol{\kappa}\mathfrak{F}$ and the identity map on V is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$. It is of interest to note that in the intuitionistic case (cofinal) subreductions are dual to IC(N)-subalgebras of Heyting algebras which preserve only implication, conjunction (and negation or $\bot$) but do not necessarily preserve disjunction.

Second, we have to change the definition of open domains. Now we say an antichain $\mathfrak{a}$ (of at least two points) is an *open domain* in an intuitionistic model $\mathfrak{N}$ relative to a formula φ if there ia a pair $t_\mathfrak{a} = (\Gamma_\mathfrak{a}, \Delta_\mathfrak{a})$ such that $\Gamma_\mathfrak{a} \cup \Delta_\mathfrak{a} = \mathbf{Sub}\varphi$, $\bigwedge \Gamma_\mathfrak{a} \to \bigvee \Delta_\mathfrak{a} \notin \mathbf{Int}$ and

- $\psi \in \Gamma_\mathfrak{a}$ iff $a \models \psi$ for all $a \in \mathfrak{a}$.

It is worth noting that in any intuitionistic model every antichain $\mathfrak{a}$ is open relative to every disjunction free formula φ. Indeed, let $\Gamma_\mathfrak{a}$ be defined by condition above and $\Delta_\mathfrak{a} = \mathbf{Sub}\varphi - \Gamma_\mathfrak{a}$. It should be clear that $\psi \wedge \chi \in \Gamma_\mathfrak{a}$ iff $\psi \in \Gamma_\mathfrak{a}$ and $\chi \in \Gamma_\mathfrak{a}$. And if $\psi \to \chi \in \Gamma_\mathfrak{a}$, $\psi \in \Gamma_\mathfrak{a}$ but $\chi \in \Delta_\mathfrak{a}$ then $a \models \psi$ for every $a \in \mathfrak{a}$ and $b \not\models \chi$ for some $b \in \mathfrak{a}$, whence $b \not\models \psi \to \chi$, which is a contradiction. It follows that $\bigwedge \Gamma_\mathfrak{a} \to \bigvee \Delta_\mathfrak{a} \notin \mathbf{Int}$.

EXAMPLE 154. Let us try to characterize the class of intuitionistic refutation frames for the *Weak Kreisel–Putnam Formula*

$$\boldsymbol{wkp} = (\neg p \to \neg q \vee \neg r) \to (\neg p \to \neg q) \vee (\neg p \to \neg r).$$

First we construct its simplest countermodel; it is depicted in Fig. 14, where by putting a formula to the left (right) of a point we mean that it is true

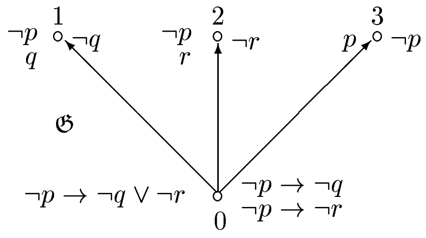


Figure 14.

(not true) at the point. Then we observe that every frame $\mathfrak{F}$ refuting $\boldsymbol{wkp}$ is cofinally subreducible to the frame $\mathfrak{G}$ underlying this countermodel by the map f defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \models \neg p \to \neg q \vee \neg r,\ x \not\models (\neg p \to \neg q) \vee (\neg p \to \neg r) \\ 1 & \text{if } x \models \neg p \to \neg q \vee \neg r,\ x \models \neg p \text{ and } x \models q \\ 2 & \text{if } x \models \neg p \to \neg q \vee \neg r,\ x \models \neg p \text{ and } x \models r \\ 3 & \text{if } x \models p \text{ or } x \models \neg p \wedge \neg q \wedge \neg r \\ \text{undefined} & \text{otherwise.} \end{cases}$$

However, the cofinal subreducibility to $\mathfrak{G}$ is only a necessary condition for $\mathfrak{F} \not\models \boldsymbol{wkp}$, witness the frame having the form of the three-dimensional Boolean cube with the top point deleted. The reason for this is that the antichain $\{1, 2\}$ is a closed domain in $\mathfrak{N}$: it is impossible to insert a point a between 0 and $\{1, 2\}$ and extend to it consistently the truth-sets for the depicted formulas. Indeed, otherwise we would have $a \models \neg p \to \neg q \vee \neg r$, $a \not\models \neg q \vee \neg r$ and so $a \not\models \neg p$, i.e., there must be a point $x \in a{\uparrow}$ such that $x \models p$, but such a point does not exist. In fact, $\mathfrak{F} \not\models \boldsymbol{wkp}$ iff there is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{G}$ satisfying (CDC) for $\{\{1, 2\}\}$.

Now, as in the modal case, with every finite rooted intuitionistic frame $\mathfrak{F} = \langle W, R \rangle$ and a set $\mathfrak{D}$ of antichains in it we can associate two formulas $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\beta(\mathfrak{F}, \mathfrak{D})$, called the *canonical* and *negation free canonical formulas*, respectively, so that $\mathfrak{G} \not\models \beta(\mathfrak{F}, \mathfrak{D}, \bot)$ ($\mathfrak{G} \not\models \beta(\mathfrak{F}, \mathfrak{D})$) iff there is a (cofinal) subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. For instance, if $a_0, \ldots, a_n$ are all points in $\mathfrak{F}$ and a_0 is its root, then one can take

$$\beta(\mathfrak{F}, \mathfrak{D}, \bot) = \bigwedge_{a_i R a_j} \psi_{ij} \wedge \bigwedge_{\mathfrak{d} \in \mathfrak{D}} \psi_{\mathfrak{d}} \wedge \psi_{\bot} \to p_0,$$

where

$$\begin{aligned}
\psi_{ij} &= (\bigwedge_{\neg a_j R a_k} p_k \to p_j) \to p_i, \\
\psi_{\mathfrak{d}} &= \bigwedge_{a_i \in W - \mathfrak{d}\uparrow} (\bigwedge_{\neg a_i R a_k} p_k \to p_i) \to \bigvee_{a_j \in \mathfrak{d}} p_j, \\
\psi_{\perp} &= \bigwedge_{i=0}^{n} (\bigwedge_{\neg a_i R a_k} p_k \to p_i) \to \perp.
\end{aligned}$$

$\beta(\mathfrak{F}, \mathfrak{D})$ is obtained from $\beta(\mathfrak{F}, \mathfrak{D}, \perp)$ by deleting the conjunct $\psi_\perp$.

THEOREM 155. *There is an algorithm which, given an intuitionistic φ, returns canonical formulas $\beta(\mathfrak{F}_1, \mathfrak{D}_1, \perp), \ldots, \beta(\mathfrak{F}_n, \mathfrak{D}_n, \perp)$ such that*

$$\mathbf{Int} + \varphi = \mathbf{Int} + \beta(\mathfrak{F}_1, \mathfrak{D}_1, \perp) + \cdots + \beta(\mathfrak{F}_n, \mathfrak{D}_n, \perp).$$

So the set of intuitionistic canonical formulas is complete for Ext**Int**. *If φ is negation free then one can use only negation free canonical formulas. And if φ is disjunction free then all $\mathfrak{D}_i$ are empty.*

Table 6 and Theorem 156 show canonical axiomatizations of the si-logics in Table 5. Using this "geometrical" representation it is not hard to see, for instance, that **SmL**, known as the *Smetanich logic*, is the greatest consistent extension of **Int** different from **Cl**; it is the logic of the two-point rooted frame. **KC**, the logic of the *Weak Law of the Excluded Middle*, is characterized by the class of directed frames. It is the greatest si-logic containing the same negation free formulas as **Int** (see [Jankov 1968a]). **LC**, the *Dummett* or *chain logic*, is characterized by the class of linear frames (see [Dummett 1959]). $\mathbf{BD}_n$ and $\mathbf{BW}_n$ are the minimal logics of depth n and width n, respectively (see [Hosoi 1967] and [Smoryński 1973]). Finite frames for $\mathbf{BTW}_n$ contain $\leq n$ top points [Smoryński 1973] and finite frames for $\mathbf{T}_n$ are of branching $\leq n$, i.e., no point has more than n immediate successors.

THEOREM 156 (Nishimura 1960, Anderson 1972). *Every extension L of* **Int** *by formulas in one variable can be represented either as*

$$L = \mathbf{Int} + \boldsymbol{nf}_{2n} = \mathbf{Int} + \beta^\sharp(\mathfrak{H}_n, \perp)$$

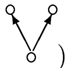
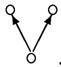
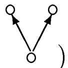
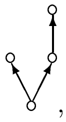
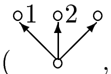
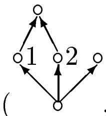
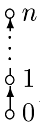
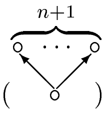
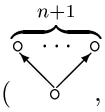
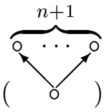
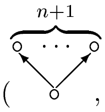
or as

$$L = \mathbf{Int} + \boldsymbol{nf}_{2n-1} = \mathbf{Int} + \beta^\sharp(\mathfrak{H}_{n+1}, \perp) + \beta^\sharp(\mathfrak{H}_{n+2}, \perp),$$

where $\mathfrak{H}_n$, $\mathfrak{H}_{n+1}$, $\mathfrak{H}_{n+2}$ are the subframes of the frame in Fig. 13 generated by the points n, $n+1$ and $n+2$, respectively, and $\beta^\sharp(\mathfrak{F}, \perp)$ is an abbreviation for $\beta(\mathfrak{F}, \mathfrak{D}^\sharp, \perp)$, $\mathfrak{D}^\sharp$ the set of all antichains in $\mathfrak{F}$.

Jankov [1969] proved in fact that logics of the form $\mathbf{Int} + \beta^\sharp(\mathfrak{F}, \perp)$ and only them are splittings of Ext**Int**. However, not every si-logic is a union-splitting of Ext**Int** which means that this class has no axiomatic basis.

Table 6. Canonical axioms of standard superintuitionistic logics

For	=	$\mathbf{Int} + \beta(\circ)$
Cl	=	$\mathbf{Int} + \beta($  $)$
SmL	=	$\mathbf{Int} + \beta($  $) + \beta($  $)$
KC	=	$\mathbf{Int} + \beta($  $, \bot)$
LC	=	$\mathbf{Int} + \beta($  $)$
SL	=	$\mathbf{Int} + \beta^\sharp($  $, \bot)$
KP	=	$\mathbf{Int} + \beta($  $, \{\{1,2\}\}, \bot) + \beta($  $, \{\{1,2\}\}, \bot)$
$\mathbf{BD}_n$	=	$\mathbf{Int} + \beta($  $)$
$\mathbf{BW}_n$	=	$\mathbf{Int} + \beta($  $)$
$\mathbf{BTW}_n$	=	$\mathbf{Int} + \beta($  $, \bot)$
$\mathbf{T}_n$	=	$\mathbf{Int} + \beta^\sharp($  $)$
$\mathbf{B}_n$	=	$\mathbf{Int} + \beta^\sharp($  $, \bot)$

3.3 Modal companions and preservation theorems

The fact that the Gödel translation T embeds **Int** into **S4** and the relationship between intuitionistic and modal frames established in Section 3.1 can be used to reduce various problems concerning **Int** (e.g. proving completeness or FMP) to those for **S4** and vice versa. Moreover, it turns out that each logic in Ext**Int** is embedded by T into some logics in NExt**S4**, and for each logic in NExt**S4** there is one in Ext**Int** embeddable in it.

We say a modal logic $M \in \mathrm{NExt}\mathbf{S4}$ is a *modal companion* of a si-logic L if L is embedded in M by T, i.e., if for every intuitionistic formula φ,

$$\varphi \in L \text{ iff } T(\varphi) \in M.$$

If M is a modal companion of L then L is called the *si-fragment* of M and denoted by $\boldsymbol{\rho} M$. The reason for denoting the operator "modal logic $\mapsto$ its si-fragment" by the same symbol we used for the skeleton operator is explained by the following

THEOREM 157. *For every $M \in \mathrm{NExt}\mathbf{S4}$, $\boldsymbol{\rho} M = \{\varphi : T(\varphi) \in M\}$. Moreover, if M is characterized by a class $\mathcal{C}$ of modal frames then $\boldsymbol{\rho} M$ is characterized by the class $\boldsymbol{\rho}\mathcal{C} = \{\boldsymbol{\rho}\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$ of intuitionistic frames.*

Proof. It suffices to show that $\{\varphi : T(\varphi) \in M\} = \mathrm{Log}\boldsymbol{\rho}\mathcal{C}$. Suppose that $T(\varphi) \in M$. Then $\mathfrak{F} \models T(\varphi)$ and so, by the Skeleton Lemma, $\boldsymbol{\rho}\mathfrak{F} \models \varphi$ for every $\mathfrak{F} \in \mathcal{C}$, i.e., $\varphi \in \mathrm{Log}\boldsymbol{\rho}\mathcal{C}$. Conversely, if $\boldsymbol{\rho}\mathfrak{F} \models \varphi$ for all $\mathfrak{F} \in \mathcal{C}$ then, by the same lemma, $T(\varphi)$ is valid in all frames in $\mathcal{C}$ and so $T(\varphi) \in M$. ■

Thus, $\boldsymbol{\rho}$ maps NExt**S4** into Ext**Int**. The following simple observation shows that actually $\boldsymbol{\rho}$ is a surjection. Given a logic $L \in \mathrm{Ext}\mathbf{Int}$, we put

$$\boldsymbol{\tau} L = \mathbf{S4} \oplus \{T(\varphi) : \varphi \in L\}.$$

THEOREM 158 (Dummett and Lemmon 1959). *For every si-logic L, $\boldsymbol{\tau} L$ is a modal companion of L.*

Proof. Clearly, $L \subseteq \boldsymbol{\rho\tau} L$. To prove the converse inclusion, suppose $\varphi \notin L$, i.e., there is a frame $\mathfrak{F}$ for L refuting φ. Since $\mathfrak{F} \cong \boldsymbol{\rho\sigma}\mathfrak{F}$, by the Skeleton Lemma we have $\boldsymbol{\sigma}\mathfrak{F} \models \boldsymbol{\tau} L$ and $\boldsymbol{\sigma}\mathfrak{F} \not\models T(\varphi)$. Therefore, $T(\varphi) \notin \boldsymbol{\tau} L$ and so $\varphi \notin \boldsymbol{\rho\tau} L$. ■

Now we use the language of canonical formulas to obtain a general characterization of all modal companions of a given si-logic L. Our presentation follows [Zakharyaschev 1989, 1991]. Notice first that for every modal frame $\mathfrak{G}$ and every intuitionistic canonical formula $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$, $\mathfrak{G} \models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ iff $\boldsymbol{\rho}\mathfrak{G} \models \beta(\mathfrak{F}, \mathfrak{D}, \bot)$ and so $\mathbf{S4} \oplus T(\beta(\mathfrak{F}, \mathfrak{D}, \bot)) = \mathbf{S4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. The same concern, of course, the negation free canonical formulas.

THEOREM 159. *A logic $M \in$ NExt**S4** is a modal companion of a si-logic $L = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\}$ iff M can be represented in the form*

$$M = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\} \oplus \{\alpha(\mathfrak{F}_j, \mathfrak{D}_j, \bot) : j \in J\},$$

where every frame $\mathfrak{F}_j$, for $j \in J$, contains a proper cluster.

Proof. ($\Leftarrow$) We must show that for every intuitionistic formula φ, $\varphi \in L$ iff $T(\varphi) \in M$. Suppose that $\varphi \notin L$ and $\mathfrak{F} = \langle W, R, P\rangle$ is a frame separating φ from L. We prove that $\boldsymbol{\sigma}\mathfrak{F}$ separates $T(\varphi)$ from M. As was observed above, $\boldsymbol{\sigma}\mathfrak{F} \not\models T(\varphi)$ and $\boldsymbol{\sigma}\mathfrak{F} \models \alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$ for any $i \in I$. So it remains to show that $\boldsymbol{\sigma}\mathfrak{F} \models \alpha(\mathfrak{F}_j, \mathfrak{D}_j, \bot)$ for every $j \in J$.

Suppose otherwise. Then, for some $j \in J$, we have a subreduction f of $\boldsymbol{\sigma}\mathfrak{F}$ to $\mathfrak{F}_j$. Let a_1 and a_2 be distinct points belonging to the same proper cluster in $\mathfrak{F}_j$. By the definition of subreduction, $f^{-1}(a_1) \subseteq f^{-1}(a_2){\downarrow}$ and $f^{-1}(a_2) \subseteq f^{-1}(a_1){\downarrow}$, and so there is an infinite chain $x_1Ry_1Rx_2Ry_2R\dots$ in $\boldsymbol{\sigma}\mathfrak{F}$ such that $\{x_1, x_2, \dots\} \subseteq f^{-1}(a_1)$ and $\{y_1, y_2, \dots\} \subseteq f^{-1}(a_2)$. And since R is a partial order, all the points x_i and y_i are distinct.

Since $f^{-1}(a_1) \in \boldsymbol{\sigma}P$, there are $X_i, Y_i \in P$ such that

$$f^{-1}(a_1) = (-X_1 \cup Y_1) \cap \dots \cap (-X_n \cup Y_n).$$

And since $f^{-1}(a_1) \cap f^{-1}(a_2) = \emptyset$, for every point y_i there is some number n_i such that $y_i \in X_{n_i}$ and $y_i \notin Y_{n_i}$. But then, for some distinct l and m, the numbers n_l and n_m must coincide, and so if, say, y_lRy_m then $x_m \notin Y_{n_m}$ and $x_m \in X_{n_l}$ (for $y_lRx_mRy_m$, $X_i = X_i\uparrow$, $Y_i = Y_i\uparrow$). Therefore, $x_m \notin f^{-1}(a_1)$, which is a contradiction.

The rest of the proof presents no difficulties. ■

This proof does not touch upon the cofinality condition. So along with canonical formulas in Theorem 159 we can use negation free canonical formulas. Thus, we have:

$$\boldsymbol{\rho}\mathbf{S4} = \boldsymbol{\rho}\mathbf{S4.1} = \boldsymbol{\rho}\mathbf{Dum} = \boldsymbol{\rho}\mathbf{Grz} = \mathbf{Int},$$

$$\boldsymbol{\rho}\mathbf{S4.2} = \boldsymbol{\rho}(\mathbf{S4.2} \oplus \mathbf{Grz}) = \mathbf{KC},$$

$$\boldsymbol{\rho}\mathbf{S4.3} = \boldsymbol{\rho}(\mathbf{S4.3} \oplus \mathbf{Grz}) = \mathbf{LC},$$

$$\boldsymbol{\rho}\mathbf{S5} = \boldsymbol{\rho}(\mathbf{S5} \oplus \mathbf{Grz}) = \mathbf{Cl}.$$

COROLLARY 160. *The set of modal companions of every consistent si-logic L forms the interval*

$$\boldsymbol{\rho}^{-1}(L) = [\boldsymbol{\tau}L, \boldsymbol{\tau}L \oplus \alpha(\boxed{\circ\,\circ})] = \{M \in \mathrm{NExt}\mathbf{S4} : \boldsymbol{\tau}L \subseteq M \subseteq \boldsymbol{\tau}L \oplus \mathbf{Grz}\}$$

and contains an infinite descending chain of logics.

Proof. Notice first that $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ and $\alpha(\mathfrak{F},\mathfrak{D})$ are in $\mathbf{Grz}$ iff $\mathfrak{F}$ contains a proper cluster. So $\boldsymbol{\rho}^{-1}(L) \subseteq [\boldsymbol{\tau}L,\ \boldsymbol{\tau}L \oplus \alpha(\circ\,\circ)]$. On the other hand, the si-fragments of all logics in the interval are the same, namely L. Therefore, $\boldsymbol{\rho}^{-1}(L) = [\boldsymbol{\tau}L, \boldsymbol{\tau}L \oplus \alpha(\circ\,\circ)]$. Now, if L is consistent then $\beta(\circ) \notin L$ and so we have

$$\boldsymbol{\tau}L \subset \cdots \subset \boldsymbol{\tau}L \oplus \alpha(\mathfrak{C}_n) \subset \cdots \subset \boldsymbol{\tau}L \oplus \alpha(\mathfrak{C}_2) \subset \boldsymbol{\tau}L \oplus \alpha(\mathfrak{C}_1) = \mathbf{For},$$

where $\mathfrak{C}_i$ is the non-degenerate cluster with i points. ■

This result is due to Maksimova and Rybakov [1974], Blok [1976] and Esakia [1979b].

Thus, all modal companions of every si-logic L are contained between the least companion $\boldsymbol{\tau}L$ and the greatest one, viz., $\boldsymbol{\tau}L \oplus \alpha(\circ\,\circ)$, which will be denoted by $\boldsymbol{\sigma}L$. Using Theorems 159 and 44, we obtain

COROLLARY 161. *There is an algorithm which, given a modal formula φ, returns an intuitionistic formula ψ such that $\boldsymbol{\rho}(\mathbf{S4} \oplus \varphi) = \mathbf{Int} + \psi$.*

The following theorem, which is also a consequence of Theorem 159, describes lattice-theoretic properties of the maps $\boldsymbol{\rho}$, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. Items (i), (ii) and (iv) in it were first proved by Maksimova and Rybakov [1974], and (iii) is due to Blok [1976] and Esakia [1979b] and known as the Blok–Esakia Theorem.

