Modal Decision Problems

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Contents

1 Modal Logic as 'die Klassentheorie'

There are different views on the subject of Modal Logic. For the purpose of this chapter it is important to distinguish between two of them.

According to the *local* view, Modal Logic deals with a number of *concrete modal logics*. Since the beginning of the 20th century developers and users of Modal Logic from philosophy, mathematics, computer science, artificial intelligence, linguistics and other fields have introduced and investigated dozens of particular modal logics suitable for their needs: *epistemic*, *provability*, *temporal*, *dynamic*, *description*, *spatial* to mention just a few.

With the number of concrete modal logics introduced in the literature growing, there came an understanding that it may be interesting and important to formulate general abstract notions of what a modal logic could be and to investigate these modal logics *per se*.| The pioneers of this *global* approach were Scroggs [127] who considered all extensions of **S5**, Dummett and Lemmon [33] who studied all logics between **S4** and **S5**, Bull [14] and Fine [40] who investigated the logics containing **S4.3**, and Lemmon [86,87,88] and Segerberg [129] who launched a systematical investigation of various classes of modal logics. Two other influential figures that should also be mentioned here are Kuznetsov [81,84,85] and Jankov [67,66,68,69] who investigated the class of all extensions of intuitionistic propositional logic which is closely related to the class of modal logics containing **S4**; see Section 9. Not clear?

Although not formulated explicitly, the 'globalist's' dream research programme was to develop a mathematical machinery that could provide general solutions to the following major problems: [1]

(i) given a class of models/structures, axiomatise the modal logic it determines, decide in an effective way whether it has certain important properties, say, decidability, compactness, interpolation, etc., and determine its computational complexity.

(ii) given a modal logic in the form of a finite set of axioms and inference rules, characterise the (simplest, smallest, largest, etc.) class of models/structures with respect to which this logic is sound and complete, decide in an effective way whether it has important properties as above, and determine its computational complexity.

This research programme is formulated in quite general terms and therefore can be interpreted in various ways. For example, it is not specified what kind of classes of frames/models we consider and what kind of axiomatic systems we take into account. Of course, different interpretations may lead to different solutions, but anyway first results within this ambitious programme looked very promising indeed! For example, Bull [14] proved that all extensions of **S4.3** have the finite model property and Fine [40] showed that all of them are finitely axiomatisable, and so decidable. (Actually, Dummett and Lemmon [33] claimed that all logics between **S4** and **S5** have the finite model property, but their proof was wrong: ten years later Jankov [68] constructed a counterexample.) In view of Makinson's theorem [94], one can effectively decide whether a given logic above **K** is consistent. Maksimova [95,97]

[1] Kuznetsov did formulate such problems *explicitly* in the context of superintuitionistic logics; e.g., given an axiomatisation of a superintuitionistic logic, can we decide in an effective way whether the logic is characterised by a finite algebra?

proved that two properties of logics containing **S4**—tabularity and interpolation—are decidable as well. It seems that many modal logicians did believe in an eventual success of this Big Programme.

In this chapter we analyse the development of Modal Logic within the research framework formulated above, starting from the beginning of the 1970s, although not necessarily in chronological order; for a historical analysis of mathematical modal logic the reader is referred to the recent paper of Goldblatt [57] and notes in [26]. Because of space limitations, we mainly concentrate on normal (multi-) modal logics and their decidability and completeness (in particular, with respect to Kripke or finite frames).

Roughly, our plan is as follows. We start in Section 2 with Thomason's explication (i′) of the *semantical* part (i) of the research programme above. Then, in Section 3, we lay the foundation for the most important *syntactical* notion of Modal Logic, namely, that of a *normal modal logic*. Having introduced an adequate semantics for normal modal logics in terms of *general frames*, we discuss in detail Blok's dichotomy in order to clarify the difference between Thomason's semantical definition of modal logics and the syntactically defined normal modal logics. Based on this discussion, we then come to the appropriate refinement (ii′) of the *syntactical* part (ii) of the research programme for normal modal logics and solutions to it given by Chagrov and Thomason.

Although beautiful from a mathematical point of view, the results of Thomason and Chagrov are 'negative' in the sense that almost all general algorithmic problems formulated in the Big Research Programme turn out to be undecidable. In the same way as the negative solution to the *classical decision problem* of Hilbert transformed the original problem into a *classification* problem, the 'negative' solution to the modal decision problems brings us down to a more 'modest' and realistic 'relativisation' of the programme to various syntactically or semantically defined classes of modal logics.

In Section 4, we consider logics axiomatised by formulas satisfying certain syntactical constraints, in particular, Sahlqvist formulas, uniform formulas, modal reduction principles, etc., and see whether such constraints allow us to prove general decidability/completeness results. In Section 5, we survey the literature on general decidability/completeness results for logics with some 'strong' axioms, say, extensions of tabular and pretabular logics, logics of finite depth and width, extensions of **S4.3**, **K5**, etc.

Then, in Section 6, we discuss an attempt to attack the Big Research Programme for normal extensions of **K4** (that is, unimodal logics with transitive general frames) and the tense logic **Lin** (of linear flows of time) by means of finite representations of modally definable classes of frames via frame and subframe formulas of Jankov and Fine [69,41,45] and more general 'canonical' formulas of [172,174,163]. This technique will be also used to draw and discuss connections between extensions of **S4** and superintuitionistic logics in Section 9.

In Section 7 we provide a 'positive' solution to the Big Research Programme for the class of all tense logics of linear flows of time. In fact, it turns out that for this class of logics all the questions posed in the programme are decidable (sometimes even in nondeterministic polynomial time).

In Section 8, we consider the class of subframe logics—i.e., logics determined by classes of (general) frames closed under the formation of substructures in the standard model-theoretic sense—and explore to what extent the research programme can be realised for this semantically defined class of modal logics.

A number of important open problems are formulated throughout the chapter.

2 Thomason's analysis

As we saw in Chapter **??**, the standard propositional modal language with a countably infinite set of propositional variables (say, $p_0, p_1, \dots$), the Boolean connectives $\wedge$, $\neg$ (and their derivatives $\to$, $\vee$, etc.) and unary modal operators $\Box_1, \dots, \Box_n$ can be regarded as a basic tool for talking about relational structures $\mathfrak{F} = \langle W, R_1, \dots, R_n\rangle$, where the R_i are binary relations on $W \neq \emptyset$. We denote this n-*modal language* by $\mathcal{ML}_n$ and call $\mathfrak{F}$ an n-*frame* or simply a (*Kripke*) *frame*, if n is understood.

$\mathcal{ML}_n$ is interpreted in n-frames by means of *valuations* $\mathfrak{V}$ which associate with every propositional variable p_i a subset $\mathfrak{V}(p_i)$ of W. The pair $\mathfrak{M} = (\mathfrak{F}, \mathfrak{V})$ is called a (*Kripke*) *model* based on $\mathfrak{F}$. Given an $\mathcal{ML}_n$-formula φ, the *truth-relation* $(\mathfrak{M}, x) \models \varphi$, read as '*$\varphi$ is true at x in $\mathfrak{M}$*' (for $x \in W$), is defined by induction on the construction of φ as follows:

$$
\begin{array}{lll}
(\mathfrak{M}, x) \models p_i & \text{iff} & x \in \mathfrak{V}(p_i), \\
(\mathfrak{M}, x) \models \psi \wedge \chi & \text{iff} & (\mathfrak{M}, x) \models \psi \text{ and } (\mathfrak{M}, x) \models \chi, \\
(\mathfrak{M}, x) \models \neg\psi & \text{iff} & \text{not } (\mathfrak{M}, x) \models \psi, \\
(\mathfrak{M}, x) \models \Box_j\psi & \text{iff} & (\mathfrak{M}, y) \models \psi \text{ for all } y \in W \text{ such that } xR_jy.
\end{array}
$$

If $(\mathfrak{M}, x) \models \varphi$ does not hold then we write $(\mathfrak{M}, x) \not\models \varphi$ and say that $\mathfrak{M}$ *refutes* φ at x. Instead of $(\mathfrak{M}, x) \models \varphi$ and $(\mathfrak{M}, x) \not\models \varphi$ we write simply $x \models \varphi$ and $x \not\models \varphi$, if $\mathfrak{M}$ is understood. A formula φ is said to be *true in* $\mathfrak{M}$ ($\mathfrak{M} \models \varphi$, in symbols) if $x \models \varphi$ for all $x \in W$; φ is *satisfied in* $\mathfrak{M}$ if $x \models \varphi$ for some $x \in W$. We say that φ is *valid* in the frame $\mathfrak{F}$ (or $\mathfrak{F}$ *validates* φ) and write $\mathfrak{F} \models \varphi$ if φ is true in all models based on $\mathfrak{F}$; φ is *satisfiable in* $\mathfrak{F}$ if it is satisfied in some model based on $\mathfrak{F}$. For a set Γ of $\mathcal{ML}_n$-formulas, we say that $\mathfrak{F}$ is a *frame for* Γ if all formulas from Γ are valid in $\mathfrak{F}$. In this case we write $\mathfrak{F} \models \Gamma$. A formula φ is Γ-*satisfiable* if it is satisfiable in a frame for Γ. Finally, we write $\Gamma \models \varphi$ if φ is valid in every frame for Γ (that is, if φ is a *semantic consequence of* Γ *over Kripke frames*).

As in classical first-order logic, given a class $\mathcal{K}$ of structures for our language—that is, a class of n-frames—we define the *theory* $\mathsf{Th}\,\mathcal{K}$ of $\mathcal{K}$ by the equation

$$\mathsf{Th}\,\mathcal{K} \quad = \quad \{\varphi \in \mathcal{ML}_n \mid \varphi \text{ is valid in every frame from } \mathcal{K}\}.$$

For example, as we know from Chapter **??**,| $\mathbf{K}_n$ is the theory of the class of all n-frames and **S4** is the theory of the class of all partial orders (or all quasi-orders). Ref missing

Conversely, given a set Γ of $\mathcal{ML}_n$-formulas, denote by $\mathsf{Fr}\,\Gamma$ the class of frames for Γ. For example, $\mathsf{Fr}\,\{\Box p \to \Box\Box p\}$ is the class of all transitive frames $\mathfrak{F} = \langle W, R\rangle$. This and other similar results were in fact obtained by Kripke in his seminal paper [79].

Being equipped with these notions and notations, let us take a closer look at the research programme from Section 1. Clearly, problem (i) depends on *how exactly* the class of models/frames we are interested in is presented. For example, it can be given as the class of structures satisfying certain first- or second-order sentences. This understanding of (i) would lead us to the branch of Modal Logic known as *correspondence theory* (which was partially considered in Chapters **??** and **??**;| see also [156]). In the pure modal perspective, it makes sense to describe frame classes by means of modal formulas, namely as $\mathsf{Fr}\,\{\varphi\}$, for $\varphi \in \mathcal{ML}_n$. Thus, we arrive to our first precise approximation of (i): Ref correct?

(i$'$) given an arbitrary $\mathcal{ML}_n$-formula φ, axiomatise $\mathsf{Th}\,\mathsf{Fr}\,\{\varphi\} = \{\psi \in \mathcal{ML}_n \mid \varphi \models \psi\}$, decide in an effective way whether $\mathsf{Th}\,\mathsf{Fr}\,\{\varphi\}$ has certain important properties, say, consistency, decidability, interpolation, etc., and determine its computational complexity.

Note that this problem is not as trivial as it might look from first sight. Of course, for numerous formulas φ the theory $\mathsf{Th}\,\mathsf{Fr}\,\{\varphi\}$ was axiomatised and thoroughly investigated a long time ago. Well-known examples are, for instance, $\mathbf{T} = \mathsf{Th}\,\mathsf{Fr}\,\{\Box p \to p\}$, the theory of the reflexive frames, and $\mathbf{K4} = \mathsf{Th}\,\mathsf{Fr}\,\{\Box p \to \Box\Box p\}$, the theory of the transitive frames; see Chapter **??**.| Our concern here, however, is not some particular formulas, but effective axiomatisation procedures which could work for *all* modal formulas. Such a procedure is available, for example, for first-order logic: by Gödel's completeness theorem, given a first-order sentence φ, we can axiomatise the theory of structures validating φ by adding φ as an extra axiom to any standard Hilbert-style first-order system. But do we have a Gödel-type completeness theorem for $\mathcal{ML}_n$ with respect to Kripke frames? Ref correct?

A comprehensive analysis of problem (i′) was launched by S. Thomason in his series of papers [140,141,143,145,144].

Theorem 2.1 (Thomason) (a) *There is an $\mathcal{ML}_1$-formula φ such that the set*

$$\mathsf{Th}\,\mathsf{Fr}\,\{\varphi\} \quad = \quad \{\psi \in \mathcal{ML}_1 \mid \varphi \models \psi\}$$

is Π^1_1-complete (in particular, it is not recursively enumerable).

(b) *There is no algorithm which is capable of deciding, given an $\mathcal{ML}_2$-formula φ, whether the theory $\mathsf{Th}\,\mathsf{Fr}\,\{\varphi\}$ is consistent, or, equivalently, whether there exists a 2-frame validating φ.*

(c) *There is a set Γ of $\mathcal{ML}_1$-formulas and an $\mathcal{ML}_1$-formula φ such that $\Gamma \models \varphi$, but $\Delta \not\models \varphi$ for any finite $\Delta \subset \Gamma$.*

(d) *For every $n < \omega + \omega$, there is an $\mathcal{ML}_2$-formula γ_n such that every rooted frame validating γ_n is of cardinality $\beth_n$ and, moreover, all such frames are isomorphic.*[2] *(Here $\beth_0 = \aleph_0$, $\beth_{m+1} = 2^{\beth_m}$, and $\beth_\omega = \lim\{\beth_m \mid m < \omega\}$.)*

Theorem 2.1 (a) shows that the first part of research problem (i′) cannot be solved, while both (a) and (b) indicate that the second part is also hopeless. In fact, this theorem clearly shows that modal theories behave similarly to second-order logic: we do not have any of the classical properties of first-order logic such as compactness, recursive enumerability of valid formulas or the Löwenheim–Skolem theorem. Moreover, as was observed by Segerberg [129], there are $\mathcal{ML}_1$-formulas φ such that $\mathsf{Fr}\,\{\varphi\}$ is not first-order definable. Perhaps, the best known example is the *Löb formula*

$$\boldsymbol{la} = \Box(\Box p \to p) \to \Box p$$

which is valid in a frame $\mathfrak{F} = \langle W, R\rangle$ iff $\mathfrak{F}$ is transitive and contains no infinite ascending chain $x_1 R x_2 R x_3 \dots$. The latter condition is not definable in first-order logic.

Thomason proved his results (a) and (c)| by first constructing (rather complex and 'artificial') multi-modal formulas with the required properties and then by showing that multi-modal logic can be reduced to unimodal logic. (For a discussion of this kind of reductions see Chapter **??**.)| Now (in 2005) we know more simple and natural bimodal formulas similar to φ in (a) of Theorem 2.1. Take, for example, the conjunction φ of the following formulas: Correct? Ref correct?

$$\Box_1(\Box_1 p \to p) \to \Box_1 p, \qquad \Box_2(\Box_2 p \to p) \to \Box_2 p, \tag{1}$$

$$\Box_1(\Box_1 p \to q) \lor \Box_1(\Box_1 q \to p), \qquad \Box_2(\Box_2 p \to q) \lor \Box_2(\Box_2 q \to p), \tag{2}$$

$$\Diamond_2\Diamond_1 p \leftrightarrow \Diamond_1\Diamond_2 p, \qquad \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p. \tag{3}$$

(1) says that both boxes satisfy the Löb axiom above, (2) ensures that frames for $\Box_1$ and $\Box_2$ satisfy the connectedness condition

$$\forall x, y, z\ (xR_i y \land xR_i z \land y \neq z \to yR_i z \lor zR_i y),$$

while (3) says that R_1 and R_2 commute and satisfy the Church–Rosser property, i.e.,

$$\forall x, z\ \big(\exists y\,(xR_2 y \land yR_1 z) \leftrightarrow \exists y\,(xR_1 y \land yR_2 z)\big),$$
$$\forall x \forall y \forall z\ \big(xR_2 y \land xR_1 z \to \exists u\,(yR_1 u \land zR_2 u)\big).$$

Typical frames validating φ are products of finite strict linear orders or infinite reverse well-founded linear orders like $\langle\{0, 1, \dots, n\}, <\rangle$ or $\langle\omega + 1, >\rangle$, where

$$\begin{aligned}
\langle W_1, R_1\rangle \times \langle W_2, R_2\rangle \quad &= \quad \langle W_1 \times W_2, R_h, R_v\rangle,\\
(u, v) R_h (u', v') \quad &\text{iff} \quad uR_1 u' \text{ and } v = v',\\
(u, v) R_v (u', v') \quad &\text{iff} \quad vR_2 v' \text{ and } u = u'.
\end{aligned}$$

[2] See also [16] and [75].

(For more details about product frames and logics see Chapter **??**). To show that validity in frames for φ is Π^1_1-hard, one can reduce the following Σ^1_1-complete *recurrent tiling problem* [62] to the satisfiability problem for $\mathcal{ML}_2$-formulas in frames for φ: 'given a finite set T of tile types (1×1-squares with colours along their edges) and a $t_0 \in T$, can T tile the $\mathbb{N}\times\mathbb{N}$-grid in such a way that colours on adjacent edges of adjacent tiles match and t_0 appears infinitely often in the first column?' For details the reader is referred to [49,120].

Of course, the notion of validity in Kripke frames is of a second-order nature. Indeed, we can easily define a translation $\cdot^s$ of $\mathcal{ML}_n$ into monadic second-order logic with n extra binary relations $R_1, \dots, R_n$ by taking inductively

$$\begin{array}{lcl} p_i^s & = & P_i(x) \\ (\varphi \wedge \psi)^s & = & \varphi^s \wedge \psi^s \\ (\neg\varphi)^s & = & \neg\varphi^s \\ (\Box_i \varphi)^s & = & \forall y\ (xR_iy \to \varphi^s[y/x]), \end{array}$$

where y is a fresh variable. And then, for every frame $\mathfrak{F} = \langle W, R_1, \dots, R_n\rangle$ and every $\mathcal{ML}_n$-formula $\varphi(p_1, \dots, p_k)$, we have $\mathfrak{F} \models \varphi$ iff $\forall P_1 \dots \forall P_k \forall x\ \varphi^s$ is true in $\mathfrak{F}$. This means that for every set $\Gamma \cup \{\varphi\}$ of $\mathcal{ML}_n$-formulas, $\Gamma \models\!\!\!\!\asymp \varphi$ iff φ^s is a logical consequence of $\{\gamma^s \mid \gamma \in \Gamma\}$ in second-order logic.

What is more surprising is that the full monadic second-order theory of a binary predicate can be reduced to propositional modal logic:

Theorem 2.2 (Thomason) *There is an effective translation $\cdot^\sharp$ of the language $\mathcal{MSO}$ of monadic second-order logic with one extra binary relation into $\mathcal{ML}_1$, and there is an $\mathcal{ML}_1$-formula τ such that, for every set $\Xi \cup \{\zeta\}$ of $\mathcal{MSO}$-formulas,*

$$\zeta \text{ is a logical consequence of } \Xi \quad \text{iff} \quad \{\tau\} \cup \{\xi^\sharp \mid \xi \in \Xi\} \models\!\!\!\!\asymp \zeta^\sharp.$$

Thus, Thomason's [145] conclusion was that $\mathcal{ML}_n$ or even $\mathcal{ML}_1$ can be regarded as a rather strong fragment of second-order predicate logic and that the modal consequence relation $\models\!\!\!\!\asymp$ is as complex as it could be. In particular, there is an $\mathcal{ML}_1$-formula φ such that the set $\{\psi \in \mathcal{ML}_1 \mid \varphi \models\!\!\!\!\asymp \psi\}$ is not definable in number theory of any finite order.

3 Normal modal logics

The analogy with second-order logic discussed above also indicated a way of 'regaining' nice properties of first-order logic (compactness, recursive enumerability and Löwenheim–Skolem). Recall (see, e.g., [34]) that following Henkin's idea of introducing a special universe over which the predicates can be interpreted, one can obtain an essentially first-order semantics—known as *general structures*—for second-order logic. Actually, this appears to be Thomason's [141] motivation for introducing *general frames* (which he called *first-order semantics* for modal logic). [3] The idea is very simple: to restrict the range of the valuation function $\mathfrak{V}$ in the definition of Kripke models to some subset of 2^W that is closed under the available operations. This leads us to the following definition.

A *general n-frame* is a structure of the form

$$\mathfrak{G} \quad = \quad \langle W, R_1, \dots, R_n, P\rangle\,,$$

where $\langle W, R_1, \dots, R_n\rangle$ is an ordinary Kripke frame and P is a subset of 2^W containing W and closed under set-theoretic intersection and complementation as well as under the operations

$$\Box_i X \quad = \quad \{x \in W \mid \forall y\ (xR_iy \to y \in X)\},$$

[3] In fact, general frames were also introduced in 1951 by Jónsson and Tarski [71] as Stone-like representations of Boolean algebras with operators; see [57] for more references, details and discussion.

for all $i = 1, \dots, n$. As before, the language $\mathcal{ML}_n$ is interpreted in general frames by means of *valuations* $\mathfrak{V}$ which associate with every propositional variable p_i a set $\mathfrak{V}(p_i)$. The only difference is that now $\mathfrak{V}(p_i)$ *must belong to* P. The pair $\mathfrak{M} = \langle \mathfrak{G}, \mathfrak{V} \rangle$ is called then a *model* based on the general frame $\mathfrak{G}$. The remaining semantical notions are defined in precisely the same way as for Kripke models, e.g., $\mathsf{Th}\,\mathcal{K}$, a general frame for a logic, etc.| To simplify notation, we denote general frames of the form $\mathfrak{F} = \left\langle W, R_1, \dots, R_n, 2^W \right\rangle$ by $\mathfrak{F} = \langle W, R_1, \dots, R_n \rangle$ (because the theories of these frames are the same). Changed

The first radical difference between Kripke and general frames is that the set of theories of classes of general frames can be characterised *syntactically*.| Say that a subset L of $\mathcal{ML}_n$ is a *normal n-modal logic* if it contains the tautologies of classical propositional logic, the formulas Ian?

$$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$$

for all $i = 1, \dots, n$, and is closed under the rules of uniform substitution, modus ponens and *necessitation* $\varphi/\Box_i\varphi$. The smallest normal n-modal logic is known to be $\mathbf{K}_n$; it clearly coincides with the theory of the class GFr of all general n-frames.

Now, given a set Γ of $\mathcal{ML}_n$-formulas, denote by $\mathbf{K}_n \oplus \Gamma$ the minimal normal n-modal logic containing Γ, and by $\mathsf{GFr}\,\Gamma$ the class of general frames for Γ.