THEOREM 162. (i) *The map $\boldsymbol{\rho}$ is a homomorphism of the lattice* NExt$\mathbf{S4}$ *onto the lattice* Ext$\mathbf{Int}$.

(ii) *The map $\boldsymbol{\tau}$ is an isomorphism of* Ext$\mathbf{Int}$ *into* NExt$\mathbf{S4}$.

(iii) *The map $\boldsymbol{\sigma}$ is an isomorphism of* Ext$\mathbf{Int}$ *onto* NExt$\mathbf{Grz}$.

(iv) *All these maps preserve infinite sums and intersections of logics.*

Now we give frame-theoretic characterizations of the operators $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. Note first that the following evident relations between frames for si-logics and their modal companions hold:

$$\mathfrak{F} \models \boldsymbol{\rho}M \text{ iff } \boldsymbol{\sigma}\mathfrak{F} \models M, \quad \mathfrak{F} \models L \text{ iff } \boldsymbol{\sigma}\mathfrak{F} \models \boldsymbol{\sigma}L,$$

$$\boldsymbol{\rho}\mathfrak{F} \models L \text{ iff } \mathfrak{F} \models \boldsymbol{\tau}L, \quad \mathfrak{F} \models L \text{ iff } \boldsymbol{\tau}_k\mathfrak{F} \models \boldsymbol{\tau}L.$$

THEOREM 163 (Maksimova and Rybakov 1974). *A si-logic L is characterized by a class $\mathcal{C}$ of intuitionistic frames iff $\boldsymbol{\sigma}L$ is characterized by the class $\boldsymbol{\sigma}\mathcal{C} = \{\boldsymbol{\sigma}\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$.*

Proof. ($\Rightarrow$) It suffices to show that any canonical formula $\alpha(\mathfrak{F},\mathfrak{D},\bot) \notin \boldsymbol{\sigma}L$ is refuted by some frame in $\boldsymbol{\sigma}\mathcal{C}$. Since $\mathfrak{F}$ is partially ordered, $\beta(\mathfrak{F},\mathfrak{D},\bot) \notin L$, i.e., there is $\mathfrak{F} \in \mathcal{C}$ refuting $\beta(\mathfrak{F},\mathfrak{D},\bot)$ and so $\boldsymbol{\sigma}\mathfrak{F} \not\models \alpha(\mathfrak{F},\mathfrak{D},\bot)$. ($\Leftarrow$) is straightforward. ■

To characterize $\boldsymbol{\tau}$ we require

LEMMA 164. *For any canonical formula $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ built on a quasi-ordered frame $\mathfrak{F}$, $\alpha(\mathfrak{F},\mathfrak{D},\bot) \in \mathbf{S4} \oplus \alpha(\boldsymbol{\rho}\mathfrak{F},\boldsymbol{\rho}\mathfrak{D},\bot)$, where $\boldsymbol{\rho}\mathfrak{D} = \{\boldsymbol{\rho}\mathfrak{d} : \mathfrak{d} \in \mathfrak{D}\}$ and $\boldsymbol{\rho}\mathfrak{d} = \{C(x) : x \in \mathfrak{d}\}$.*

Proof. Let $\mathfrak{G}$ be a quasi-ordered frame refuting $\alpha(\mathfrak{F},\mathfrak{D},\bot)$. Then there is a cofinal subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. The map h from $\mathfrak{F}$ onto $\boldsymbol{\rho}\mathfrak{F}$ defined by $h(x) = C(x)$, for every x in $\mathfrak{F}$, is clearly a reduction of $\mathfrak{F}$ to $\boldsymbol{\rho}\mathfrak{F}$. So the composition hf is a cofinal subreduction of $\mathfrak{G}$ to $\boldsymbol{\rho}\mathfrak{F}$, and it is easy to verify that it satisfies (CDC) for $\boldsymbol{\rho}\mathfrak{D}$. ■

THEOREM 165. *A si-logic L is characterized by a class $\mathcal{C}$ of frames iff $\boldsymbol{\tau}L$ is characterized by the class $\bigcup_{0<k<\omega} \boldsymbol{\tau}_k\mathcal{C}$, where $\boldsymbol{\tau}_k\mathcal{C} = \{\boldsymbol{\tau}_k\mathfrak{F} : \mathfrak{F} \in \mathcal{C}\}$.*

Proof. ($\Rightarrow$) As was noted above, if $\mathfrak{F}$ is a frame for L then $\boldsymbol{\tau}_k\mathfrak{F}$ is a frame for $\boldsymbol{\tau}L$. So suppose that a formula $\alpha(\mathfrak{F},\mathfrak{D},\bot)$, built on a quasi-ordered frame $\mathfrak{F} = \langle W, R\rangle$, does not belong to $\boldsymbol{\tau}L$ and show that it is refuted by some frame in $\bigcup_{0<k<\omega} \boldsymbol{\tau}_k\mathcal{C}$. By Lemma 164, $\alpha(\boldsymbol{\rho}\mathfrak{F},\boldsymbol{\rho}\mathfrak{D},\bot) \notin \boldsymbol{\tau}L$ and so $\beta(\boldsymbol{\rho}\mathfrak{F},\boldsymbol{\rho}\mathfrak{D},\bot) \notin L$. Hence there is a frame $\mathfrak{G} = \langle V, S, Q\rangle$ in $\mathcal{C}$ which refutes $\beta(\boldsymbol{\rho}\mathfrak{F},\boldsymbol{\rho}\mathfrak{D},\bot)$. But then $\boldsymbol{\sigma}\mathfrak{G} \models \boldsymbol{\tau}L$ and $\boldsymbol{\sigma}\mathfrak{G} \not\models \alpha(\boldsymbol{\rho}\mathfrak{F},\boldsymbol{\rho}\mathfrak{D},\bot)$. Let f be a subreduction of $\boldsymbol{\sigma}\mathfrak{G}$ to $\boldsymbol{\rho}\mathfrak{F}$ satisfying (CDC) for $\boldsymbol{\rho}\mathfrak{D}$ and let $k = \max\{|C(x)| : x \in W\}$. Define a partial map h from $\boldsymbol{\tau}_k\mathfrak{G} = \langle kV, kS, kQ\rangle$ onto $\mathfrak{F}$ as follows: if $x \in V$, $y_0 \in W$, $f(x) = C(y_0)$ and $C(y_0) = \{y_0, \dots, y_n\}$ then we put $h(\langle i, x\rangle) = y_i$, for $i = 0, \dots, n$. By the definition of $\boldsymbol{\tau}_k$, for any $i \in \{0, \dots, n\}$ we have

$$h^{-1}(y_i) = \{\langle i, x\rangle : \ x \in f^{-1}(C(y_0))\} = \{i\} \times f^{-1}(C(y_0)) \in kQ.$$

Now, one can readily prove that h is a cofinal subreduction of $\boldsymbol{\tau}_k\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. So $\boldsymbol{\tau}_k\mathfrak{G} \not\models \alpha(\mathfrak{F},\mathfrak{D},\bot)$. ($\Leftarrow$) is obvious. ■

It is worth noting that this proof will not change if we put in it $k = \omega$.

COROLLARY 166. *A logic $L \in$ Ext**Int** is characterized by a class $\mathcal{C}$ of frames iff $\boldsymbol{\tau}L$ is characterized by the class $\boldsymbol{\tau}_\omega\mathcal{C}$.*

The following theorem provides a deductive characterization of the maps $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$.

THEOREM 167. *For every si-logic L and every modal canonical formula $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ built on a quasi-ordered frame $\mathfrak{F}$,*

(i) *$\alpha(\mathfrak{F},\mathfrak{D},\bot) \in \boldsymbol{\tau}L$ iff $\beta(\boldsymbol{\rho}\mathfrak{F},\boldsymbol{\rho}\mathfrak{D},\bot) \in L$;*

(ii) *$\alpha(\mathfrak{F},\mathfrak{D},\bot) \in \boldsymbol{\sigma}L$ iff either $\mathfrak{F}$ is partially ordered and $\beta(\mathfrak{F},\mathfrak{D},\bot) \in L$ or $\mathfrak{F}$ contains a proper cluster.*

Proof. (i) The implication ($\Rightarrow$) was actually established in the proof of Theorem 165, and the converse one follows from Lemma 164.

(ii) Suppose $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \boldsymbol{\sigma} L$. Then either $\mathfrak{F}$ is partially ordered, and so $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \in L$, or $\mathfrak{F}$ contains a proper cluster. The converse implication follows from (i) and the fact that $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \mathbf{Grz}$ for every frame $\mathfrak{F}$ with a proper cluster. ■

The results obtained in this section not only establish some structural correspondences between logics in Ext**Int** and NExt**S4** and their frames, but may be also used for transferring various properties of modal logics to their si-fragments and back. A few results of that sort are collected in Table 7; we shall cite them as the Preservation Theorem. The preservation of decidability follows from the definition of $\boldsymbol{\rho}$ and Theorem 167. That $\boldsymbol{\rho}$ preserves Kripke completeness, FMP and tabularity is a consequence of Theorem 157. The map $\boldsymbol{\tau}$ preserves Kripke completeness and FMP, since we can define $\boldsymbol{\tau}_k$ in Theorem 165 so that $\boldsymbol{\tau}_k \langle W, R\rangle = \langle kW, kR\rangle$; however, $\boldsymbol{\tau}$ does not in general preserve the tabularity, because $\boldsymbol{\tau}\mathbf{Cl} = \mathbf{S5}$ is not tabular. The preservation of FMP and tabularity under $\boldsymbol{\sigma}$ follows from Theorem 163. On the other hand, Shehtman [1980] proved that $\boldsymbol{\sigma}$ does not preserve Kripke completeness (since $\boldsymbol{\tau}$ preserves it and **Grz** is complete, this means in particular that Kripke completeness is not preserved under sums of logics in NExt**S4**). Some other preservation results in Table 7 will be discussed later. For references see [Chagrov and Zakharyaschev 1992, 1997].

3.4 Completeness

In this section we briefly discuss the most important results concerning completeness of si-logics with respect to various classes of Kripke frames.

Kripke completeness That not all si-logics are complete with respect to Kripke frames was discovered by Shehtman [1977], who found a way to adjust Fine's [1974b] idea to the intuitionistic case (which was not so easy because intuitionistic formulas do not "feel" infinite ascending chains essential in Fine's construction; see Section 20 of *Basic Modal Logic*). Note however that Kuznetsov's [1975] question whether all si-logics are complete with respect to the topological semantics (see *Intuitionistic Logic*) is still open.

As to general positive results, notice first that the Preservation Theorem yields the following translation of Fine's [1974c] Theorem on finite width logics (si-logics of finite width were studied by Sobolev [1977a]).

THEOREM 168. *Every si-logic of width n (i.e., a logic in* Ext$\mathbf{BW}_n$*; see Table 5) is characterized by a class of Noetherian Kripke frames of width $\leq n$.*

The translation of Sahlqvist's Theorem gives nothing interesting for si-logics. A sort of intuitionistic analog of this theorem has been recently

Table 7. Preservation Theorem

Property of logics	Preserved under ρ	τ	σ
Decidability	Yes	Yes	Yes
Kripke completeness	Yes	Yes	No
Strong completeness	Yes	Yes	No
Finite model property	Yes	Yes	Yes
Tabularity	Yes	No	Yes
Pretabularity	Yes	No	Yes
$\mathcal{D}$-persistence	Yes	Yes	No
Local tabularity	Yes	No	No
Disjunction property	Yes	Yes	Yes
Halldén completeness	Yes	No	No
Interpolation property	Yes	No	No
Elementarity	Yes	Yes	No
Independent axiomatizability	No	Yes	Yes

proved by Ghilardi and Meloni [1997]. Here is a somewhat simplified variant of their result in which $\overline{p}$, $\overline{q}$, $\overline{r}$, $\overline{s}$ denote tuples of propositional variables and $\overline{\psi}$, $\overline{\chi}$ tuples of formulas of the same length as $\overline{r}$ and $\overline{s}$, respectively.

THEOREM 169 (Ghilardi and Meloni 1997). *Suppose $\varphi(\overline{p},\overline{q},\overline{r},\overline{s})$ is an intuitionistic formula in which the variables $\overline{r}$ occur positively and the variables $\overline{s}$ occur negatively, and which does not contain any $\to$, except for negations and double negations of atoms, in the premise of a subformula of the form $\varphi' \to \varphi''$. Assume also that $\overline{\psi}(\overline{p},\overline{q})$ and $\overline{\chi}(\overline{p},\overline{q})$ are formulas such that $\overline{p}$ occur positively in $\overline{\psi}$ and negatively in $\overline{\chi}$, while $\overline{q}$ occur negatively in $\overline{\psi}$ and positively in $\overline{\chi}$. Then the logic*

$$\mathbf{Int} + \varphi(\overline{p},\overline{q},\overline{\psi}(\overline{p},\overline{q}),\overline{\chi}(\overline{p},\overline{q}))$$

is canonical.

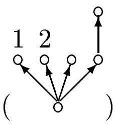
The preservation of $\mathcal{D}$-persistence under $\boldsymbol{\rho}$ (see [Zakharyaschev 1996]) and the fact (discovered by Chagrova [1990]) that $\boldsymbol{\tau}L$ is characterized by an elementary class of Kripke frames whenever L is determined by such a class provide us with an intuitionistic variant of the Fine–van Benthem Theorem.

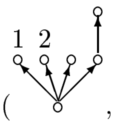
THEOREM 170. *If a si-logic is characterized by an elementary class of Kripke frames then it is $\mathcal{D}$-persistent.*

As in the modal case, it is unknown whether the converse of this theorem holds. All known non-elementary si-logics, for instance the Scott logic **SL** and the logics $\mathbf{T}_n$ of finite n-ary trees (see [Rodenburg 1986]) are not canonical and even strongly complete either, as was shown by Shimura [1995]. (Actually he proved that no logic in the intervals $[\mathbf{SL}, \mathbf{SL} + \boldsymbol{bd}_3]$ and $[\mathbf{Int}, \mathbf{T}_2]$, save of course **Int**, is strongly complete.)

As far as we know, there are no examples of si-logics separating canonicity, $\mathcal{D}$-persistence and strong completeness. (Ghilardi, Meloni and Miglioli have recently showed that **SL** in any language with finitely many variables is canonical). Theorem 40 which holds in the intuitionistic case as well gives an algebraic counterpart of strong Kripke completeness.

The finite model property The first example of an infinitely axiomatizable si-logic without FMP was constructed by Jankov [1968b]—that was in fact the starting point of a long series of "negative" results in modal logic. A finitely axiomatizable logic without FMP appeared two years later in [Kuznetsov and Gerchiu 1970]. The reader can get some impression about this and other examples of that sort by proving (it is really not hard) that

$\varphi = \beta($ 

$) \notin L = \mathbf{Int} + \boldsymbol{bw}_4 + \beta($ 

$, \{\{1,2\}\})$

but no finite frame can separate φ from L. (Notice by the way that $\boldsymbol{\tau}L$ is axiomatizable by Sahlqvist formulas; see [Chagrov and Zakharyaschev 1995b].)

FMP of a good many si-logics was proved using various forms of filtration; see e.g. [Gabbay 1970], [Ono 1972], [Smoryński 1973], [Ferrari and Miglioli 1993]. As an illustration of a rather sophisticated selective filtration we present here the following

THEOREM 171 (Gabbay and de Jongh 1974). *The logic $\mathbf{T}_n$ (see Table 5) is characterized by the class of finite n-ary trees.*

Proof. First we prove that $\mathbf{T}_n$ is characterized by the class of finite frames of branching $\leq n$. Suppose $\varphi \notin \mathbf{T}_n$ and $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ is a model for $\mathbf{T}_n$ refuting φ. Without loss of generality we may assume that $\mathfrak{F} = \langle W, R \rangle$ is a tree. Let $\Sigma = \mathbf{Sub}\varphi$ and $\Gamma_x = \{\psi \in \Sigma : x \models \psi\}$, for every point x in $\mathfrak{F}$.

Given x in $\mathfrak{F}$, put $rg(x) = \{[y] : y \in x\uparrow\}$ and say that x is of *minimal range* if $rg(x) = rg(y)$ for every $y \in [x] \cap x\uparrow$. Since there are only finitely many distinct Σ-equivalence classes in $\mathfrak{M}$, every $y \in [x]$ sees a point $z \in [x]$ of minimal range. Now we extract from $\mathfrak{M}$ a finite refutation frame $\mathfrak{G} = \langle V, S \rangle$ for φ of branching $\leq n$. To begin with, we select some point x of minimal range at which φ is refuted and put $V_0 = \{x\}$.

Suppose V_k has already been defined. If $|rg(x)| = 1$ for every $x \in V_k$, then we put $\mathfrak{G} = \langle V, S\rangle$, where $V = \bigcup_{i=0}^{k} V_k$ and S is the restriction of R to V. Otherwise, for each $x \in V_k$ with $|rg(x)| > 1$ and each $[y] \in rg(x)$ different from $[x]$ and such that $\Gamma_z \subset \Gamma_y$ for no $[z] \in rg(x) - \{[x]\}$, we select a point $u \in [y] \cap x\uparrow$ of minimal range. Let U_x be the set of all selected points for x and $V_{k+1} = \bigcup_x U_x$. It should be clear that $\Gamma_x \subset \Gamma_u$ (and $rg(x) \supset rg(u)$), for every $u \in U_x$, and so the inductive process must terminate. Consequently $\mathfrak{G} \not\models \varphi$.

It remains to establish that $\mathfrak{G} \models \mathbf{T}_n$, i.e., $\mathfrak{G}$ is of branching $\leq n$. Suppose otherwise. Then there is a point x in $\mathfrak{G}$ with $m \geq n+1$ immediate successors $x_0, \dots, x_m$, which are evidently in U_x because $\mathfrak{F}$ is a tree. We are going to construct a substitution instance of $\mathbf{T}_n$'s axiom $\boldsymbol{bb}_n$ which is refuted at x in $\mathfrak{M}$.

Denote by δ_i the conjunction of the formulas in Γ_{x_i}. Since all of them are true at x_i in $\mathfrak{M}$, we have $x_i \models \delta_i$; and since $\Gamma_i \subseteq \Gamma_j$ for no distinct i and j, we have $x_j \not\models \chi_i$ if $i \neq j$. Put $\chi_i = \delta_i$, for $0 \leq i < n$, $\chi_n = \delta_n \vee \cdots \vee \delta_m$ and consider the truth-value of the formula $\psi = \boldsymbol{bb}_n\{\chi_0/p_0, \dots, \chi_n/p_n\}$ at x in $\mathfrak{M}$.

Since xRx_i for every $i = 0, \dots, m$, we have $x \not\models \bigvee_{i=0}^{n} \chi_i$. Suppose that $x \not\models \bigwedge_{i=0}^{n}((\chi_i \to \bigvee_{i\neq j} \chi_j) \to \bigvee_{i\neq j} \chi_j)$. Then $y \models \chi_i \to \bigvee_{i\neq j} \chi_j$ and $y \not\models \bigvee_{i\neq j} \chi_j$, for some $y \in x\uparrow$ and some $i \in \{0, \dots, n\}$, and hence $y \not\models \chi_i$. Since $x_i \models \chi_i$ and $x_i \not\models \bigvee_{i\neq j} \chi_j$, y sees no point in $[x_i]$ and so $y \not\sim_\Sigma x$ (for otherwise x would not be of minimal range). Therefore, $\Gamma_{x_j} \subseteq \Gamma_y$ for some $j \in \{0, \dots, m\}$, and then $y \models \chi_j$ if $j < n$ and $y \models \chi_n$ if $j \geq n$, which is a contradiction.

It follows that $x \models \bigwedge_{i=0}^{n}((\chi_i \to \bigvee_{i\neq j} \chi_j) \to \bigvee_{i\neq j} \chi_j)$, from which $x \not\models \psi$, contrary to $\mathfrak{M}$ being a model for $\boldsymbol{bb}_n$. It remains to notice that every finite frame of branching $\leq n$ is a reduct of a finite n-ary tree, which clearly validates $\mathbf{T}_n$. ∎

Another way of obtaining general results on FMP of si-logics is to translate the corresponding results in modal logic with the help of the Preservation Theorem.

THEOREM 172. *Every si-logic of finite depth (i.e., every logic in* Ext$\mathbf{BD}_n$*, for $n < \omega$) is locally tabular.*

Note, however, that unlike NExt**K4**, the converse does not hold: the Dummett logic **LC**, characterized by the class of finite chains (or by the infinite ascending chain), is locally tabular. As we saw in Section 1.7, every non-locally tabular in NExt**S4** logic is contained in **Grz.3**, the only *pre-locally tabular logic* in NExt**S4**. But in Ext**Int** this way of determining local tabularity does not work:

THEOREM 173 (Mardaev 1984). *There is a continuum of pre-locally tabular logics in* Ext**Int**.

Besides, it is not clear whether every locally tabular logic in Ext**Int** (or NExt**K4**) is contained in a pre-locally tabular one.