Theorem 3.1 (a) *For every class $\mathcal{K}$ of general n-frames, $\mathsf{Th}\,\mathcal{K}$ is a normal n-modal logic. More precisely,*

$$\mathsf{Th}\,\mathcal{K} \quad = \quad \mathbf{K}_n \ \oplus \ \{\varphi \in \mathcal{ML}_n \mid \varphi \textit{ is valid in every general frame from } \mathcal{K}\}.$$

(b) *For every set Γ of $\mathcal{ML}_n$-formulas,*

$$\mathbf{K}_n \ \oplus \ \Gamma \quad = \quad \mathsf{Th}\,\mathsf{GFr}\,\Gamma.$$

(Some comments on the proof of this theorem will be given later on in this section; see also Chapters **??** and **??**.)| Refs correct?

From now on, instead of $\mathsf{Th}\,\mathcal{K}$ we will write $\mathsf{Log}\,\mathcal{K}$ and call it the (*normal n-modal*) *logic characterised* (or *determined*) *by* $\mathcal{K}$. The set of all normal n-modal logics containing a logic L will be denoted by $\mathrm{NExt}\,L$. In particular, $\mathrm{NExt}\,\mathbf{K}_n$ is the set of all normal n-modal logics. Thus,

$$\mathrm{NExt}\,\mathbf{K}_n \quad = \quad \{\mathbf{K}_n \oplus \Gamma \mid \Gamma \subseteq \mathcal{ML}_n\} \quad = \quad \{\mathsf{Log}\,\mathcal{K} \mid \mathcal{K} \subseteq \mathsf{GFr}\}.$$

Let Kripke_n be the set of all *Kripke complete* normal n-modal logics, that is

$$\mathsf{Kripke}_n \quad = \quad \{\mathsf{Log}\,\mathcal{K} \mid \mathcal{K} \text{ a class of Kripke } n\text{-frames}\}.$$

As follows from Theorem 2.1 (a), Kripke_n is a *proper* subset of $\mathrm{NExt}\,\mathbf{K}_n$. Indeed, take a formula φ such that $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$ is not recursively enumerable and consider the normal modal logic $\mathbf{K}_n \oplus \varphi$. Then

$$\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\} \quad \supsetneqq \quad \mathsf{Log}\,\mathsf{GFr}\,\{\varphi\} \quad = \quad \mathbf{K}_n \ \oplus \ \varphi$$

simply because $\mathbf{K}_n \oplus \varphi$ is recursively enumerable. On the other hand, $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$ and $\mathsf{Log}\,\mathsf{GFr}\,\{\varphi\}$ have precisely the same Kripke frames, and so the latter cannot be Kripke complete. The logic $\mathbf{K}_2 \oplus (1) \oplus (2) \oplus (3)$ is probably the 'simplest natural' Kripke incomplete normal modal logic.

So what is the relation between the classes Kripke_n and $\mathrm{NExt}\,\mathbf{K}_n$? What can be said about the class $\mathrm{NExt}\,\mathbf{K}_n - \mathsf{Kripke}_n$?

3.1 Blok's dichotomy

The first examples of Kripke incomplete logics were discovered by Thomason [141,142] and Fine [42]. In order to understand the phenomenon of Kripke incompleteness more deeply, Fine proposed to investigate how many logics may share the same Kripke frames with a given normal (uni)modal logic L. The cardinality of the set $\{L' \in \mathrm{NExt}\,\mathbf{K} \mid \mathsf{Fr}\,L = \mathsf{Fr}\,L'\}$ was called by Fine the *degree of Kripke incompleteness* of L. A very interesting complete solution to this problem was found by Blok [9]. The key player in his solution was the concept of splitting originating in lattice theory [106] (for details see Chapter **??**).| Ref correct? Added!

To explain the idea behind Blok's result informally, let us| observe first that a Kripke complete logic L is always the maximal logic in the set $\{L' \mid \mathsf{Fr}\,L' = \mathsf{Fr}\,L\}$. Now suppose that L is a Kripke complete logic with the following property: there exists a Kripke frame $\mathfrak{F}$ such that L is the smallest logic in the class $\{L' \in \mathrm{NExt}\,\mathbf{K} \mid \mathfrak{F} \notin \mathsf{Fr}\,L'\}$. Then the degree of Kripke incompleteness of L is 1. Indeed, assume that L' is a normal modal logic with $\mathsf{Fr}\,L = \mathsf{Fr}\,L'$. Then $\mathfrak{F} \notin \mathsf{Fr}\,L'$, and so $L \subseteq L'$. To prove $L' \subseteq L$, assume $\varphi \notin L$. As L is Kripke complete, there exists a Kripke frame $\mathfrak{F}' \in \mathsf{Fr}\,L$ such that $\mathfrak{F}' \not\models \varphi$. And since $\mathfrak{F}' \in \mathsf{Fr}\,L'$, we then have $\varphi \notin L'$. Of course, the same argument goes through if instead of just one frame $\mathfrak{F}$ we take some set of frames.

Thus, we can try to generate modal logics with degree of Kripke incompleteness 1 by taking sets $\mathcal{F}$ of frames, proving that the smallest normal logic $L_{\mathcal{F}}$ in the class $\{L' \in \mathrm{NExt}\,\mathbf{K} \mid \exists \mathfrak{F} \in \mathcal{F}\, \mathfrak{F} \notin \mathsf{Fr}\,L'\}$ *exists* and showing its Kripke completeness. Blok's achievement was that he (a) characterised sets $\mathcal{F}$ of frames for which the logics $L_{\mathcal{F}}$ exist, (b) proved that all these $L_{\mathcal{F}}$ are Kripke complete (actually, have the finite model property), and (c) showed that any normal modal logic different from such $L_{\mathcal{F}}$ has degree of Kripke incompleteness $2^{\aleph_0}$.

We now introduce the notions required to explain Blok's result in more detail. Given a logic $L_0 \in \mathrm{NExt}\,\mathbf{K}$, we say that a (finite rooted) frame $\mathfrak{F}$ *splits* $\mathrm{NExt}\,L_0$ if $\mathfrak{F}$ is not a frame for the normal modal logic

$$L_1 \quad = \quad \bigcap\{L \in \mathrm{NExt}\,L_0 \mid \mathfrak{F} \not\models L\}.$$

In this case we denote L_1 by $L_0/\mathfrak{F}$ and call it the *splitting* of $\mathrm{NExt}\,L_0$ *by* $\mathfrak{F}$. This notation reflects the fact that $L_0/\mathfrak{F}$ is the *smallest* logic in $\mathrm{NExt}\,L_0$ which does not have $\mathfrak{F}$ as its frame. If all frames in a set $\mathcal{F}$ split $\mathrm{NExt}\,L_0$, we call $\bigoplus\{L_0/\mathfrak{F} \mid \mathfrak{F} \in \mathcal{F}\}$—i.e., the smallest normal modal logic containing $\bigcup\{L_0/\mathfrak{F} \mid \mathfrak{F} \in \mathcal{F}\}$—the *union-splitting of* $\mathrm{NExt}\,L_0$ *by* $\mathcal{F}$ and denote it by $L_0/\mathcal{F}$.

Example 3.2 Denote by $\bullet$ the Kripke frame which consists of a single irreflexive point. A frame comprised of a single reflexive point is denoted by $\circ$.

(a) $\bullet$ splits $\mathrm{NExt}\,\mathbf{K}$ and $\mathbf{D} = \mathbf{K}/\bullet$ (we remind the reader that $\mathbf{D} = \mathbf{K} \oplus \Diamond\top$ is characterised by the class of *serial* Kripke frames in which every point has a successor). To see this, set

$$L_1 \quad = \quad \bigcap\{L \in \mathrm{NExt}\,\mathbf{K} \mid \bullet \not\models L\}.$$

Since, for every $L \in \mathrm{NExt}\,\mathbf{K}$, $\bullet \not\models L$ iff $\Diamond\top \in L$, L_1 is the intersection of all normal modal logics containing $\Diamond\top$. But $\mathbf{D}$ is the smallest such logic and therefore $L_1 = \mathbf{D}$.

(b) $\circ$ does not split $\mathrm{NExt}\,\mathbf{K}$. To see this recall that $\mathbf{K}$ is determined by the class of finite frames $\mathfrak{F} = \langle W, R\rangle$ without cycles (i.e., R-paths from a point to itself); see, e.g., Chapter **??**. For every such $\mathfrak{F}$, we have $\circ \not\models \mathsf{Log}\,\{\mathfrak{F}\}$ because there exists $n < \omega$ such that $\Box^n\bot \in \mathsf{Log}\,\{\mathfrak{F}\}$, but $\circ \not\models \Box^n\bot$. Therefore,

$$\bigcap\{L \in \mathrm{NExt}\,\mathbf{K} \mid \circ \not\models L\} \quad \subseteq \quad \bigcap\{\mathsf{Th}\,\mathfrak{F} \in \mathrm{NExt}\,\mathbf{K} \mid \mathfrak{F} \text{ finite and cycle free }\} \quad = \quad \mathbf{K}$$

which means that there does *not* exist a smallest normal modal logic without $\circ$ among its frames.

(c) No frame with cycles splits $\mathrm{NExt}\,\mathbf{K}$. The argument is similar to that in (b): just use the fact that no $\Box^n\bot$ is valid in a frame with cycles.

(d) Every finite cycle-free rooted frame splits NExt $\mathbf{K}$. To prove this, we associate with every finite rooted frame $\mathfrak{F} = \langle W, R\rangle$ the formula

$$\delta_{\mathfrak{F}} \quad = \quad \bigwedge_{xRy}(p_x \to \Diamond p_y) \wedge \bigwedge_{\neg xRy}(p_x \to \neg\Diamond p_y) \wedge \bigwedge_{x\neq y}(p_x \to \neg p_y) \wedge \bigvee_{x\in W} p_x.$$

Suppose now that $\mathfrak{F}$ is cycle free, r is a root of $\mathfrak{F}$, $d(\mathfrak{F})$ is the *depth* of $\mathfrak{F}$ (i.e., the length of the longest R-path in $\mathfrak{F}$), and $\Box^{\leq n}\varphi = \varphi \wedge \Box\varphi \wedge \cdots \wedge \Box^n\varphi$. It is not hard to see then that a (general) frame $\mathfrak{G}$ satisfies $\Box^{\leq d(\mathfrak{F})}\delta_{\mathfrak{F}} \wedge p_r$ iff there is a generated subframe $\mathfrak{H}$ of $\mathfrak{G}$ which can be p-morphically mapped onto $\mathfrak{F}$. It follows that the smallest normal logic without $\mathfrak{F}$ among its frames exists and can be axiomatised as

$$\mathbf{K}/\mathfrak{F} \quad = \quad \mathbf{K} \oplus \Box^{\leq d(\mathfrak{F})}\delta_{\mathfrak{F}} \to \neg p_r. \tag{4}$$

(e) The *inconsistent logic*—i.e., $\mathcal{ML}_1$—can be represented as $\mathbf{D}/\circ = (\mathbf{K}/\bullet)/\circ$ (actually this is a variant of Makinson's theorem [94]).

(f) Note by the way that if $L_0 \in$ NExt $\mathbf{K4}$ then *every* finite rooted transitive frame $\mathfrak{F}$ for L_0 splits NExt L_0 and $L_0/\mathfrak{F} = L_0 \oplus \Box^{\leq 1}\delta_{\mathfrak{F}} \to \neg p_r$ (a (general) transitive frame $\mathfrak{G}$ satisfies $\Box^{\leq 1}\delta_{\mathfrak{F}} \wedge p_r$ iff there is a generated subframe $\mathfrak{H}$ of $\mathfrak{G}$ which can be p-morphically mapped onto $\mathfrak{F}$).

Now, returning back to the degree of Kripke incompleteness, we obtain the first part of Blok's dichotomy:

Theorem 3.3 (Blok) (i) *A finite rooted frame $\mathfrak{F}$ splits* NExt $\mathbf{K}$ *iff it is cycle free. In this case we have* $\mathbf{K}/\mathfrak{F} = \mathbf{K} \oplus \Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r$, *where* $n = d(\mathfrak{F})$.

(ii) *Every union-splitting of* NExt $\mathbf{K}$ *has the finite model property, and so its degree of Kripke incompleteness is* 1.

The proof of (ii) is by a variant of standard filtration (see Chapter ??).| By (e) from the example above, the inconsistent logic $\mathcal{ML}_1$ also has degree of Kripke incompleteness 1 (it may be of interest to note that the degree of Kripke incompleteness of $\mathcal{ML}_2$ in NExt $\mathbf{K}_2$ is $2^{\aleph_0}$). The second part of Blok's dichotomy states that all normal modal logics not covered by Theorem 3.3 have degree of Kripke incompleteness $2^{\aleph_0}$: Ref missing?

Theorem 3.4 (Blok) *If a logic L is inconsistent or a union-splitting of* NExt $\mathbf{K}$, *then L has degree of Kripke incompleteness* 1. *Otherwise L has degree of Kripke incompleteness $2^{\aleph_0}$ in* NExt $\mathbf{K}$.

Before we sketch a proof of this result it is worth spending some time on its interpretation. First, it means that $\mathbf{D}$ is the only 'standard' normal modal logic with degree of Kripke incompleteness 1. Logics like $\mathbf{S5}$, $\mathbf{T}$, $\mathbf{K4}$, and $\mathbf{S4}$ have degree of incompleteness $2^{\aleph_0}$. In fact, every consistent normal logic containing $\mathbf{K4}$ or containing $\mathbf{D}$ properly as well as every consistent tabular normal modal logic (a logic is *tabular* if it is determined by a single finite frame; see Section 5), has degree of Kripke incompleteness $2^{\aleph_0}$. Second, in frame-theoretic terms it means that for every modally definable class $\mathcal{F}$ of, say, transitive frames (that is, $\mathcal{F} = \mathsf{Fr}\,\Gamma$ for some set Γ of modal formulas containing $\Box p \to \Box\Box p$) there exist uncountably many different $L \in$ NExt $\mathbf{K}$ such that

$$\mathcal{F} \quad = \quad \mathsf{Fr}\,L.$$

This applies, for example, to the class of all frames based on equivalence relations, quasi-orders, linear orders, and so on.

Proof Suppose that a consistent L is not a union-splitting and L' is the greatest union-splitting contained in L. Since L' has the finite model property, there is a finite rooted frame $\mathfrak{F} = \langle W, R\rangle$ for L' refuting some $\varphi \in L$ and such that every proper generated subframe of $\mathfrak{F}$ validates L. Clearly, $\mathfrak{F}$ is not cycle free. Let $x_1Rx_2R\dots Rx_nRx_1$ be the shortest cycle in $\mathfrak{F}$ and $k = md(\varphi)+1$. We construct a new frame $\mathfrak{F}'$ by extending the cycle $x_1, \dots, x_n, x_1$ as shown in Fig. 1 ((a) for $n = 1$ and (b) for

$n > 1$). More precisely, we add to $\mathfrak{F}$ copies $x_i^1, \dots, x_i^k$ of x_i for each $i \in \{1, \dots, n\}$, organise them into the nontransitive cycle shown in Fig. 1 and draw an arrow from x_i^j to $y \in W - \{x_1, \dots, x_n\}$ iff $x_i R y$. Denote the resulting frame by $\mathfrak{F}' = \langle W', R' \rangle$ and let $x' = x_n^k$. By the construction, $\mathfrak{F}$ is a p-morphic

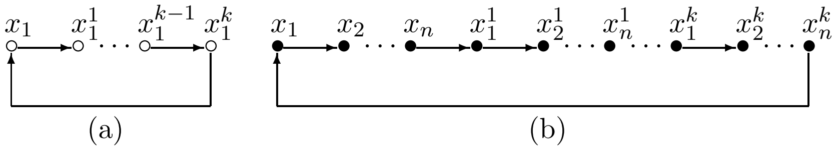

Fig. 1.

image of $\mathfrak{F}'$. Therefore, for all models $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ and $\mathfrak{M}' = \langle \mathfrak{F}', \mathfrak{V}' \rangle$ such that

$$\mathfrak{V}'(p) \quad = \quad \mathfrak{V}(p) \cup \{x_i^j \mid x_i \in \mathfrak{V}(p),\ j < k\}$$

and for every $x \in W$ and every subformula ψ of φ, we have $(\mathfrak{M}, x) \models \psi$ iff $(\mathfrak{M}', x) \models \psi$. So we can hook some other model on x', and points in W will not feel its presence by means of φ's subformulas. The frame to be hooked on x' depends on whether $\bullet \models L$ or $\circ \models L$. We consider only the former alternative.

Fix some $m > |W'|$. For each $I \subseteq \omega - \{0\}$, let $\mathfrak{F}_I = \langle W_I, R_I, P_I \rangle$ be the frame whose diagram is shown in Fig. 2 (d_0 sees the root of $\mathfrak{F}'$, all points e_i and e'_j and is seen from x'; the subframes in dashed boxes are transitive, $e'_i \in W_I$ iff $i \in I$, and P_I consists of sets of the form $X \cup Y$ such that X is a finite or cofinite subset of $W_I - \{b, a_i \mid i < \omega\}$ and Y is either a finite subset of $\{a_i \mid i < \omega\}$ or is of the form $\{b\} \cup Y'$, where Y' is a cofinite subset of $\{a_i \mid i < \omega\}$. It is not hard to see that the

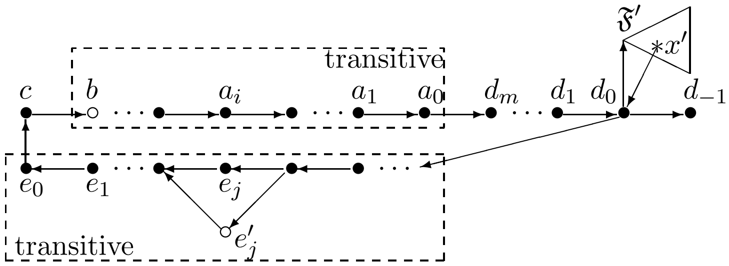

Fig. 2.

points a_i, c, e_i and e'_i are characterised by the variable free formulas

$$\alpha_0 = \Diamond(\delta_m \wedge \Diamond(\delta_{m-1} \wedge \cdots \wedge \Diamond\delta_0)\dots) \wedge \neg\Diamond^2(\delta_m \wedge \Diamond(\delta_{m-1} \wedge \cdots \wedge \Diamond\delta_0)\dots), \quad \alpha_{i+1} = \Diamond\alpha_i \wedge \neg\Diamond^2\alpha_i,$$
$$\gamma = \Diamond^2\alpha_0 \wedge \neg\Diamond\alpha_0, \qquad \epsilon_0 = \Diamond\gamma, \qquad \epsilon_{i+1} = \Diamond\epsilon_i \wedge \neg\Diamond^2\epsilon_i, \qquad \epsilon'_{i+1} = \Diamond\epsilon_i \wedge \neg\Diamond^+\epsilon_{i+1},$$

(in the sense that $x \models \alpha_i$ iff $x = a_i$, etc.), where

$$\delta_0 = \Diamond\Box\bot, \qquad \delta_1 = \Diamond\delta_0 \wedge \neg\delta_0, \qquad \delta_2 = \Diamond\delta_1 \wedge \neg\delta_1 \wedge \neg\Diamond^+\delta_0,$$
$$\delta_{k+1} = \Diamond\delta_k \wedge \neg\delta_k \wedge \neg\Diamond^+\delta_{k-1} \wedge \cdots \wedge \neg\Diamond^+\delta_0.$$

Define L_I to be the logic determined by the class of frames for L and $\mathfrak{F}_I$, i.e., $L_I = L \cap \mathsf{Log}\,\mathfrak{F}_I$. Since $\neg(\epsilon'_i \wedge \Diamond^{m+6}\neg\varphi) \in L_J - L_I$ for $i \in I - J$ (φ is refuted at the root of $\mathfrak{F}'$), $\left|\{L_I \mid I \subseteq \omega - \{0\}\}\right| = 2^{\aleph_0}$.

Let us show now that L_I has the same Kripke frames as L. Since $L_I \subseteq L$, we must prove that every Kripke frame for L_I validates L. Suppose there is a rooted Kripke frame $\mathfrak{G}$ such that $\mathfrak{G} \models L_I$ but $\mathfrak{G} \not\models \psi$, for some $\psi \in L$. Since ψ is in L, it is valid in all frames for L, in particular, $\bullet \models \psi$. And since $\psi \notin L_I$, ψ is refuted in $\mathfrak{F}_I$. Moreover, by the construction of $\mathfrak{F}_I$, it is refuted at a point from which the root of $\mathfrak{F}'$ can be reached by a finite number of steps. Therefore, the following formulas are valid in $\mathfrak{F}_I$ and so belong to L_I and are valid in $\mathfrak{G}$:

$$\neg\psi \to \bigvee_{i=0}^{l} \Diamond^i \gamma, \tag{5}$$

$$\neg\psi \to \bigwedge_{i=0}^{l} \Box^i \big(\gamma \to \Box(\Box_0(\Box_0 p \to p) \to p)\big), \tag{6}$$

where p does not occur in ψ and l is a sufficiently big number so that any point in $\mathfrak{F}_I$ is accessible by $\leq l$ steps from every point in the selected cycle and every point at which ψ may be false, and $\Box_0\chi = \Box(\Diamond\alpha_0 \to \chi)$. According to (5), $\mathfrak{G}$ contains a point where γ is true. By the construction of γ, this point has a successor y where, by (6), $\Box_0(\Box_0 p \to p) \to p$ is true *under any valuation* in $\mathfrak{G}$ and $y \models \Diamond\alpha_0$. Define a valuation $\mathfrak{U}$ in $\mathfrak{G}$ by taking $\mathfrak{U}(p) = y\!\uparrow$, where $y\!\uparrow$ is the set of all points accessible from y. Then $y \models \Box_0(\Box_0 p \to p)$, from which $y \models p$ and so $y \in y\!\uparrow$. Now define another valuation $\mathfrak{U}'$ so that $\mathfrak{U}'(p) = y\!\uparrow - \{y\}$. Since y is reflexive, we again have $y \models \Box_0(\Box_0 p \to p)$, whence $y \models p$, which is a contradiction. ❑

Blok's dichotomy can be generalised in various directions. First, it holds for the languages $\mathcal{ML}_n$ and the corresponding classes NExt $\mathbf{K}_n$ as well; see [92]. And second, it can be extended to completeness with respect to the neighbourhood semantics [29] as well as some other, algebraically motivated, semantics for normal modal logics [92]. On the other hand, the following major problem remains open:

Problem 3.5 *Characterise the degree of Kripke incompleteness of 'transitive' logics in the classes* NExt $\mathbf{K4}$*,* NExt $\mathbf{S4}$*, etc., where Theorem 3.3 does not hold.*| Numbering?

One conclusion to be drawn from these results is that Kripke complete logics are rather exceptional, that Kripke completeness of syntactically defined 'standard' modal logics is a kind of good luck. Another conclusion is that instead of considering logics in the class Kripke_n it may be worthwhile to move to the larger class NExt $\mathbf{K}_n$. First, as we know from other disciplines, more general settings can be very useful (for example, various problems about natural or rational numbers can only be analysed in the framework of real numbers). The second reason is that NExt $\mathbf{K}_n$ is quite natural not only from the syntactical point of view. In fact, as follows from Chapters **??** and **??**,| the lattice $\langle \text{NExt}\,\mathbf{K}_n, \subseteq\rangle$ is dually isomorphic to the lattice of varieties (alias equational theories) of Boolean algebras with operators. This means, in particular, that ideas and techniques from universal algebra are more suitable for investigating NExt $\mathbf{K}_n$ rather than Kripke_n. Refs correct?

Thus, it makes sense to extend the research programme above to the class of all normal n-modal logics.

Research programme for normal modal logics

Within the framework of normal modal logics, the original research programme can be interpreted as follows. By Thomason's Theorem 2.1, we know that (i′) is not realisable. The reformulation of (i′) for general frames has a trivial solution—just remember that $\mathsf{Log}\,\mathsf{GFr}\,\{\varphi\}$ is axiomatised as $\mathbf{K}_n \oplus \varphi$. Instead, we suggest the following reformulation a solution to which would clearly show how complex it is to axiomatise logics determined by classes of (general) frames:

(i″) Characterise those modal formulas φ for which we can effectively recognise whether $\mathbf{K} \oplus \psi$ axiomatises $\mathsf{Log}\,\mathsf{GFr}\,\{\varphi\}$. Characterise those formulas φ for which we can effectively recognise whether $\mathbf{K}\oplus\psi$ axiomatises $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$. For example, is there an algorithm which decides, given a formula ψ, whether $\mathbf{K}\oplus\psi$ axiomatises the logic of all transitive frames (i.e., $\mathsf{Log}\,\mathsf{Fr}\,\{\Box p \to \Box\Box p\}$), reflexive frames (i.e., $\mathsf{Log}\,\mathsf{Fr}\,\{\Box p \to p\}$), etc.?

The first part of (i″) can be reformulated as an axiomatisation problem. Given a modal formula φ and a logic $L_0 \in$ NExt $\mathbf{K}$, we say that the *axiomatisation problem* for $L_0 \oplus \varphi$ is *decidable above* L_0 if the set $\{\psi \in \mathcal{ML}_1 \mid L_0 \oplus \psi = L_0 \oplus \varphi\}$ is recursive. Then the first part of (i″) asks for a characterisation of those modal formulas φ for which the axiomatisation problem for $\mathbf{K} \oplus \varphi$ is decidable.

Being equipped with the notion of a normal modal logic, we can give a precise interpretation of the syntactically formulated problem (ii) from Section 1:

(ii″) Given a modal formula φ, characterise the (simplest, smallest, largest, etc.) class of frames with respect to which $\mathbf{K} \oplus \varphi$ is sound and complete. In particular, is it decidable whether the logic $\mathbf{K} \oplus \varphi$ is Kripke complete, has the finite model property, is determined by a finite frame? Furthermore, can we effectively recognise, given a modal formula φ, whether $\mathbf{K} \oplus \varphi$ is decidable, compact, has interpolation, etc.?

3.2 Chagrov's classification

Rather surprisingly, the *partition* of NExt $\mathbf{K}$ into union-splittings and non-union-splittings not only gives a concise solution to the problem of locating Kripke_n within NExt $\mathbf{K}_n$, but also provides means to attack (i″). A comprehensive solution was found by A. Chagrov.

To explain the intuition behind Chagrov's classification result for the axiomatisation problem, consider the logic $\mathbf{D} = \mathbf{K}/\bullet$ and suppose that we want to decide, for a given formula ψ, whether $\mathbf{K}/\bullet = \mathbf{K} \oplus \psi$ or, equivalently, whether

(a) $\mathbf{K} \oplus \psi \ \subseteq \ \mathbf{K}/\bullet$ and

(b) $\mathbf{K}/\bullet \ \subseteq \ \mathbf{K} \oplus \psi$.

Now, (a) is equivalent to the problem '$\psi \in \mathbf{K}/\bullet$?' which is decidable because the modal logic $\mathbf{D}$ is decidable. And (b) can be checked effectively because, by the definition of splittings, it is equivalent to the problem '$\bullet \not\models \psi$?'. Thus we have proved the decidability of the axiomatisation problem for $\mathbf{D}$ using the fact that $\mathbf{D}$ is decidable and is a union-splitting of NExt $\mathbf{K}$. By Theorem 3.3, all union-splittings have the finite model property and, therefore, are decidable if finitely axiomatisable. So this proof shows the decidability of the axiomatisation problem for every union-splitting $\mathbf{K}/\mathcal{F}$, where $\mathcal{F}$ is a finite set of finite rooted cycle free frames. Chagrov's achievement was to show that for no other logic is the axiomatisation problem decidable:

Theorem 3.6 (Chagrov) *The axiomatisation problem for a consistent logic* $\mathbf{K} \oplus \varphi \supsetneq \mathbf{K}$ *is decidable iff* $\mathbf{K} \oplus \varphi$ *is a union-splitting.*

Hence, the axiomatisation problem is undecidable for such intuitively simple logics as **S4** or **K4**. Again, $\mathbf{D}$ is the only 'standard' modal logic for which the problem is decidable. Before explaining the proof in some detail, we draw conclusions regarding the second part of (i″) and formulate and discuss solutions for (ii″).

Corollary 3.7 *If* $\mathsf{Fr}\,\{\varphi\}$ *is nonempty, then the problem '*$\mathbf{K} \oplus \psi = \mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$*?' is decidable iff*

- $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$ *is a union-splitting of* NExt $\mathbf{K}$ *or*
- $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$ *is not finitely axiomatisable.*

Similarly to Blok's dichotomy, this means, for example, that for any nonempty modally definable class $\mathcal{F}$ of transitive frames such that $\mathsf{Log}\,\mathcal{F}$ is finitely axiomatisable, no algorithm can recognise whether a given formula axiomatises $\mathsf{Log}\,\mathcal{F}$.

The following result due to Thomason [146] and Chagrov [18,19,22] (for more details see [26] and references therein) gives 'negative' solutions to (ii″):

Theorem 3.8 (Thomason & Chagrov) *The following sets are undecidable:*

(a) $\{\varphi \in \mathcal{ML}_1 \mid \mathbf{K} \oplus \varphi$ *is Kripke complete*$\}$,

(b) $\{\varphi \in \mathcal{ML}_1 \mid \mathbf{K} \oplus \varphi$ *is decidable*$\}$,

(c) $\{\varphi \in \mathcal{ML}_1 \mid \mathbf{K} \oplus \varphi$ *has the fmp*$\}$,

(d) $\{\varphi \in \mathcal{ML}_1 \mid \mathbf{K} \oplus \varphi$ *is tabular*$\}$,

(e) $\{\varphi \in \mathcal{ML}_1 \mid \mathbf{K} \oplus \varphi = L\}$, *where L is an arbitrarily fixed consistent tabular logic.*

Proof We begin by showing how to prove Theorem 3.6. The implication ($\Leftarrow$) should be clear from the example $\mathbf{D} = \mathbf{K}/\bullet$ discussed above.

($\Rightarrow$) We show that if $L = \mathbf{K} \oplus \varphi \neq \mathbf{K}$ is a consistent logic that is not a union-splitting then the axiomatisation problem for L is undecidable. The proof is by reduction of the undecidable configuration problem for Minsky (alias register) machines with two tapes (registers).

We remind the reader that a *Minsky machine* (with two tapes) is a finite set of instructions for transforming triples $\langle s, m, n\rangle$ of natural numbers, called *configurations*. The intended meaning of the current configuration $\langle s, m, n\rangle$ is as follows: s is the number (label) of the current machine state and m, n represent the current state of information. Each instruction has one of the four possible forms:

$$s \to \langle t, 1, 0\rangle\,, \quad s \to \langle t, 0, 1\rangle\,, \quad s \to \langle t, -1, 0\rangle\,(\langle t', 0, 0\rangle), \quad s \to \langle t, 0, -1\rangle\,(\langle t', 0, 0\rangle).$$

The last of them, for instance, means: transform $\langle s, m, n\rangle$ into $\langle t, m, n-1\rangle$ if $n > 0$ and into $\langle t', m, n\rangle$ if $n = 0$. For a Minsky machine $\boldsymbol{P}$, we write $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$ if starting with $\langle s, m, n\rangle$ and applying the instructions in $\boldsymbol{P}$, in finitely many steps (possibly, in 0 steps) we can reach $\langle t, k, l\rangle$.

We use the well known fact (see, e.g., [102]) that the following *configuration problem* is undecidable: given a program $\boldsymbol{P}$ and configurations $\langle s, m, n\rangle$, $\langle t, k, l\rangle$, determine whether $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$.

Now let $L = \mathbf{K} \oplus \varphi$. Similarly to the proof of Blok's Theorem 3.4, we analyse two cases: $\bullet \models \varphi$ and $\circ \models \varphi$. Here we only show that the axiomatisability problem for $L = \mathbf{K} \oplus \varphi$ with $\bullet \models \varphi$ is undecidable and leave the remaining case to the reader.

We will use a modification of the (general) frame $\mathfrak{F}_I$ constructed in the proof of Blok's theorem. Let us take another look at this frame. The root of $\mathfrak{F}'$ refutes some formula in L; without loss of generality we may assume that this formula is φ. The part of $\mathfrak{F}_I$ comprising points a_i, b, c was used to ensure that the logic of $\mathfrak{F}_I$ is Kripke incomplete, while the points e_0, e_j, e'_j ($j > 0$) were used to produce a continuum of frames of the form $\mathfrak{F}_I$, i.e., to make sure that $\mathsf{Log}\,\{\mathfrak{F}_I\} \neq \mathsf{Log}\,\{\mathfrak{F}_J\}$ whenever $I \neq J$. Finally, the points d_k ($-1 \leq k \leq m$) ensured, in particular, that a_i, b, c, e_0, e_j, e'_j have no impact on $\mathfrak{F}'$ and the other way round.

For our present aims the subframe consisting of points a_i, b, c, e_0, e_j, e'_j is not required and we can modify it. Namely, we remove from $\mathfrak{F}_I$ the points e_j, e'_j ($j \geq 1$) and replace them with the frame shown in Fig. 3. More precisely, the frame in Fig. 3 is transitive, e^2_1 is its only reflexive point, all points are accessible from d_0 and 'see' e_0; the points of the form $s(t, k, l)$ to not see each other and occur in the frame iff $\boldsymbol{P} : \langle s, m, n\rangle \to \langle t, k, l\rangle$, where $\boldsymbol{P}$, $\langle s, m, n\rangle$ are some fixed Minsky machine and configuration. Denote the resulting frame by $\mathfrak{F}(\boldsymbol{P}, \langle s, m, n\rangle)$.

Now, starting with the formula ϵ_0 from the proof of Blok's theorem, we define the following formulas

$$\begin{array}{rclrcl}
\epsilon^1_1 & = & \Diamond\epsilon_0 \wedge \neg\Diamond\Diamond\epsilon_0, & \epsilon^2_1 & = & \Diamond\Diamond\epsilon_0 \wedge \neg\Diamond\epsilon^1_1,\\
\epsilon^1_2 & = & \Diamond\epsilon^1_1 \wedge \neg\Diamond\Diamond\epsilon^1_1 \wedge \neg\Diamond\epsilon^2_1, & \epsilon^2_2 & = & \Diamond\epsilon^1_1 \wedge \Diamond\Diamond\epsilon^2_1 \wedge \neg\Diamond\Diamond\epsilon^1_1,\\
\epsilon^1_3 & = & \Diamond\epsilon^1_2 \wedge \neg\Diamond\Diamond\epsilon^1_2 \wedge \neg\Diamond\epsilon^2_2, & \epsilon^1_4 & = & \Diamond\epsilon^1_3 \wedge \neg\Diamond\Diamond\epsilon^1_3 \wedge \neg\Diamond\epsilon^2_2,\\
\epsilon^2_3 & = & \Diamond\epsilon^2_2 \wedge \neg\Diamond\Diamond\epsilon^2_2 \wedge \neg\Diamond\epsilon^1_2, & \epsilon^2_4 & = & \Diamond\epsilon^2_3 \wedge \neg\Diamond\Diamond\epsilon^2_3 \wedge \neg\Diamond\epsilon^1_2.
\end{array}$$

These variable free formulas characterise points e^i_j in $\mathfrak{F}(\boldsymbol{P}, \langle s, m, n\rangle)$ in the sense that ϵ^i_j is true in

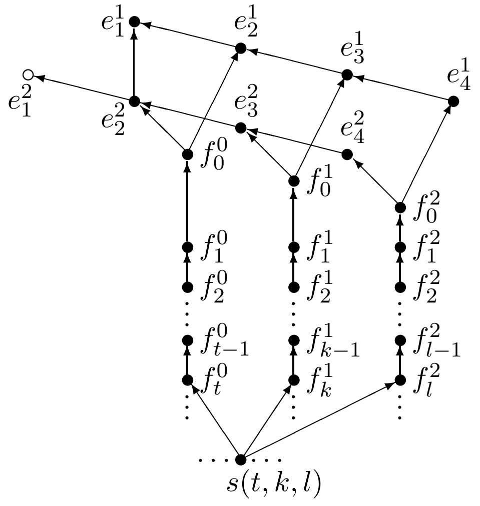

Fig. 3.

$\mathfrak{F}(\boldsymbol{P}, \langle s, m, n\rangle)$ only at the point denoted by e_j^i. Further, let

$$\begin{aligned}
\phi_0^0 &= \Diamond\epsilon_2^1 \wedge \Diamond\epsilon_2^2 \wedge \neg\Diamond\Diamond\epsilon_2^1 \wedge \neg\Diamond\Diamond\epsilon_2^2,\\
\phi_0^1 &= \Diamond\epsilon_3^1 \wedge \Diamond\epsilon_3^2 \wedge \neg\Diamond\Diamond\epsilon_3^1 \wedge \neg\Diamond\Diamond\epsilon_3^2,\\
\phi_0^2 &= \Diamond\epsilon_4^1 \wedge \Diamond\epsilon_4^2 \wedge \neg\Diamond\Diamond\epsilon_4^1 \wedge \neg\Diamond\Diamond\epsilon_4^2,\\
\phi_{j+1}^i &= \Diamond\phi_j^i \wedge \neg\Diamond\Diamond\phi_j^i \wedge \bigwedge_{i\neq k=0}^{2} \neg\Diamond\phi_0^k
\end{aligned}$$

for $i \in \{0,1,2\}$, $j \geq 0$. The formula ϕ_j^i is true only at f_j^i. The formulas characterising $s(t,k,l)$ are denoted by $\sigma(t, \phi_k^1, \phi_l^2)$, where

$$\sigma(t,\psi,\chi) \quad = \quad \bigwedge_{i=0}^{t} \Diamond\phi_i^0 \wedge \neg\Diamond\phi_{t+1}^0 \wedge \Diamond\psi \wedge \neg\Diamond^2\psi \wedge \Diamond\chi \wedge \neg\Diamond^2\chi.$$

We also require formulas characterising not only fixed but arbitrary configurations:

$$\begin{aligned}
\kappa_1 &= (\Diamond\phi_0^1 \vee \phi_0^1) \wedge \neg\Diamond\phi_0^0 \wedge \neg\Diamond\phi_0^2 \wedge p_1 \wedge \neg\Diamond p_1,\\
\kappa_2 &= \Diamond\phi_0^1 \wedge \neg\Diamond\phi_0^0 \wedge \neg\Diamond\phi_0^2 \wedge \Diamond p_1 \wedge \neg\Diamond^2 p_1,\\
\pi_1 &= (\Diamond\phi_0^2 \vee \phi_0^2) \wedge \neg\Diamond\phi_0^0 \wedge \neg\Diamond\phi_0^1 \wedge p_2 \wedge \neg\Diamond p_2,\\
\pi_2 &= \Diamond\phi_0^2 \wedge \neg\Diamond\phi_0^0 \wedge \neg\Diamond\phi_0^1 \wedge \Diamond p_2 \wedge \neg\Diamond^2 p_2,
\end{aligned}$$

where p_1 and p_2 are fresh variables.

Now we are fully equipped to simulate the behaviour of Minsky machines by means of modal formulas. Let

$$\blacklozenge\psi \quad = \quad \psi \vee \Diamond\psi \vee \cdots \vee \Diamond^n\psi,$$

where n is a sufficiently large number such that if xR^*y in $\mathfrak{F}(\boldsymbol{P}, \langle s, m, n\rangle)$ then xR^ky for some $k \leq n$. (Note that in the proof of Blok's theorem we took $n = m + 6$.)

With each instruction I in $\boldsymbol{P}$ we associate a formula AxI by taking:

$$AxI \quad = \quad \neg\varphi \wedge \blacklozenge\sigma(t,\pi_1,\kappa_1) \rightarrow \neg\varphi \wedge \blacklozenge\sigma(t',\pi_2,\kappa_1)$$

if I has the form $t \rightarrow \langle t', 1, 0\rangle$,

$$AxI \quad = \quad \neg\varphi \wedge \blacklozenge\sigma(t,\pi_1,\kappa_1) \rightarrow \neg\varphi \wedge \blacklozenge\sigma(t',\pi_1,\kappa_2)$$

if I is $t \to \langle t', 0, 1\rangle$,

$$AxI \quad = \quad (\neg\varphi \wedge \blacklozenge\sigma(t,\pi_2,\kappa_1) \to \neg\varphi \wedge \blacklozenge\sigma(t',\pi_1,\kappa_1)) \wedge (\neg\varphi \wedge \blacklozenge\sigma(t,\phi_0^1,\kappa_1) \to \neg\varphi \wedge \blacklozenge\sigma(t'',\phi_0^1,\kappa_1))$$

if I is $t \to \langle t', -1, 0\rangle$ ($\langle t'', 0, 0\rangle$),

$$AxI \quad = \quad (\neg\varphi \wedge \blacklozenge\sigma(t,\pi_1,\kappa_2) \to \neg\varphi \wedge \blacklozenge\sigma(t',\pi_1,\kappa_1)) \wedge (\neg\varphi \wedge \blacklozenge\sigma(t,\pi_1,\phi_0^2) \to \neg\varphi \wedge \blacklozenge\sigma(t'',\pi_1,\phi_0^2))$$

if I is $t \to \langle t', 0, -1\rangle$ ($\langle t'', 0, 0\rangle$). The formula simulating $\boldsymbol{P}$ as a whole is

$$Ax\boldsymbol{P} \quad = \quad \bigwedge_{I \in \boldsymbol{P}} AxI.$$

Now, by induction on the length of computations one can show that, for every program $\boldsymbol{P}$ and all configurations $\langle s,m,n\rangle$, $\langle t,k,l\rangle$, we have the following property ($\Downarrow$)

$$\boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle$$
$$\Downarrow$$
$$\neg\varphi \wedge \blacklozenge\sigma(s,\phi_m^1,\phi_n^2) \to \neg\varphi \wedge \blacklozenge\sigma(t,\phi_k^1,\phi_l^2) \quad \in \quad \mathbf{K} \oplus Ax\boldsymbol{P}.$$

Finally, define a logic $L(\boldsymbol{P}, \langle s,m,n\rangle, \langle t,k,l\rangle)$ by taking

$$L(\boldsymbol{P}, \langle s,m,n\rangle, \langle t,k,l\rangle) \quad = \quad \mathbf{K} \oplus Ax\boldsymbol{P} \oplus (\neg\varphi \wedge \blacklozenge\sigma(s,\phi_m^1,\phi_n^2) \to \neg\varphi \wedge \blacklozenge\sigma(t,\phi_k^1,\phi_l^2)) \to \varphi.$$

Clearly, for given $\boldsymbol{P}$ and configurations $\langle s,m,n\rangle$ and $\langle t,k,l\rangle$, the logic $L(\boldsymbol{P}, \langle s,m,n\rangle, \langle t,k,l\rangle)$ is constructed effectively. We claim that

$$\boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle \qquad \text{iff} \qquad L(\boldsymbol{P}, \langle s,m,n\rangle, \langle t,k,l\rangle) = \mathbf{K} \oplus \varphi.$$

The implication ($\Rightarrow$) is proved using ($\Downarrow$) above and the obvious $L(\boldsymbol{P}, \langle s,m,n\rangle, \langle t,k,l\rangle) \subseteq \mathbf{K} \oplus \varphi$. To show ($\Leftarrow$), it suffices to observe that if $\boldsymbol{P} : \langle s,m,n\rangle \not\to \langle t,k,l\rangle$ then $\mathfrak{F}(\boldsymbol{P}, \langle s,m,n\rangle)$ validates all the axioms of $L(\boldsymbol{P}, \langle s,m,n\rangle, \langle t,k,l\rangle)$ and refutes φ.

To prove Theorem 3.8, we modify the definition of the logic $L(\boldsymbol{P}, \langle s,m,n\rangle, \langle t,k,l\rangle)$ above. First we take a formula φ such that $\mathbf{K} \oplus \varphi$ is a consistent tabular logic. One can show that every such logic is not a union-splitting.

Now observe that there exist a program $\boldsymbol{P}$ and a configuration $\langle s,m,n\rangle$ such that no algorithm is capable of deciding, given a configuration $\langle t,k,l\rangle$, whether $\boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle$ (for details see [26]). Fix some $\boldsymbol{P}$ and $\langle s,m,n\rangle$ satisfying this condition, and let

$$\begin{aligned} L'(\langle t,k,l\rangle) \quad = \quad & \mathbf{K} \oplus Ax\boldsymbol{P} \oplus \\ & \big(\neg\varphi \wedge \blacklozenge\sigma(s,\phi_m^1,\phi_n^2) \to \neg\varphi \wedge \blacklozenge\sigma(t,\phi_k^1,\phi_l^2)\big) \to \varphi \oplus \\ & \neg\varphi \to \bigvee_{i=0}^{l} \Diamond^i\gamma \oplus \\ & \neg\varphi \to \bigwedge_{i=0}^{l} \Box^i\big(\gamma \to \Box(\Box_0(\Box_0 p \to p) \to p)\big), \end{aligned}$$

where p is a fresh variable.

If $\boldsymbol{P} : \langle s,m,n\rangle \to \langle t,k,l\rangle$ then, as in the proof of Theorem 3.6, it is easy to see that we have $L'(\langle t,k,l\rangle) = \mathbf{K} \oplus \varphi$. Thus, this logic is tabular (coincides with the chosen tabular logic, to be more precise), and so it is decidable, Kripke complete and has the fmp.

If $\boldsymbol{P} : \langle s, m, n \rangle \not\to \langle t, k, l \rangle$ then $L'(\langle t, k, l \rangle) \neq \mathbf{K} \oplus \varphi$, which can be shown with the help of the frame $\mathfrak{F}(\boldsymbol{P}, \langle s, m, n \rangle)$. Thus, our logic is different from the chosen tabular logic $\mathbf{K} \oplus \varphi$. Moreover, using the last two axioms (cf. formulas (5), (6) in the proof of Blok's theorem) one can show that although $\varphi \notin L'(\langle t, k, l \rangle)$, no Kripke frame for $L'(\langle t, k, l \rangle)$ can refute φ. It follows that $L'(\langle t, k, l \rangle)$ is Kripke incomplete and does not have the finite model property. Next we use the properties of $\boldsymbol{P}$ and $\langle s, m, n \rangle$ to show that

$$\boldsymbol{P} : \langle s, m, n \rangle \not\to \langle t', k', l' \rangle \quad \text{iff} \quad L'(\langle t, k, l \rangle) \vdash \neg\varphi \wedge \blacklozenge\sigma(s, \phi^1_m, \phi^2_n) \to \neg\varphi \wedge \blacklozenge\sigma(t', \phi^1_{k'}, \phi^2_{l'}).$$

The implication ($\Rightarrow$) is proved by induction on the length of computation and ($\Leftarrow$) is shown using the frame $\mathfrak{F}(\boldsymbol{P}, \langle s, m, n \rangle)$. It follows that $L'(\langle t, k, l \rangle)$ is undecidable. ❑

In fact, using the technique above one can prove undecidability of many other important properties of modal logics such as first-order definability (i.e., whether $\mathsf{Fr}\,\{\varphi\}$ is definable by first-order formulas, for an arbitrarily given φ), canonicity, the interpolation and the disjunction properties, etc.; see [26] and references therein. Actually, we know only two interesting decidable properties of finitely axiomatisable logics in NExt $\mathbf{K}$: consistency and coincidence with $\mathbf{K}$. However, even consistency becomes undecidable in NExt $\mathbf{K}_2$ [146].