An intuitionistic formula is said to be *essentially negative* if every occurrence of a variable in it is in the scope of some $\neg$. If φ is essentially negative then $T(\varphi)$ is a $\Box\Diamond$-formula, which yields

THEOREM 174 (McKay 1971, Rybakov 1978). *If a si-logic L is decidable (or has FMP) and φ is an essentially negative formula then $L+\varphi$ is decidable (has FMP).*

Originally this result was proved with the help of Glivenko's Theorem (see Section 7 in *Intuitionistic Logic*). Say that an occurrence of a variable in a formula is *essential* if it is not in the scope of any $\neg$. A formula φ is *mild* if every two essential occurrences of the same variable in φ are either both positive or both negative. Kuznetsov [1972] claimed (we have not seen the proof) that all si-logics whose extra axioms do not contain negative occurrences of essential variables have FMP. And Wroński [1989] announced that if L is a decidable si-logic and φ a mild formula then $L+\varphi$ is also decidable.

Subframe and cofinal subframe si-logics—that is logics axiomatizable by canonical formulas of the form $\beta(\mathfrak{F})$ and $\beta(\mathfrak{F},\bot)$, respectively—can be characterized both syntactically and semantically (see [Zakharyaschev 1996]).

THEOREM 175. *The following conditions are equivalent for every si-logic L:*

(i) *L is a (cofinal) subframe logic;*

(ii) *L is axiomatizable by implicative (respectively, disjunction free) formulas;*

(iii) *L is characterized by a class of finite frames closed under the formation of (cofinal) subframes.*

That all si-logics with disjunction free axioms have FMP was first proved by McKay [1968] with the help of Diego's [1966] Theorem according to which there are only finitely many pairwise non-equivalent in **Int** disjunction free formulas in variables $p_1,\dots,p_n$ (see also [Urquhart 1974]).

Since frames for **Int** contain no clusters, Theorem 58 and its analog for cofinal subframe logics reduce in the intuitionistic case to the following result which is due to Chagrova [1986], Rodenburg [1986], Shimura [1993] and Zakharyaschev [1996].

THEOREM 176. *All si-logics with disjunction free axioms are elementary (definable by $\forall\exists$-sentences) and $\mathcal{D}$-persistent.*

Theorem 68 is translated into the intuitionistic case simply by replacing **K4** with **Int**, $\oplus$ with $+$ and α with β. As a consequence we obtain, for instance, that Ono's [1972] $\mathbf{B}_n$ and all other logics whose canonical axioms are built on trees have FMP. Moreover, we also have

THEOREM 177 (Sobolev 1977b, Nishimura 1960). *All si-logics with extra axioms in one variable have FMP and are decidable.*

In fact Sobolev [1977b] proved a more general (but rather complicated) syntactical sufficient condition of FMP and constructed a formula in two variables axiomatizing a si-logic without FMP (Shehtman's [1977] incomplete si-logic has also axioms in two variables).

Tabularity By the Blok–Esakia and Preservation Theorems, the situation with tabular logics in Ext**Int** is the same as in NExt**Grz**. In particular, $L \in \mathrm{Ext}\mathbf{Int}$ is tabular iff $\mathbf{BD}_n + \mathbf{BW}_n \subseteq L$ for some $n < \omega$ iff L is not a sublogic of one of the three pretabular logics in Ext**Int**, namely **LC**, $\mathbf{BD}_2$ and $\mathbf{KC} + \boldsymbol{bd}_3$. (The pretabular si-logics were described by Maksimova [1972].) The tabularity problem is decidable in Ext**Int**.

3.5 Disjunction property

One of the aims of studying extensions of **Int**, which may be of interest for applications in computer science, is to describe the class of constructive si-logics. At the propositional level a consistent logic $L \in \mathrm{Ext}\mathbf{Int}$ is regarded to be constructive if it has the *disjunction property* (DP, for short) which means that for all formulas φ and ψ,

$$\varphi \vee \psi \in L \text{ implies } \varphi \in L \text{ or } \psi \in L.$$

That intuitionistic logic itself is constructive in this sense was proved in a syntactic way by Gentzen [1934–1935]. However, Łukasiewicz (1952) conjectured that no proper consistent extension of **Int** has DP.

A similar property was introduced for modal logics (see e.g. [Lemmon and Scott 1977]): $L \in \mathrm{NExt}\mathbf{K}$ has the *(modal) disjunction property* if, for every $n \geq 1$ and all formulas $\varphi_1, \ldots, \varphi_n$,

$$\Box\varphi_1 \vee \cdots \vee \Box\varphi_n \in L \text{ implies } \varphi_i \in L, \text{ for some } i \in \{1, \ldots, n\}.$$

The following theorem (in a somewhat different form it was proved in [Hughes and Cresswell 1984] and [Maksimova 1986]) provides a semantic criterion of DP.

THEOREM 178. *Suppose a modal or si-logic L is characterized by a class $\mathcal{C}$ of descriptive rooted frames closed under the formation of rooted generated subframes. Then L has DP iff, for every $n \geq 1$ and all $\mathfrak{F}_1, \ldots, \mathfrak{F}_n \in \mathcal{C}$ with roots $x_1, \ldots, x_n$, there is a frame $\mathfrak{F}$ for L with root x such that the disjoint union $\mathfrak{F}_1 + \cdots + \mathfrak{F}_n$ is a generated subframe of $\mathfrak{F}$ with $\{x_1, \ldots, x_n\} \subseteq x{\uparrow}$.*

Proof. We consider only the modal case. ($\Rightarrow$) Let $\mathfrak{F}_L = \langle W_L, R_L, P_L \rangle$ be a universal frame for L, big enough to contain $\mathfrak{F}_1 + \cdots + \mathfrak{F}_n$ as its generated subframe. Assuming that $\mathfrak{F}_L$ is associated with a suitable canonical model for L, we show that there is a point x in $\mathfrak{F}_L$ such that $x{\uparrow} = W_L$. The set

$$\Delta' = \{\neg\Box\varphi : \exists y \in W_L \ y \not\models \varphi\}$$

is L-consistent (for otherwise $\Box\varphi_1 \vee \cdots \vee \Box\varphi_n \in L$ for some $\varphi_1, \ldots, \varphi_n \notin L$). Let Δ be a maximal L-consistent extension of Δ' and x the point in $\mathfrak{F}_L$ where Δ is true. Then xR_Ly, for every $y \in W_L$.

($\Leftarrow$) Suppose otherwise. Then there are formulas $\varphi_1, \ldots, \varphi_n \notin L$ such that $\Box\varphi_1 \vee \cdots \vee \Box\varphi_n \in L$. Take frames $\mathfrak{F}_1, \ldots, \mathfrak{F}_n \in \mathcal{C}$ refuting $\varphi_1, \ldots, \varphi_n$ at their roots, respectively, and let $\mathfrak{F}$ be a rooted frame for L containing $\mathfrak{F}_1 + \cdots + \mathfrak{F}_n$ as a generated subframe and such that its root x sees the roots of $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$. Then all the formulas $\Box\varphi_1, \ldots, \Box\varphi_n$ are refuted at x and so $\Box\varphi_1 \vee \cdots \vee \Box\varphi_n \notin L$, which is a contradiction. ■

It should be clear that if we use only the sufficient condition of Theorem 178, the requirement that frames in $\mathcal{C}$ are descriptive is redundant. Furthermore, it is easy to see that for $L \in \mathrm{NExt}\mathbf{K4}$ we may assume $n \leq 2$. And clearly a logic $L \in \mathrm{NExt}\mathbf{S4}$ has DP iff, for all φ and ψ, $\Box\varphi \vee \Box\psi \in L$ implies $\Box\varphi \in L$ or $\Box\psi \in L$.

As a direct consequence of the proof above we obtain

COROLLARY 179. *A modal or si-logic L has DP iff the canonical frame $\mathfrak{F}_L = \langle W_L, R_L\rangle$ contains a point x such that $x{\uparrow} = W_L$.*

Using the semantic criterion above it is not hard to show that DP is preserved under $\boldsymbol{\rho}$, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. It is also a good tool for proving and disproving DP of logics with transparent semantics.

EXAMPLE 180. (i) Let $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ be serial rooted Kripke frames. Then the frame obtained by adding a root to $\mathfrak{F}_1 + \cdots + \mathfrak{F}_n$ is also serial. Therefore, **D** has DP. In the same way one can show that **K**, **K4**, **T**, **S4**, **Grz**, **GL** and many other modal logics have DP.

(ii) Since no rooted symmetrical frame can contain a proper generated subframe, no consistent logic in $\mathrm{NExt}\mathbf{KB}$ has DP.

The first proper extensions of **Int** with DP were constructed by Kreisel and Putnam [1957]: these were **KP** (now called the *Kreisel–Putnam logic*) and **SL** (known as the *Scott logic*). We present here Gabbay's [1970] proof that **KP** has DP.

THEOREM 181 (Kreisel and Putnam 1957). **KP** *has DP.*

Proof. Using filtration one can show that **KP** is characterized by the class of finite rooted frames $\mathfrak{F} = \langle W, R\rangle$ satisfying the condition

$$\begin{aligned} &\forall x,y,z\ (xRy \wedge xRz \wedge \neg yRz \wedge \neg zRy \to \exists u\ (xRu \wedge uRy \wedge uRz \wedge \\ (15)\qquad &\qquad\qquad \forall v\ (uRv \to \exists w\ (vRw \wedge (yRw \vee zRw))))). \end{aligned}$$

If $\mathfrak{F}$ is such a frame then for each non-empty $X \subseteq W^{\leq 1}$, the generated subframe of $\mathfrak{F}$ based on the set $W - (W^{\leq 1} - X)\downarrow$ is rooted; we denote its root by $r(X)$.

Let $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$ be finite rooted frames satisfying (15). We construct from them a frame $\mathfrak{F} = \langle W, R \rangle$ by taking

$$W = W_1 \cup W_2 \cup U,$$

where $U = \{X_1 \cup X_2 : X_1 \subseteq W_1^{\leq 1},\ X_2 \subseteq W_2^{\leq 1},\ X_1, X_2 \neq \emptyset\}$, and

$$\begin{aligned} xRy \quad \text{iff} \quad & (x, y \in W_i \wedge xR_iy) \vee (x, y \in U \wedge x \supseteq y) \vee \\ & (x = X_1 \cup X_2 \in U \wedge y \in W_i \wedge r(X_i)R_iy). \end{aligned}$$

It follows from the given definition that $\mathfrak{F}_1 + \mathfrak{F}_2$ is a generated subframe of $\mathfrak{F}$, $W_1 \cup W_2$ is a cover for $\mathfrak{F}$ and $W_1^{\leq 1} \cup W_2^{\leq 1}$ is its root. So our theorem will be proved if we show that (15) holds.

Suppose $x, y, z \in W$ satisfy the premise of (15). Since (15) holds for $\mathfrak{F}_1$ and $\mathfrak{F}_2$, we can assume that $x = X_1 \cup X_2 \in U$. Let $Y_1 \cup Y_2$ and $Z_1 \cup Z_2$ be the sets of final points in $y\uparrow$ and $z\uparrow$, respectively, with $Y_i, Z_i \subseteq W_i$. By the definition of R, we have $Y_i, Z_i \subseteq X_i$. Consider $u = (Y_1 \cup Z_1) \cup (Y_2 \cup Z_2)$. Clearly, xRu, uRy and uRz. Suppose now that $v \in u\uparrow$. Let w be any final point in $v\uparrow$. Then $v \in (Y_1 \cup Z_1) \cup (Y_2 \cup Z_2)$ and so either yRw or zRw. ∎

Other examples of constructive si-logics were constructed by Ono [1972] and Gabbay and de Jongh [1974], namely, $\mathbf{B}_n$ and $\mathbf{T}_n$. Anderson [1972] proved that among the consistent si-logics with extra axioms in one variable only those of the form $\mathbf{Int} + \boldsymbol{nf}_{2n+2}$, for $n \geq 5$, have DP (for $n = 6$ the proof was found by Wroński [1974]; see also [Sasaki 1992]). Finally, Wroński [1973] showed that there is a continuum of si-logics with DP.

The additional axioms of logics in all these examples contained occurrences of $\vee$; on the other hand, known examples of si-logics with disjunction free extra axioms, say **LC**, **KC**, **Cl**, $\mathbf{BW}_n$ or $\mathbf{BD}_n$, were not constructive. This observation led Hosoi and Ono [1973] to the conjecture that the disjunction free fragment of every consistent si-logic with DP coincides with that of **Int**. We present a proof of this conjecture following [Zakharyaschev 1987].

First we describe the cofinal subframe logics in NExt**S4** with DP, assuming that every such logic L is represented by its independent canonical axiomatization

(16) $L = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i, \perp) : i \in I\}$.

All frames in the rest of this section are assumed to be quasi-ordered.

Say that a finite rooted frame $\mathfrak{F}$ with ≥ 2 points is *simple* if its root cluster and at least one of the final clusters are simple. Suppose $\mathfrak{F} = \langle W, R \rangle$ is a simple frame, $a_0, a_1, \ldots, a_m, a_{m+1}, \ldots, a_n$ are all its points, with a_0 being the root, $C(a_1), \ldots, C(a_m)$ all the distinct immediate cluster-successors of a_0, and a_n a final point with simple $C(a_n)$. For every $k = 1, \ldots, n$, define a formula ψ_k by taking

$$\psi_k = \bigwedge_{a_i R a_j, i \neq 0} \varphi_{ij} \wedge \bigwedge_{i=1}^{n} \varphi_i \wedge \varphi'_\bot \to p_k$$

where φ_{ij}, φ_i were defined in Section 3.2 and $\varphi'_\bot = \Box(\bigwedge_{i=1}^{n} \Box p_i \to \bot)$. Now we associate with $\mathfrak{F}$ the formula $\gamma(\mathfrak{F}) = \Box p_0 \vee \Box\psi_1$ if $m = 1$, and the formula $\gamma(\mathfrak{F}) = \Box\psi_1 \vee \cdots \vee \Box\psi_m$ if $m > 1$.

LEMMA 182. *For every simple frame $\mathfrak{F}$, $\gamma(\mathfrak{F}) \in \mathbf{S4} \oplus \alpha(\mathfrak{F}, \bot)$.*

Proof. It is enough to show that $\mathfrak{G} \not\models \gamma(\mathfrak{F})$ implies $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \bot)$, for any finite $\mathfrak{G}$. So suppose $\gamma(\mathfrak{F})$ is refuted in a finite frame $\mathfrak{G}$ under some valuation. Define a partial map f from $\mathfrak{G}$ onto $\mathfrak{F}$ by taking

$$f(x) = \begin{cases} a_0 & \text{if } x \not\models \gamma(\mathfrak{F}) \\ a_i & \text{if } x \not\models \psi_i,\ 1 \leq i \leq n \\ \text{undefined} & \text{otherwise.} \end{cases}$$

One can readily check that f is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}$. However it is not necessarily cofinal. So we extend f by putting $f(x) = a_n$, for every x of depth 1 in $\mathfrak{G}$ such that $f(x{\downarrow}) = \{a_0\}$. Clearly, the improved map is still a subreduction of $\mathfrak{G}$ to $\mathfrak{F}$, and $\varphi'_\bot$ ensures its cofinality. ∎

Using the semantical properties of the canonical formulas it is a matter of routine to prove the following

LEMMA 183. *Suppose $i \in \{1, \ldots, m\}$ and $\mathfrak{G}$ is the subframe of $\mathfrak{F}$ generated by a_i. Then $\alpha(\mathfrak{G}, \bot) \in \mathbf{S4} \oplus \psi_i$.*

We are in a position now to prove a criterion of DP for the cofinal subframe logics in NExt**S4**.

THEOREM 184. *A consistent cofinal subframe logic $L \in$ NExt**S4** has the disjunction property iff no frame $\mathfrak{F}_i$ in its independent axiomatization (16) is simple, for $i \in I$.*

Proof. ($\Rightarrow$) Suppose, on the contrary, that $\mathfrak{F}_i$ is simple, for some $i \in I$. Since the axiomatization (16) is independent, every proper generated subframe of $\mathfrak{F}_i$ validates L. By Lemma 182, $\gamma(\mathfrak{F}_i) \in L$ and so either $p_0 \in L$ or $\psi_j \in L$. However, both alternatives are impossible: the former means that L is inconsistent, while the latter, by Lemma 183, implies $\alpha(\mathfrak{G}, \bot) \in L$, where $\mathfrak{G}$ is the subframe of $\mathfrak{F}_i$ generated by an immediate successor of $\mathfrak{F}_i$'s root.

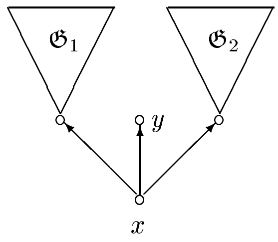


Figure 15.

($\Leftarrow$) Given two finite rooted frames $\mathfrak{G}_1$ and $\mathfrak{G}_2$ for L, we construct the frame $\mathfrak{F}$ as shown in Fig. 15 and prove that $\mathfrak{F} \models L$. Suppose otherwise, i.e., there exists a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{F}_i$, for some $i \in I$. Let x_i be the root of $\mathfrak{F}_i$. Since $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are not cofinally subreducible to $\mathfrak{F}_i$ and since L is consistent, $f^{-1}(x_i) = \{x\}$. By the cofinality condition, it follows in particular that $y \in \mathrm{dom} f$. But then $\mathfrak{F}_i$ is simple, which is a contradiction. Thus, by Theorem 178, L has DP. ■

Note that in fact the proof of ($\Rightarrow$) shows that if $L \in \mathrm{NExt}\mathbf{S4}$, $\mathfrak{F}$ is a simple frame, $\alpha(\mathfrak{F}, \bot) \in L$ and $\alpha(\mathfrak{G}, \bot) \notin L$ for any proper generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ then L does not have DP. Transferring this observation to the intuitionistic case, we obtain

THEOREM 185 (Minari 1986, Zakharyaschev 1987). *If a si-logic is consistent and has DP then the disjunction free fragments of L and* **Int** *are the same.*

Sufficient conditions of DP in terms of canonical formulas can be found in [Chagrov and Zakharyaschev 1993, 1997].

Since classical logic is not constructive, it is of interest to find maximal consistent si-logics with DP. That they exist follows from Zorn's Lemma. Here is a concrete example of such a logic.

Trying to formalize the proof interpretation of intuitionistic logic, Medvedev [1962] proposed to treat intuitionistic formulas as finite problems. Formally, a *finite problem* is a pair $\langle X, Y\rangle$ of finite sets such that $Y \subseteq X$ and $X \neq \emptyset$; elements in X are called *possible solutions* and elements in Y *solutions* to the problem. The operations on finite problems, corresponding to the logical connectives, are defined as follows:

$$\langle X_1, Y_1\rangle \wedge \langle X_2, Y_2\rangle = \langle X_1 \times X_2, Y_1 \times Y_2\rangle\,,$$

$$\langle X_1, Y_1\rangle \vee \langle X_2, Y_2\rangle = \langle X_1 \sqcup X_2, Y_1 \sqcup Y_2\rangle\,,$$

$$\langle X_1, Y_1\rangle \to \langle X_2, Y_2\rangle = \left\langle X_2^{X_1}, \{f \in X_2^{X_1} : f(Y_1) \subseteq Y_2\}\right\rangle,$$

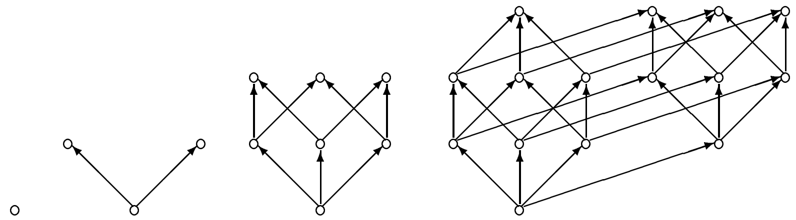


Figure 16.

$$\bot = \langle X, \emptyset \rangle .$$

Here $X \sqcup Y = (X \times \{1\}) \cup (Y \times \{2\})$ and X^Y is the set of all functions from X into Y. Note that in the definition of $\bot$ the set X is fixed, but arbitrary; for definiteness one can take $X = \{\emptyset\}$.

Now we can interpret formulas by finite problems. Namely, given a formula φ, we replace its variables by arbitrary finite problems and perform the operations corresponding to the connectives in φ. If the result is a problem with a non-empty set of solutions no matter what finite problems are substituted for the variables in φ, then φ is called *finitely valid*. One can show that the set of all finitely valid formulas is a si-logic; it is called *Medvedev's logic* and denoted by **ML**.

In fact, **ML** can be defined semantically. Medvedev [1966] showed that **ML** coincides with the set of formulas that are valid in all frames $\mathfrak{B}_n$ having the form of the n-ary Boolean cubes with the topmost point deleted; for $n = 1, 2, 3, 4$, the Medvedev frames are shown in Fig. 16. Since $\mathfrak{B}_n + \mathfrak{B}_m$ is a generated subframe of $\mathfrak{B}_{n+m}$, **ML** has DP. Moreover, Levin [1969] proved that it has no proper consistent extension with DP. The following proof of this result is due to Maksimova [1986].