Theorems 3.6 and 3.8 give rise to further interesting questions. First, we still do not know a solution to the following open problem:

Problem 3.9 *Is the set* $\{\varphi \in \mathcal{ML}_1 \mid \mathbf{K} \oplus \varphi$ *is a union-splitting*$\}$ *decidable?*

Second, some of the undecidable problems formulated above may turn out to be *recursively enumerable*, so that we can at least effectively enumerate finitely axiomatisable logics with this or that property. For example, it is easy to show recursive enumerability of the set $\{\varphi \in \mathcal{ML}_n \mid \mathbf{K}_n \oplus \varphi = L\}$, where L is fixed consistent tabular logic (just use the fact that L is finitely axiomatisable, say, by a formula ψ and enumerate those φ from which ψ is derivable in $\mathbf{K}_n$ and vice versa). However, for the majority of important properties of modal logics this problem remains open:

Problem 3.10 *Is it possible to effectively enumerate* $\mathcal{ML}_n$*-formulas* φ *for which* $\mathbf{K}_n \oplus \varphi$ *is decidable (Kripke complete, has the finite model property, interpolation, etc.)?*

Finally, one may wonder what happens if we consider the decision problems above for *recursively axiomatisable* modal logics, i.e., those that are given by programs generating their formulas. In this case we have the following analogue of the Rice theorem from general recursion theory:

Theorem 3.11 (Kuznetsov) *No nontrivial property of recursively axiomatisable logics is decidable in any of the classes of logics considered above.*

In particular, this result applies even to NExt $\mathbf{S5}$ (where all logics are finitely axiomatisable)—provided that its logics are represented as programs computing their formulas. Of course, we can recognise, say, consistency in NExt $\mathbf{S5}$ if all logics in the class are given by finite sets of axioms. The situation is different in, e.g., NExt $\mathbf{S4}$ where there exist recursively enumerable logics that are not finitely axiomatisable. The proof of this theorem (Kuznetsov left it unpublished) is very simple. In fact, it has nothing to do with modal logics; it is rather about effective computations. The reader can find it in [26].

3.3 Postmortem

So, was the Big Research Programme a failure or a success? Or simply lost illusions?

Dealing with individual systems like $\mathbf{K}_n$, $\mathbf{S4}$ or even $\mathbf{PDL}$, one might think that Modal Logic is 'harmless,' that it is a reasonable compromise between expressiveness and effectiveness, especially in various application areas in computer science and artificial intelligence. Looking at modal logics from a more general perspective, we see, however, that the propositional modal language is extremely expressive, even if we have a single box operator. Modal Logic has been praised by 'users' for being *robustly decidable*. The analysis above shows that, when put in a more general setting, Modal Logic is rather *robustly* ***un****decidable*. (However, we can take comfort in the mathematical beauty of the splitting-based dichotomy between Kripke completeness and Kripke incompleteness and its transparent repercussions for modal decision problems.)

The outcomes of the Big Research Programme discussed above appear to be similar to the negative solution to the Classical Decision Problem, *das Entscheidungsproblem*, of Hilbert; see [11] and references therein. According to [11], 'the reaction of logicians to the discoveries of Church and Turing was that the classical decision problem was wider than the yes/no version of it ... The logicians started to think about the classical decision problem as a classification problem. Which fragments are decidable for satisfiability and which are undecidable? Which fragments are decidable for finite satisfiability and which are undecidable? Which fragments have the finite model property and which contain axioms of infinity (that is satisfiable formulae without finite models)?'

Similar questions make sense in Modal Logic as well. The modal decision problems considered above can be transformed into *modal classifications problems*:

(iii) determine (in some sense) maximal classes of modal logics with the desirable properties.

Of course, particularly interesting are natural classes like

- extensions of certain logics, e.g., $\mathbf{K4}$, $\mathbf{S5} \times \mathbf{S5}$, $\mathbf{K4}_t$;
- logics axiomatised by certain 'normal' formulas (e.g., reductions of modalities, Sahlqvist or uniform formulas);
- logics whose classes of frames are closed under certain natural operations (e.g., taking subframes).

To understand the landscape of modal logics in this respect, a variety of different methodologies are required. One established path is to look 'outside' and, e.g., employ modal logics' relation to finite variable/guarded fragments of first-order logic (see Chapter ??),| or their relation to languages recognised by tree automata (see Chapter ??).| In this chapter we follow the 'internal' approach and analyse how different 'modal' syntactic or semantic restrictions can guarantee this or that desirable property.

Ref missing?

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4 Syntactical classes of modal logics

To understand a modal logic is, to a large extent, to understand the structure of its frames, in particular, Kripke frames. An obvious way of doing this is to try to characterise frames by means of first-order formulas in some suitable signature. Classical observations going back to Kripke [79] are as follows, where $\mathfrak{F} = \langle W, R\rangle$ is treated as a Kripke frame in the left-hand column and as a first-order structure in the right-hand one:

$$\mathfrak{F} \models \Box p \to p \quad \text{iff} \quad \mathfrak{F} \models \forall x\ R(x,x),$$

$$\mathfrak{F} \models \Box p \to \Box\Box p \quad \text{iff} \quad \mathfrak{F} \models \forall x,y,z\ \big(R(x,y) \wedge R(y,z) \to R(x,z)\big),$$

etc.

A nice first-order characterisation not only helps in understanding the structure of frames. First, using the standard translation $\cdot^s$ of modal formulas into the first-order language from Section 2 (see also Chapters **??** and **??**)| and Gödel's completeness theorem, it is readily seen that if $\mathsf{Fr}\,\{\varphi\}$ is definable by a first-order formula as above, then $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$ is recursively enumerable, while in general, by Theorem 2.1 (a), $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$ might be Π^1_1-hard and even more complex. Second, as was proved by Fine [44] (see also [154,155]), we have the following:

Refs correct?

Theorem 4.1 (Fine) *If a logic $L \in \mathrm{NExt}\,\mathbf{K}$ is determined by a first-order definable class of frames then L is $\mathcal{D}$-persistent and, in particular, canonical.*|

one or both?

This means that to investigate a first-order definable $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$, we can use the well-known methods of canonical models and filtration developed in the 1960–1970s (see, e.g., [26] and references therein). Although the converse of Fine's theorem does not hold, as has been recently shown in [58], it is nevertheless a kind of empiric rule that logics not determined by first-order definable classes are *not* canonical, and therefore, the standard way of proving completeness or the finite model property is blocked for them.

By Chagrova's theorem [28], there is no effective way of deciding, given a formula φ, whether $\mathsf{Fr}\,\{\varphi\}$ is first-order definable. However, one can try to find and describe *syntactically* some classes of formulas φ for which $\mathsf{Fr}\,\{\varphi\}$ is first-order definable. In fact, this approach has been the driving force behind much research in Modal Logic since the 1960s (see, e.g., [88]). The (so far) most general syntactically defined class of formulas for which this holds true was discovered by H. Sahlqvist [125].

4.1 Sahlqvist logics

Sahlqvist's theorem [125] (see also [53])—perhaps the most celebrated general result in Modal Logic—gives a sufficient condition for first-order definability and Kripke completeness of logics in $\mathrm{NExt}\,\mathbf{K}_n$. To formulate it we require the following definitions.

Say that a formula is *positive* if it is constructed from variables and the constants $\top$, $\bot$ using $\wedge$, $\vee$, $\Diamond_i$ and $\Box_i$. An arbitrary finite sequence of boxes $\Box_i$, $i = 1, \dots, n$ will be denoted by $\Box^*$.

A formula $\varphi \in \mathcal{ML}_n$ is called a *Sahlqvist formula* if it is equivalent in $\mathbf{K}_n$ to a formula of the form $\Box^*(\psi \to \chi)$, where χ is positive and ψ is constructed from variables and their negations, $\bot$ and $\top$ with the help of $\wedge$, $\vee$, $\Box_i$ and $\Diamond_i$ in such a way that no subformula of ψ of the form $\psi_1 \vee \psi_2$ or $\Diamond_i\psi_1$, containing an occurrence of a variable without $\neg$, is in the scope of some $\Box_j$. For example, formulas (2)–(3) from Section 2 are Sahlqvist, while the Löb axiom (1) and the McKinsey axiom

$$\boldsymbol{ma} \quad = \quad \Box\Diamond p \to \Diamond\Box p \tag{7}$$

are not.

Theorem 4.2 (Sahlqvist) (a) *Given a Sahlqvist formula $\varphi \in \mathcal{ML}_n$, one can effectively construct a first-order formula $\phi(x)$ in $R_1, \dots, R_n$ and $=$ having x as its only free variable and such that, for every descriptive| or Kripke frame $\mathfrak{F}$ and every point a in $\mathfrak{F}$,* Give reference?

$$(\mathfrak{F}, a) \models \varphi \qquad \textit{iff} \qquad \mathfrak{F} \models \phi(x)[a].$$

(*Here $(\mathfrak{F}, a) \models \varphi$ means that φ is true at a in $\mathfrak{F}$ under any valuation.*)

(b) *If Γ is a set of Sahlqvist $\mathcal{ML}_n$-formulas and $L \in \mathrm{NExt}\,\mathbf{K}_n$ is a $\mathcal{D}$-persistent logic then the logic $L \oplus \Gamma$ (in particular, $\mathbf{K}_n \oplus \Gamma$) is $\mathcal{D}$-persistent as well. Moreover, $L \oplus \Gamma$ is elementary (in the sense that the class of Kripke frames for it coincides with the class of all models for some set of first-order formulas in R_i and $=$) whenever L is so.*

Various detailed proofs of this result can be found in [126,70,7] (for some generalisations see, e.g., [30,60,72,50]). So, instead of going into technical details, we will concentrate on the meaning of Sahlqvist's theorem.

First, it gives much more than just first-order definability of $\mathsf{Fr}\,\{\varphi\}$, for a Sahlqvist formula φ, and, therefore, recursive enumerability of $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$. In fact, we also obtain an axiomatisation, namely, that $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\} = \mathbf{K} \oplus \varphi$. As we know from Blok's theorem, this is much stronger than just first-order definability. (For example, there are a lot of formulas φ such that $\mathsf{Fr}\,\{\varphi\}$ is the class of transitive frames, but $\mathbf{K} \oplus \varphi \neq \mathbf{K4}$.) Thus, Sahlqvist's theorem has two aspects: the correspondence part (stating first-order definability of $\mathsf{Fr}\,\{\varphi\}$) and the completeness part (stating that $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\} = \mathbf{K} \oplus \varphi$).

However, Sahlqvist axioms do not guarantee good computational properties of modal logics. For example, there are finitely axiomatisable Sahlqvist logics without the finite model property in $\mathrm{NExt}\,\mathbf{S4}$

[25] (see also [65]). There are undecidable finitely axiomatisable Sahlqvist logics in NExt $\mathbf{K}$. Such a logic can easily be constructed if we have more than one box [23]. For instance, consider the undecidable associative calculus T of [148] with the axioms

$$ac = ca, \quad ad = da, \quad bc = cb, \quad bd = db, \quad edb = be, \quad eca = ae, \quad abac = abacc.$$

The reader will notice immediately an analogy between these axioms and the axioms of the following modal logic with five necessity operators:

$$\begin{aligned} L \quad = \quad & \mathbf{K}_5 \oplus \Box_1\Box_3 p \leftrightarrow \Box_3\Box_1 p \;\oplus\; \Box_1\Box_4 p \leftrightarrow \Box_4\Box_1 p \;\oplus\; \Box_2\Box_3 p \leftrightarrow \Box_3\Box_2 p \;\oplus\; \Box_2\Box_4 p \leftrightarrow \Box_4\Box_2 p \;\oplus \\ & \Box_5\Box_4\Box_2 p \leftrightarrow \Box_2\Box_5 p \;\oplus\; \Box_5\Box_3\Box_1 p \leftrightarrow \Box_1\Box_5 p \;\oplus\; \Box_1\Box_2\Box_1\Box_3 p \leftrightarrow \Box_1\Box_2\Box_1\Box_3\Box_3 p. \end{aligned}$$

Moreover, it is not hard to see that words x, y in the alphabet $\{a, b, c, d, e\}$ are equivalent in T iff $f(x)p \leftrightarrow f(y)p \in \mathbf{K}_5$, where f is the natural one-to-one correspondence between such words and modalities in language $\{\Box_1, \dots, \Box_5\}$ under which, for instance, $f(cadedb) = \Box_3\Box_1\Box_4\Box_5\Box_4\Box_2$. It follows immediately that the Sahlqvist 5-modal logic L is undecidable. An even simpler example of an undecidable finitely axiomatisable Sahlqvist logic is the bimodal product $\mathbf{K4} \times \mathbf{K4}$; for details see Chapter **??**. Now, using the reduction of multi-modal logics to those in NExt $\mathbf{K}$ [77] one can construct an undecidable finitely axiomatisable Sahlqvist logic from NExt $\mathbf{K}$.

Problem 4.3 *Are finitely axiomatisable Sahlqvist logics in* NExt $\mathbf{K4}$ *decidable?*

It is also worth noting that there is no effective way of recognising whether a given modal formula is (deductively equivalent to) a Sahlqvist formula; in particular, the set

$$\{\varphi \in \mathcal{ML}_1 \mid \mathbf{S4} \oplus \varphi \text{ is a Sahlqvist logic}\}$$

is not recursive [25].

The simplest formula not covered by Sahlqvist's theorem is the Mckinsey axiom $\boldsymbol{ma}$ (see (7) above). It is neither first-order definable| [56,152] [4], nor canonical [55]. The problem whether the following equality holds Recall def?

$$\mathbf{K} \;\oplus\; \Box\Diamond p \to \Diamond\Box p \quad = \quad \mathsf{Log}\,\mathsf{Fr}\,\{\Box\Diamond p \to \Diamond\Box p\}$$

and whether the logic $\mathbf{K} \oplus \boldsymbol{ma}$ is decidable had resisted all attempts based on the standard methods of canonical models and filtration until Fine [44] introduced a new proof technique based on certain normal forms to be considered in the next section.| New

Another logic not covered by Sahlqvist's theorem is the logic $\mathbf{KM}^\infty = \mathbf{K} \oplus \{\boldsymbol{ma}_k \mid k \geq 1\}$ defined in [88], where

$$\boldsymbol{ma}_k \quad = \quad \Diamond \bigwedge_{1 \leq i \leq k} (\Diamond p_i \to \Box p_i).$$

$\boldsymbol{ma}_1$ is $\mathbf{K}$-equivalent to $\boldsymbol{ma}$, so $\mathbf{KM}^\infty \supseteq \mathbf{K} \oplus \boldsymbol{ma}$. In fact, $\mathbf{KM}^\infty$ is the logic of the class of frames satisfying

$$\forall x \exists y \; \big(R(x,y) \land \forall z, z' \; (R(y,z) \land R(y,z') \to z = z')\big), \tag{8}$$

so by Theorem 4.1 it is canonical.

In [64], the proof of Sahlqvist's theorem is extended to $\mathbf{KM}^\infty$ and other logics. The method uses *'quasipositive' hybrid sentences;* see Chapter **??**| for full details of hybrid logic. In these formulas, REF?? existential and relativised universal quantifiers over nominals are allowed, negation can only occur in the latter, and there are no free nominals or propositional variables. From any quasipositive sentence φ, an infinite set of *modal* axioms can be obtained effectively. The modal axioms approximate φ, by

[4] This result was first proved by R. Goldblatt in his PhD thesis in 1974.

treating nominals as propositional variables ranging over the partition sets of finite partitions of the worlds of a model. Each partition is defined by the truth values of an arbitrary finite set of modal formulas. Existential and universal quantification are simulated by disjunctions and conjunctions over partition sets. The axioms obtained in this way axiomatise a modal logic L_φ, which is shown to be the logic of the class of frames validating φ: i.e., $L_\varphi = \mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$.

For example, let $\varphi = \Diamond\exists i\Box i$, where i is a nominal. Then φ is valid in precisely the frames satisfying (8). The axioms obtained from φ are equivalent to substitution instances of the $\boldsymbol{ma}_k$ above, and so $L_\varphi = \mathbf{KM}^\infty$. For instance, the axiom obtained by approximating φ with respect to the finite set $X = \{p_1, \dots, p_k\}$ is

$$\Diamond \bigvee_{Y\subseteq X} \Box\Big(\bigwedge_{p\in Y} p \wedge \bigwedge_{p\in X-Y} \neg p\Big),$$

and this is $\mathbf{K}$-equivalent to $\boldsymbol{ma}_k$.

The method extends to sets Φ of quasipositive sentences. Every L_Φ is the logic of an elementary class of frames, namely, $\mathsf{Fr}\,\Phi$. [64] shows that the modal logics of elementary classes of frames are *precisely* those of the form Λ_Φ. The result applies to multi-modal logics and to logics with polyadic modalities.

This result gives an interesting link between modal and hybrid logic. It is analogous to Sahlqvist's completeness theorem, since $\mathsf{Log}\,\mathsf{Fr}\,\Phi = L_\Phi$. An analogue of Sahlqvist's correspondence theorem would state that $\mathsf{Fr}\,L_\Phi$ is first-order definable (by Φ), but this cannot be achieved in general, since in many cases, $\mathsf{Fr}\,L_\Phi$ is non-elementary.

4.2 Uniform logics

Fine [44] used a modal analogue of the full disjunctive normal form for constructing finite models and proving the fmp of a family of logics in NExt $\mathbf{D}$ (containing, in particular, $\mathbf{K} \oplus \boldsymbol{ma}$).

Observe first that every modal formula $\varphi(p_1, \dots, p_m)$ is equivalent in $\mathbf{K}$ either to $\bot$ or to a disjunction of normal forms (in the variables $p_1, \dots, p_m$) of degree $md(\varphi)$ (the modal depth of φ), which are defined inductively in the following way. $\mathbf{NF}_0$, the set of *normal forms of degree* 0, contains all formulas of the form $\neg_1 p_1 \wedge \cdots \wedge \neg_m p_m$, where each $\neg_i$ is either blank or $\neg$. $\mathbf{NF}_{n+1}$, the set of *normal forms of degree* $n+1$, consists of formulas of the form

$$\theta \wedge \neg_1\Diamond\theta_1 \wedge \cdots \wedge \neg_k\Diamond\theta_k,$$

where $\theta \in \mathbf{NF}_0$ and $\theta_1, \dots, \theta_k$ are all distinct normal forms in $\mathbf{NF}_n$. Put $\mathbf{NF} = \bigcup_{n<\omega} \mathbf{NF}_n$. Using the fact that $\bigvee\{\Diamond\theta \mid \theta \in \mathbf{NF}_n\} \in \mathbf{D}$, it is not hard to see also that in $\mathbf{D}$ every formula φ with $md(\varphi) \le n$ is equivalent either to $\bot$ or to a disjunction of normal forms of degree n such that at least one of $\neg_1, \dots, \neg_k$ in the inductive step of the definition above is blank. Such normal forms are called $\mathbf{D}$-*suitable*. It should be clear that, for any distinct $\theta', \theta'' \in \mathbf{NF}_n$, $\neg(\theta' \wedge \theta'') \in \mathbf{K}$. Consequently, for every $\theta \in \mathbf{NF}_n$ and every $\varphi(p_1, \dots, p_m)$ with $md(\varphi) \le n$, we have either $\theta \to \varphi \in \mathbf{K}$ or $\theta \to \neg\varphi \in \mathbf{K}$.

With each $\mathbf{D}$-suitable normal form θ we associate a model $\mathfrak{M}_\theta = \langle \mathfrak{F}_\theta, \mathfrak{V}_\theta \rangle$ based on $\mathfrak{F}_\theta = \langle W_\theta, R_\theta \rangle$ by taking

$$\begin{array}{lcl} W_\theta & = & \{\top\} \cup \{\theta' \in \mathbf{NF} \mid \theta' <^n \theta, \text{ for some } n \ge 0\}, \\ \theta' < \theta'' & \text{iff} & \Diamond\theta' \text{ is a conjunct of } \theta'', \\ \theta' R_\theta \theta'' & \text{iff} & \text{either } \theta' > \theta'' \text{ or } md(\theta') = 0 \text{ and } \theta'' = \top, \\ \mathfrak{V}_\theta(p) & = & \{\theta' \in W_\theta \mid p \text{ is a conjunct of } \theta'\}. \end{array}$$

According to the definition, $\top$ is the reflexive end-point in $\mathfrak{F}_\theta$, and so $\mathfrak{F}_\theta$ is serial. By a straightforward induction on the degree of $\theta' \in W_\theta$ one can show that $(\mathfrak{M}_\theta, \theta') \models \theta'$. It follows immediately that $\mathbf{D}$ has the finite model property (fmp, for short). Indeed, given $\varphi \notin \mathbf{D}$, we reduce $\neg\varphi$ to a disjunction of $\mathbf{D}$-suitable normal forms with at least one disjunct θ, and then $(\mathfrak{M}_\theta, \theta) \models \theta$.

It turns out that in the same way we can prove the fmp of all logics in NExt $\mathbf{D}$ axiomatisable by *uniform formulas* which are defined as follows. Every φ without modal operators is a *uniform formula of degree* 0; and if $\varphi = \psi(\blacklozenge_1\chi_1, \dots, \blacklozenge_m\chi_m)$, where $\blacklozenge_i \in \{\Box, \Diamond\}$, $md(\psi(p_1, \dots, p_m)) = 0$ and $\chi_1, \dots, \chi_m$ are uniform formulas of degree n, then φ is a *uniform formula of degree* $n+1$. A remarkable property of uniform formulas is the following. Suppose that φ is a uniform formula of degree n and $\mathfrak{M}$, $\mathfrak{N}$ are models based on the same frame $\mathfrak{F} = \langle W, R\rangle$ and such that, for some point x, $(\mathfrak{M}, y) \models p$ iff $(\mathfrak{N}, y) \models p$ for every $y \in x{\uparrow}^n$ and every variable p in φ, where

$$X{\uparrow} \ = \ X{\uparrow}^1 \ = \ \{y \in W \mid \exists x \in X \ xRy\}, \qquad X{\uparrow}^{n+1} \ = \ (X{\uparrow}^n){\uparrow} \,.$$

Then $(\mathfrak{M}, x) \models \varphi$ iff $(\mathfrak{N}, x) \models \varphi$.

Given a logic L, we call a normal form θ *L-suitable* if $\mathfrak{F}_\theta \models L$.

Theorem 4.4 (Fine) *Every logic $L \in$ NExt$\mathbf{D}$ axiomatisable by uniform formulas has the fmp.*

Proof It suffices to prove that each formula φ with $md(\varphi) \le n$ is equivalent in L either to $\bot$ or to a disjunction of L-suitable normal forms of degree n. And this fact will be established if we show that every $\mathbf{D}$-suitable normal form θ such that $\theta \to \bot \notin L$ is L-suitable. Suppose otherwise. Let θ be an L-consistent and $\mathbf{D}$-suitable normal form of the least possible degree under which it is not L-suitable. Then there are a uniform formula $\psi \in L$ of some degree m and a model $\mathfrak{M} = \langle \mathfrak{F}_\theta, \mathfrak{V}\rangle$ such that $(\mathfrak{M}, \theta) \not\models \psi$.

For every variable p in ψ, let $\Gamma_p = \{\theta' \in \theta{\uparrow}^m \mid (\mathfrak{M}, \theta') \models p\}$ and let $\delta_p = \bigvee \Gamma_p$ (if $\Gamma_p = \emptyset$ then $\delta_p = \bot$). Observe that for every $\theta' \in \theta{\uparrow}^m$ we have $(\mathfrak{M}_\theta, \theta') \models \delta_p$ iff $\theta' \in \Gamma_p$ iff $(\mathfrak{M}, \theta') \models p$. Therefore, the formula ψ' that results from ψ by replacing each p with δ_p is false at θ in $\mathfrak{M}_\theta$. Now, if $md(\psi') > n$ then $m > n$, and so $\delta_p = \bot$ for every p in ψ, i.e., ψ' is variable free. But then ψ' is equivalent in $\mathbf{D}$ to $\top$ or $\bot$, contrary to $\mathfrak{F}_\theta \not\models \psi'$ and L being consistent. And if $md(\psi') \le n$ then either $\theta \to \psi' \in \mathbf{K}$, which is impossible, since $(\mathfrak{M}_\theta, \theta) \not\models \theta \to \psi'$, or $\theta \to \neg\psi' \in \mathbf{K}$, from which $\psi' \to \neg\theta \in \mathbf{K}$ and so $\neg\theta \in L$, contrary to θ being L-consistent. ❑

It is not hard to extend Fine's theorem to the multi-modal case, namely, to those logics that contain $\Diamond_i\top$, for all $i = 1, \dots, n$, and are axiomatisable by formulas φ in which all maximal sequences of nested modal operators coincide with respect to the distribution of the indices i of $\Box_i$ and $\Diamond_i$.

As a consequence of Theorem 4.4 we obtain that $\mathbf{KM} = \mathbf{K} \oplus \boldsymbol{ma}$ enjoys the fmp and so is decidable. Strange as it may seem, the following problem is still open:| New

Problem 4.5 *What is the computational complexity of* $\mathbf{KM}$*?*

4.3 Logics with $\Box\Diamond$-axioms

Another result, connecting the fmp of logics with the distribution of $\Box$ and $\Diamond$ over their axioms, is based on the following observation which can be regarded as the modal analogue of Glivenko's theorem for intuitionistic logic (see, e.g., [26]): for any $\varphi, \psi \in \mathcal{ML}_1$, we have $\Diamond\varphi \leftrightarrow \Diamond\psi \in \mathbf{S5}$ iff $\Box\Diamond\varphi \leftrightarrow \Box\Diamond\psi \in \mathbf{K4}$. The proof of this observation is almost trivial. Suppose $\Box\Diamond\varphi \to \Box\Diamond\psi \notin \mathbf{K4}$. Then there exist a finite model $\mathfrak{M}$, based on a transitive frame, and a point x in it such that $x \models \Box\Diamond\varphi$ and $x \not\models \Box\Diamond\psi$. It follows from the former that every final cluster accessible from x, if any, is non-degenerate and contains a point where φ is true. The latter means that x 'sees' a final cluster C at all points of which ψ is false. Now, by taking the generated submodel of $\mathfrak{M}$ based on C, we obtain a model for $\mathbf{S5}$ refuting $\Diamond\varphi \to \Diamond\psi$. The rest is obvious, since $\Diamond p \leftrightarrow \Box\Diamond p$ is in $\mathbf{S5}$ and $\mathbf{K4} \subseteq \mathbf{S5}$.

$\mathcal{ML}_1$-formulas in which every occurrence of a variable is in the scope of a modality $\Box\Diamond$ will be called *$\Box\Diamond$-formulas*. The next theorem is due to Rybakov [124].

Theorem 4.6 (Rybakov) *If a logic $L \in$ NExt$\mathbf{K4}$ is decidable (or has the fmp) and ψ is a $\Box\Diamond$-formula then $L \oplus \psi$ is also decidable (has the fmp).*

Proof Let $\psi = \psi'(\Box\Diamond\chi_1, \dots, \Box\Diamond\chi_n)$, for some formula $\psi'(q_1, \dots, q_n)$. If $\varphi(p_1, \dots, p_m) \in L \oplus \psi$ then there exists a derivation of φ in $L \oplus \psi$ in which substitution instances of ψ contain no variables different from $p_1, \dots, p_m$. Each of these instances has the form $\psi'(\Box\Diamond\chi'_1, \dots, \Box\Diamond\chi'_n)$, where every χ'_i is some substitution instance of χ_i containing only $p_1, \dots, p_m$. Now, it is not hard to see that, similarly to classical propositional logic, there are finitely many pairwise nonequivalent formulas in **S5** built from $p_1, \dots, p_m$ (for more details see, e.g., [129] or [26]). In view of the observation above, there are finitely many pairwise nonequivalent in **K4** substitution instances of $\Box\Diamond\chi_i$ of that sort (the reader can easily estimate the number of them). So there exist only finitely many pairwise nonequivalent in **K4** substitution instances of ψ containing $p_1, \dots, p_m$, say, $\psi_1, \dots, \psi_k$, and we can effectively construct them. Then, by the deduction theorem,

$$\varphi \in L \oplus \psi \qquad \text{iff} \qquad \Box^+(\psi_1 \wedge \dots \wedge \psi_k) \to \varphi \in L,$$

where $\Box^+\chi = \chi \wedge \Box\chi$. Thus, $L \oplus \psi$ is decidable (or has the fmp) whenever L is decidable (has the fmp). ❑

It should be noted that by adding infinitely many $\Box\Diamond$-formulas to a logic L with the fmp one can construct a Kripke incomplete logic; for a concrete example see [123].

4.4 Logics with noniterative axioms

Lewis [91] considered those logics in $\mathrm{NExt}\,\mathbf{K}_n$ that can be axiomatised by $\mathcal{ML}_n$-formulas without nested modal operators. We call such logics *noniterative*. Examples of noniterative logics are

$$\mathbf{T} \;=\; \mathbf{K} \oplus \Box p \to p \qquad \text{or} \qquad \mathbf{K}_2 \oplus \Box_2 p \to \Box_1 p.$$

Theorem 4.7 (Lewis) *All noniterative logics in* $\mathrm{NExt}\,\mathbf{K}_n$ *have the fmp.*

Proof Suppose that the axioms of $L = \mathbf{K}_n \oplus \Gamma$ have no nested modal operators and $\varphi \notin L$. Let $\mathsf{sub}\,\varphi$ be the set of all subformulas of φ. By a *φ-description* we mean any set of subformulas of φ together with the negations of the remaining formulas in $\mathsf{sub}\,\varphi$. For each L-consistent φ-description Θ, take a maximal L-consistent set Δ_Θ containing Θ. Denote by W the (finite) set of the selected Δ_Θ and define $\mathfrak{F} = \langle W, \langle R_i \mid i \in I\rangle\rangle$ and $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ by taking

$$\Delta_\Theta R_i \Delta_\Psi \qquad \text{iff} \qquad \Diamond_i \bigwedge \Psi \in \Delta_\Theta$$

and $\mathfrak{V}(p) = \{\Delta_\Theta \in W \mid p \in \Delta_\Theta\}$. It is easily proved that $(\mathfrak{M}, \Delta_\Theta) \models \psi$ iff $\psi \in \Delta_\Theta$, for all subformulas ψ of φ and $\Delta_\Theta \in W$. Hence $\mathfrak{F} \not\models \varphi$. It is also easy to see that for all truth-functional compounds ψ of subformulas in φ,

$$(\mathfrak{M}, \Delta_\Theta) \models \Diamond_i\psi \qquad \text{iff} \qquad \Diamond_i\psi \in \Delta_\Theta. \tag{9}$$

Consider now a model $\mathfrak{M}' = \langle \mathfrak{F}, \mathfrak{V}'\rangle$ and $\chi \in \Gamma$. For each variable p put

$$\psi_p \quad = \quad \bigvee \left\{ \bigwedge \Theta \mid \Delta_\Theta \in \mathfrak{V}'(p) \right\}$$

and denote by χ' the result of substituting ψ_p for p, for each p in χ. Then $\mathfrak{M}' \models \chi$ iff $\mathfrak{M} \models \chi'$. In view of (9), we have $\mathfrak{M} \models \chi'$ because χ' has no nested modalities. Thus, $\mathfrak{F} \models \chi$ and so $\mathfrak{F} \models L$. ❑

4.5 Modal reduction principles

Modal reduction principles—that is, formulas of the form $\boldsymbol{M}p \to \boldsymbol{N}p$, where $\boldsymbol{M}$ and $\boldsymbol{N}$ are strings of $\Box_i$ and $\Diamond_j$—have always attracted the attention of modal logicians, with the aim being to reduce the number of nested modal operators. (For example, both $\Box\Diamond p \leftrightarrow \Diamond p$ and $\Diamond\Box p \leftrightarrow \Box p$ are in **S5**.) In the context of this chapter, we are interested in completeness and decidability of normal modal logics axiomatised by modal reduction principles.

As we know from Section 4.1, there are undecidable logics in NExt $\mathbf{K}_2$ with finitely many modal reduction principles as their axioms. But it seems that nearly nothing is known about the behaviour of logics with such axioms in NExt $\mathbf{K}$. Perhaps one of the most intriguing open problems in Modal Logic is the following:

Problem 4.8 *Do the logics of the form* $\mathbf{K} \oplus \Box^n p \to \Box^m p$ *have the finite model property?*

Note, however, that by Sahlqvist theorem all logics of the form $\mathbf{K} \oplus \Box^n p \to \Box^m p$ are characterised by their Kripke frames that are definable by first-order formulas (which are similar to transitivity: if y is accessible from x in m steps, then y is accessible from x in n steps as well).

Van Benthem [153] showed that modal reduction principles in $\mathcal{ML}_1$ are all first-order definable over *transitive* frames (this is not the case in general: e.g., the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$ is not first-order definable over arbitrary frames; see [153] for a complete characterisation).

The following result was proved in [175] using the method of canonical formulas to be discussed in Section 6:

Theorem 4.9 *All logics in* NExt $\mathbf{K4}$ *axiomatisable by modal reduction principles have the fmp and are decidable.*

Problem 4.10 *Are extensions of* $\mathbf{K}_n$ *with modal reduction principles Kripke complete?*

Problem 4.11 *Are extensions of* $\mathbf{K}$ *with modal reduction principles decidable?*

4.6 Logics with n-variable axioms

A very natural syntactical parameter of a modal logic $L \oplus \Gamma$ is the number of variables in its extra axioms Γ over L. For example, $\mathbf{D} = \mathbf{K} \oplus \Diamond\top$ is axiomatised by a *variable-free* formula over $\mathbf{K}$ and almost all standard modal logics—e.g., **K4**, **S4**, **S5**, **Grz**, **GL**—can be axiomatised by adding axioms with only one variable to $\mathbf{K}$. (A notable exception is **K4.3** whose axiomatisation requires two variables; see [119]).

We start our discussion of variable-free axioms with a simple observation that the truth of a variable-free formula φ does not depend on the valuation, i.e., for every model $\mathfrak{M}$ based on a frame $\mathfrak{F}$, we have $\mathfrak{M} \models \varphi$ iff $\mathfrak{F} \models \varphi$. Therefore, we can reduce deduction in $L \oplus \varphi$ to 'global' deduction in L. More precisely, we say that a formula ψ *follows globally* from a set of formulas Γ in a logic L if $\mathfrak{M} \models \Gamma$ implies $\mathfrak{M} \models \psi$ for every model $\mathfrak{M}$ based on a frame for L. Now, if global deduction is decidable for L, then $L \oplus \varphi$ is decidable: indeed, $\psi \in L \oplus \varphi$ iff ψ follows globally from φ in L.

To make use of this observation we need to know how to prove decidability of 'global deducibility' for modal logics L. For $L \in$ NExt $\mathbf{K4}$ this is simple because ψ follows globally from φ in L iff $\varphi \wedge \Box\varphi \to \psi \in L$.

Theorem 4.12 *If* φ *is variable-free and* $L \in$ NExt $\mathbf{K4}$ *is decidable, then* $L \oplus \varphi$ *is decidable.*

Note that there are extensions of **K4** with *infinitely many* variable free formulas which are undecidable and do not have the fmp; for a concrete example and further details see [26]. This cannot happen in NExt **GL**, however, since each variable-free formula is deductively equivalent in **GL** to one of the formulas $\top$, $\Box^n\bot$, for $n < \omega$. As $\Box^i\bot \to \Box^j\bot \in \mathbf{K4} \subseteq \mathbf{GL}$, for $i \leq j$, all extensions of **GL** with variable-free formulas are finitely axiomatisable and decidable.

The simple reduction of global deducibility above does not apparently work for 'non-transitive' logics in NExt $\mathbf{K}$. In fact, there are decidable normal modal logics L such that global deducibility is undecidable for L. (The first example of such a logic was constructed by Spaan [139]. Another natural example is $\mathbf{K} \times \mathbf{K}$; for details see Chapter ??| or [48].) It should be clear that global deducibility in a finitely axiomatisable modal logic L is decidable if L enjoys the so-called *global finite model property* (global fmp, for short): for every finite set $\Gamma \cup \{\psi\}$ of formulas, ψ does not follow globally from Γ iff there exists a finite model $\mathfrak{M}$ based on a frame for L such that $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \not\models \varphi$. The global Ref?

fmp of many standard modal logics like $\mathbf{K}_n$, $\mathbf{K4}_n$, $\mathbf{S5}_n$ can be proved by filtration: just start with a possibly infinite model $\mathfrak{M}$ such that $\mathfrak{M} \models \Gamma$ and $\mathfrak{M} \not\models \psi$, and then filtrate it through the subformulas of $\Gamma \cup \{\psi\}$; see [59] for details. For example, we have the following:

Theorem 4.13 *If φ is variable-free, then $\mathbf{K}_n \oplus \varphi$ is decidable.*

The undecidable Sahlqvist logic on page 19 shows that this result does not hold for axioms with one variable. Chagrov [21] constructed a one-variable formula φ such that $\mathbf{GL} \oplus \varphi$ is undecidable. It turns out, however, that we have the following theorem which was proved in [175] using the method of canonical formulas (to be discussed in Section 6):

Theorem 4.14 *For every one-variable formula φ, $\mathbf{S4} \oplus \varphi$ has the fmp and is decidable.*

On the other hand, an infinite number of one-variable axioms can yield an extension of **S4** without the fmp [133]. A Kripke incomplete extension of **S4** with a two-variable axiom was constructed by Shehtman [132] and an undecidable logic above **S4** with a three-variable axiom by Chagrov [21].

Problem 4.15 *Are extensions of* **S4** *with a two-variable axiom decidable?*

5 Semantically constrained classes of modal logics

In the previous section we considered 'nice' classes of modal logics defined in terms of the form of the logics' axioms. Here we give a brief overview of well-behaved classes of modal logics determined by imposing some natural constraints on the form of their frames.

Tabular logics

A normal modal logic L is said to be *tabular* if it is determined by a *finite* set of *finite* frames. Since the class of frames for a normal modal logic is closed under disjoint unions, L is tabular iff there exists a single finite frame that determines L. In many respects tabular logics are easy to deal with. For instance, the problem of deciding whether a formula φ belongs to a tabular logic is trivially decided in NP by considering all possible valuations in the finite frame characterising L. Moreover, it is not difficult to provide a finite axiomatisation for a tabular logic; for details see, e.g., [177]. Thus, we arrive at the following:

Theorem 5.1 *Every tabular normal modal logic is NP-complete and finitely axiomatisable.*

What is the position of tabular logics within the lattices $\mathrm{NExt}\,\mathbf{K}_n$? First, it is easy to see that every normal modal logic containing a tabular modal logic L is tabular as well and is determined by frames that are p-morphic images of generated subframes of any frame which determines L. Therefore, we have:

Theorem 5.2 *If L is tabular then* $\mathrm{NExt}\,L$ *is finite and contains only tabular logics.* $\mathrm{NExt}\,L$ *can be effectively computed.*

On the other hand, according to Theorem 3.8, the axiomatisation problem for tabular modal logics is always undecidable in $\mathrm{NExt}\,\mathbf{K}_n$. The situation is not so hopeless if we consider the following relativised version of the axiomatisation problem for tabular logics above some sufficiently 'strong' logic $L_0 \supset \mathbf{K}_n$: given a tabular logic $L \supset L_0$ and an arbitrary formula φ, decide whether $L_0 \oplus \varphi = L$. For example, one can easily show (see, e.g., [26,177] and references therein) that every tabular logic containing **K4** is a union-splitting of **K4** and that a logic is tabular in $\mathrm{NExt}\,\mathbf{K4}$ iff it has finitely many normal extensions. Moreover, the following holds:

Theorem 5.3 *If $L \in \mathrm{NExt}\,\mathbf{K4}$ is tabular then $\{\varphi \mid \mathbf{K4} \oplus \varphi = L\}$ is decidable.*

How to determine whether a given logic is tabular? The key idea suggested by Kuznetsov [82] is to consider the so-called pretabular logics.

A logic $L \in \mathrm{NExt}\, L_0$ is said to be *pretabular* if L is not tabular but every proper extension of L in $\mathrm{NExt}\, L_0$ is tabular. In other words, a pretabular logic is a maximal nontabular logic in $\mathrm{NExt}\, L_0$. Using Zorn's lemma it is easily seen that, in $\mathrm{NExt}\, \mathbf{K}_n$, every non-tabular logic is contained in a pretabular one. It is also known that every pretabular logic in $\mathrm{NExt}\, \mathbf{K4}$ has the fmp (for proofs and references consult [26]). Moreover, Maksimova [96] and Esakia and Meskhi [38] showed that there are only five (pretty simple) pretabular logics in $\mathrm{NExt}\, \mathbf{S4}$. Using this result one can show the following:

Theorem 5.4 *The set $\{\varphi \mid \mathbf{S4} \oplus \varphi$ is tabular$\}$ is decidable.*

Note that there are a continuum of pretabular logics in $\mathrm{NExt}\, \mathbf{K4}$, while $\mathrm{NExt}\, \mathbf{GL}$ contains countably many of them [10,17], and the set $\{\varphi \mid \mathbf{GL} \oplus \varphi$ is tabular$\}$ is decidable.

Transitive logics of finite depth and width

A very natural semantical constraint on logics from $\mathrm{NExt}\, \mathbf{K4}$ is the length of maximal chains and antichains in their rooted frames. Say that a logic $L \in \mathrm{NExt}\, \mathbf{K4}$ is of *depth* $n < \omega$ if L has a frame $\langle W, R\rangle$ with a chain $x_1 R x_2 R \dots R x_n$ of points from distinct clusters, but no frame with such chains of greater length validates L. Syntactically, logics of depth n can be defined as those extensions of $\mathbf{K4}$ that contain the formula $\boldsymbol{bd}_n$ but not $\boldsymbol{bd}_{n+1}$, where

$$\begin{array}{rcl} \boldsymbol{bd}_1 & = & \Diamond\Box p_1 \to p_1, \\ \boldsymbol{bd}_{n+1} & = & \Diamond(\Box p_{n+1} \wedge \neg \boldsymbol{bd}_n) \to p_{n+1}. \end{array}$$

The following theorem was proved in [129]:

Theorem 5.5 (Segerberg) *Every logic of finite depth has the finite model property (in fact, is locally tabular in the sense that it has only finitely many nonequivalent formulas with variables $p_1, \dots, p_n$), and so is decidable if finitely axiomatisable.*

It is to be noted that there are a continuum of logics of depth 3 [84].

Say that a logic $L \in \mathrm{NExt}\, \mathbf{K4}$ is of *width* $n < \omega$ if it has a rooted frame $\langle W, R\rangle$ with an antichain $x_1, \dots, x_n$ (i.e., $x_i R x_j$ does not hold for any distinct $i, j \le n$) but no rooted frame with an $n+1$-point antichain validates L. Syntactically, logics of width n can be described as those extensions of $\mathbf{K4}$ that contain the formula $\boldsymbol{bw}_n$ but not $\boldsymbol{bw}_{n+1}$, where

$$\boldsymbol{bw}_n \quad = \quad \bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0 \le i \ne j \le n} \Diamond\big(p_i \wedge (p_j \vee \Diamond p_j)\big).$$

The logics of width 1 are precisely the extensions of $\mathrm{NExt}\, \mathbf{K4.3}$.

The following theorem was proved in [43]:

Theorem 5.6 (Fine) *All logics of finite width are Kripke complete.*

There are a continuum of logics of width 1. However, those of them that are finitely axiomatisable behave quite nicely as was shown in [176,93]:

Theorem 5.7 *All finitely axiomatisable logics in* $\mathrm{NExt}\, \mathbf{K4.3}$ *are decidable (in fact coNP-complete), though not necessarily have the finite model property.*

Nothing is known about decidability of finitely axiomatisable logics of width $n > 1$ (our conjecture is that all of them are decidable):

Problem 5.8 *Are finitely axiomatisable logics of width $n > 1$ decidable? What is their computational complexity?*

For logics above **S4.3** we have the following classical result of [14,40,139]:

Theorem 5.9 (Bull, Fine, Spaan) *All logics in* NExt **S4.3** *are finitely axiomatisable, have the finite model property and are coNP-complete.*

Problem 5.10 *Does there exist an algorithm that decides, given a formula φ, whether the logic* $\mathbf{K4} \oplus \varphi$ *is of finite width/depth?*

Logics containing **K5**

Recall that $\mathbf{K5} = \mathbf{K} \oplus \Diamond p\Box \to \Box p$ is the logic determined by all *Euclidean* frames $\langle W, R \rangle$, where Euclidean means that

$$\forall x \forall y \forall z (xRz \wedge xRy \to yRz).$$

The papers [113,114] investigate the (possibly non-normal) extensions of **K5**. The following theorem summarises the results for logics in NExt **K5**:

Theorem 5.11 (Nagle, Thomason) *All logics in* NExt **K5** *have the finite model property, are finite axiomatisable, and so decidable. The lattice* NExt **K5** *can be computed effectively.*

It is not difficult to see that actually all logics in NExt **K5** have the polynomial finite model property and are coNP-complete.