THEOREM 186 (Levin 1969). **ML** *is a maximal si-logic with DP.*

Proof. Suppose, on the contrary, that there exists a proper consistent extension L of **ML** having DP. Then we have a formula $\varphi \in L - \mathbf{ML}$. We show first that there is an essentially negative substitution instance φ^* of φ such that $\varphi^* \notin \mathbf{ML}$. Since $\varphi(p_1, \dots, p_n) \notin \mathbf{ML}$, there is a Medvedev frame $\mathfrak{B}_m$ refuting φ under some valuation $\mathfrak{V}$. With every point x in $\mathfrak{B}_m$ we associate a new variable q_x and extend $\mathfrak{V}$ to these variables by taking $\mathfrak{V}(q_x)$ to be the set of final points in $\mathfrak{B}_m$ that are not accessible from x. By the construction of $\mathfrak{B}_m$, we have $y \models \neg q_x$ iff $y \in x{\uparrow}$, from which

$$\mathfrak{V}(\bigvee_{x \in \mathfrak{V}(p_i)} \neg q_x) = \mathfrak{V}(p_i).$$

Let $\varphi^* = \varphi(\bigvee_{x\in\mathfrak{V}(p_1)} \neg q_x, \ldots, \bigvee_{x\in\mathfrak{V}(p_n)} \neg q_x)$. It follows that $\mathfrak{V}(\varphi^*) = \mathfrak{V}(\varphi)$ and so $\varphi^* \notin \mathbf{ML}$.

Thus, we may assume that φ is an essentially negative formula. Since $\mathbf{KP} \subseteq \mathbf{ML}$, $\mathbf{ML}$ contains the formulas

$$\boldsymbol{nd}_k = (\neg p \to \neg q_1 \vee \cdots \vee \neg q_k) \to (\neg p \to \neg q_1) \vee \cdots \vee (\neg p \to \neg q_k)$$

which, as is easy to see, belong to $\mathbf{KP}$. Let us consider the logic

$$\mathbf{ND} = \mathbf{Int} + \{\boldsymbol{nd}_k : \ k \geq 1\}.$$

Using the fact that the outermost $\to$ in $\boldsymbol{nd}_k$ can be replaced with $\leftrightarrow$ and that $(\neg p \to \neg q) \leftrightarrow \neg(\neg p \wedge q) \in \mathbf{Int}$, one can readily show that every essentially negative formula is equivalent in $\mathbf{ND}$ to the conjunction of formulas of the form $\neg\chi_1 \vee\cdots\vee \neg\chi_l$. So $L - \mathbf{ML}$ contains a formula of the form $\neg\chi_1 \vee\cdots\vee \neg\chi_l$. Since L has DP, $\neg\chi_i \in L$ for some i. But then, by Glivenko's Theorem, $\neg\chi_i \in \mathbf{ML}$, which is a contradiction. ■

Remark. **ML** is not finitely axiomatizable, as was shown by Maksimova *et al.* [1979]. Nobody knows whether it is decidable.

It turns out, however, that **ML** is not the unique maximal logic with DP in Ext**Int**. Kirk [1982] noted that there is no greatest consistent si-logic with DP. Maksimova [1984] showed that there are infinitely many maximal constructive si-logics, and Chagrov [1992a] proved that in fact there are a continuum of them; see also Ferrari and Miglioli [1993, 1995a, 1995b]. Galanter [1990] claims that each si-logic characterized by the class of frames of the form

$$\langle \{W : W \subseteq \{1, \ldots, n\},\ W \neq \emptyset,\ |W| \notin N\}, \supseteq \rangle\,,$$

where $n = 1, 2, \ldots$ and N is some fixed infinite set of natural numbers, is a maximal si-logic with DP.

3.6 Intuitionistic Modal Logics

All modal logics we have dealt with so far were constructed on the classical non-modal basis. It can be replaced by logics of other types. For instance, one can consider modal logics based on relevant logic (see e.g. [Fuhrmann 1989]) or many-valued logics (see e.g. [Segerberg 1967], [Morikawa 1989], [Ostermann 1988]), and many others. In this section we briefly discuss modal logics with the intuitionistic basis.

Unlike the classical case, the intuitionistic $\Box$ and $\Diamond$ are not supposed to be dual, which provides more possibilities for defining intuitionistic modal logics. For a non-empty set M of modal operators, let $\mathcal{L}_{\mathsf{M}}$ be the standard propositional language augmented by the connectives in M. By an *intuitionistic modal logic* in the language $\mathcal{L}_{\mathsf{M}}$ we understand any subset of $\mathcal{L}_{\mathsf{M}}$ containing **Int** and closed under modus ponens, substitution and the regularity rule $\varphi \to \psi / \bigcirc \varphi \to \bigcirc \psi$, for every $\bigcirc \in \mathsf{M}$.

There are three ways of defining intuitionistic analogues of (classical) normal modal logics. First, one can take the family of logics extending the basic system $\mathbf{IntK}_{\Box}$ in the language $\mathcal{L}_{\Box}$ which is axiomatized by adding to **Int** the standard axioms of **K**

$$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q \text{ and } \Box\top.$$

An example of a logic in this family is Kuznetsov's [1985] intuitionistic provability logic $\mathbf{I}^{\triangle}$ (Kuznetsov used $\triangle$ instead of $\Box$), the intuitionistic analog of the provability logic **GL**. It can be obtained by adding to $\mathbf{IntK}_{\Box}$ (and even to **Int**) the axioms

$$p \to \Box p,\ (\Box p \to p) \to p,\ ((p \to q) \to p) \to (\Box q \to p).$$

A model theory for logics in NExt$\mathbf{IntK}_{\Box}$ was developed by Ono [1977], Božić and Došen [1984], Došen [1985a], Sotirov [1984] and Wolter and Zakharyaschev [1997, 1999a]; we discuss it below. Font [1984, 1986] considered these logics from the algebraic point of view, and Luppi [1996] investigated their interpolation property by proving, in particular, that the superamalgamability of the corresponding varieties of algebras is equivalent to interpolation.

A possibility operator $\Diamond$ in logics of this sort can be defined in the classical way by taking $\Diamond\varphi = \neg\Box\neg\varphi$. Note, however, that in general this $\Diamond$ does not distribute over disjunction and that the connection via negation between $\Box$ and $\Diamond$ is too strong from the intuitionistic standpoint (actually, the situation here is similar to that in intuitionistic predicate logic where $\exists$ and $\forall$ are not dual.)

Another family of "normal" intuitionistic modal logics can be defined in the language $\mathcal{L}_{\Diamond}$ by taking as the basic system the smallest logic in $\mathcal{L}_{\Diamond}$ to contain the axioms

$$\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q \text{ and } \neg\Diamond\bot;$$

it will be denoted by $\mathbf{IntK}_{\Diamond}$. Logics in NExt$\mathbf{IntK}_{\Diamond}$ were studied by Božić and Došen [1984], Došen [1985a], Sotirov [1984] and Wolter [1997e].

Finally, we can define intuitionistic modal logics with independent $\Box$ and $\Diamond$. These are extensions of $\mathbf{IntK}_{\Box\Diamond}$, the smallest logic in the language $\mathcal{L}_{\Box\Diamond}$ containing both $\mathbf{IntK}_{\Box}$ and $\mathbf{IntK}_{\Diamond}$. Fischer Servi [1980, 1984] constructed a logic in NExt$\mathbf{IntK}_{\Box\Diamond}$ by imposing a weak connection between the necessity and possibility operators:

$$\mathbf{FS} = \mathbf{IntK}_{\Box\Diamond} \oplus \Diamond(p \to q) \to (\Box p \to \Diamond q) \oplus (\Diamond p \to \Box q) \to \Box(p \to q).$$

A remarkable feature of **FS** is that the standard translation ST of modal formulas into first order ones (see *Correspondence Theory*) not only embeds **K** into classical predicate logic but also **FS** into intuitionistic first order logic: φ belongs to the former iff $ST(\varphi)$ is a theorem of the latter. According to Simpson [1994], this result was proved by C. Stirling; see also Grefe [1997].

Various extensions of **FS** were studied by Bull [1966a], Ono [1977], Fischer Servi [1977, 1980, 1984], Amati and Pirri [1994], Ewald [1986], Wolter and Zakharyaschev [1997], Wolter [1997e]. The best known one is probably the logic

$$\begin{aligned}\mathbf{MIPC} \quad = \quad & \mathbf{FS} \oplus \Box p \to p \oplus \Box p \to \Box\Box p \oplus \Diamond p \to \Box\Diamond p \oplus \\ & p \to \Diamond p \oplus \Diamond\Diamond p \to \Diamond p \oplus \Diamond\Box p \to \Box p\end{aligned}$$

introduced by Prior [1957]. Bull [1966a] noticed that the translation * defined by

$$\begin{aligned}&(p_i)^* = P_i(x), \ \ \bot^* = \bot, \\ &(\psi \odot \chi)^* = \psi^* \odot \chi^*, \text{ for } \odot \in \{\wedge, \vee, \to\}, \\ &(\Box\psi)^* = \forall x\ \psi^*, \ \ (\Diamond\psi)^* = \exists x\ \psi^*\end{aligned}$$

is an embedding of **MIPC** into the monadic fragment of intuitionistic predicate logic. Ono [1977], Ono and Suzuki [1988], Suzuki [1990], and Bezhanishvili [1998] investigated the relations between logics in NExt**MIPC** and superintuitionistic predicate logics induced by that translation.

In what follows we restrict attention only to the classes of intuitionistic modal logics introduced above. An interesting example of a system not covered here was constructed by Wijesekera [1990]. A general model theory for such logics is developed by Sotirov [1984] and Wolter and Zakharyaschev [1997].

Let us consider first the algebraic and relational semantics for the logics introduced above. All the semantical concepts to be defined below turn out to be natural combinations of the corresponding notions developed for classical modal and si-logics. For details and proofs we refer the reader to Wolter and Zakharyaschev [1997, 1999a].

From the algebraic point of view, every logic $L \in \mathrm{NExt}\mathbf{IntK}_{\mathsf{M}}$, for $\mathsf{M} \subseteq \{\Box, \Diamond\}$, corresponds to the variety of Heyting algebras with one or two operators validating L. The variety of algebras for $\mathbf{IntK}_{\mathsf{M}}$ will be called the *variety of* M*-algebras*.

To construct the relational representations of M-algebras, we define a $\Box$-*frame* to be a structure of the form $\langle W, R, R_\Box, P\rangle$ in which $\langle W, R, P\rangle$ is an intuitionistic frame, $R_\Box$ a binary relation on W such that

$$R \circ R_\Box \circ R \ = R_\Box$$

and P is closed under the operation

$$\Box X = \{x \in W : \forall y \in W\ (xR_\Box y \to y \in X)\}.$$

A $\Diamond$-*frame* has the form $\langle W, R, R_\Diamond, P\rangle$, where $\langle W, R, P\rangle$ is again an intuitionistic frame, $R_\Diamond$ a binary relation on W satisfying the condition

$$R^{-1} \circ R_\Diamond \circ R^{-1} = R_\Diamond$$

and P is closed under

$$\Diamond X = \{x \in W : \exists y \in X \; xR_\Diamond y\}.$$

Finally, a $\Box\Diamond$-*frame* is a structure $\langle W, R, R_\Box, R_\Diamond, P\rangle$ the unimodal reducts $\langle W, R, R_\Box, P\rangle$ and $\langle W, R, R_\Diamond, P\rangle$ of which are $\Box$- and $\Diamond$-frames, respectively. (To see why the intuitionistic and modal accessibility relations are connected by the conditions above the reader can construct in the standard way the canonical models for the logics under consideration. The important point here is that we take the Leibnizean definition of the truth-relation for the modal operators. Other definitions may impose different connecting conditions; see below.)

Given a $\Box\Diamond$-frame $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$, it is easy to check that its *dual*

$$\mathfrak{F}^+ = \langle P, \cap, \cup, \to, \emptyset, \Box, \Diamond\rangle$$

is a $\Box\Diamond$-algebra. Conversely, for each $\Box\Diamond$-algebra $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot, \Box, \Diamond\rangle$ we can define the *dual frame*

$$\mathfrak{A}_+ = \langle W, R, R_\Box, R_\Diamond, P\rangle$$

by taking $\langle W, R, P\rangle$ to be the dual of the Heyting algebra $\langle A, \wedge, \vee, \to, \bot\rangle$ and putting

$$\nabla_1 R_\Box \nabla_2 \text{ iff } \forall a \in A \; (\Box a \in \nabla_1 \to a \in \nabla_2),$$

$$\nabla_1 R_\Diamond \nabla_2 \text{ iff } \forall a \in A \; (a \in \nabla_2 \to \Diamond a \in \nabla_1).$$

$\mathfrak{A}_+$ is a $\Box\Diamond$-frame and, moreover, $\mathfrak{A} \cong (\mathfrak{A}_+)^+$. Using the standard technique of the model theory for classical modal and si-logics, one can show that a $\Box\Diamond$-frame $\mathfrak{F}$ is isomorphic to its bidual $(\mathfrak{F}^+)_+$ iff $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ is *descriptive*, i.e., $\langle W, R, P\rangle$ is a descriptive intuitionistic frame and, for all $x, y \in W$,

$$xR_\Box y \text{ iff } \forall X \in P \; (x \in \Box X \to y \in X),$$

$$xR_\Diamond y \text{ iff } \forall X \in P \; (y \in X \to x \in \Diamond X).$$

Thus we get the following completeness theorem.

THEOREM 187. *Every logic* $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$ *is characterized by a suitable class of (descriptive)* $\Box\Diamond$*-frames, e.g. by the class* $\{\mathfrak{A}_+ : \mathfrak{A} \models L\}$.

Similar results hold for logics in $\mathrm{NExt}\mathbf{IntK}_\Box$ and $\mathrm{NExt}\mathbf{IntK}_\Diamond$.

As usual, by a *Kripke frame* we understand a frame $\langle W, R, R_\Box, R_\Diamond, P\rangle$ in which P consists of all R-cones; in this case we omit P. An intuitionistic modal logic L is *$\mathcal{D}$-persistent* if the underlying Kripke frame of each descriptive frame for L validates L. For example, **FS** as well as the logics

$$\mathbf{L}(k,l,m,n) = \mathbf{IntK}_{\Box\Diamond} \oplus \Diamond^k\Box^l p \to \Box^m\Diamond^n p, \text{ for } k,l,m,n \geq 0$$

are $\mathcal{D}$-persistent and so Kripke complete (see Wolter and Zakharyaschev [1997]). Descriptive frames validating **FS** satisfy the conditions

$$xR_\Diamond y \;\to\; \exists z\ (yRz \wedge xR_\Box z \wedge xR_\Diamond z),$$

$$xR_\Box y \;\to\; \exists z\ (xRz \wedge zR_\Box y \wedge zR_\Diamond y),$$

and those for $\mathbf{L}(k,l,m,n)$ satisfy

$$xR_\Diamond^k y \wedge xR_\Box^m y \to \exists u\ (yR_\Box^l u \wedge zR_\Diamond^n u).$$

It follows, in particular, that **MIPC** is $\mathcal{D}$-persistent; its Kripke frames have the properties: $R_\Box$ is a quasi-order, $R_\Diamond = R_\Box^{-1}$ and $R_\Box = R \circ (R_\Box \cap R_\Diamond)$. On the contrary, $\mathbf{I}^\triangle$ is not $\mathcal{D}$-persistent, although it is complete with respect to the class of Kripke frames $\langle W, R, R_\Box\rangle$ such that $\langle W, R_\Box\rangle$ is a frame for **GL** and R the reflexive closure of $R_\Box$.

The next step in constructing duality theory of M-algebras and M-frames is to find relational counterparts of the algebraic operations of forming homomorphisms, subalgebras and direct products. Let $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ be a $\Box\Diamond$-frame and V a non-empty subset of W such that

$$\forall x \in V \forall y \in W\ (xR_\Box y \vee xRy \to y \in V),$$

$$\forall x \in V \forall y \in W\ (xR_\Diamond y \to \exists z \in V\ (xR_\Diamond z \wedge yRz)).$$

Then $\mathfrak{G} = \langle V, R \restriction V, R_\Box \restriction V, R_\Diamond \restriction V, \{X \cap V : X \in P\}\rangle$ is also a $\Box\Diamond$-frame which is called the *subframe of $\mathfrak{F}$ generated by V*. The former of the two conditions above is standard: it requires V to be upward closed with respect to both R and $R_\Box$. However, the latter one does not imply that V is upward closed with respect to $R_\Diamond$: the frame $\mathfrak{G}$ in Fig. 17 is a generated subframe of $\mathfrak{F}$, although the set $\{x, z\}$ is not an $R_\Diamond$-cone in $\mathfrak{F}$. This is one difference from the standard (classical modal or intuitionistic) case. Another one arises when we define the relational analog of subalgebras.

Given $\Box\Diamond$-frames $\mathfrak{F} = \langle W, R, R_\Box, R_\Diamond, P\rangle$ and $\mathfrak{G} = \langle V, S, S_\Box, S_\Diamond, Q\rangle$, we say a map f from W onto V is a *reduction* of $\mathfrak{F}$ to $\mathfrak{G}$ if $f^{-1}(X) \in P$ for every $X \in Q$ and, for all $x, y \in W$ and $u \in V$,

xRy implies $f(x)Sf(y)$,

$xR_\bigcirc y$ implies $f(x)S_\bigcirc f(y)$, for $\bigcirc \in \{\Box, \Diamond\}$,

$f(x)Su$ implies $\exists z \in f^{-1}(u)\ xRz$,

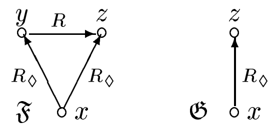


Figure 17.

Figure 18.

$f(x)S_\Box u$ implies $\exists z \in f^{-1}(u)\ xR_\Box z$,
$f(x)S_\Diamond u$ implies $\exists z \in W\ (xR_\Diamond z \wedge uSf(z))$.

Again, the last condition differs from the standard one: given $f(x)S_\Diamond f(y)$, in general we do not have a point z such that $xR_\Diamond z$ and $f(y) = f(z)$, witness the map gluing 0 and 1 in the frame $\mathfrak{F}$ in Fig. 18 and reducing it to $\mathfrak{G}$.

Note that both these concepts coincide with the standard ones in classical modal frames, where R and S are the diagonals. The relational counterpart of direct products—disjoint unions of frames—is defined as usual.

THEOREM 188. (i) *If $\mathfrak{G}$ is the subframe of a $\Box\Diamond$-frame $\mathfrak{F}$ generated by V then the map h defined by $h(X) = X \cap V$, for X an element in $\mathfrak{F}^+$, is a homomorphism from $\mathfrak{F}^+$ onto $\mathfrak{G}^+$.*

(ii) *If h is a homomorphism from a $\Box\Diamond$-algebra $\mathfrak{A}$ onto a $\Box\Diamond$-algebra $\mathfrak{B}$ then the map h_+ defined by $h_+(\nabla) = h^{-1}(\nabla)$, ∇ a prime filter in $\mathfrak{B}$, is an isomorphism from $\mathfrak{B}_+$ onto a generated subframe of $\mathfrak{A}_+$.*

(iii) *If f is a reduction of a $\Box\Diamond$-frame $\mathfrak{F}$ to a $\Box\Diamond$-frame $\mathfrak{G}$ then the map f^+ defined by $f^+(X) = f^{-1}(X)$, X an element in $\mathfrak{G}^+$, is an embedding of $\mathfrak{G}^+$ into $\mathfrak{F}^+$.*

(iv) *If $\mathfrak{B}$ is a subalgebra of a $\Box\Diamond$-algebra $\mathfrak{A}$ then the map f defined by $f(\nabla) = \nabla \cap B$, ∇ a prime filter in $\mathfrak{A}$ and B the universe of $\mathfrak{B}$, is a reduction of $\mathfrak{A}_+$ to $\mathfrak{B}_+$.*

This duality can be used for proving various results on modal definability. For instance, a class $\mathcal{C}$ of $\Box\Diamond$-frames is of the form $\mathcal{C} = \{\mathfrak{F} : \mathfrak{F} \models \Gamma\}$, for some set Γ of $\mathcal{L}_{\Box\Diamond}$-formulas, iff $\mathcal{C}$ is closed under the formation of generated subframes, reducts, disjoint unions, and both $\mathcal{C}$ and its complement are closed under the operation $\mathfrak{F} \mapsto (\mathfrak{F}^+)_+$ (see Wolter and Zakharyaschev [1997]). Moreover, one can extend Fine's Theorem connecting the first order definability and $\mathcal{D}$-persistence of classical modal logics to the intuitionistic modal case:

THEOREM 189. *If a logic* $L \in \mathrm{NExt}\mathbf{IntK}_{\Box\Diamond}$ *is characterized by an elementary class of Kripke frames then* L *is* $\mathcal{D}$*-persistent.*

These results may be regarded as a justification for the relational semantics introduced in this section. However, it is not the only possible one. For example, Božić and Došen [1984] impose a weaker condition on the connection between R and $R_\Box$ in $\Box$-frames. Fisher Servi [1980] interprets **FS** in birelational Kripke frames of the form $\langle W, R, S\rangle$ in which R is a partial order, $R \circ S \subseteq S \circ R$, and

$$xRy \wedge xSz \to \exists u\ (ySu \wedge zRu).$$

The intuitionistic connectives are interpreted by R and the truth-conditions for $\Box$ and $\Diamond$ are defined as follows

$$\Box X = \{x \in W : \forall y, z\ (xRySz \to z \in X\},$$

$$\Diamond X = \{x \in W : \exists y \in X\ xSy\}.$$

In birelational frames for **MIPC** S is an equivalence relation and

$$xSyRz \to \exists u\ xRuSz.$$

These frames were independently introduced by L. Esakia who also established duality between them and “monadic Heyting algebras”.

There are two ways of investigating various properties of intuitionistic modal logics. One is to continue extending the classical methods to logics in $\mathrm{NExt}\mathbf{IntK}_{\mathsf{M}}$. Another one uses those methods indirectly via embeddings of intuitionistic modal logics into classical ones. That such embeddings are possible was noticed by Shehtman [1979], Fischer Servi [1980, 1984], and Sotirov [1984]. Our exposition here follows Wolter and Zakharyaschev [1997, 1999a]. For simplicity we confine ourselves only to considering the class $\mathrm{NExt}\mathbf{IntK}_\Box$ and refer the reader to the cited papers for information about more general embeddings.

Let T be the translation of $\mathcal{L}_\Box$ into $\mathcal{L}_{\Box_I\Box}$ prefixing $\Box_I$ to every subformula of a given $\mathcal{L}_\Box$-formula. Thus, we are trying to embed intuitionistic modal logics in $\mathrm{NExt}\mathbf{IntK}_\Box$ into classical bimodal logics with the necessity operators $\Box_I$ (of **S4**) and $\Box$. Say that T *embeds* $L \in \mathrm{NExt}\mathbf{IntK}_\Box$ into $M \in \mathrm{NExt}(\mathbf{S4} \otimes \mathbf{K})$ (**S4** in $\mathcal{L}_{\Box_I}$ and **K** in $\mathcal{L}_\Box$) if, for every $\varphi \in \mathcal{L}_\Box$,

$$\varphi \in L \text{ iff } T(\varphi) \in M.$$

In this case M is called a *bimodal* (or BM-) *companion* of L.

For every logic $M \in \mathrm{NExt}(\mathbf{S4} \otimes \mathbf{K})$ put

$$\boldsymbol{\rho} M = \{\varphi \in \mathcal{L}_\Box : T(\varphi) \in M\},$$

and let $\boldsymbol{\sigma}$ be the map from NExt$\mathbf{IntK}_\Box$ into NExt$(\mathbf{S4} \otimes \mathbf{K})$ defined by

$$\boldsymbol{\sigma}(\mathbf{IntK}_\Box \oplus \Gamma) = (\mathbf{Grz} \otimes \mathbf{K}) \oplus \boldsymbol{mix} \oplus T(\Gamma),$$

where $\Gamma \subseteq \mathcal{L}_\Box$ and $\boldsymbol{mix} = \Box_I \Box \Box_I p \leftrightarrow \Box p$. (The axiom $\boldsymbol{mix}$ reflects the condition $R \circ R_\Box \circ R = R_\Box$ of $\Box$-frames.) Then we have the following extension of the embedding results of Maksimova and Rybakov [1974], Blok [1976] and Esakia [1979a,b]:

THEOREM 190. (i) *The map* $\boldsymbol{\rho}$ *is a lattice homomorphism from the lattice* NExt$(\mathbf{S4} \otimes \mathbf{K})$ *onto* NExt$\mathbf{IntK}_\Box$ *preserving decidability, Kripke completeness, tabularity and the finite model property.*

(ii) *Each logic* $\mathbf{IntK}_\Box \oplus \Gamma$ *is embedded by* T *into any logic* M *in the interval*

$$(\mathbf{S4} \otimes \mathbf{K}) \oplus T(\Gamma) \subseteq M \subseteq (\mathbf{Grz} \otimes \mathbf{K}) \oplus \boldsymbol{mix} \oplus T(\Gamma).$$

(iii) *The map* $\boldsymbol{\sigma}$ *is an isomorphism from the lattice* NExt$\mathbf{IntK}_\Box$ *onto the lattice* NExt$(\mathbf{Grz} \otimes \mathbf{K}) \oplus \boldsymbol{mix}$ *preserving FMP and tabularity.*

Note that Fischer Servi [1980] used another generalization of the Gödel translation. She defined

$$T(\Diamond\varphi) = \Diamond T(\varphi),$$
$$T(\Box\varphi) = \Box_I \Box T(\varphi)$$

and showed that this translation embeds **FS** into the logic

$$(\mathbf{S4} \otimes \mathbf{K}) \oplus \Diamond \Box_I p \to \Box_I \Diamond p \oplus \Diamond \Diamond_I p \to \Diamond_I \Diamond p.$$

It is not clear, however, whether all extensions of **FS** can be embedded into classical bimodal logics via this translation.

Let us turn now to completeness theory of intuitionistic modal logics. As to the standard systems $\mathbf{I}^\triangle$, **FS**, and **MIPC**, their FMP can be proved by using (sometimes rather involved) filtration arguments; see Muravitskij [1981], Simpson [1994] and Grefe [1997], and Ono [1977], respectively. Further results based on the filtration method were obtained by Sotirov [1984] and Ono [1977]. However, in contrast to classical modal logic, only a few general completeness results covering interesting classes of intuitionistic modal logics are known. The proofs of the following two theorems are based on the translation into classical bimodal logics discussed above.

THEOREM 191. *Suppose that a si-logic* $\mathbf{Int} + \Gamma$ *has one of the properties: decidability, Kripke completeness, FMP. Then the logics* $\mathbf{IntK}_\Box \oplus \Gamma$ *and* $\mathbf{IntK}_\Box \oplus \Gamma \oplus \Box p \to p$ *also have the same property.*

Proof. It suffices to show that there is a BM-companion of each of these systems satisfying the corresponding property. Notice that

$$\boldsymbol{\rho}((\mathbf{S4} \oplus T(\Gamma)) \otimes \mathbf{K}) = \mathbf{IntK}_\Box \oplus \Gamma,$$

$$\boldsymbol{\rho}((\mathbf{S4} \oplus T(\Gamma)) \otimes (\mathbf{K} \oplus \Box p \to p)) = \mathbf{IntK}_\Box \oplus \Gamma \oplus \Box p \to p.$$

So it remains to use the fact that if $\mathbf{Int} + \Gamma$ has one of the properties under consideration then its smallest modal companion $\mathbf{S4} \oplus T(\Gamma)$ has this property as well (Table 7), and if L_1, L_2 are unimodal logics having one of those properties then the fusion $L_1 \otimes L_2$ also enjoys the same property (Theorem 111). ■

Such a simple reduction to known results in classical modal logic is not available for logics containing $\mathbf{IntK4}_\Box = \mathbf{IntK}_\Box \oplus \Box p \to \Box\Box p$. However, by extending Fine's [1974] method of maximal points to bimodal companions of extensions of $\mathbf{IntK4}_\Box$ Wolter and Zakharyaschev [1999a] proved the following:

THEOREM 192. *Suppose $L \supseteq \mathbf{IntK4}_\Box$ has a $\mathcal{D}$-persistent BM-companion $M \supseteq (\mathbf{S4} \otimes \mathbf{K4}) \oplus \boldsymbol{mix}$ whose Kripke frames are closed under the formation of substructures. Then*

(i) *for every set Γ of intuitionistic negation and disjunction free formulas, $L \oplus \Gamma$ has FMP;*

(ii) *for every set Γ of intuitionistic disjunction free formulas and every $n \geq 1$,*

$$L \oplus \Gamma \oplus \bigvee_{i=0}^{n} (p_i \to \bigvee_{j \neq i} p_j)$$

has the finite model property.

One can use this result to show that the following (and many other) intuitionistic modal logics enjoy FMP:

(1) $\mathbf{IntK4}_\Box$;

(2) $\mathbf{IntS4}_\Box = \mathbf{IntK4}_\Box \oplus \Box p \to p$ ($R_\Box$ is reflexive);

(3) $\mathbf{IntS4.3}_\Box = \mathbf{IntS4}_\Box \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$ ($R_\Box$ is reflexive and connected);

(4) $\mathbf{IntK4}_\Box \oplus p \vee \Box\neg\Box p$ ($R_\Box$ is symmetrical);

(5) $\mathbf{IntK4}_\Box \oplus \Box p \vee \Box\neg\Box p$ ($R_\Box$ is Euclidean);

(6) $\mathbf{IntK4}_\Box \oplus \Box p \vee \neg\Box p$ ($xRy \wedge xR_\Box z \to yR_\Box z$).

We conclude this section with some remarks on lattices of intuitionistic modal logics. Wolter [1997e] uses duality theory to study splittings of lattices of intuitionistic modal logics. For example, he showed that each finite rooted frame splits $\mathrm{NExt}(L \oplus \Box^{\leq n} p \to \Box^{n+1} p)$, for $L = \mathbf{IntK}_\Box$ and $L = \mathbf{FS}$, and each $R_\Box$-cycle free finite rooted frame splits the lattices of extensions of $\mathbf{IntK}_\Box$ and $\mathbf{FS}$. No positive results are known, however, for the lattice $\mathrm{NExt}\mathbf{IntK}_\Diamond$. In fact, the behavior of $\Diamond$-frames is quite different from that of frames for $\mathbf{FS}$. For instance, in classical modal logic we have $\mathsf{RG}\mathcal{F} = \mathsf{GR}\mathcal{F}$, for each class of frames (or even $\Box$-frames) $\mathcal{F}$, where G and R are the operations of forming generated subframes and reducts, respectively.

But this does not hold for $\Diamond$-frames. More precisely, there exists a finite $\Diamond$-frame $\mathfrak{G}$ such that $\mathsf{RG}\{\mathfrak{G}\} \not\supseteq \mathsf{GR}\{\mathfrak{G}\}$. In other terms, the variety of modal algebras for **K** has the *congruence extension property* (i.e., each congruence of a subalgebra of a modal algebra can be extended to a congruence of the algebra itself) but this is not the case for the variety of $\Diamond$-algebras.

Vakarelov [1981, 1985] and Wolter [1997e] investigate how logics having **Int** as their non-modal fragment are located in the lattices of intuitionistic modal logics. It turns out, for instance, that in NExt$\mathbf{IntK}_\Diamond$ the inconsistent logic has a continuum of immediate predecessors all of which have **Int** as their non-modal fragment, but no such logic exists in the lattice of extensions of $\mathbf{IntK}_\Box$.

4 ALGORITHMIC PROBLEMS

All algorithmic results considered in the previous sections were positive: we presented concrete procedures for deciding whether an arbitrary given formula belongs to a given logic in some class or whether it axiomatizes a logic with a certain property. What is the complexity of those decision algorithms? Do there exist undecidable calculi[18] and properties? These are the main questions we address in this chapter.

4.1 Undecidable calculi

The first undecidable modal and si-calculi were constructed by Thomason [1975c] (polymodal and unimodal), Isard [1977] (unimodal) and Shehtman [1978c] (superintuitionistic). However, we begin with the very simple example of [Shehtman 1982] which is a modal reformulation of the undecidable associative calculus T of [Tseitin 1958]. The axioms of T are

$$\begin{array}{ll} ac = ca, & ad = da, \\ bc = cb, & bd = db, \\ edb = be, & eca = ae, \\ abac = abacc. & \end{array}$$

The reader will notice immediately an analogy between them and the axioms of the following modal calculus with five necessity operators:

$$\begin{aligned} L = \mathbf{K}_5 \oplus\ & \Box_1\Box_3 p \leftrightarrow \Box_3\Box_1 p \oplus \Box_1\Box_4 p \leftrightarrow \Box_4\Box_1 p \oplus \\ & \Box_2\Box_3 p \leftrightarrow \Box_3\Box_2 p \oplus \Box_2\Box_4 p \leftrightarrow \Box_4\Box_2 p \oplus \\ & \Box_5\Box_4\Box_2 p \leftrightarrow \Box_2\Box_5 p \oplus \Box_5\Box_3\Box_1 p \leftrightarrow \Box_1\Box_5 p \oplus \\ & \Box_1\Box_2\Box_1\Box_3 p \leftrightarrow \Box_1\Box_2\Box_1\Box_3\Box_3 p. \end{aligned}$$

[18] By a calculus we mean a logic with finitely many axioms (inference rules in our case are fixed).

Moreover, it is not hard to see that words x, y in the alphabet $\{a, b, c, d, e\}$ are equivalent in T[19] iff $f(x)p \leftrightarrow f(y)p \in \mathbf{K}_5$, where f is the natural one-to-one correspondence between such words and modalities in language $\{\Box_1, \dots, \Box_5\}$ under which, for instance, $f(cadedb) = \Box_3\Box_1\Box_4\Box_5\Box_4\Box_2$. It follows immediately that L is undecidable. Using the undecidable associative calculus of Matiyasevich [1967], one can construct in the same way an undecidable bimodal calculus having three reductions of modalities as its axioms. It is unknown whether there is an undecidable unimodal calculus axiomatizable by reductions of modalities.

Another simple way of proving undecidability, known as the *domino* or *tiling technique*, was suggested by Harel [1983]. It is particularly useful in the case of multi-dimensional modal logics, say Cartesian products.

Tiles can be thought of as 4-tuples of colours

$$t = \langle left(t), right(t), up(t), down(t) \rangle \,.$$

A finite set T of tiles is said to *tile* $\mathbb{N} \times \mathbb{N}$ if there is a map $\tau : \mathbb{N} \times \mathbb{N} \mapsto T$ such that for all $i, j \in \mathbb{N}$,

- $up(\tau(i, j)) = down(\tau(i, j + 1))$ and
- $right(\tau(i, j)) = left(\tau(i + 1, j))$.

If we think of a tile as a physical 1×1-square with colours along its four edges, then a tiling of $\mathbb{N} \times \mathbb{N}$ is just a way of placing an infinite number of tiles, each of a type from T, together to cover the first quarter of the infinite plane, with no rotation of the tiles allowed and the colours on adjacent edges of adjacent tiles matching.

The *tiling problem* for $\mathbb{N} \times \mathbb{N}$ is formulated as follows: "given a finite set T of tiles, does T tile $\mathbb{N} \times \mathbb{N}$?" Robinson [1971] proved that this problem is undecidable (in fact, co-r.e.-complete).

We will demonstrate the use of tiling to show the undecidability of the logic $(\mathbf{K} \times \mathbf{K})_u$, i.e., the square of $\mathbf{K}$ (with boxes $\boxminus$ and $\boxbar$) extended with the universal modality $\Box$ (see Section 2.2); this result is due to Spaan [1993].

Given a finite set T of tiles, construct a formula φ_T as the conjunction of the following formulas:

$$\begin{array}{l}
\Box \bigvee_{t \in T} p_t, \\
\Box \bigwedge_{t \neq t'} \neg(p_t \wedge p_{t'}), \\
\Box \bigwedge_{t \in T}(p_t \to \bigvee_{up(t)=down(t')} \boxbar p_{t'}), \\
\Box \bigwedge_{t \in T}(p_t \to \bigvee_{right(t)=left(t')} \boxminus p_{t'}), \\
\Box(\diamondbar \top \wedge \diamondminus \top).
\end{array}$$

[19] I.e., they can be obtained from each other by a finite number of transformations of the form $w_1ww_2 \to w_1vw_2$, where $w = v$ or $v = w$ is an axiom of T.

It is easily seen (see e.g. [Spaan 1993] or [Marx 1999]) that φ_T is satisfiable in the product of two frames iff T tiles $\mathbb{N} \times \mathbb{N}$. It follows that $(\mathbf{K} \times \mathbf{K})_u$ is undecidable.

Thomason's simulation and the undecidable polymodal calculi mentioned above provide us with examples of undecidable calculi in NExt$\mathbf{K}$. However, to find axioms of undecidable unimodal calculi with transitive frames, as well as undecidable si-calculi, a more sophisticated construction is required.

Instead of associative calculi, let us use now Minsky machines with two tapes (or register machines with two registers). A *Minsky machine* is a finite set (program) of instructions for transforming triples $\langle s, m, n\rangle$ of natural numbers, called *configurations*. The intended meaning of the current configuration $\langle s, m, n\rangle$ is as follows: s is the number (label) of the current machine state and m, n represent the current state of information. Each instruction has one of the four possible forms:

$$s \to \langle t, 1, 0\rangle\,,\ \ s \to \langle t, 0, 1\rangle\,,$$

$$s \to \langle t, -1, 0\rangle\,(\langle t', 0, 0\rangle),\ \ s \to \langle t, 0, -1\rangle\,(\langle t', 0, 0\rangle).$$

The last of them, for instance, means: transform $\langle s, m, n\rangle$ into $\langle t, m, n-1\rangle$ if $n > 0$ and into $\langle t', m, n\rangle$ if $n = 0$. For a Minsky machine $\boldsymbol{P}$, we shall write $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ if starting with $\langle s, m, n\rangle$ and applying the instructions in $\boldsymbol{P}$, in finitely many steps (possibly, in 0 steps) we can reach $\langle t, k, l\rangle$.

We shall use the well known fact (see e.g. [Mal'cev 1970]) that the following *configuration problem* is undecidable: given a program $\boldsymbol{P}$ and configurations $\langle s, m, n\rangle$, $\langle t, k, l\rangle$, determine whether $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$.

With every program $\boldsymbol{P}$ and configuration $\langle s, m, n\rangle$ we associate the transitive frame $\mathfrak{F}$ depicted in Fig. 19. Its points $e(t, k, l)$ represent configurations $\langle t, k, l\rangle$ such that $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$; $e(t, k, l)$ sees the points a^0_t, a^1_k, a^2_l representing the components of $\langle t, k, l\rangle$. The following variable free formulas characterize points in $\mathfrak{F}$ in the sense that each of these formulas, denoted by Greek letters with subscripts and/or superscripts, is true in $\mathfrak{F}$ only at the point denoted by the corresponding Roman letter with the same subscript and/or superscript:

$$\alpha = \Diamond\top \wedge \Box\Diamond\top,\ \ \beta = \Box\bot,\ \ \gamma = \Diamond\alpha \wedge \Diamond\beta \wedge \neg\Diamond^2\beta,$$

$$\delta = \neg\gamma \wedge \Diamond\beta \wedge \neg\Diamond^2\beta,\ \ \delta_1 = \Diamond\delta \wedge \neg\Diamond^2\delta,\ \ \delta_2 = \Diamond\delta_1 \wedge \neg\Diamond^2\delta_1,$$

$$\gamma_1 = \Diamond\gamma \wedge \neg\Diamond^2\gamma \wedge \neg\Diamond\delta,\ \ \gamma_2 = \Diamond\gamma_1 \wedge \neg\Diamond^2\gamma_1 \wedge \neg\Diamond\delta,$$

$$\alpha^0_0 = \Diamond\gamma \wedge \Diamond\delta \wedge \neg\Diamond^2\gamma \wedge \neg\Diamond^2\delta,$$

$$\alpha^1_0 = \Diamond\gamma_1 \wedge \Diamond\delta_1 \wedge \neg\Diamond^2\gamma_1 \wedge \neg\Diamond^2\delta_1,$$

$$\alpha^2_0 = \Diamond\gamma_2 \wedge \Diamond\delta_2 \wedge \neg\Diamond^2\gamma_2 \wedge \neg\Diamond^2\delta_2,$$

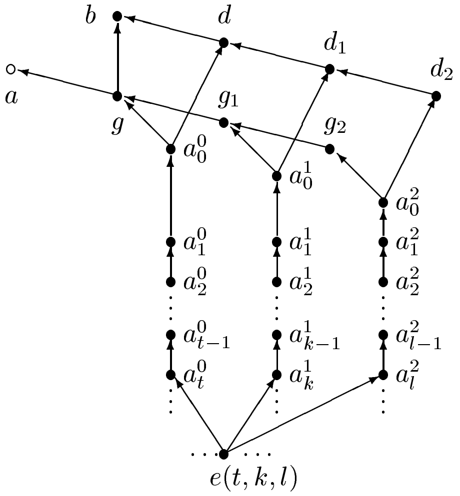


Figure 19.

$$\alpha_{j+1}^i = \Diamond\alpha_j^i \wedge \neg\Diamond^2\alpha_j^i \wedge \bigwedge_{i\neq k} \neg\Diamond\alpha_0^k,$$

where $i \in \{0,1,2\}$, $j \geq 0$. The formulas characterizing $e(t,k,l)$ are denoted by $\epsilon(t,\alpha_k^1,\alpha_l^2)$, where

$$\epsilon(t,\varphi,\psi) = \bigwedge_{i=0}^{t} \Diamond\alpha_i^0 \wedge \neg\Diamond\alpha_{t+1}^0 \wedge \Diamond\varphi \wedge \neg\Diamond^2\varphi \wedge \Diamond\psi \wedge \neg\Diamond^2\psi.$$

We require also formulas characterizing not only fixed but arbitrary configurations:

$$\pi_1 = (\Diamond\alpha_0^1 \vee \alpha_0^1) \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^2 \wedge p_1 \wedge \neg\Diamond p_1,$$
$$\pi_2 = \Diamond\alpha_0^1 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^2 \wedge \Diamond p_1 \wedge \neg\Diamond^2 p_1,$$
$$\tau_1 = (\Diamond\alpha_0^2 \vee \alpha_0^2) \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge p_2 \wedge \neg\Diamond p_2,$$
$$\tau_2 = \Diamond\alpha_0^2 \wedge \neg\Diamond\alpha_0^0 \wedge \neg\Diamond\alpha_0^1 \wedge \Diamond p_2 \wedge \neg\Diamond^2 p_2.$$