Logics containing $\mathbf{S5} \times \mathbf{S5}$

$\mathbf{S5} \times \mathbf{S5}$ is the bimodal logic determined by product frames of the form $\langle W \times W, R_1, R_2 \rangle$, where $(w_1, w_2) R_1 (w_1', w_2')$ iff $w_2 = w_2'$, and $(w_1, w_2) R_2 (w_1', w_2')$ iff $w_1 = w_1'$. It was introduced and investigated because of its close relation to the two-variable fragment of first-order logic [130,48]; see also Chapter **??**. $\mathbf{S5} \times \mathbf{S5}$ can be axiomatised by adding to the fusion| of **S5** with **S5** the modal axiom $\Diamond_1 \Diamond_2 p \leftrightarrow \Diamond_2 \Diamond_1 p$ saying that R_1 and R_2 commute. Logics containing $\mathbf{S5} \times \mathbf{S5}$ are surprisingly well-behaved. Indeed, while $\mathbf{S5} \times \mathbf{S5}$ itself is NEXPTIME-complete [103], it was proved in [6,5] that we have the following: Reference or def?

Theorem 5.12 *Every normal bimodal logic properly containing* $\mathbf{S5} \times \mathbf{S5}$ *has the poly-size model property (i.e., every refutable formula φ is refutable in a model of size polynomial in φ), it is finitely axiomatisable, and coNP-complete.*

The axiomatisation result is based on a variant of Kruskal's tree theorem. The picture is different if a constant for the diagonal $\{(w, w) \mid w \in W\}$ (motivated by its interpretation as 'equality' in first-order logic) is added to the bimodal language. In this case there exist uncountably many normal modal logics extending $\mathbf{S5} \times \mathbf{S5}$ with the diagonal, and it is open whether all of them have the finite model property [3,4].

Logics containing $\mathbf{K} \oplus \mathbf{alt}_n$

A frame $\mathfrak{F} = \langle W, R \rangle$ validates the formula

$$\boldsymbol{alt}_n \quad = \quad \Box p_1 \vee \Box(p_1 \to p_2) \vee \cdots \vee \Box(p_1 \wedge \cdots \wedge p_n \to p_{n+1}),$$

where $n \geq 0$, iff each point in $\mathfrak{F}$ has at most n distinct R-successors. Segerberg [131] proved the following:

Theorem 5.13 *All logics in* NExt $(\mathbf{K} \oplus \boldsymbol{alt}_1)$ *have the finite model property, are finitely axiomatisable, and so decidable. The lattice* NExt $(\mathbf{K} \oplus \boldsymbol{alt}_1)$ *can be computed effectively.*

It is not difficult to see that actually all extensions of $\mathbf{K} \oplus \boldsymbol{alt}_1$ have the polynomial finite model property and are coNP-complete. Extensions of $\mathbf{K} \oplus \boldsymbol{alt}_n$ for $n > 1$ are investigated in [1]:

Theorem 5.14 *All logics in* NExt $(\mathbf{K} \oplus \boldsymbol{alt}_n)$ *are Kripke complete and their frames are first-order definable.*

An analysis of extensions of fusions of NExt $(\mathbf{K} \oplus \boldsymbol{alt}_n)$ is given in [76,61].

6 Frame-theoretic characterisation

Finding characterisations of those classes of structures that can be defined by (sets of) formulas of a given language $\mathcal{L}$ is one of the central research problems in the development of a model theory for the language. This is often achieved by introducing certain truth-preserving operators on classes of structures (e.g., the formation of p-morphic images, generated subframes, disjoint unions, ultraproducts, etc.) and then proving that the $\mathcal{L}$-definable classes are precisely those that are closed under these operators—a kind of Birkhoff-type theorem for varieties of abstract algebras. Such characterisations for modal logics are discussed in Chapters **??** and **??**| on modal model theory. Unfortunately, abstract characterisations of this sort are of limited use when we deal with modal decision problems. In this context, what we need is not characterisations that are 'as abstract as possible,' but rather *explicit finitely presentable* ones. Refs correct?

Of course, modal formulas themselves can be regarded as a 'finitely presented characterisation' of modally definable classes of frames. However, in general, the information contained in formulas is rather *implicit* and non-structural (or 'non-geometric')—one has to work hard to learn how to decipher their meaning.

As a first step towards more informative finite presentations of modally definable classes of general frames, let us find out which of these classes $\mathcal{F}$ cannot be *decomposed* in the sense that whenever $\mathcal{F} = \mathsf{GFr}\,\Gamma$, for some set Γ of modal formulas, then there is a $\psi \in \Gamma$ such that $\mathcal{F} = \mathsf{GFr}\,\{\psi\}$. In other words, this means that we cannot make the information provided by such a formula ψ 'more explicit' by replacing it with two (or more) formulas ψ_1 and ψ_2 such that $\mathsf{GFr}\,\{\psi\} = \mathsf{GFr}\,\{\psi_1, \psi_2\}$, but $\mathsf{GFr}\,\{\psi\} \subsetneqq \mathsf{GFr}\,\{\psi_i\}$ for $i = 1, 2$.

Again, as in Blok's dichotomy and Chagrov's classification, it is the notion of splittings that provides a proper framework for investigating indecomposability. In fact, one can show that, for every normal modal logic L, the class $\mathsf{GFr}\,L$ is indecomposable iff there exists a finite rooted (cycle free) frame $\mathfrak{F}$ such that $L = \mathbf{K}/\mathfrak{F}$. Indeed, suppose that $L = \mathbf{K}/\mathfrak{F} = \mathbf{K} \oplus \Gamma$ for some set of formulas Γ. Then there is a $\psi \in \Gamma$ such that $\mathfrak{F} \not\models \psi$ (for otherwise $\mathfrak{F} \models \Gamma$ and therefore $\mathfrak{F} \in \mathsf{GFr}\,L$). But then $L \subseteq \mathbf{K} \oplus \psi$, from which (since $\psi \in \Gamma$) $L = \mathbf{K} \oplus \psi$. (For the other direction and further details see Chapter **??**.|) Thus, we can say that the formula ψ describes $\mathcal{F}$. Moreover, in view of (4), ψ is deductively equivalent to the formula $\Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r$, where $n = d(\mathfrak{F})$, which explicitly says: '$\mathfrak{G} \in \mathcal{F}$ *iff there does not exist a generated subframe of* $\mathfrak{G}$ *having* $\mathfrak{F}$ *as its p-morphic image.*' The formula $\Box^{\leq n}\delta_{\mathfrak{F}} \to \neg p_r$ can be regarded as a *modal diagram* of $\mathfrak{F}$. Ref cortrect?

It follows from these considerations that we would have a kind of optimal explicit and finite presentation of modally definable classes if we could prove that for every formula φ there exists a set $\mathcal{F}$ of finite rooted frames such that

$$\mathsf{GFr}\,\varphi \quad = \quad \mathsf{GFr}\,(\mathbf{K}/\mathcal{F}).$$

Then every modally definable class could be presented by means of a set of indecomposable geometrically explicit formulas as above.

Now, the *bad news* is that we know from the proof of Blok's dichotomy that this is far from being the case: $\mathbf{D}$ is the only standard modal system that can be represented in this way.

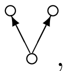
But the *good news* is that this situation changes drastically as soon as we confine ourselves to 'transitive' modal logics, in particular, unimodal normal extensions of $\mathbf{K4}$, linear tense logics, or bimodal provability logics. Although, as we shall see below, it is impossible to represent all normal modal logics extending, say, $\mathbf{K4}$ in the form $L = \mathbf{K4}/\mathcal{F}$, for some set $\mathcal{F}$ of finite rooted frames, we can still introduce appropriate modifications and extensions of the notion of splitting which allow geometric finite representations of every normal extension of **K4**. In this chapter we will show such representations first for normal extensions of **K4** and then for extensions of linear tense logics. The case of bimodal provability logics is considered in [167].

6.1 Canonical formulas for **K4**

This frame-theoretic or 'geometric' approach to investigating modal logics in NExt **K4** and similar classes was launched by Jankov [66,69] (in the framework of extensions of intuitionistic logic[5]), Blok [8], Fine [45] and Zakharyaschev [172,174,175]. Let us observe first that a number of standard logics in NExt **K4** are indeed union-splittings, and so their frames can be elegantly characterised in frame-theoretic terms. For example,

$$\mathbf{S4} = \mathbf{K4} \oplus \Box p \to p = \mathbf{K4}/\{\bullet,$$  $$\},$$

$$\mathbf{S4.1} = \mathbf{S4} \oplus \Box\Diamond p \to \Diamond\Box p = \mathbf{S4}/②,$$

$$\mathbf{S4.2} = \mathbf{S4} \oplus \Diamond\Box p \to \Box\Diamond p = \mathbf{S4}/$$  ,

where ② is a two-point cluster. As we saw above, this means that, e.g., for every (general) frame $\mathfrak{F}$ for **S4**, $\mathfrak{F} \not\models \Box\Diamond p \to \Diamond\Box p$ iff there is a generated subframe of $\mathfrak{F}$ which can be p-morphically mapped onto ②. To appreciate the elegance of this frame-theoretic language, compare the purely geometric characterisation above with the standard first-order description of the Kripke frames for **S4.1**:

$$\langle W, R\rangle \models \Box\Diamond p \to \Diamond\Box p \quad \text{iff} \quad \forall x \exists y\ \big(xRy \wedge \forall z\ (yRz \to y = z)\big).$$

This observation leads to the following natural questions:

(A) Is it possible to characterise transitive frames for arbitrary formulas in a similar way?

(B) If this is indeed the case, then perhaps the decision problem (as well as many other problems) could be reduced to 'comparing' some finite frames? (E.g., $\mathbf{K4}/\mathfrak{F} \subseteq \mathbf{K4}/\mathfrak{G}$ iff $\mathfrak{G}$ is a p-morphic image of some generated subframe of $\mathfrak{F}$.)

We analyse these questions using a number of simple examples. Consider first the Gödel–Löb provability logic $\mathbf{GL} = \mathbf{K4} \oplus \boldsymbol{la}$, where

$$\boldsymbol{la} = \Box(\Box p \to p) \to \Box p.$$

It is well-known that a Kripke frame $\mathfrak{F}$ validates $\boldsymbol{la}$ iff $\mathfrak{F}$ is transitive, irreflexive (i.e., a strict partial order) and *Noetherian* in the sense that it contains no infinite ascending chain. It is also well-known that the condition of Noetherianness is not a first-order one. But what is more important in the present context, the frame

refutes $\boldsymbol{la}$ and yet contains no generated subframe that can be p-morphically mapped onto a finite frame refuting $\boldsymbol{la}$. This means that **GL** is not a union-splitting of NExt **K4** by means of finite frames. To find an explicit finitely presentable geometric characterisation of frames for **GL** some other frame-theoretic constructions are needed. Let us have another look at the structure of countermodels for $\boldsymbol{la}$.

[5] Jankov [69] described all 'conjunctively indecomposable' intuitionistic formulas—i.e., splittings of the extensions of intuitionistic logic—and promised to investigate 'decomposable formulas' in his next paper which has never appeared. At the beginning of the 1980s he was arrested by the KGB for his support of the Solidarity movement in Poland.

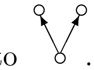
Suppose that a general frame $\mathfrak{F} = \langle W, R, P \rangle$ refutes $\boldsymbol{la}$ under some valuation. Then the set $V = \{x \in W \mid x \not\models \boldsymbol{la}\}$ is in P and $V \subseteq V{\downarrow} = \{w \in W \mid \exists v \in V \ wRv\}$. It follows from the former that $\mathfrak{G} = \langle V, R{\restriction}V, \{X \cap V \mid X \in P\}\rangle$ is a frame—we call it the *subframe of* $\mathfrak{F}$ *induced by* V. And the latter condition means that there is a p-morphism from $\mathfrak{G}$ onto a single reflexive point $\circ$, which is the simplest refutation frame for $\boldsymbol{la}$. Moreover, one can readily check that the converse also holds: if there is a subframe $\mathfrak{G}$ of $\mathfrak{F}$ which can be p-morphcally mapped onto $\circ$ then $\mathfrak{F} \not\models \boldsymbol{la}$.

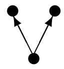
This example motivates the following definitions. Given frames $\mathfrak{F} = \langle W, R, P \rangle$ and $\mathfrak{G} = \langle V, S, Q \rangle$, a partial (i.e., not totally defined, in general) map f from W onto V is called a *subreduction* of $\mathfrak{F}$ to $\mathfrak{G}$ if it satisfies the following conditions, for all $x, y \in \operatorname{dom} f = f^{-1}(V)$ and all $X \in Q$,

- xRy implies $f(x)Sf(y)$;
- $f(x)Sf(y)$ implies $\exists z \in W \ \big(xRz \wedge f(z) = f(y)\big)$;
- $f^{-1}(X) \in P$.

In other words, an *f-subreduct* of $\mathfrak{F}$ is a p-morphic image—or a *reduct*—of the subframe of $\mathfrak{F}$ induced by $\operatorname{dom} f$. A frame $\mathfrak{G} = \langle V, S, Q \rangle$ is a *subframe* of $\mathfrak{F} = \langle W, R, P \rangle$ if $V \subseteq W$ and the identity map on V is a subreduction of $\mathfrak{F}$ to $\mathfrak{G}$, i.e., if $S = R{\restriction}V$ and $Q \subseteq P$. Note that a generated subframe $\mathfrak{G}$ of $\mathfrak{F}$ is not in general a subframe of $\mathfrak{F}$, since V may be not in P.

Thus, the characterisation of frames for $\mathbf{GL}$ can be reformulated like this: $\mathfrak{F} \not\models \boldsymbol{la}$ iff $\mathfrak{F}$ is subreducible to $\circ$. Here are two more examples:

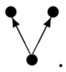
- A frame $\mathfrak{F}$ refutes the *Grzegorczyk axiom* $\Box(\Box(p \to \Box p) \to p) \to p$ iff is subreducible to $\bullet$ or to ②.
- A quasi-order $\mathfrak{F}$ refutes the *Dummett axiom* $\Box(\Box p \to q) \vee \Box(\Box q \to p)$ iff $\mathfrak{F}$ is subreducible to .

Now let us consider the logic $\mathbf{GL.2} = \mathbf{GL} \oplus \boldsymbol{ga}$, where $\boldsymbol{ga}$ is the *Geach axiom* $\Diamond\Box p \to \Box\Diamond p$. It is easy to see that every Kripke frame refuting $\boldsymbol{ga}$ must contain the fork  as a subframe, and in general, if $\mathfrak{F} \not\models \boldsymbol{ga}$ then $\mathfrak{F}$ is subreducible to this fork. However, the converse does not hold—just add a point $\bullet$ above the spikes of the fork to obtain a counterexample. What we actually need is a fork-like subframe with its spikes having no common successor. A good mathematical notion that is capable of describing this and other similar cases is the notion of cofinality.

A subreduction f of $\mathfrak{F}$ to $\mathfrak{G}$ is called *cofinal* if

$$\operatorname{dom} f{\uparrow} \quad \subseteq \quad \operatorname{dom} f \cup \operatorname{dom} f{\downarrow}$$

or in English: if a point x is accessible from the domain of f then either x belongs to the domain of f itself or 'sees' a point in $\operatorname{dom} f$. For example, if we add to the fork a top point as above, then the resulting frame is subreducible to the original fork, but *not cofinally* because the top point cannot belong to the domain of the subreduction.

Returning back to the Geach axiom $\boldsymbol{ga}$, it is an easy exercise to show that a frame $\mathfrak{F}$ for $\mathbf{GL}$ refutes $\boldsymbol{ga}$ iff $\mathfrak{F}$ is cofinally subreducible to . Another example: a transitive $\mathfrak{F}$ refutes $\Diamond\top$ iff $\mathfrak{F}$ is cofinally subreducible to $\bullet$.

For the majority of standard modal axioms these two notions—plain and cofinal subreductions—are enough. But not for all. The simplest counterexample is the *density axiom* $\boldsymbol{den} = \Box\Box p \to \Box p$. It is refuted by the chain $\mathfrak{H}$ of two irreflexive points but becomes valid if we insert between them a reflexive one. In fact, $\mathfrak{F} \not\models \boldsymbol{den}$ iff there is a subreduction f of $\mathfrak{F}$ to $\mathfrak{H}$ such that $f(x{\uparrow}) = \{a\}$ for no point x in $\operatorname{dom} f{\uparrow} - \operatorname{dom} f$, where a is the final point in $\mathfrak{H}$.

Intuitively, every refutation frame for formulas like $\boldsymbol{la}$ can be constructed by adding new points to a frame $\mathfrak{G}$ that is reducible to some finite refutation frame of fixed size. For formulas like $\boldsymbol{ga}$ we have to take into account the cofinality condition and do not place new points 'above' $\mathfrak{G}$. And formulas like ***den*** impose another restriction: some places inside $\mathfrak{G}$ may be 'closed' for inserting new points. These 'closed domains' can be singled out in the following way.

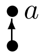
Suppose $\mathfrak{N} = \langle \mathfrak{H}, \mathfrak{U} \rangle$ is a model and $\mathfrak{a}$ an *antichain* in $\mathfrak{H}$—i.e., the points in $\mathfrak{a}$ do not see each other. Say that $\mathfrak{a}$ is an *open domain* in $\mathfrak{N}$ relative to a formula φ if there is a pair $t_\mathfrak{a} = (\Gamma_\mathfrak{a}, \Delta_\mathfrak{a})$ such that $\Gamma_\mathfrak{a} \cup \Delta_\mathfrak{a} = \mathsf{sub}\,\varphi$, $\bigwedge \Gamma_\mathfrak{a} \to \bigvee \Delta_\mathfrak{a} \notin \mathbf{K4}$ and

- $\Box\psi \in \Gamma_\mathfrak{a}$ implies $\psi \in \Gamma_\mathfrak{a}$,
- $\Box\psi \in \Gamma_\mathfrak{a}$ iff $a \models \Box^+\psi$ for all $a \in \mathfrak{a}$.

Otherwise $\mathfrak{a}$ is called a *closed domain* in $\mathfrak{N}$ relative to φ. A reflexive singleton $\mathfrak{a} = \{a\}$ is always open: just take $t_\mathfrak{a} = (\{\psi \in \mathsf{sub}\,\varphi \mid a \models \psi\}, \{\psi \in \mathsf{sub}\,\varphi \mid a \not\models \psi\})$. It is easy to see also that antichains consisting of points from the same clusters are open or closed simultaneously; we will not distinguish between such antichains.

Given a frame $\mathfrak{H}$ and a (possibly empty) set $\mathfrak{D}$ of antichains in $\mathfrak{H}$, we say that a subreduction f of $\mathfrak{F}$ to $\mathfrak{H}$ satisfies the *closed domain condition* for $\mathfrak{D}$ if

$$\text{(CDC)} \qquad \neg\exists x \in \mathrm{dom}\, f{\uparrow} - \mathrm{dom}\, f \ \exists\, \mathfrak{d} \in \mathfrak{D} \quad f(x{\uparrow}) \quad = \quad \mathfrak{d} \cup \mathfrak{d}{\uparrow}.$$

In terms of (CDC) refutation frames for the density axiom ***den*** can be characterised as follows: $\mathfrak{F} \not\models$ ***den*** iff there is a subreduction of $\mathfrak{F}$ to 

satisfying (CDC) for $\{\{a\}\}$.

Suppose now that $\mathfrak{N} = \langle \mathfrak{H}, \mathfrak{U} \rangle$ is a finite countermodel for φ and $\mathfrak{D}$ is the set of all closed domains in $\mathfrak{N}$ relative to φ. We claim that in this case $\mathfrak{F} \not\models \varphi$ whenever there is a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{H}$ satisfying (CDC) for $\mathfrak{D}$. Moreover, if φ is *negation free* (i.e., contains no $\bot$, $\neg$, $\Diamond$) then a plain subreduction satisfying (CDC) for $\mathfrak{D}$ is enough. Indeed, if f is cofinal and $\mathfrak{F} = \langle W, R, P \rangle$ then we can assume that $\mathrm{dom}\, f \cup \mathrm{dom} \uparrow = W$. Define a valuation $\mathfrak{V}$ in $\mathfrak{F}$ as follows. If $x \in \mathrm{dom}\, f$ then we take $x \models p$ iff $f(x) \models p$, for every variable p in φ. If $x \notin \mathrm{dom}\, f$ then $f(x{\uparrow}) \neq \emptyset$, since f is cofinal. Let $\mathfrak{a}$ be an antichain in $\mathfrak{H}$ such that $\mathfrak{a} \cup \mathfrak{a}{\uparrow} = f(x{\uparrow})$. By (CDC), $\mathfrak{a}$ is an open domain in $\mathfrak{N}$, and we put $y \models p$ iff $p \in \Gamma_\mathfrak{a}$, for every $y \notin \mathrm{dom}\, f$ such that $f(y{\uparrow}) = f(x{\uparrow})$. It is easy to check that under this valuation $x \models \psi$ iff $f(x) \models \psi$ in the case $x \in \mathrm{dom}\, f$, and $x \models \psi$ iff $\psi \in \Gamma_\mathfrak{a}$, where $\mathfrak{a}$ is the open domain in $\mathfrak{N}$ associated with x, in the case $x \notin \mathrm{dom}\, f$, for every $\psi \in \mathsf{sub}\,\varphi$. If φ is negation free and f is a plain subreduction then $f(x{\uparrow})$ may be empty. In such a case we just put $x \models p$, for all variables p.

Moreover, given an arbitrary formula φ, one can effectively construct a finite collection of finite rooted frames $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ (of some fixed size that depends on the size of φ) and select in them sets $\mathfrak{D}_1, \dots, \mathfrak{D}_n$ of antichains such that, for any frame $\mathfrak{F}$, $\mathfrak{F} \not\models \varphi$ iff there is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{F}_i$, for some i, satisfying (CDC) for $\mathfrak{D}_i$. If φ is negation free then a plain subreduction satisfying (CDC) is enough. Details can be found in [172,26].