Now we are fully equipped to simulate the behavior of Minsky machines by means of modal formulas. Let us consider for simplicity only tense logics and observe that $\mathfrak{F}$ satisfies the condition

$$\forall x \forall y \exists z\ (xRzR^{-1}y \vee xR^{-1}zRy \vee xRy \vee xR^{-1}y \vee x = y).$$

So, for every valuation in $\mathfrak{F}$, a formula φ is true at some point in $\mathfrak{F}$ iff the formula

$$\bigcirc\varphi = \Diamond\Diamond^{-1}\varphi \vee \Diamond^{-1}\Diamond\varphi \vee \Diamond\varphi \vee \Diamond^{-1}\varphi \vee \varphi$$

is true at all points in $\mathfrak{F}$, i.e., the modal operator $\bigcirc$ can be understood as "omniscience". Let χ be a formula which is refuted in $\mathfrak{F}$ and does not contain p_1 and p_2. With each instruction I in $\boldsymbol{P}$ we associate a formula AxI by taking:

$$AxI = \neg\chi \wedge \bigcirc\epsilon(t, \pi_1, \tau_1) \to \neg\chi \wedge \bigcirc\epsilon(t', \pi_2, \tau_1)$$

if I has the form $t \to \langle t', 1, 0\rangle$,

$$AxI = \neg\chi \wedge \bigcirc\epsilon(t, \pi_1, \tau_1) \to \neg\chi \wedge \bigcirc\epsilon(t', \pi_1, \tau_2)$$

if I is $t \to \langle t', 0, 1\rangle$,

$$\begin{aligned} AxI = \quad & (\neg\chi \wedge \bigcirc\epsilon(t, \pi_2, \tau_1) \to \neg\chi \wedge \bigcirc\epsilon(t', \pi_1, \tau_1)) \wedge \\ & (\neg\chi \wedge \bigcirc\Diamond\epsilon(t, \alpha_0^1, \tau_1) \to \neg\chi \wedge \bigcirc\epsilon(t'', \alpha_0^1, \tau_1)) \end{aligned}$$

if I is $t \to \langle t', -1, 0\rangle\,(\langle t'', 0, 0\rangle)$,

$$\begin{aligned} AxI = \quad & (\neg\chi \wedge \bigcirc\epsilon(t, \pi_1, \tau_2) \to \neg\chi \wedge \bigcirc\epsilon(t', \pi_1, \tau_1)) \wedge \\ & (\neg\chi \wedge \bigcirc\epsilon(t, \pi_1, \alpha_0^2) \to \neg\chi \wedge \bigcirc\epsilon(t'', \pi_1, \alpha_0^2)) \end{aligned}$$

if I is $t \to \langle t', 0, -1\rangle\,(\langle t'', 0, 0\rangle)$. The formula simulating $\boldsymbol{P}$ as a whole is

$$AxP = \bigwedge_{I \in \boldsymbol{P}} AxI.$$

Now, by induction on the length of computations and using the frame $\mathfrak{F}$ in Fig. 19 one can show that for every program $\boldsymbol{P}$ and configurations $\langle s, m, n\rangle$, $\langle t, k, l\rangle$, we have $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ iff

$$\neg\chi \wedge \bigcirc\epsilon(s, \alpha_m^1, \alpha_n^2) \to \neg\chi \wedge \bigcirc\epsilon(t, \alpha_k^1, \alpha_l^2) \in \mathbf{K4}.t \oplus AxP.$$

Thus, if the configuration problem is undecidable for $\boldsymbol{P}$ then the tense calculus $\mathbf{K4}.t \oplus AxP$ is undecidable too. In the same manner (but using somewhat more complicated frames and formulas) one can construct undecidable calculi in NExt**K4** and even Ext**Int**; for details consult [Chagrova, 1991] and [Chagrov and Zakharyaschev, 1997]. The following table presents some "quantitative characteristics" of known undecidable calculi in various classes of logics. Its first line, for instance, means that there is an undecidable si-calculus with axioms in 4 variables and the derivability problem in it is undecidable in the class of formulas in 2 variables; = means that the number of variables is optimal, and $\leq$ indicates that the optimal number is still unknown.

Class of logics	The number of variables in undecidable calculi	separated formulas
Ext**Int**	$\leq 4,\ \geq 2$	$= 2$
NExt**S4**	$\leq 3,\ \geq 2$	$= 1$
Ext**S4**	≤ 3	$= 1$
NExt**GL**	$= 1$	$= 1$
Ext**GL**	$= 1$	$= 1$
Ext**S**	$= 1$	$= 1$
NExt**K4**	$= 1$	$= 0$
Ext**K4**	$= 1$	$= 0$

These observations follow from [Anderson, 1972; Chagrov, 1994; Sobolev, 1977a] and [Zakharyaschev, 1997a]. Say that a formula ψ is *undecidable* in (N)Ext$L$ if no algorithm can determine for an arbitrary given φ whether $\psi \in L + \varphi$ (respectively, $\psi \in L \oplus \varphi$). For example, formulas in one variable, the axioms of $\mathbf{BW}_n$ and $\mathbf{BD}_n$ are decidable in Ext**Int**. On the other hand, there are purely implicative undecidable formulas in Ext**Int**, and

$$\neg(p \wedge q) \vee \neg(\neg p \wedge q) \vee \neg(p \wedge \neg q) \vee \neg(\neg p \wedge \neg q)$$

is the shortest known undecidable formula in this class. Here are some modal examples: the formula $\Box(\Box^2\bot \to \Box p \vee \Box\neg p)$ is undecidable in NExt**GL**, $\Box^+\neg\Box^+ p \vee \Box^+\neg\Box^+\neg\Box^+ p$ in Ext**S**, $\bot$ in Ext**K4** and NExt**K4**.t; in NExt**K** and NExt**K4**.t undecidable is the conjunction of axioms of any consistent tabular logic in these classes. However, no non-trivial criteria are known for a formula to be decidable; it is unclear also whether one can effectively recognize the decidability of formulas in the classes Ext**Int**, (N)Ext**S4**, (N)Ext**GL**, Ext**S**, (N)Ext**K4**.

4.2 Admissibility and derivability of inference rules

Another interesting algorithmic problem for a logic L is to determine whether an arbitrary given inference rule $\varphi_1, \ldots, \varphi_n/\varphi$ is *derivable* in L, i.e., φ is derivable in L from the assumptions $\varphi_1, \ldots, \varphi_n$, and whether it is *admissible* in L, i.e., for every substitution $\boldsymbol{s}$, $\varphi\boldsymbol{s} \in L$ whenever $\varphi_1\boldsymbol{s}, \ldots, \varphi_n\boldsymbol{s} \in L$. (Note that derivability depends on the postulated inference rules in L, while admissibility depends only on the set of formulas in L.) Admissible and derivable rules are used for simplifying the construction of derivations. Derivable rules, like the well known rule of syllogism

$$\frac{\varphi \to \psi,\ \psi \to \chi}{\varphi \to \chi},$$

may replace some fragments of fixed length in derivations, thereby shortening them linearly. Admissible rules in principle may reduce derivations more drastically. Since $\varphi \in L$ iff the rule $\top/\varphi$ is derivable (or admissible) in L, the derivability and admissibility problems for inference rules may be regarded as generalizations of the decidability problem.

If the only postulated rules in L are substitution and modus ponens, the Deduction Theorem reduces the derivability problem for inference rules in L to its decidability:

$$\frac{\varphi_1, \dots, \varphi_n}{\psi} \text{ is derivable in } L \text{ iff } \varphi_1 \wedge \dots \wedge \varphi_n \to \psi \in L.$$

However, if the rule of necessitation $\varphi/\Box\varphi$ is also postulated in L, we have only

$$\frac{\varphi_1, \dots, \varphi_n}{\psi} \text{ is derivable in } L \text{ iff } \varphi_1, \dots, \varphi_n \vdash_L^* \psi.$$

For n-transitive L this is equivalent to $\Box^{\leq n}(\varphi_1 \wedge \dots \wedge \varphi_n) \to \psi \in L$, and so the derivability problem for inference rules in n-transitive logics is decidable iff the logics themselves are decidable. In general, in view of the existential quantifier in Theorem 1, the situation is much more complicated.

Notice first that similarly to Harrop's Theorem, a sufficient condition for the derivability problem to be decidable in a calculus is its global FMP (see Section 1.5). Thus we have

THEOREM 193. *The derivability problem for inference rules in* **K**, **T**, **D**, **KB** *is decidable.*

Moreover, sometimes we can obtain an upper bound for the parameter m in the Deduction Theorem, which also ensures the decidability of the derivability problem for inference rules. One can prove, for instance, that for **K** it is enough to take $m = 2^{|\mathbf{Sub}\varphi \cup \mathbf{Sub}\psi|}$. In general, however, the derivability problem for inference rules in a logic L turns out to be more complex than the decidability problem for L. (Recall, by the way, that there are logics with FMP but not global FMP.)

THEOREM 194 (Spaan 1993). *There is a decidable calculus in* NExt**K** *the derivability problem for inference rules in which is undecidable.*

Spaan proves this result by simulating in $\vdash_L^*$, L the decidable logic defined below, the tiling problem for $\mathbb{N} \times \mathbb{N}$. The logic L is surprisingly simple:

$$L = \mathbf{Alt}_2 \oplus \bigwedge_{1 \leq i \leq 4} \Diamond\Diamond p_i \to \bigvee_{1 \leq i < j \leq 4} \Diamond\Diamond(p_i \wedge p_j).$$

It is a subframe logic, so it is $\mathcal{D}$-persistent and has FMP (because $\mathbf{Alt}_2 \subseteq L$; see Theorem 22 and Proposition 59). Note also that the bimodal logic L_u (see Section 2.2) is a complete and elementary subframe logic which is undecidable because $\vdash_L^*$ is undecidable. Using this observation one can construct a unimodal subframe logic in NExt**K** with the same properties.

Let us turn now to the admissibility problem. It is not hard to see that the rules

$$\frac{(\neg\neg p \to p) \to p \vee \neg p}{\neg p \vee \neg\neg p} \quad \text{and} \quad \frac{\neg p \to q \vee r}{(\neg p \to q) \vee (\neg p \to r)}$$

are admissible but not derivable in **Int** and $\Diamond p \wedge \Diamond\neg p/\bot$ is admissible but not derivable in any extension of **S4.3** save those containing $\Box\Diamond p \to \Diamond\Box p$, in which it is derivable. (Recall that a logic L is said to be *structurally complete* if every admissible inference rule in L is derivable in L. We have just seen that **Int** as well as **S4.3** are not structurally complete. For more information on structural completeness see e.g. [Tsytkin 1978, 1987] and [Rybakov 1995].) The following result strengthens Fine's [1971] Theorem according to which all logics in Ext**S4.3** are decidable.

THEOREM 195 (Rybakov 1984a). *The admissibility problem for inference rules is decidable in every logic containing* **S4.3**.

An impetus for investigations of admissible inference rules in various logics was given by Friedman's [1975] problem 40 asking whether one can effectively recognize admissible rules in **Int**. This problem turned out to be closely connected to the admissibility problem in suitable modal logics. We demonstrate this below for the logic **GL** following [Rybakov 1987, 1989].

First we show that dealing with logics in NExt**K**, it is sufficient to consider inference rules of a rather special form. Let $\varphi(q_1, \dots, q_{2n+2})$ be a formula containing no $\Box$ and $\Diamond$ and represented in the full disjunctive normal form. Say that an inference rule is *reduced* if it has the form

$$\varphi(p_0, \dots, p_n, \Diamond p_0, \dots, \Diamond p_n)/p_0.$$

THEOREM 196. *For every rule φ/ψ one can effectively construct a reduced rule φ'/ψ' such that φ/ψ is admissible in a logic $L \in$ NExt**K** iff φ'/ψ' is admissible in L.*

Proof. Observe first that if φ and ψ do not contain p then φ/ψ is admissible in L iff $\varphi \wedge (\psi \leftrightarrow p)/p$ is admissible in L. So we can consider only rules of the form φ/p_0. Besides, without loss of generality we may assume that φ does not contain $\Box$. With every non-atomic subformula χ of φ we associate the new variable p_χ. For convenience we also put $p_\chi = p_i$ if $\chi = p_i$ and $p_\chi = \bot$ if $\chi = \bot$. We show now that the rule

$$p_\varphi \wedge \bigwedge\{p_\chi \leftrightarrow p_{\chi_1} \odot p_{\chi_2} : \chi = \chi_1 \odot \chi_2 \in \mathbf{Sub}\varphi,\ \odot \in \{\wedge, \vee, \to\}\} \wedge$$
$$\bigwedge\{p_\chi \leftrightarrow \Diamond p_{\chi_1} :\ \chi = \Diamond\chi_1 \in \mathbf{Sub}\varphi\}/p_0$$

is admissible in L iff φ/p_0 is admissible in L. For brevity we denote the antecedent of that rule by φ''.

($\Rightarrow$) Since every substitution instance of φ''/p_0 is admissible in L, the rule $\varphi \wedge \bigwedge_{\chi\in\ \mathbf{Sub}\varphi}(\chi \leftrightarrow \chi)/p_0$ and so φ/p_0 are also admissible in L.

($\Leftarrow$) Suppose φ/p_0 is admissible in L and $\varphi''\boldsymbol{s}$ is in L, for some substitution $\boldsymbol{s} = \{\alpha_\chi/p_\chi : \chi \in \mathbf{Sub}\varphi\}$. By induction on the construction of χ one can readily show that $\alpha_\chi \leftrightarrow \chi\boldsymbol{s} \in L$. Therefore, $\alpha_\varphi \leftrightarrow \varphi\boldsymbol{s} \in L$. Since $\varphi''\boldsymbol{s} \in L$, we must have $p_\varphi s = \alpha_\varphi \in L$, from which $\varphi\boldsymbol{s} \in L$ and so $p_0\boldsymbol{s} \in L$. Thus φ''/p_0 is admissible in L.

The rule φ''/p_0 is not reduced, but it is easy to make it so simply by representing φ'' in its full disjunctive normal form φ', treating subformulas $\Diamond p_i$ as variables. ■

From now on we will deal with only reduced rules different from $\bot/p_0$ (which is clearly admissible in any logic). Let $\bigvee_j \varphi_j/p_0$ be a reduced rule in which every disjunct φ_j is the conjunction of the form

(17) $\neg_0 p_0 \wedge \cdots \wedge \neg_m p_m \wedge \neg^0\Diamond p_0 \wedge \cdots \wedge \neg^m\Diamond p_m,$

where each $\neg_i$ and $\neg^j$ is either blank or $\neg$. We will identify such conjunctions with the sets of their conjuncts. Now, given a non-empty set W of conjunctions of the form (17), we define a frame $\mathfrak{F} = \langle W, R\rangle$ and a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ by taking

$$\begin{aligned}\varphi_i R \varphi_j \quad \text{iff} \quad & \forall k \in \{0,\dots,m\}(\neg\Diamond p_k \in \varphi_i \to \neg\Diamond p_k \in \varphi_j \wedge \neg p_k \in \varphi_j) \wedge \\ & \exists k \in \{0,\dots,m\}(\neg\Diamond p_k \in \varphi_j \wedge \Diamond p_k \in \varphi_i),\end{aligned}$$

$$\mathfrak{V}(p_k) = \{\varphi_i \in W : \ p_k \in \varphi_i\}.$$

It should be clear that $\mathfrak{F}$ is finite, transitive and irreflexive.

THEOREM 197. *A reduced rule $\bigvee_j \varphi_j/p_0$ is not admissible in* **GL** *iff there is a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ defined as above on a set W of conjunctions of the form (17) and such that*

(i) *$\neg p_0 \in \varphi_i$ for some $\varphi_i \in W$;*

(ii) *$\varphi_i \models \varphi_i$ for every $\varphi_i \in W$;*

(iii) *for every antichain $\mathfrak{a}$ in $\mathfrak{F}$ there is $\varphi_j \in W$ such that, for every $k \in \{0,\dots,m\}$, $\varphi_j \models \Diamond p_k$ iff $\varphi_i \models \Diamond^+ p_k$ for some $\varphi_i \in \mathfrak{a}$.*

Proof. ($\Rightarrow$) We are given that there are formulas $\psi_0,\dots,\psi_m$ in variables $q_1,\dots,q_n$ such that $\bigvee_j \varphi_j^* \in$ **GL** and $p_0^* \notin$ **GL**, where by χ^* we denote $\chi\{\psi_0/p_0,\dots,\psi_m/p_m\}$. This is equivalent to $\mathfrak{M}_{\mathbf{GL}}(n) \models \bigvee_j \varphi_j^*$ and $\mathfrak{M}_{\mathbf{GL}}(n) \not\models p_0^*$. Define W to be the set of those disjuncts φ_j in $\bigvee_j \varphi_j$ whose substitution instances φ_j^* are satisfied in $\mathfrak{M}_{\mathbf{GL}}(n)$. Clearly $W \neq \emptyset$. Let us check (i) – (iii).

(i) Take a point x in $\mathfrak{M}_{\mathbf{GL}}(n)$ at which p_0^* is false. As $\mathfrak{M}_{\mathbf{GL}}(n) \models \bigvee_j \varphi_j^*$, we must have $x \models \varphi_i^*$ for some i. One of the formulas p_0^* or $\neg p_0^*$ is a conjunct of φ_i^*. Clearly it is not p_0^*. Therefore, $\neg p_0 \in \varphi_i$.

(ii) It suffices to show that, for all $\varphi_i \in W$ and $k \in \{0, \ldots, m\}$, $\varphi_i \models \Diamond p_k$ iff $\Diamond p_k \in \varphi_i$. Suppose $\varphi_i \models \Diamond p_k$. Then there is $\varphi_j \in W$ such that $\varphi_i R \varphi_j$ and $\varphi_j \models p_k$. By the definition of $\mathfrak{V}$ and R, this means that $p_k \in \varphi_j$ and $\Diamond p_k \in \varphi_i$. Conversely, suppose $\Diamond p_k \in \varphi_i$. Then $x \models \varphi_i^*$ and in particular $x \models \Diamond p_k^*$ for some x in $\mathfrak{M}_{\mathbf{GL}}(n)$. Let y be a final point in the set $\{z \in x\uparrow : z \models p_k^*\}$. Since $\mathfrak{M}_{\mathbf{GL}}(n)$ is irreflexive, we have $y \models p_k^*$, $y \not\models \Diamond p_k^*$ and $y \models \varphi_j^*$ for some $\varphi_j \in W$. It follows that $\varphi_i R \varphi_j$ and $\varphi_j \models p_k$, from which $\varphi_i \models \Diamond p_k$.

(iii) Let $\mathfrak{a}$ be an antichain in $\mathfrak{F}$. For every $\varphi_i \in \mathfrak{a}$, let x_i be a final point in the set $\{y \in W_{\mathbf{GL}}(n) : y \models \varphi_i^*\}$. It should be clear that the points $\{x_i : \varphi_i \in \mathfrak{a}\}$ form an antichain $\mathfrak{b}$ in $\mathfrak{F}_{\mathbf{GL}}(n)$ and so, by the construction of $\mathfrak{F}_{\mathbf{GL}}(n)$, there is a point y in $\mathfrak{F}_{\mathbf{GL}}(n)$ such that $y\uparrow = \mathfrak{b}\underline{\uparrow}$. Then the formula $\varphi_j \in W$ we are looking for is any one satisfying the condition $y \models \varphi_j^*$, as can be easily checked by a straightforward inspection.

($\Leftarrow$) The proof in this direction is rather technical; we confine ourselves to just a few remarks. Let $\mathfrak{M}$ be a model satisfying (i)–(iii). To prove that $\bigvee_j \varphi_j / p_0$ is not admissible in **GL** we require once again the n-universal model $\mathfrak{M}_{\mathbf{GL}}(n)$, but this time we take n to be the number of symbols in the rule. By induction on the depth of points in $\mathfrak{M}$ one can show that $\mathfrak{M}$ is a generated submodel of $\mathfrak{M}_{\mathbf{GL}}(n)$.

Our aim is to find formulas $\psi_0, \ldots, \psi_m$ such that $\mathfrak{M}_{\mathbf{GL}}(n) \models \bigvee_j \varphi_j^*$ and $\mathfrak{M}_{\mathbf{GL}}(n) \not\models p_0^*$ (here again $\chi^* = \chi\{\psi_0/p_0, \ldots, \psi_m/p_m\}$). Loosely, we need to extend the properties of $\mathfrak{M}$ to the whole model $\mathfrak{M}_{\mathbf{GL}}(n)$. To this end we can take the sets $\{\varphi_i\}$ in $\mathfrak{F}_{\mathbf{GL}}(n)$ and augment them inductively in such a way that we could embrace all points in $\mathfrak{F}_{\mathbf{GL}}(n)$. At the induction step we use the condition (iii), and the required $\psi_0, \ldots, \psi_m$ are constructed with the help of (i) and (ii); roughly, they describe in $\mathfrak{M}_{\mathbf{GL}}(n)$ the analogues of the truth-sets in $\mathfrak{M}$ of the variables in our rule. ■

A remarkable feature of this criterion is that it can be effectively checked. Thus we have

THEOREM 198. *There is an algorithm which, given an inference rule, can decide whether it is admissible in* **GL**.