This 'explicit finitely presentable' characterisation of the constitution of refutation transitive frames can be expressed in the language of modal formulas similarly to the equation

$$\mathbf{K}/\mathfrak{F} \quad = \quad \mathbf{K} \ \oplus \ \Box^{\leq n}\delta_\mathfrak{F} \to \neg p_r.$$

Indeed, with every finite frame $\mathfrak{F} = \langle W, R \rangle$ with root r and every (possibly empty) set $\mathfrak{D}$ of antichains in $\mathfrak{F}$ we can associate formulas $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha(\mathfrak{F}, \mathfrak{D})$ such that $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ ($\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathfrak{D})$) iff there is a cofinal (respectively, plain) subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$. Consider, for example, the following formulas

$$\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \quad = \quad \Box^+\delta(\mathfrak{F}, \mathfrak{D}) \to \neg p_r,$$

where

$$
\begin{array}{lcl}
\mathbf{D4} & = & \mathbf{K4} \oplus \alpha(\bullet, \bot) \\
\mathbf{S4} & = & \mathbf{K4} \oplus \alpha(\bullet) \\
\mathbf{GL} & = & \mathbf{K4} \oplus \alpha(\circ) \\
\mathbf{Grz} & = & \mathbf{K4} \oplus \alpha(\bullet) \oplus \alpha(\textcircled{2}) \\
\mathbf{K4.1} & = & \mathbf{K4} \oplus \alpha(\bullet, \bot) \oplus \alpha(\textcircled{2}, \bot) \\
\mathbf{Triv} & = & \mathbf{K4} \oplus \alpha(\bullet) \oplus \alpha(\textcircled{2}) \oplus \alpha\left(\begin{smallmatrix}\circ\\ \uparrow\\ \circ\end{smallmatrix}\right) \\
\mathbf{Verum} & = & \mathbf{K4} \oplus \alpha(\circ) \oplus \alpha\left(\begin{smallmatrix}\bullet\\ \uparrow\\ \bullet\end{smallmatrix}\right) \\
\mathbf{S5} & = & \mathbf{S4} \oplus \alpha\left(\begin{smallmatrix}\circ\\ \uparrow\\ \circ\end{smallmatrix}\right) \\
\mathbf{K4B} & = & \mathbf{K4} \oplus \alpha\left(\begin{smallmatrix}*\\ \uparrow\\ *\end{smallmatrix}\right) \text{ (4 axioms)} \\
\mathbf{K4.2} & = & \mathbf{K4} \oplus \alpha\left(\begin{smallmatrix}\bullet\\ \uparrow\\ \bullet\\ \uparrow\\ \bullet\end{smallmatrix}, \bot\right) \oplus \alpha\left(\begin{smallmatrix}\bullet\\ \uparrow\\ \circ\end{smallmatrix}, \bot\right) \oplus \alpha\left(\begin{smallmatrix}* & & *\\ \nwarrow & & \nearrow\\ & * & \end{smallmatrix}, \bot\right) \text{ (8 axioms)} \\
\mathbf{K4.3} & = & \mathbf{K4} \oplus \alpha\left(\begin{smallmatrix}* & & *\\ \nwarrow & & \nearrow\\ & * & \end{smallmatrix}\right) \text{ (6 axioms)} \\
\mathbf{Dum} & = & \mathbf{S4} \oplus \alpha\left(\begin{smallmatrix}\textcircled{2} & & \circ\\ \nwarrow & & \nearrow\\ & \circ & \end{smallmatrix}\right) \oplus \alpha\left(\begin{smallmatrix}\circ\\ \uparrow\\ \textcircled{2}\end{smallmatrix}\right)
\end{array}
$$

Table 1
Canonical axioms of standard modal logics

$$
\begin{aligned}
\delta(\mathfrak{F}, \mathfrak{D}) \quad = \quad & \bigwedge_{xRy} (p_x \to \Diamond p_y) \wedge \bigwedge_{\neg xRy} (p_x \to \neg\Diamond p_y) \wedge \bigwedge_{x \neq y} (p_x \to \neg p_y) \wedge \\
& \bigwedge_{\mathfrak{d} \in \mathfrak{D}} \Big(\bigwedge_{\substack{x \in \mathfrak{d} \cup \mathfrak{d}\uparrow \\ y \notin \mathfrak{d} \cup \mathfrak{d}\uparrow}} (\Diamond p_x \wedge \neg\Diamond p_y) \to \bigvee_{z \in W} p_z \Big) \wedge \\
& \bigwedge_{x \in W} (p_x \to \Diamond \bigvee_{y \in W} p_y)
\end{aligned}
$$

and $\alpha(\mathfrak{F}, \mathfrak{D})$ is obtained by omitting the last conjunct from $\delta(\mathfrak{F}, \mathfrak{D})$. The formulas $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha(\mathfrak{F}, \mathfrak{D})$ (or any other deductively equivalent formulas) are called the *canonical* and *negation free canonical formulas* for $\mathfrak{F}$ and $\mathfrak{D}$, respectively (it is not hard to get rid of $\neg$ and $\Diamond$ in the latter formula; see, e.g., [26]). The semantical meaning of these formulas should be clear: $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is refuted in a frame $\mathfrak{G}$ iff there is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.

Theorem 6.1 *There is an algorithm which, given an $\mathcal{ML}_1$-formula φ, returns canonical formulas $\alpha(\mathfrak{F}_1, \mathfrak{D}_1, \bot), \dots, \alpha(\mathfrak{F}_n, \mathfrak{D}_n, \bot)$ such that*

$$\mathbf{K4} \quad \oplus \quad \varphi \quad = \quad \mathbf{K4} \quad \oplus \quad \alpha(\mathfrak{F}_1, \mathfrak{D}_1, \bot) \oplus \cdots \oplus \alpha(\mathfrak{F}_n, \mathfrak{D}_n, \bot).$$

If φ is negation free then one can use negation free canonical formulas.

Table 1 shows canonical axiomatisations of some standard modal logics in the field of **K4**. For brevity we write $\alpha(\mathfrak{F}, \bot)$ instead of $\alpha(\mathfrak{F}, \emptyset, \bot)$ and $\alpha(\mathfrak{F})$ instead of $\alpha(\mathfrak{F}, \emptyset)$. Each $*$ in the table is to be replaced by both $\circ$ and $\bullet$.

Theorem 6.1 provides a solution to problem (A) formulated at the beginning of this section. It shows that as far as such properties of modal logics (from NExt **K4**) as decidability, completeness, the fmp, etc. are concerned, we can always deal with canonical formulas—which explicitly describe their frames. And the following observation gives a *partial* solution to problem (B).

Theorem 6.2 (1) *For every logic $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset) \mid i \in I\}$ and every canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, we have $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ iff $\mathfrak{F}$ is subreducible to $\mathfrak{F}_i$ for some $i \in I$.*

(2) *For every logic $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset, \bot) \mid i \in I\}$ and every canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, we have $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ iff $\mathfrak{F}$ is cofinally subreducible to $\mathfrak{F}_i$ for some $i \in I$.*

(3) *For every logic $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) \mid i \in I\}$ and every canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$ where $\mathfrak{D}^\sharp$ is the set of all antichains in $\mathfrak{F}$, we have $\alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot) \in L$ iff there is a generated subframe of $\mathfrak{F}$ that is reducible to $\mathfrak{F}_i$ for some $i \in I$.*

It follows from (3) that $\mathbf{K4} \oplus \varphi$ is a splitting of NExt **K4** iff φ is deductively equivalent in NExt **K4** to a formula of the form $\alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$, where $\mathfrak{D}^\sharp$ is the set of all antichains in $\mathfrak{F}$ (in this case $\mathbf{K4}/\mathfrak{F} = \mathbf{K4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$). Such formulas are known as *Jankov formulas* (Jankov [66] introduced them for intuitionistic logic in the algebraic setting), or *frame formulas* (used by Fine [43]), or *Jankov–Fine formulas.*

But the most interesting consequence can be drawn from (1) or (2): all logics mentioned in (1) and (2) of Theorem 6.2 have the fmp, and so are decidable if finitely axiomatisable. It follows, for instance, that all logics in NExt **S4.3**—which can be represented in the form (2) because antichains in frames for **S4.3** are reflexive singletons—have the fmp. It is not hard to see also that all these logics are finitely axiomatisable which gives the well-known result of Bull [14] and Fine [40] from Theorem 5.9.

As we have already mentioned, practically all 'standard' modal logics in the field of **K4** can be axiomatised by canonical formulas of the form $\alpha(\mathfrak{F}, \emptyset, \bot)$ or $\alpha(\mathfrak{F}, \emptyset)$. This yields an answer to the question 'why modal logic is so robustly decidable?' of Vardi [158] for the case of transitive unimodal logics. Although it is impossible to effectively recognise whether a logic $\mathbf{K4} \oplus \varphi$ can be axiomatised by such formulas [24], there is a simple model-theoretic characterisation of logics from (1) and (2) of Theorem 6.2 discovered in [45,174]:

Theorem 6.3 (1) *A logic $L \in$ NExt **K4** is axiomatisable by canonical formulas of the form $\alpha(\mathfrak{F}, \emptyset)$ iff L is characterised by a class of (general) frames that is closed under the formation of subframes.*

(2) *A logic $L \in$ NExt **K4** is axiomatisable by canonical formulas of the form $\alpha(\mathfrak{F}, \emptyset, \bot)$ iff L is characterised by a class of (general) frames that is closed under the formation of cofinal subframes.*

The logics from (1) and (2) are called *subframe* and *cofinal subframe logics*, respectively. It turns out that for these logics the notions of first-order definability, canonicity and strong Kripke completeness are equivalent; see [45,174] and Theorem 8.1 below. It is worth noting that the fmp of (cofinal) subframe logics and the decidability of those of them that are finitely axiomatisable (there are a continuum of subframe logics [174]) is obtained from Theorems 6.1 and 6.2 for free: it suffices to check whether the frame of the tested canonical formula is (cofinally) subreducible to the frame of one of the canonical axioms of a given logic. This provides us with another general method of proving that a given logic is Kripke complete, decidable, canonical, has the finite model property, etc.: usually it is much easier to check that the class of general frames for a given logic is closed under cofinal subframes and to find the logic's canonical axioms than to use filtration and/or canonical models.

Problem 6.4 *Characterise the computational complexity of finitely axiomatisable cofinal subframe logics (e.g., all such extensions of **K4.3** are coNP-complete).*

Problem 6.5 *Give a syntactical characterisation of subframe and cofinal subframe logics (cf. Theorem 9.2).*

Note that, for every (cofinal) subframe logic L and every formula $\varphi \notin L$, there is a frame for L refuting φ whose size is exponential in the length of φ.

In general, question (B) does not seem to have such an elegant answer as for the case of cofinal subframe logics (see, however, [147] for a recent attempt to introduce 'canonical formulas' for NExt $\mathbf{K}$). We can only console ourselves with a number of sufficient conditions of decidability, the fmp (in particular, Theorems 4.9 and 4.14) and some other properties imposed on the canonical axioms; for details consult [24,175,26,177].

6.2 Canonical formulas for tense logics of linear time flows

In this section we consider normal bimodal logics containing the tense logic **Lin** of linear orders (the class of all frames $\langle W, R_1, R_2\rangle$ in which R_2 is the converse of R_1 and R_1 is a linear order, i.e., R_1 is transitive and connected $(\forall x \forall y x R_1 y \vee y R_1 x \vee x = y)$). These logics are of interest in this chapter for two reasons. First, many natural and useful| modal logics are contained in this class, e.g., the logics determined by the flows of time $\langle \mathbb{N}, <\rangle$, $\langle \mathbb{Z}, <\rangle$, $\langle \mathbb{Q}, <\rangle$, and $\langle \mathbb{R}, <\rangle$ [15,128,135]. And second, this class is of exceptional interest because currently NExt **Lin** is the only lattice of modal logics for which *almost every decision problem is decidable* and which, nevertheless, contains numerous Kripke incomplete modal logics—where the standard techniques based on proving the finite model property or tree model property do not work. Frank?

The decidability results for logics in NExt **Lin** are based on two ingredients. First, they can be axiomatised by canonical formulas which explicitly show the geometrical and topological conditions they define. And second, a general completeness result establishes that they are determined by general frames which are composed from a set of rather simple 'atomic' general frames. In this part we introduce the canonical formulas. In the next section we discuss the general completeness theorem and survey its consequences regarding decision problems for logics in NExt **Lin**.

The logic **Lin** is obviously axiomatised as

$$\mathbf{Lin} \quad = \quad \mathbf{K4}_2 \;\oplus\; p \to \Box_1 \Diamond_2 p \;\oplus\; p \to \Box_2 \Diamond_1 p \;\oplus\; \Diamond_1 \Diamond_2 p \vee \Diamond_2 \Diamond_1 p \to p \vee \Diamond_1 p \vee \Diamond_2 p.$$

To begin our discussion of canonical formulas for logics extending NExt **Lin** observe that Dedekind cuts can be characterised by means of splittings with the frame $\circ\!\!\to\!\!\circ$. In fact, one can show that

$$\mathsf{Log}\,\{\langle \mathbb{R}, <, >\rangle\} \quad = \quad \mathsf{Log}\,\{\langle \mathbb{Q}, <, >\rangle\}/\ \circ\!\!\to\!\!\circ.$$

The intuition behind this equation should be clear from the observation that the class of rooted general frames for $\mathsf{Log}\,\{\langle \mathbb{Q}, <, >\rangle\}/\ \circ\!\!\to\!\!\circ$ consists of all general frames $\left\langle W, R, R^{-1}, P\right\rangle$ such that

- $\langle W, R\rangle$ is a dense linear order without endpoints (i.e., it validates $\forall x \forall y (xRy \to \exists z (xRz \wedge zRy))$, $\forall x \exists y\; xRy$ and $\forall x \exists y\; yRx$) and
- there is no p-morphism from $\left\langle W, R, R^{-1}, P\right\rangle$ onto $\circ\!\!\to\!\!\circ$.

But such a p-morphism exists iff there is a partition $X, Y \in P$ of W such that

$$\forall x \in X \forall y \in Y\; xRy, \qquad \forall x \in X \exists y \in X\; xRy, \qquad \forall x \in Y \exists y \in Y\; yRx. \tag{10}$$

Observe that $\langle \mathbb{Q}, <, >\rangle \notin \mathsf{Fr}\,\mathsf{Log}\,\{\langle \mathbb{Q}, <, >\rangle\}/\ \circ\!\!\to\!\!\circ$ since, for example, $X = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$ and $Y = \{y \in \mathbb{Q} \mid y > \sqrt{2}\}$ form such a partition.

To characterise the class of *all linear orders* without partitions satisfying (10), we have to weaken the splitting formula. For example, the frame $\langle \{x \geq 0 \mid x \in \mathbb{Q}\}, <, >\rangle$ (which obviously has a partition satisfying (10)) validates

$$\mathbf{Lin}/\circ\!\!\to\!\!\circ,$$

simply because it contains an irreflexive left endpoint and, therefore, cannot be p-morphically mapped onto $\circ\!\!\to\!\!\circ$. To obtain the class of all linear orders without partitions satisfying (10) the point a in $\overset{a}{\circ}\!\!\to\!\!\overset{b}{\circ}$

should be regarded as reflexive by the operator $\Box_1$ for the future but as an arbitrary linear order by the operator $\Box_2$ for the past. To make this idea precise, we introduce the notion of a *type assignment* $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ which is a map from the set of clusters C of a finite linear order into the set of pairs over $\{\mathsf{j}, \mathsf{m}\}$ such that $\mathbf{t}C = (\mathbf{t}_1 C, \mathbf{t}_2 C) = (\mathsf{m}, \mathsf{m})$ for every cluster C consisting of an irreflexive point. Here j stands for 'joker' and m stands for 'maximal.' For example, $\mathbf{t}C = (\mathsf{m}, \mathsf{j})$ means intuitively that cluster C should be regarded as 'what it is' by $\Box_1$ and as an arbitrary linear order by $\Box_2$. The condition that irreflexive points are mapped to (m, m) means that they are always regarded as what they are, that is, irreflexive points. Before we associate with every finite linear order with type assignment $\langle \mathfrak{F}, \mathbf{t} \rangle$ a formula $\alpha(\mathfrak{F}, \mathbf{t})$ with the corresponding meaning some notation is required.

Given a finite sequence $\overline{\mathfrak{F}} = \left\langle \mathfrak{F}_i = \left\langle W_i, R_i, R_i^{-1}, P_i \right\rangle \mid 1 \leq i \leq n \right\rangle$ of disjoint frames, we denote by $[\overline{\mathfrak{F}}] = \mathfrak{F}_1 \lhd \cdots \lhd \mathfrak{F}_n$ the ordered sum of them, i.e., the frame $\left\langle W, R, R^{-1}, P \right\rangle$ in which

$$W = \bigcup_{i=1}^{n} W_i, \qquad R = \bigcup_{i=1}^{n} R_i \cup \bigcup_{1 \leq i < j \leq n} (W_i \times W_j)$$

and $P = \{X_1 \cup \cdots \cup X_n \mid X_i \in P_i\}$. Each finite frame can be represented then as the ordered sum $C_1 \lhd \cdots \lhd C_n$ of its clusters.

With every finite tense frame $\mathfrak{F} = \langle W, R, \rangle = C_1 \lhd \cdots \lhd C_n$ with cluster assignment $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2)$ we associate the formula

$$\alpha(\mathfrak{F}, \mathbf{t}) = \neg \overline{\delta}(\mathfrak{F}, \mathbf{t}),$$

where

$$\overline{\delta}(\mathfrak{F}, \mathbf{t}) = \delta(\mathfrak{F}, \mathbf{t}) \wedge \Box_F \delta(\mathfrak{F}, \mathbf{t}) \wedge \Box_P \delta(\mathfrak{F}, \mathbf{t})),$$

and

$$\begin{aligned}
\delta(\mathfrak{F}, \mathbf{t}) \quad = \quad & \bigwedge \{p_x \to \neg p_y \mid x \neq y\} \wedge \\
& \bigwedge \{p_x \to \neg \Diamond_P p_y \mid \neg(xRy)\} \wedge \\
& \bigwedge \{p_x \to \Diamond_F p_y \mid \exists i \leq n \ (\mathbf{t}_1 C_i = \mathsf{m} \wedge x, y \in C_i \wedge xRy)\} \wedge \\
& \bigwedge \{p_x \to \Diamond_P p_y \mid \exists i \leq n \ (\mathbf{t}_2 C_i = \mathsf{m} \wedge x, y \in C_i \wedge xR^{-1}y)\} \wedge \\
& \bigvee \{p_y \mid y \in W\} \wedge \\
& \bigwedge \{\Diamond_F p_y \vee p_y \vee \Diamond_P p_y \mid y \in W\}.
\end{aligned}$$

To explain the semantical meaning of these formulas, notice first that if $\mathbf{t}C = (\mathsf{m}, \mathsf{m})$ for all clusters C then $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathbf{t})$ is iff $\mathfrak{G}$ there exists a p-morphism from $\mathfrak{G}$ onto $\mathfrak{F}$. Therefore,

$$\mathbf{Lin}/\mathfrak{F} = \mathbf{Lin} \oplus \alpha(\mathfrak{F}, \mathbf{t}).$$

If $\mathbf{t}_i C = \mathsf{j}$ for some $i \in \{1, 2\}$ and some cluster C in $\mathfrak{F}$, then the formula $\alpha(\mathfrak{F}, \mathbf{t})$ can be refuted in frames that do not necessarily have $\mathfrak{F}$ as a p-morphic image. In this case $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \mathbf{t})$ iff there exist frames $\mathfrak{G}_i$, for $1 \leq i \leq n$, such that $\mathfrak{G} = \mathfrak{G}_1 \lhd \cdots \lhd \mathfrak{G}_n$ and $\mathfrak{G}_i \not\models \alpha(C_i, \mathbf{t} \restriction C_i)$ for all $1 \leq i \leq n$. So it suffices to consider $\mathfrak{G} \not\models \alpha(C, \mathbf{t})$ for a cluster C. Assume for simplicity that $\mathfrak{G}$ is a Kripke frame. *Case 1:* $\mathbf{t}C = (\mathsf{j}, \mathsf{j})$. Then $\mathfrak{G} \not\models \alpha(C, \mathbf{t})$ iff $|\mathfrak{G}| \geq |C|$. *Case 2:* $\mathbf{t}C = (\mathsf{m}, \mathsf{j})$. Then C is nondegenerate and $\mathfrak{G} \not\models \alpha(C, \mathbf{t})$ iff either $\mathfrak{G}$ contains an R-final cluster of cardinality $\geq |C|$ or it has no R-final point at all. *Case 3:* $\mathbf{t}C = (\mathsf{j}, \mathsf{m})$. This is the mirror image of Case 2. *Case 4:* $\mathbf{t}C = (\mathsf{m}, \mathsf{m})$. If C is an irreflexive point then $\mathfrak{G}$ is an irreflexive point as well whenever $\mathfrak{G} \not\models \alpha(C, \mathbf{t})$. If C is non-degenerate and $\mathfrak{G} \not\models \alpha(C, \mathbf{t})$ then $\mathfrak{G}$ satisfies the conditions of Cases 2 and 3.

Now, the following is proved in [163]:

Theorem 6.6 *There exists an algorithm which, given a formula φ, returns canonical formulas $\alpha(\mathfrak{F}_1, \mathbf{t}_1), \ldots, \alpha(\mathfrak{F}_n, \mathbf{t}_n)$ such that*

$$\mathbf{Lin} \oplus \varphi = \mathbf{Lin} \oplus \alpha(\mathfrak{F}_1, \mathbf{t}_1) \oplus \ldots \oplus \alpha(\mathfrak{F}_n, \mathbf{t}_n).$$

$$
\begin{array}{lcl}
\mathbf{Ord}_t & = & \mathsf{Log}\,\{\langle \xi, <, > \rangle \mid \xi \text{ an ordinal}\} = \mathbf{Lin} \oplus \alpha(-, (\circ, (\mathsf{j}, \mathsf{m}))) \\[1ex]
\mathbf{E}_t & = & \mathbf{Lin} \oplus \Diamond_1 \top \oplus \Diamond_2 \top = \\[1ex]
 & & \mathbf{Lin} \oplus \alpha(-, (\bullet, (\mathsf{m}, \mathsf{m}))) \oplus \alpha((\bullet, (\mathsf{m}, \mathsf{m})), -) \\[1ex]
\mathbf{RD} & = & \mathsf{Log}\,\{\mathfrak{G} \mid \forall x (\neg x R x \to \exists y (x R y \wedge \{z \mid x R z R y\} = \emptyset))\} = \\[1ex]
 & & \mathbf{Lin} \oplus \alpha(-, (\bullet, (\mathsf{m}, \mathsf{m}))) \oplus \alpha(-, (\bullet, (\mathsf{m}, \mathsf{m})) \lhd (\circ, (\mathsf{m}, \mathsf{j}))) \\[1ex]
\mathbf{LD} & = & \text{the mirror image of } \mathbf{RD} \\[1ex]
\mathbf{Z}_t & = & \mathsf{Log}\,\langle \mathbb{Z}, <, > \rangle = \\[1ex]
 & & \mathbf{RD} \oplus \mathbf{LD} \oplus \alpha((\circ, (\mathsf{j}, \mathsf{j})) \lhd (\circ, (\mathsf{j}, \mathsf{m}))) \oplus \alpha((\circ, (\mathsf{m}, \mathsf{j})) \lhd (\circ, (\mathsf{j}, \mathsf{j}))) \\[1ex]
\mathbf{Ds}_n & = & \mathbf{Lin} \oplus \Box_1^{n+1} p \to \Box_1^n p = \\[1ex]
 & & \mathbf{Lin} \oplus \alpha(-, \underbrace{(\bullet, (\mathsf{m}, \mathsf{m}) \lhd \cdots \lhd (\bullet, (\mathsf{m}, \mathsf{m}))}_{n+1}, -) \\[1ex]
\mathbf{Q}_t & = & \mathsf{Log}\,\langle \mathbb{Q}, <, > \rangle = \mathbf{Ds}_1 \oplus \mathbf{E}_t \\[1ex]
\mathbf{R}_t & = & \mathsf{Log}\,\langle \mathbb{R}, <, > \rangle = \mathbf{Q}_t \oplus \alpha((\circ, (\mathsf{m}, \mathsf{j})) \lhd (\circ, (\mathsf{j}, \mathsf{m})))
\end{array}
$$

Table 2
Axiomatisations of standard tense logics

Canonical axiomatisations of some standard linear tense logics are shown in Table 2, where we use the following notation. Given a finite frame $\mathfrak{F} = C_1 \lhd \cdots \lhd C_n$, we write $\alpha((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n))$ instead of $\alpha(\mathfrak{F}, \mathbf{t})$ and $\alpha(-, (C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n))$ instead of

$$\alpha((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n)) \;\oplus\; \alpha((\circ, (\mathsf{j}, \mathsf{j})) \lhd (C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n)).$$

$\alpha((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n), -)$ is defined analogously.