In a similar way one can prove

THEOREM 199 (Rybakov 1987). *The admissibility problem in* **Grz** *is decidable.*

We show now that the admissibility problem in **Int** can be reduced to the same problem in **Grz** and so is also decidable. To this end we require the following

THEOREM 200 (Rybakov 1984b). *A rule φ/ψ is admissible in* **Int** *iff the rule $T(\varphi)/T(\psi)$ is admissible in* **Grz**.

As a consequence of Theorems 199 and 200 we obtain

THEOREM 201 (Rybakov 1984b). *The admissibility problem in* **Int** *is decidable.*

Although there are many other examples of logics in which the admissibility problem is decidable and the scheme of establishing decidability is quite similar to the argument presented above,[20] proofs are rather difficult and only in few cases they work for big families of logics as in [Rybakov 1994]. Besides, all these results hold only for extensions of **K4** and **Int**. For logics with non-transitive frames, even for **K**, the admissibility problem is still waiting for a solution. The same concerns polymodal, in particular tense logics. Chagrov [1992b] constructed a decidable infinitely axiomatizable logic in NExt**K4** for which the admissibility problem is undecidable. It would be of interest to find modal and si-calculi of that sort.

A close algorithmic problem for a logic L is to determine, given an arbitrary formula $\varphi(p_1, \dots, p_n)$, whether there exist formulas $\psi_1, \dots, \psi_n$ such that $\varphi(\psi_1, \dots, \psi_n) \in L$. Note that an "equation" $\varphi(p_1, \dots, p_n)$ has a solution in L iff the rule $\varphi(p_1, \dots, p_n)/\bot$ is not admissible in L. This observation and Theorem 195 provide us with examples of logics in which the substitution problem is decidable (see e.g. [Rybakov 1993]). We do not know, however, if there is a logic such that the substitution problem in it is decidable, while the admissibility one is not.

The inference rules we have dealt with so far were *structural* in the sense that they were "closed" under substitution. An interesting example of a nonstructural rule was considered by Gabbay [1981a]:

$$\frac{\varphi \vee (\Box p \to p), \text{ where } p \notin \mathbf{Sub}\varphi}{\varphi}.$$

It is readily seen that this rule holds in a frame $\mathfrak{F}$ (in the sense that for every formula φ and every variable p not occurring in φ, φ is valid in $\mathfrak{F}$ whenever $(\Box p \to p) \vee \varphi$ is valid in $\mathfrak{F}$) iff $\mathfrak{F}$ is irreflexive and that **K** is closed under it (since **K** is characterized by the class of irreflexive frames). We refer the reader to [Venema 1991] and [Marx and Venema 1997] for more information about rules of this type.

4.3 Properties of recursively axiomatizable logics

Dealing with infinite classes of logics, we can regard questions like "Is a logic L decidable?", "Does L have FMP?", etc., as mass algorithmic problems. But to formulate such problems properly we should decide first how

[20] Quite recently S. Ghilardi [1999a,b] has found another way of recognizing admissibility of inference rules. He showed that certain si- and modal logics L (in particular, **Int**, **K4**, **S4**, **GL**, **Grz**) have the following property. Given an L-consistent formula φ, one can effectively compute substitutions $\sigma_1, \dots, \sigma_n$ such that $\sigma_i \varphi \in L$ for every $i = 1, \dots, n$, and if $\sigma\varphi \in L$ for some substitution σ, then σ is, up to provable equivalence, an instantiation of some of the σ_i. A rule φ/ψ is then admissible in L iff $\sigma_i \psi \in L$ for all $i = 1, \dots, n$.

to represent the input data of algorithms recognizing properties of logics. One can, for instance, consider the class of recursively axiomatizable logics (which, by Craig's [1953] Theorem, coincides with that of recursively enumerable ones) and represent them as programs generating their axioms. However, this approach turns out to be too general because the following analog of the Rice–Uspenskij Theorem holds.

THEOREM 202 (Kuznetsov). *No nontrivial property of recursively axiomatizable si-logics is decidable.*

Of course, nothing will change if we take some other family of logics, say NExt**K4**. The proof of this theorem (Kuznetsov left it unpublished) is very simple; we give it even in a more general form than required.

PROPOSITION 203. *Suppose L_1 and L_2 are logics in some family $\mathcal{L}$, L_1 is recursively axiomatizable, $L_1 \subset L_2$, L_2 is finitely axiomatizable (say, by a formula γ), and a property $\mathcal{P}$ holds for only one of L_1, L_2. Then no algorithm can recognize $\mathcal{P}$, given a program enumerating axioms of a logic in $\mathcal{L}$.*

Proof. Let $\alpha_0, \alpha_1, \dots$ be a recursive sequence of axioms for L_1. Given an arbitrary (Turing, Minsky, Pascal, etc.) program $\boldsymbol{P}$ having natural numbers as its input, we define the following recursive sequence of formulas (where $(n)_1$ and $(n)_2$ are the first and second components of the pair of natural numbers with code n under some fixed effective encoding):

$$\beta_n = \begin{cases} \alpha_n & \text{if } \boldsymbol{P} \text{ does not come to a stop on input } (n)_1 \text{ in } (n)_2 \text{ steps} \\ \gamma & \text{otherwise.} \end{cases}$$

This sequence axiomatizes L_1 if $\boldsymbol{P}$ does not come to a stop on any input and L_2 otherwise. It is well known in recursion theory that the halting problem is undecidable, and so the property $\mathcal{P}$ is undecidable in $\mathcal{L}$ as well. ■

The reader must have already noticed that this proof has nothing to do with modal and si-logics; it is rather about effective computations. To avoid this unpleasant situation let us confine ourselves to the smaller class of *finitely axiomatizable* modal and si-logics and try to find algorithms recognizing properties of the corresponding calculi. However, even in this case we should be very careful. If arbitrary finite axiomatizations are allowed then we come across the following

THEOREM 204 (Kuznetsov 1963). *For every finitely axiomatizable si-logic L (in particular,* **Int**, **Cl**, *inconsistent logic), there is no algorithm which, given an arbitrary finite list of formulas, can determine whether its closure under substitution and modus ponens coincides with L.*

Needless to say that the same holds for (normal) modal logics as well. Fortunately, the situation is not so hopeless if we consider finite axiomatizations over some basic logics. For instance, by Makinson's Theorem, one can effectively recognize, given a formula φ, whether the logic $\mathbf{K} \oplus \varphi$ is consistent. Other examples of decidable properties in various lattices of modal logics were presented in Theorems 89, 93, 101, and 142. In the next section we consider those properties that turn out to be undecidable in various classes of modal and si-calculi.

4.4 Undecidable properties of calculi

The first "negative" algorithmic results concerning properties of modal calculi were obtained by Thomason [1982] who showed that FMP and Kripke completeness are undecidable in NExt$\mathbf{K}$, and consistency is undecidable in NExt$\mathbf{K}.t$. Later Thomason's discovery has been extended to other properties and narrower classes of logics. In fact, a good many standard properties of modal and si-calculi (in reasonably big classes) proved to be undecidable; decidable ones are rather exceptional.

In this section we present three known schemes of proving such kind of undecidability results. Each of them has its advantages (as well as disadvantages) and can be adjusted for various applications. The first one is due to Thomason [1982].

Let $L(n)$ be a recursive sequence of normal bimodal calculi such that no algorithm can decide, given n, whether $L(n)$ is consistent. Such sequences, as we shall see a bit later, exist even in NExt$\mathbf{K4}.t$. Suppose also that L^* is a normal unimodal calculus which does not have some property, say, FMP, decidability or Kripke completeness. Consider now the recursive sequence of logics $L(n) \otimes L^*$ with three necessity operators. If $L(n)$ is inconsistent then the fusion $L(n) \otimes L^*$ is inconsistent too and so has the properties mentioned above. And if $L(n)$ is consistent then, in accordance with Proposition 110, $L(n) \otimes L^*$ is a conservative extension of both $L(n)$ and L^*, which means that it is Kripke incomplete, undecidable and does not have FMP whenever L^* is so. Consequently, the three properties under consideration cannot be decidable in the class NExt$\mathbf{K}_3$, for otherwise the consistency of $L(n)$ would be decidable. By Theorem 123, these properties are undecidable in NExt$\mathbf{K}$ as well. Note however that, since Thomason's simulation embeds polymodal logics only into "non-transitive" unimodal ones, this very simple scheme does not work if we want to investigate algorithmic aspects of properties of calculi in NExt$\mathbf{K4}$ and Ext$\mathbf{Int}$.

To illustrate the second scheme let us recall the construction of the undecidable calculus in NExt$\mathbf{K4}.t$ discussed in Section 4.1. First, we choose a Minsky program $\boldsymbol{P}$ and a configuration $\mathfrak{a} = \langle s, m, n\rangle$ so that no algorithm can decide, given a configuration $\mathfrak{b}$, whether $\boldsymbol{P} : \mathfrak{a} \to \mathfrak{b}$. (That they exist is shown in [Chagrov 1990b].) Then we put $\chi = \bot$ and add to $\mathbf{K4}.t \oplus AxP$ one more axiom

$$(\neg\chi \wedge \bigcirc\epsilon(s, \alpha_m^1, \alpha_n^2) \to \neg\chi \wedge \bigcirc\epsilon(t, \alpha_k^1, \alpha_l^2)) \to \chi,$$

where $\mathfrak{c} = \langle t, k, l \rangle$ is an arbitrary fixed configuration. The resulting calculus is denoted by $L(\mathfrak{c})$. Suppose that $\boldsymbol{P} : \mathfrak{a} \not\to \mathfrak{c}$. Then one can readily check that the new axiom is valid in the frame $\mathfrak{F}$ shown in Fig. 19 and prove that $\boldsymbol{P} : \langle s, m, n \rangle \to \langle t', k', l' \rangle$ iff

$$\neg\chi \wedge \bigcirc\epsilon(s, \alpha^1_m, \alpha^2_n) \to \neg\chi \wedge \bigcirc\epsilon(t', \alpha^1_{k'}, \alpha^2_{l'}) \in L(\mathfrak{c}).$$

Therefore, $L(\mathfrak{c})$ is undecidable, consistent and does not have FMP. And if $\boldsymbol{P} : \mathfrak{a} \to \mathfrak{c}$ then $L(\mathfrak{c})$ is clearly inconsistent. It follows by the choice of $\boldsymbol{P}$ and $\mathfrak{a}$ that consistency, decidability and FMP are undecidable in NExt$\mathbf{K4}.t$. In fact, the argument will change very little if we take as χ the axiom of some tabular logic in NExt$\mathbf{K4}.t$. So we obtain

THEOREM 205. *The properties of tabularity and coincidence with an arbitrary fixed tabular logic (in particular, inconsistent) are undecidable in* NExt$\mathbf{K4}.t$

Moreover, these results (except the consistency problem, of course) can be transferred to logics in NExt$\mathbf{K}$. We demonstrate this by an example; complete proofs can be found in [Chagrov 1996].

We require the frame which results from that in Fig. 19 by adding to it a reflexive point c_0 and an irreflexive one c_1 so that c_1 sees all other points save a and b and is seen itself only from a and b. As before, we denote the frame by $\mathfrak{F}$.

PROPOSITION 206. *Let χ be a formula refutable at some point in $\mathfrak{F}$ different from c_0 and $\Diamond\top \in \mathbf{K} \oplus \chi$. Then the problem of deciding, for an arbitrary formula φ, whether $\mathbf{K} \oplus \varphi = \mathbf{K} \oplus \chi$ is undecidable.*

Proof. It should be clear that χ contains at least one variable, say r, and there are points in $\mathfrak{F}$ at which r has distinct truth-values (under the valuation refuting χ); c_0 and c_1 are then the only points in $\mathfrak{F}$ where the formulas $\sigma_0 = \Box^3 r \vee \Box^3\neg r$ and

$$\sigma_1 = \Diamond\sigma_0 \wedge (r \vee \Diamond r \vee \Diamond^2 r) \wedge (\neg r \vee \Diamond\neg r \vee \Diamond^2\neg r)$$

are true, respectively. Observe that from every point in $\mathfrak{F}$ save c_0 we can reach all points in $\mathfrak{F}$ by ≤ 3 steps. So we can take $\bigcirc = \Diamond^{\leq 3}$. The formulas α and β should be replaced with $\alpha = \Diamond\sigma_1 \wedge \Diamond^2\sigma_1$, $\beta = \Diamond\sigma_1 \wedge \neg\Diamond^2\sigma_1$ which (under the valuation refuting χ) are true only at a and b, respectively. Now consider the logic

$$L(\mathfrak{c}) = \mathbf{K} \oplus AxP \oplus (\neg\chi \wedge \bigcirc\epsilon(s, \alpha^1_m, \alpha^2_n) \to \neg\chi \wedge \bigcirc\epsilon(t, \alpha^1_k, \alpha^2_l)) \to \chi.$$

If $\boldsymbol{P} : \mathfrak{a} \to \mathfrak{c}$ then $L(\mathfrak{c}) = \mathbf{K} \oplus \chi$. And if $\boldsymbol{P} : \mathfrak{a} \not\to \mathfrak{c}$ then, using the fact that the set of points in $\mathfrak{F}$ where χ is refutable coincides with the set of points from which every point of the form $e(x, y, z)$ is accessible by three steps, one can show that $\mathfrak{F} \models L(\mathfrak{c})$ and so $L(\mathfrak{c}) \neq \mathbf{K} \oplus \chi$. ■

Putting, for instance, $\chi = \Box p \leftrightarrow p$, we obtain then that the problem of coincidence with Log∘ is undecidable in NExt**K**. Likewise one can prove the following

THEOREM 207. (i) *If a consistent finitely axiomatizable logic L is not a union-splitting of* NExt**K** *then the axiomatization problem for L above* **K** *is undecidable.*

(ii) *The properties of tabularity and coincidence with an arbitrary fixed consistent tabular logic are undecidable in* NExt**K**.

(iii) *The problem of coincidence with an arbitrary fixed consistent calculus in* NExt**D4** *or in* NExt**GL** *is undecidable in* NExt**K**.

(iv) *The properties of tabularity and coincidence with an arbitrary fixed tabular (in particular, inconsistent) logic are undecidable in* Ext**K4**.

Of the algorithmic problems concerning tabularity that remain open the most intriguing are undoubtedly the tabularity and local tabularity problems in NExt**K4**. Note that a positive solution to the former implies a positive solution to the latter.

Now we present the second scheme in a more general form used in [Chagrov 1990b] and [Chagrov and Zakharyaschev 1993]. Assume again that the second configuration problem is undecidable for $\boldsymbol{P}$ and $\mathfrak{a}$, and let χ be a formula such that $L_0 \oplus \chi$ has some property $\mathcal{P}$, where L_0 is the minimal logic in the class under consideration. Associate with $\boldsymbol{P}$, $\mathfrak{a}$ and a configuration $\mathfrak{b}$ formulas AxP and $\psi(\mathfrak{a}, \mathfrak{b})$ such that $\psi(\mathfrak{a}, \mathfrak{b}) \in L_0 \oplus AxP$ iff $\boldsymbol{P} : \mathfrak{a} \to \mathfrak{b}$. Besides, χ and AxP are chosen so that $AxP \in L_0 \oplus \chi$. Now consider the calculus

$$L(\mathfrak{b}) = L_0 \oplus AxP \oplus \psi(\mathfrak{a}, \mathfrak{b}) \to \chi \oplus \gamma,$$

where γ is some formula such that $\gamma \in L_0 \oplus \chi$. If $\boldsymbol{P} : \mathfrak{a} \to \mathfrak{b}$ then we clearly have $L(\mathfrak{b}) = L_0 \oplus \chi$ and so $L(\mathfrak{b})$ has $\mathcal{P}$; but if $\boldsymbol{P} : \mathfrak{a} \not\to \mathfrak{b}$ then the fact that $L(\mathfrak{b})$ does not have $\mathcal{P}$ must be ensured by an appropriate choice of γ. (In the considerations above we did not need γ, i.e., it was sufficient to put $\gamma = \top$). With the help of this scheme one can prove the following

THEOREM 208. (i) *The properties of decidability, Kripke completeness as well as FMP are undecidable in the classes* Ext**Int**, (N)Ext**Grz**, (N)Ext**GL**.

(ii) *The interpolation property is undecidable in* (N)Ext**GL**.

(iii) *Halldén completeness is undecidable in* Ext**Int**, (N)Ext**Grz**, Ext**S**.

These and some other results of that sort can be found in [Chagrov 1990b,c, 1994, 1996], [Chagrova 1991], [Chagrov and Zakharyaschev 1993, 1995b].

The third scheme was developed in [Chagrova 1989, 1991] and [Chagrov and Chagrova 1995] for establishing the undecidability of certain first order properties of modal calculi (or formulas). The difference of this scheme from the previous one is that now we use calculi of the form

$$L(\mathfrak{b}) = L_0 \oplus AxP \oplus \psi(\mathfrak{a}, \mathfrak{b}) \vee \gamma,$$

where AxP satisfies one more condition besides those mentioned above: it must be first order definable on Kripke frames for L_0. If $P : \mathfrak{a} \to \mathfrak{b}$ then the formula $AxP \wedge (\psi(\mathfrak{a}, \mathfrak{b}) \vee \gamma)$ is equivalent to AxP in the class of Kripke frames for L_0 and so is first order definable on that class or its any subclass. And if $P : \mathfrak{a} \not\to \mathfrak{b}$ then by choosing an appropriate γ one can show that $AxP \wedge (\psi(\mathfrak{a}, \mathfrak{b}) \vee \gamma)$ is not first order definable on, say, countable Kripke frames for L_0, as in [Chagrova 1989], or on finite frames for L_0, as in [Chagrov and Chagrova 1995]. In this way the following theorem is proved:

THEOREM 209. (i) *No algorithm is able to recognize the first order definability of modal formulas on the class of Kripke frames for* **S4** *and even the first order definability on countable (finite) Kripke frames for* **S4**. *The properties of first order definability and definability on countable (finite) Kripke frames of intuitionistic formulas are undecidable as well.*

(ii) *The set of modal or intuitionistic formulas that are first order definable on countable (finite) frames but are not first order definable on the class of all (respectively, countable) Kripke frames mentioned in (i) is undecidable.*

We conclude this section with two remarks. First, all undecidability results above can be formulated in the stronger form of recursive inseparability. For instance, the set of inconsistent calculi in NExt**K4**.t and the set of calculi without FMP are recursively inseparable. And second, some properties are not only undecidable but the families of calculi having them are not recursively enumerable; for example, the set of consistent calculi in NExt**K4**.t is not enumerable. However, for the majority of other properties the problem of enumerability of the corresponding calculi is open.

4.5 Semantical consequence

So far we have dealt with only syntactical formalizations of logical entailment. However, sometimes a semantical approach is preferable. Say that a formula φ is a *semantical consequence* of a formula ψ in a class of frames $\mathcal{C}$ if φ is valid in all frames in $\mathcal{C}$ validating ψ. (One can consider also the local, i.e., point-wise variant of this relation.) Note that φ is a consequence of ψ in the class of, say, Kripke frames for **S4** iff φ is a consequence of $(\Box p \to \Box^2 p) \wedge (\Box p \to p) \wedge \psi$ in the class of all Kripke frames. But the consequence relation on finite frames is not expressible by modal formulas (as was shown in [Chagrov 1995], if $(\Box p \to \Box^2 p) \wedge \varphi$ is valid in arbitrarily large finite rooted frames then it is valid in some infinite rooted frame as well).

In parallel with constructing and proving the undecidability of modal and si-calculi we can obtain the following

THEOREM 210. *The semantical consequence relation in the class of all (***K4***-,* **S4***-,* **Int***-) Kripke frames is undecidable. Moreover, if* $\models$ *denotes one* *of these relations then there is a formula ψ (a formula φ) such that the set $\{\varphi : \psi \models \varphi\}$ is undecidable.*

In a sense, formulas ψ and φ, for which $\{\varphi : \psi \models \varphi\}$ is undecidable are analogous to undecidable calculi and formulas, respectively. However, this analogy is far from being perfect: for every formula ψ, the sets $\{\varphi : \psi \vdash \varphi\}$ and $\{\varphi : \psi \vdash^* \varphi\}$ are recursively enumerable, which contrasts with

THEOREM 211 (Thomason 1975a). *There exists a formula ψ such that $\{\varphi : \psi \models \varphi\}$ is a Π^1_1-complete set.*

Unfortunately, Thomason's [1974b, 1975b, 1975c] results have not been transferred so far to transitive frames, although this does not seem to be absolutely impossible.

Chagrov [1990a] (see also [Chagrov and Chagrova 1995]) developed a technique for proving the analog of Theorem 210 for the consequence relation on all (**K4**-, **S4**-, **GL**-, **Int**-) finite frames. Moreover, since this relation is clearly enumerable, instead of "undecidable" one can use "not enumerable".

4.6 Complexity problems

Having proved that a given logic is decidable, we are facing the problem of finding an optimal (in one sense or another) decision algorithm for it. The complexity of decision algorithms for many standard modal and si-logics is determined by the size of minimal frames separating formulas from those logics. For instance, as was shown by Jaśkowski (1936) and McKinsey (1941), for every $\varphi \notin \mathbf{S4}$ (or $\varphi \notin \mathbf{Int}$) there is a frame $\mathfrak{F} \models \mathbf{S4}$ with $\leq 2^{|\mathbf{Sub}\varphi|}$ points such that $\mathfrak{F} \not\models \varphi$. The same upper bound is usually obtained by the standard filtration. Is it possible to reduce the exponential upper bound to the polynomial one? This question was raised by Kuznetsov [1975] for **Int**. It turned out, however, that it concerns not only **Int**. First, Kuznetsov observed (for the proof see [Kuznetsov 1979]) that if the answer to his question is positive, i.e., **Int** has polynomial FMP, then the problem "Are **Int** and **Cl** polynomially equivalent?" has a positive solution as well. (Logics L_1 and L_2 are *polynomially equivalent* if there are polynomial time transformations f and g of formulas such that $\varphi \in L_1$ iff $f(\varphi) \in L_2$ and $\varphi \in L_2$ iff $g(\varphi) \in L_1$.) Then Statman [1979] showed that the problem "$\varphi \in$ **Int**?" is $\boldsymbol{PSPACE}$-complete and so Kuznetsov's problem is equivalent to one of the "hopeless" complexity problems, namely "$\boldsymbol{NP} = \boldsymbol{PSPACE}$?".

Complexity function

For a logic L with FMP, we introduce the *complexity function*

$$f_L(n) = \max_{\substack{l(\varphi)\leq n \\ \varphi\notin L}} \min_{\substack{\mathfrak{F}\models L \\ \mathfrak{F}\not\models\varphi}} |\mathfrak{F}|,$$

where $l(\varphi)$, the *length* of φ, is the number of subformulas in φ and $|\mathfrak{F}|$ the number of points in $\mathfrak{F}$. If there is a constant c such that

$$f_L(n) \leq 2^{c \cdot n} \text{ (or } f_L(n) \leq n^c \text{ or } f_L(n) \leq c \cdot n),$$

L is said to have the *exponential* (respectively, *polynomial* or *linear*) *finite model property.* The following result shows that **Int** does not have polynomial FMP.

THEOREM 212 (Zakharyaschev and Popov 1979). $\log_2 f_{\mathbf{Int}}(n) \asymp n$.

Proof. The exponential upper bound is well known and to establish the lower one it is sufficient to use the formulas

$$\beta_n = \bigwedge_{i=1}^{n-1} ((\neg p_{i+1} \to q_{i+1}) \vee (p_{i+1} \to q_{i+1}) \to q_i) \to (\neg p_1 \to q_1) \vee (p_1 \to q_1).$$

It is not hard to see that $\beta_n \notin \mathbf{Int}$ and every refutation frame for β_n contains the full binary tree of depth n as a subframe. ■

Likewise the same result can be proved for many other standard superintuitionistic and modal logics whose FMP is established by the usual filtration and whose frames contain full binary trees of arbitrary finite depth. Such are, for instance, **KC**, **SL**, **K4**, **S4**, **GL**. In the case of **K** the length of formulas that play the role of β_n is not a linear but a square function of n, which means that $f_{\mathbf{K}}(n) \geq 2^{\sqrt{c \cdot n}}$, for some constant $c > 0$, and so **K** does not have polynomial FMP either. As was shown in [Zakharyaschev 1996], all cofinal subframe modal and si-logics have exponential FMP. It seems plausible that $\log_2 f_L(n) \asymp n$ for every consistent si-logic L different from **Cl** and axiomatizable by formulas in one variable.

The construction of Theorem 212 does not work for logics whose frames do not contain arbitrarily large full binary trees. Such are, for instance, logics of finite width or of finite depth, and the following was proved in [Chagrov 1983].

THEOREM 213. (i) *The minimal logics of width $n < \omega$ in the classes* NExt**K4**, NExt**S4**, NExt**Grz**, NExt**GL**, Ext**Int** *have polynomial FMP.*

(ii) **Lin** *and all logics containing* **S4.3** *have linear FMP.*

(iii) *The minimal logics of depth n in* NExt**Grz**, NExt**GL**, Ext**Int** *have polynomial FMP, with the power of the corresponding polynomial $\leq n - 1$.*

(iv) *The minimal logics of depth n in* NExt**K4**, NExt**S4** *have polynomial FMP, with the power of the corresponding polynomial $\leq n$.*

Proof. (i) is proved by two filtrations. First, with the help of the standard filtration one constructs a finite frame separating a formula φ from the given logic L and then, using the selective filtration, extracts from it a polynomial

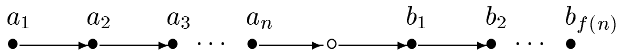


Figure 20.

separation frame: it suffices to take a point refuting φ and all maximal points at which ψ is false, for some $\Box\psi \in \mathbf{Sub}\varphi$ (in the intuitionistic case $\psi \to \chi \in \mathbf{Sub}\varphi$ should be considered). (ii) is proved analogously.

To illustrate the proof of (iii) and (iv), we consider the minimal logic L of depth 3 in NExt**GL**. Suppose $\varphi \notin L$. Then there is a transitive irreflexive model $\mathfrak{M}$ of depth ≤ 3 refuting φ at its root r. Let $\Box\psi_i$, for $1 \leq i \leq m$, be all "boxed" subformulas of φ. For every $i \in \{1, \dots, m\}$, we choose a point refuting ψ_i, if it exists. And then we do the same in the set $x\uparrow$, for every chosen point x. Let $\mathfrak{M}'$ be the submodel formed by the selected points and r. Clearly, it contains at most $1 + m + m^2$ points. And by induction on the construction of formulas in $\mathbf{Sub}\varphi$ one can easily show that $\mathfrak{M}'$ refutes φ at r.

To prove the lower bound one can use the formulas

$$\begin{aligned}\alpha_n \quad = \quad & \neg(\bigwedge_{i=1}^{n} \Box(p_{i+1} \to p_i) \wedge \bigwedge_{i=1}^{n} \Box(q_{i+1} \to q_i) \wedge \\ & \bigwedge_{i=1}^{n} \Diamond(\Diamond\top \wedge \Box^+(\neg p_{i+1} \wedge p_i)) \wedge \Box(\Diamond\bot \to \bigwedge_{i=1}^{n} \Diamond(\neg q_{i+1} \wedge q_i)))\end{aligned}$$

which are not in L and every separation frame for which contains the full n-ary tree of depth 3, i.e., at least $1 + n + n^2$ points. ■

However, even if frames for a logic with FMP do not contain full finite binary trees its complexity function can grow very fast, witness the following result of [Chagrov 1985a].

THEOREM 214. *For every arithmetic function $f(n)$, there are logics L of width 1 in* NExt**K4** *and of width* 2 *in* Ext**Int**, NExt**Grz**, NExt**GL** *having FMP and such that $f_L(n) \geq f(n)$.*

Proof. We construct a logic $L \in$ NExt**K4.3** whose complexity function grows faster than a given increasing arithmetic function $f(n)$. Define L to be the logic of all frames of the form shown in Fig. 20. To see that L satisfies the property we need, consider the sequence of formulas

$$\beta_1 = p_1 \vee \Box(\Box p_1 \to (\Box(\Box p \to p) \to p)),$$

$$\beta_{i+1} = p_{i+1} \vee \Box(\Box p_{i+1} \to \beta_i).$$

Since these formulas are refuted at points of the form a_j in sufficiently large frames depicted in Fig. 20, they are not in L. And since L contains the formulas

$$\neg\beta_n \to \Diamond(\Diamond^{f(n)-1}\top \wedge \Box^{f(n)}\bot),$$

β_n cannot be separated from L by a frame with $\leq f(n)$ points. ■

For logics of finite depth this theorem does not hold, since according to the description of finitely generated universal frames in Section 1.2, for every $L \in \mathrm{NExt}\mathbf{K4BD}_k$ ($k \geq 3$), we have

$$f_L(n) \leq \left.2^{2^{\cdot^{\cdot^{2^{c\cdot n}}}}}\right\} k-2$$

for some constant $c > 0$. And as was shown in [Chagrov 1985a], one cannot in general reduce this upper bound.

THEOREM 215. *For every $k \geq 3$, there are logics L of depth k in* NExt**Grz***,* NExt**GL***,* Ext**Int** *such that*

$$f_L(n) \geq \left.2^{2^{\cdot^{\cdot^{2^{n}}}}}\right\} k-2 \quad .$$

Proof. We illustrate the proof for $k = 3$ in NExt**GL**. Let L be the logic characterized by the class of rooted frames $\mathfrak{F}_m$ for **GL** of depth 3 defined as follows. $\mathfrak{F}_m$ contains m dead ends, every non-empty set of them has a focus, i.e., a point that sees precisely the dead ends in this set, and besides the root there are no other points in $\mathfrak{F}_m$. It should be clear that L does not contain the formulas

$$\gamma_m = \bigwedge_{i=1}^{n} \Box(p_{i+1} \to p_i) \to \bigwedge_{i=1}^{n} \Box\Box(p_i \to p_{i+1}).$$

On the other hand γ_n is not refutable in a frame for L with $< 2^m$ points because the following formulas are in L:

$$\neg\gamma_m \to \bigwedge_{X\subseteq\{1,\dots,m\},X\neq\emptyset} \Diamond(\bigwedge_{i\in X} \Diamond\delta_i \wedge \bigwedge_{i\notin X,1\leq i\leq m} \neg\Diamond\delta_i),$$

where $\delta_i = p_1 \wedge \cdots \wedge p_i \wedge \neg p_{i+1} \wedge \cdots \wedge \neg p_{m+1}$. ■

Note, however, that the logics constructed in the proofs of the last two theorems are not finitely axiomatizable. We know of only one "very complex" calculus with FMP.

THEOREM 216. $\log_2\log_2 f_{\mathbf{KP}}(n) \asymp n$.

For the proof see [Chagrov and Zakharyaschev 1997], where the reader can find also some other results in this direction.

Relation to complexity classes

Let us return to the original problem of optimizing decision algorithms for the logics under consideration. First of all, it is to be noted that there is a natural lower bound for decision algorithms which cannot be reduced—we mean the complexity of decision procedures for **Cl**. This is clear for (consistent) modal logics on the classical base; and by Glivenko's Theorem, every si-logic "contains" **Cl** in the form of the negated formulas. Thus, if we manage to construct an effective decision procedure for some of our logics then **Cl** can be decided by an equally effective algorithm. (We remind the reader that all existing decision algorithms for **Cl** require exponential time (of the number of variables in the tested formulas). On the other hand, only polynomial time algorithms are regarded to be acceptable in complexity theory.)

So, when analyzing the complexity of decision algorithms for modal and si-logics, it is reasonable to compare them with decision algorithms for **Cl**. For example, if a logic L is polynomially equivalent to **Cl** then we can regard these two logics to be of the same complexity. Moreover, provided that somebody finds a polynomial time decision procedure for **Cl**, a polynomial time decision algorithm can be constructed for L as well. The following theorem lists results obtained by [Ladner 1977], [Ono and Nakamura 1980], [Chagrov 1983], and [Spaan 1993].

THEOREM 217. *All logics mentioned in the formulation of Theorem 213 are polynomially equivalent to* **Cl**.

Proof. We illustrate the proof only for the minimal logic L of depth 3 in NExt**GL** using the method of [Kuznetsov 1979]. Suppose φ is a formula of length n. By Theorem 213, the condition $\varphi \notin L$ means that $\mathfrak{M} \not\models \varphi$, for some model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ based on a frame $\mathfrak{F}$ for **GL** of depth ≤ 3 and cardinality $\leq c \cdot n^2$. We describe this observation by means of classical formulas, understanding their variables as follows. Let x, y, z be names (numbers) of points in $\mathfrak{F}$, for $1 \leq x, y, z \leq c \cdot n^2$. With every pair $\langle x, y \rangle$ of points in $\mathfrak{F}$ we associate a variable p_{xy} whose meaning is "x sees y". And with every $\psi \in \mathbf{Sub}\varphi$ and every x we associate a variable q_x^ψ which means "ψ is true at x". Denote by α the conjunction

$$q_1^\varphi \wedge q_2^\varphi \wedge \cdots \wedge q_{c \cdot n^2}^\varphi.$$

It means that φ is true in $\mathfrak{M}$. And let β be the conjunction of the following formulas under all possible values of their subscripts:

$$\neg p_{xx}, \quad p_{xy} \wedge p_{yz} \to p_{xz}, \quad q_x^{\neg\psi} \leftrightarrow \neg q_x^\psi,$$

$$q_x^{\psi\wedge\chi} \leftrightarrow q_x^\psi \wedge q_x^\chi, \quad q_x^{\psi\vee\chi} \leftrightarrow q_x^\psi \vee q_x^\chi, \quad q_x^{\Box\psi} \leftrightarrow \bigwedge_{y=1}^{c\cdot n^2} (p_{xy} \to q_y^\psi).$$

(The first two formulas say that R is irreflexive and transitive and the rest simulate the truth-relation in $\mathfrak{M}$.) Finally, we define a formula saying that our frame is of depth ≤ 3:

$$\gamma = \bigwedge_{1\leq x,y,z,u\leq c\cdot n^2} \neg(p_{xy} \wedge p_{yz} \wedge p_{zu}).$$

The formula $\beta\wedge\gamma\wedge\neg\alpha$ is of length $\leq 50(c\cdot n^2)^5$ and can be clearly constructed by an algorithm working at most polynomial time in the length of φ. It is readily seen that $\varphi \notin L$ iff $\beta \wedge \gamma \wedge \neg\alpha$ is satisfiable in **Cl**. Thus we have polynomially reduced the derivability problem in L to that in **Cl**. Since the converse reduction is trivial, L and **Cl** are polynomially equivalent. ■

The reader must have noticed that Theorem 217 lists almost all logics known to have polynomial FMP. Kuznetsov [1975] conjectured that every calculus having polynomial FMP is polynomially equivalent to **Cl**. This conjecture is closely related to some problems in the complexity theory of algorithms. We remind the reader that $\boldsymbol{NP}$ is the class of problems that can be solved by polynomial time algorithms on nondeterministic (Turing) machines. An $\boldsymbol{NP}$-complete problem is a problem in $\boldsymbol{NP}$ to which all other problems in $\boldsymbol{NP}$ are polynomially reducible. (For more detailed definitions consult [Garey and Johnson 1979].) The most popular $\boldsymbol{NP}$-complete problem is the satisfiability problem for Boolean formulas, i.e., the nonderivability problem for **Cl**. So the nonderivability problem for all logics listed Theorem 217 is $\boldsymbol{NP}$-complete and Kuznetsov's conjecture is equivalent to a positive solution to the problem whether the nonderivability problem for every calculus with polynomial FMP is $\boldsymbol{NP}$-complete.

Note that if $\boldsymbol{coNP} = \boldsymbol{NP}$ (for the definition of the class $\boldsymbol{coNP}$ see [Garey and Johnson 1979]; we just mention that the derivability problem in **Cl** is $\boldsymbol{coNP}$-complete) then Kuznetsov's conjecture does hold. But since "$\boldsymbol{coNP} = \boldsymbol{NP}$?" belongs to the list of "unsolvable" problems under the current state of knowledge, it may be of interest to find out whether Kuznetsov's conjecture implies $\boldsymbol{coNP} = \boldsymbol{NP}$.

Another complexity class we consider here is the class $\boldsymbol{PSPACE}$ of problems that can be solved by polynomial space algorithms. A typical example of a *$\boldsymbol{PSPACE}$-complete* problem is the truth problem for quantified Boolean formulas. The following theorem (which summarizes results obtained by Ladner [1977], Statman [1979], Chagrov [1985a], Halpern and Moses [1992] and Spaan [1993]) lists some $\boldsymbol{PSPACE}$-complete logics.

THEOREM 218. *The nonderivability problem (and so the derivability problem) in the following logics is $\boldsymbol{PSPACE}$-complete:* **Int**, **KC**, **K**, $\mathbf{K}\otimes\mathbf{K}$, **S4**, $\mathbf{S4}\otimes\mathbf{S4}$, $\mathbf{S5}\otimes\mathbf{S5}$, **GL**, **Grz**, $\mathbf{K}.t$ *and* $\mathbf{K4}.t$.

It follows in particular that complexity is not preserved under the formation of fusions of logics (under the assumption $\boldsymbol{NP} \neq \boldsymbol{PSPACE}$), since nonderivability in **S5** is $\boldsymbol{NP}$-complete. For more information on the preservation of complexity under fusions consult [Spaan 1993].

Finally we note that the nonderivability problem in logics with the universal modality or common knowledge operator is mostly even $\boldsymbol{EXPTIME}$-complete, witness $\mathbf{K}_u$ [Spaan 1993] and $\mathbf{S4EC}_2$ [Halpern and Moses 1992]. The complexity of the nonderivabilty problem for Cartesian products of many standard modal logics is $\boldsymbol{NEXPTIME}$-hard; $\mathbf{S5} \times \mathbf{S5}$ and $\mathbf{K} \times \mathbf{S5}$ are examples of $\boldsymbol{NEXPTIME}$-complete logics (see [Marx 1999]). (Note, by the way, that the known upper bound for $\mathbf{K} \times \mathbf{K}$ is non-elementary.)

5 APPENDIX

We conclude this chapter with a (by no means complete) list of references for those directions of research in modal logic that were not considered above:

- *Congruential logics.* These are modal logics that do not necessarily contain the distribution axiom $\Box(p \to q) \to (\Box p \to \Box q)$ but are closed under modus ponens and the congruence rule $p \leftrightarrow q/\Box p \leftrightarrow \Box q$. Segerberg [1971] and Chellas [1980] define a semantics for these logics; Lewis [1974] proves FMP of all congruential non-iterative logics and Surendonk [1996] shows that they are canonical. Došen [1988] considers duality between algebras and neighbourhood frames and Kracht and Wolter [1999] study embeddings into normal bimodal logics.

- *Modal logics with graded modalities.* The truth-relation for their possibility operators $\Diamond_n$ is defined as follows: $x \models \Diamond_n p$ iff there exist at least n points accessible from x at which p holds. An early reference is [Fine 1972]; more recent are [van der Hoek 1992] (applications to epistemic logic) and [Cerrato 1994] (FMP and decidability).

- *Modal logics with the difference operator* or with *nominals* (or *names*). The semantics of nominals is similar to that of propositional variables; the difference is that a nominal is true at exactly one point in a frame. For the difference operator $[\neq]$, we have $x \models [\neq]p$ iff p is true everywhere except x. De Rijke [1993], Blackburn [1993] and Goranko and Gargov [1993] study the completeness and expressive power of systems of that sort. Closely related to the difference operator is the modal operator $[i]$ for inaccessible worlds: $x \models [i]p$ iff p is true in all worlds which are not accessible from x, see [Humberstone 1983] and [Goranko 1990a].

- *Modal logics with dyadic* or even *polyadic operators.* For duality theory in this case see [Goldblatt 1989]. An extensive study of Sahlqvist-type theorems with applications to polyadic logics is [Venema 1991]. For connections with the theory of relational algebras see [Mikulas

1995] and [Marx 1995]. In those dissertations the reader can find also recent results on arrow logic, i.e., a certain type of polyadic logic which is interpreted in Kripke frames built from arrows. An embedding of polyadic logics into polymodal logics is discussed in [Kracht and Wolter 1997].

- *Bisimulations.* Bisimulations were introduced in modal logic by van Benthem [1983] to characterize its expressive power; see also [de Rijke 1996]. Visser [1996] used bisimulations to prove uniform interpolation. Recently, bisimulations have attracted attention because they form a common tool in modal logic and process theory. We refer the reader to collection [Ponse *et al.* 1996] for information on this subject.

- *Modal logics with fixed point operators*, i.e., modal logics enriched by operators forming the least and greatest fixed points of monotone formulas. These systems are also called *modal μ-calculi.* Under this name they were introduced and studied by Kozen [1983, 1988]; see also [Walukiewicz 1993, 1996] and [Bosangue and Kwiatkowska 1996].

- *Proof theory.* Early references to studies of sequent calculi and natural deduction systems for a few modal logics can be found in *Basic Modal Logic.* More recently, (non-standard) sequent calculi for modal logics have been considered by Došen [1985b], Masini [1992] and Avron [1996]; see also collection [Wansing 1996] and the chapter *Sequent systems for modal logics* later in this *Handbook.* For natural deduction systems see Borghuis [1993]; tableau systems for modal and tense logics were constructed in [Fitting 1983], [Rautenberg 1983], [Gore 1994] and [Kashima 1994]. Orlowska [1996] develops *relational proof systems.* Display calculi for modal logics were introduced by Belnap [1982]; see also [Wansing 1994] and collection [Wansing 1996].

- *Description logic*, a formalism closely related to modal logic, was designed in artificial intelligence by Brachman and Schmolze [1985] as a means for knowledge representation and reasoning (for a survey see [Donini *et al.* 1996]). Schild [1991] was the first to observe that the basic description logic $\mathcal{ALC}$ is just a terminological variant of the polymodal $\mathbf{K}$. Recently, in order to represent dynamic and intensional knowledge, combinations of description and modal logics have been introduced, see e.g. Baader and Ohlbach [1995], Baader and Laux [1995], and Wolter and Zakharyaschev [1998, 1999b,c].

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