Applications of this result will be discussed in the section below.

7 Decision problems for tense logics

In this section we show that in NExt **Lin** almost all decision problems are decidable. For example, we can decide whether a finitely axiomatisable logic is Kripke complete, has the finite model property, is canonical, etc. Moreover, finitely axiomatisable linear tense logics turn out to be coNP-complete. Thus, NExt **Lin** is an example of a class of logics for which the original research programme has a complete and 'positive' solution.

The first step in understanding logics in NExt **Lin** was done in the section above, where we saw that they can be axiomatised by canonical formulas $\alpha(\mathfrak{F}, \mathbf{t})$ which show directly the geometric and topological conditions definable on linear orders by means of bimodal formulas. The second step is a completeness result which determines a class of rather simple general frames which respect to which every logic in NExt **Lin** is complete. We require the following notation:

(1) Denote by ⓚ the non-degenerate cluster with $k > 0$ points.

(2) Let $\omega^<(0)$ be the strictly ascending chain $\langle \omega, <, > \rangle$ of natural numbers, $\omega^<(1)$ the chain $\langle \omega, \leq, \geq \rangle$, $\omega^<(2)$ the ascending chain of natural numbers in which precisely the even points are reflexive, $\omega^<(3)$ the chain in which precisely the multiples of 3 are reflexive, and so on; $\omega^>(n)$ is the mirror image of $\omega^<(n)$.

(3) $\mathfrak{C}(0,①) = \langle\omega^<(0) \lhd ①, P\rangle$, where P consists of all cofinite sets containing ① and their complements. We generalise this construction to chains $\omega^<(n)$ and clusters ⓚ. Namely, for $n < \omega$, $k > 1$ and ⓚ $= \{a_0, \dots, a_{k-1}\}$, we put

$$\mathfrak{C}(n, ⓚ) \quad = \quad \langle\omega^<(n) \lhd ⓚ, P\rangle,$$

where P is the set generated by means of the Boolean operators from the set of finite subsets of $\omega^<(n)$ and the sets $\{X_i \mid 0 \le i \le k-1\}$, for $X_i = \{a_i\} \cup \{kj + i \mid j \in \omega\}$, $0 \le i \le k-1$. $\mathfrak{C}(ⓚ, n)$ denotes the mirror image of $\mathfrak{C}(n, ⓚ)$.

(4) $\mathfrak{C}(0,①,0) = \langle\omega^<(0) \lhd ① \lhd \omega^>(0), P\rangle$, where P consists of all cofinite sets containing ① and their complements.

It is easy to check that the frames defined in (3) and (4) are descriptive and a singleton $\{x\}$ is in P iff $x \notin$ ⓚ. Notice also that the logics $\mathsf{Log}(\mathfrak{C}(n, ⓚ))$, $\mathsf{Log}(\mathfrak{C}(ⓚ, n))$, $k \ge 2$, and $\mathsf{Log}\mathfrak{C}(0,①,0)$ are Kripke incomplete.

For a class of frames $\mathcal{C}$, we denote by $\mathcal{C}^*$ the class of finite sequences of frames from $\mathcal{C}$ and let $[\mathcal{C}^*] = \{[\overline{\mathfrak{F}}] \mid \overline{\mathfrak{F}} \in \mathcal{C}^*\}$. The class of finite clusters and the frames of the form (3) and (4) is denoted by $\mathcal{B}$. We are now in a position to formulate the completeness result:

Theorem 7.1 *Each logic $L \in$ NExt **Lin** is determined by a set $\mathcal{C} \subseteq [\mathcal{B}^*]$.*

Proof We briefly explain the idea of the proof. Suppose that $\mathfrak{M} = \langle\mathfrak{F}, \mathfrak{V}\rangle$ is a countermodel for $\alpha = \alpha((C_1, \mathbf{t}C_1) \lhd \cdots \lhd (C_n, \mathbf{t}C_n))$ based on a descriptive frame $\mathfrak{F} = \langle W, R, R^{-1}, P\rangle$. We must show that there exists $\mathfrak{G} \in [\mathcal{B}^*]$ refuting α and such that $\mathsf{Log}\,\mathfrak{G} \supseteq \mathsf{Log}\,\mathfrak{F}$. Consider the sets

$$W_i \quad = \quad \{y \in W \mid (\mathfrak{M}, y) \models \bigvee\{p_x \mid x \in C_i\}\}.$$

One can easily show that W_i are intervals in $\mathfrak{F}$ and $\mathfrak{F} = \mathfrak{F}_1 \lhd \cdots \lhd \mathfrak{F}_n$, for the subframes $\mathfrak{F}_i$ of $\mathfrak{F}$ induced by W_i. Moreover, $\mathfrak{G} = [\overline{\mathfrak{G}}]$ is as required if $\overline{\mathfrak{G}} = \langle\mathfrak{G}_1, \dots, \mathfrak{G}_n\rangle$ is a sequence in $\mathcal{B}^*$ such that $\mathsf{Log}\,\mathfrak{G}_i \supseteq \mathsf{Log}\,\mathfrak{F}_i$, and $\mathfrak{G}_i \not\models \alpha(C_i, \mathbf{t}C_i)$, for $1 \le i \le n$. Frames $\mathfrak{G}_i$ with those properties are constructed in [163]. ❑

Example 7.2 The logic $\mathbf{Q}_t$ of the rational numbers is determined by the frames $\mathfrak{F} \in [\mathcal{B}^*]$ which contain no pair of adjacent irreflexive points. The logic $\mathbf{R}_t$ of the real line is determined by the frames $\mathfrak{F} \in [\mathcal{B}^*]$ which contain neither a pair of adjacent irreflexive points nor a pair of adjacent non-degenerate clusters.

Now, based on both the canonical formulas for logics in NExt **Lin** and this completeness result, [93] shows:

Theorem 7.3 (i) *All finitely axiomatisable logics in* NExt **Lin** *are coNP-complete.*

(ii) *All logics determined by a frame $\mathfrak{G} \in [\mathcal{B}^*]$ are coNP-complete.*

Proof sketch. (ii) is proved by showing that, given a formula ψ, it can be checked in nondeterministic polynomial time in the length of ψ whether it is satisfiable in a given $\mathfrak{G} \in [\mathcal{B}^*]$. To this end, [93] shows that it is sufficient to check satisfiability of ψ in a certain finite subframe of $\mathfrak{G}$ (whose size is polynomial in ψ) under certain 'good' valuations.

(i) Suppose

$$L \quad = \quad \mathbf{Lin} \oplus \alpha(\mathfrak{F}_1, \mathbf{t}_1) \oplus \cdots \oplus \alpha(\mathfrak{F}, \mathbf{t}_n)$$

is given. Then [93] shows that any given formula ψ which is satisfiable in a frame $\mathfrak{G} \in [\mathcal{B}^*]$ validating L is satisfiable in a frame of this type whose parameters (i.e., the number of blocks required, the size of its clusters, and the maximal n such that $\mathfrak{C}(n, ⓚ)$ or $\mathfrak{C}(ⓚ, n)$ occurs in it) are polynomial in ψ. By the proof of (ii), it can be checked in nondeterministic polynomial time whether ψ is satisfiable in such a frame. Additionally, one can show that it can be checked in polynomial time (in the length of ψ) whether such a frame validates a formula of the form $\alpha(\mathfrak{F}, \mathbf{t})$. ❑

Given the coNP-completeness of all finitely axiomatisable tense logics, at least two questions arise: First, are there interesting classes of finitely axiomatisable linear tense logics? Second, how complex are non-finitely axiomatisable linear tense logics? Regarding the second question, consider, for any $M \subseteq \mathbb{N}$, the logic L_M determined by the class of frames

$$\{\langle\{1,\ldots,m\},<\rangle \mid m \in M\}.$$

Set $\bot = p \wedge \neg p$ and $\top = p \vee \neg p$. Then the formula

$$\varphi_m \quad = \quad \Box_2\bot \wedge \Box_1^{m+1}\bot \wedge \Diamond_1^m\top$$

is satisfiable in a frame validating L_M iff $m+1 \in M$. Thus, for any set of natural numbers M there exists a tense logic of the same complexity as M. Regarding the first question, [163] determines a number of classes of finitely axiomatisable linear tense logics. For example, the following result is proved using Kruskal's tree theorem [80]:

Theorem 7.4 *A linear tense logic L is finitely axiomatisable whenever there exists $n < \omega$ such that $\Box_1^n p \to \Box_1^{n+1} p \in L$. In particular, all linear tense logics of reflexive frames as well as all extensions of the tense logic of $\langle\mathbb{Q},<\rangle$ are finitely axiomatisable.*

Where are the Kripke incomplete modal logics in NExt **Lin**? The reader can get an impression from the following result, proved in [163] and stated here without proof:

Theorem 7.5 *Suppose that $L \in$ NExt **Lin** and there is a Kripke frame of infinite length validating L. Then there exists a Kripke incomplete logic in* NExt L.

As promised at the beginning of this section, we conclude by discussing a complete solution to our original research problem relativised to the set of logics in NExt**Lin**. The proofs can be found in [162,165].

Theorem 7.6 *There are algorithms which, given a formula φ, decide whether* $\mathbf{Lin} \oplus \varphi$

- *has the finite model property;*
- *has the interpolation property;*
- *is Kripke complete;*
- *is strongly complete;*
- *is canonical.*

We shall not go into details of the proof here. But the reader can get some impression regarding the combinatorics and methods involved from the criteria which allow us to decide whether a linear tense logic is canonical and strongly complete. Denote by $\mathcal{B}_+$ the class of frames containing $\mathcal{B}$ together with frames $\mathfrak{C}(n_1, ⓚ, n_2)$ defined as follows. Suppose $k > 1$, $n_1, n_2 < \omega$ are such that $n_1 + n_2 > 0$ and $ⓚ = \{a_0, \ldots, a_{k-1}\}$. Then

$$\mathfrak{C}(n_1, ⓚ, n_2) \quad = \quad \langle \omega^<(n_1) \lhd ⓚ \lhd \omega^>(n_2), P\rangle,$$

where P is the set of possible values generated by $\{X_i \mid 0 \le i \le k-1\}$, for

$$X_i \quad = \quad \{a_i\} \cup \{kj + i \mid j \in \omega\} \cup \{k^*j^* + i^* \mid j \in \omega\}$$

and $\{0^*, 1^*, \ldots, n^*, \ldots\}$ being the points in $\omega^>(n_2)$.

Let $\mathcal{F}$ be the class of frames of the form

$$\langle\{0,\ldots,n_1\},<,>\rangle \lhd ① \lhd \langle\{0,\ldots,n_2\},<,>\rangle \quad \text{or} \quad \langle\{0,\ldots,n\},<,>\rangle\,.$$

Theorem 7.7 (i) *A logic $L \in$ NExt**Lin** is canonical iff the underlying Kripke frame of each frame $\mathfrak{F} \in [\mathcal{B}^*_+]$ for L validates L as well.*

(ii) *A logic $L \in$ NExt**Lin** is strongly complete iff for each frame $\mathfrak{F} \in [\mathcal{B}^*_+]$ validating L, there exists a Kripke frame $\mathfrak{G}$ for L which results from $\mathfrak{F}$ by replacing*

- *every $\mathfrak{C}(n, ⓚ)$ with $\omega^<(n)$ or $\omega^<(n) \lhd \mathfrak{H} \lhd ⓚ$, for some $\mathfrak{H} \in \mathcal{F}$, and*
- *every $\mathfrak{C}(ⓚ, n)$ with $\omega^>(n)$ or $ⓚ \lhd \mathfrak{H} \lhd \omega^>(n)$, for some $\mathfrak{H} \in \mathcal{F}$, and*
- *every $\mathfrak{C}(n_1, ⓚ, n_2)$ with $\omega^<(n_1) \lhd \mathfrak{H} \lhd \omega^>(n_2)$, for some $\mathfrak{H} \in \mathcal{F}$.*

Example 7.8 The logic $\mathbf{R}_t$ of the real line is not canonical since $\mathfrak{C}(2, ②) \models \mathbf{R}_t$ but $\omega^<(2) \lhd ② \not\models \mathbf{R}_t$. However, $\mathbf{R}_t$ is strongly complete, since $\mathfrak{F} \models \mathbf{R}_t$ whenever $\mathfrak{G} \in [\mathcal{B}^*_+]$ validates $\mathbf{R}_t$ and $\mathfrak{F}$ is obtained from $\mathfrak{G}$ as in the formulation of Theorem 7.7 with $\mathfrak{H} = \bullet \in \mathcal{F}$.

8 Subframe logics

If you randomly pick a logic from the list of Modal Logic celebrities introduced in this handbook, it is very likely that its frames are closed under the *formation of subframes* (as before, a Kripke frame $\mathfrak{G} = \langle V, S_1, \ldots, S_n \rangle$ is a *subframe* of a Kripke frame $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$ if $V \subseteq W$ and $S_i = R_i \restriction V$, for $1 \leq i \leq n$). Examples are the standard unimodal logics **K** (because the class of all frames is closed under subframes), **K4**, **S5**, and **K4.3** (because transitivity, symmetry, reflexivity, and right-linearity are definable using universal first-order formulas and therefore preserved under the formation subframes), as well as **GL** and **Grz** (because the class of Noetherian frames is closed under subframes).

Moreover, many important operations on classes of frames preserve the property of being closed under forming subframes:

- the union and intersection of two classes of frames;
- the *fusion* $\mathcal{K}_1 \otimes \mathcal{K}_1$ of classes of frames $\mathcal{K}_1$ and $\mathcal{K}_2$ (see Chapter **??** which, in the unimodal case, is defined by
$$\mathcal{K}_1 \otimes \mathcal{K}_2 \quad = \quad \{\langle W, R_1, R_2 \rangle \mid \langle W, R_1 \rangle \in \mathcal{K}_1, \langle W_2, R_2 \rangle \in \mathcal{K}_2\};$$
- the *tense extension* (or addition of converse) $\mathcal{K}_t$ of a class of frames $\mathcal{K}$, where
$$\mathcal{K}_t \quad = \quad \{\langle W, R, R^{-1} \rangle \mid \langle W, R \rangle \in \mathcal{K}\};$$
- the *Boolean extension* $\mathcal{K}^{\cap,\neg}$ of a class of frames $\mathcal{K}$ which consists of all frames
$$\langle W, R_1, \ldots, R_n, R_1 \cap R_2, W - R_1, \ldots \rangle$$
with $2^n|$ relations corresponding to the Boolean combinations of the R_i, for $\langle W, R_1, \ldots, R_n \rangle \in \mathcal{K}$. 2^{2^n} Frank?

Using these operators we obtain numerous additional important modal 'subframe logics:' multimodal fusions like $\mathbf{S5}_n$, minimal tense extensions like $\mathbf{K4.3}_t$, and Boolean Modal Logics like $\mathbf{K}^{\cap,-}$ are all examples of modal logics determined by classes of frames closed under subframes.

In this section we explore the extent to which the restriction to 'subframe logics' leads to general 'positive' results for properties like axiomatisability, decidability, Kripke completeness, and the fmp. A systematic investigation of subframe logics in NExt **K4** was launched by Fine [45] (see also [174]). Subframe logics in NExt $\mathbf{K}_n$ as well as subframe tense and provability logics were investigated in [166].| Add ref

We begin by observing the fact that all the 'negative' results above were obtained using logics whose frames were not closed under subframes. So one might conjecture that 'subframe logics' behave better than arbitrary modal logics. The answer to this question is 'yes' and 'no.' Yes, because indeed there are general decidability (fmp, completeness, etc.) results explaining the nice behaviour of standard 'subframe logics.' The answer is 'no,' because still one can find intuitively 'simple' subframe logics with 'bad' properties.

First, we note, however, that for subframe logics a number of otherwise separable properties of modal logics fall together; for a proof see [166] or [177]. For logics from NExt **K4** this result was first obtained by Fine [45].

Theorem 8.1 *Suppose L is determined by a class of Kripke frames closed under subframes. Then the following conditions are equivalent:*

(i) $\mathsf{Fr}\, L$ *is universal,*

(ii) $\mathsf{Fr}\, L$ *is first-order definable,*

(iii) L *is* $\mathcal{D}$*-persistent,*

(iv) L *is strongly Kripke complete,*

(v) $\mathsf{Fr}\, L$ *has the finite embedding property—i.e.,* $\mathfrak{F} \in \mathsf{Fr}\, L$ *iff every finite subframe of* $\mathfrak{F}$ *is in* $\mathsf{Fr}\, L$.

Note that for cofinal subframe logics $L \in \mathrm{NExt}\,\mathbf{K4}$ conditions (ii)–(iv) are equivalent [174].

Thomason's analysis for subframe logics

Now let us see to which extent Thomason's results (Theorem 2.1) hold for modal logics determined by classes of frames $\mathsf{Fr}\,\varphi$ closed under subframes.

Theorem 8.2 (a) *There is an $\mathcal{ML}_6$-formula φ such that $\mathsf{Fr}\,\{\varphi\}$ is closed under subframes and the set*

$$\mathsf{Th}\,\mathsf{Fr}\,\{\varphi\} \quad = \quad \{\psi \in \mathcal{ML}_6 \mid \varphi \models \psi\}$$

is Π^1_1-complete.

(b) *It is decidable whether, given $\varphi \in \mathcal{ML}_n$ with $\mathsf{Fr}\,\{\varphi\}$ closed under subframes, $\mathsf{Log}\,\mathsf{Fr}\,\{\varphi\}$ is consistent.*

(c) *There is a set Γ of $\mathcal{ML}_1$-formulas and an $\mathcal{ML}_1$-formula φ such that $\Gamma \models \varphi$, but $\Delta \not\models \varphi$ for any finite $\Delta \subset \Gamma$.*

(d) *Given a formula φ such that $\mathsf{Fr}\,\{\varphi\}$ is closed under subframes, every ψ with $\varphi \not\models \psi$ is refutable in a countable frame validating φ.*

(a) can be proved by modifying the proof of Theorem 2.1 (a) discussed above by introducing modalities for the converse relations of the relations R_1 and R_2 and modalities for the immediate successor relations for R_1 and R_2 and describing in $\mathcal{ML}_6$ by means of a single formula φ sufficiently many properties of the class of all subframes of the product frame $\langle \omega, <, +1\rangle \times \langle \omega, <, +1\rangle$. (Notice, however, that those subframes are not necessarily product frames).

As the reduction of multimodal logics to unimodal logics employed in the proof of Theorem 2.1 (a) does not preserve the property of being a subframe logic, it is an open problem whether such examples of unimodal logics exist.

(c) still holds since the example provided in the proof of Theorem 2.1 (c) was closed under subframes. (b) and (d) are now 'positive:' (b) is trivial because it is sufficient to check whether φ is valid in at least one singleton frame. (d) can be proved by constructing inductively from a frame refuting ψ a countable subframe by selecting the witnesses required to satisfy $\Diamond$-formulas; see [166] for details.

Normal subframe logics

We have not defined yet the notion of a subframe logic in a formal way because a proper definition should be slightly more general than the one suggested by considering Kripke frames only: to be able to provide a syntactic characterisation of subframe logics, to ensure that $\mathbf{K}_n \oplus \Sigma_1 \oplus \Sigma_2$ is a subframe logic whenever both $\mathbf{K}_n \oplus \Sigma_1$ and $\mathbf{K}_n \oplus \Sigma_2$ are subframe logics, and finally, to cover a number of

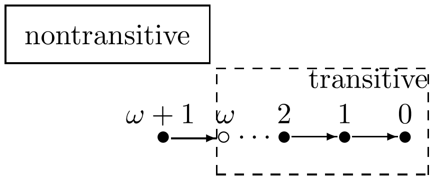


Fig. 4.

interesting Kripke incomplete logics (like bimodal provability logics), we extend the definition of a subframe to general frames. Recall that a frame $\mathfrak{G} = \langle V, S_1, \dots, S_n, P\rangle$ is a *subframe* of a frame $\mathfrak{F} = \langle W, R_1, \dots, R_n, Q\rangle$ if $V \subseteq W$, $P = \{A \cap V \mid A \in Q\}$, and $S_i = R_i \upharpoonright V$, for $1 \le i \le n$. A logic L is a *subframe logic* if its class of general frames is closed under subframes.

To describe subframe logics syntactically, define inductively the *relativisation* φ^p of a formula φ to a propositional variable p (which does not occur in φ) by taking

$$
\begin{array}{lcl}
q^p & = & q \wedge p, \quad q \text{ an atom},\\
(\psi \odot \chi)^p & = & \psi^p \odot \chi^p, \quad \text{for } \odot \in \{\wedge, \vee, \rightarrow\},\\
(\Box_i \psi)^p & = & \Box_i(p \rightarrow \psi^p) \wedge p
\end{array}
$$

and put $\varphi^{sf} = p \rightarrow \varphi^p$. It is not difficult to show that φ^{sf} is valid in a frame $\mathfrak{F}$ iff φ is valid in all subframes of $\mathfrak{F}$. (Notice that if $\mathfrak{M}$ is a model based on $\mathfrak{F}$, $\mathfrak{M}'$ a model based on the subframe of $\mathfrak{F}$ induced by $\{y \mid (\mathfrak{M}, y) \models p\}$ and $(\mathfrak{M}, x) \models q$ iff $(\mathfrak{M}', x) \models q$, for all variables q, then $(\mathfrak{M}, x) \models \varphi^p$ iff $(\mathfrak{M}', x) \models \varphi$.) Therefore, we obtain

Proposition 8.3 *The following conditions are equivalent for any $L \in \mathrm{NExt}\,\mathbf{K}_n$:*
(i) *L is a subframe logic,*
(ii) *$L = \mathbf{K}_n \oplus \{\varphi^{sf} \mid \varphi \in \Gamma\}$, for some set of formulas Γ,*
(iii) *L is characterised by a class of frames closed under subframes.*

Based on this proposition, it is not difficult to see that the class of subframe logics forms a complete sublattice of $\mathrm{NExt}\,\mathbf{K}_n$. Now, the first question to address is whether there are Kripke incomplete subframe logics at all:

Example 8.4 (Van Benthem 1979) Let $\mathfrak{F} = \langle W, R, P\rangle$ be the frame whose underlying Kripke frame is shown in Fig. 4 ($\omega + 1$ sees only ω and the subframe generated by ω is transitive) and $X \subseteq W$ is in P iff either X is finite and $\omega \notin X$ or X is cofinite in W and $\omega \in X$. It is easy to see that P is closed under $\cap$, $-$ and $\Diamond$.

Let L be the logic of the frame $\mathfrak{F}$ constructed in Example 8.4. Since every rooted subframe $\mathfrak{G}$ of $\mathfrak{F}$ is isomorphic to a generated subframe of $\mathfrak{F}$, L is a subframe logic. We show that L has the same Kripke frames as $\mathbf{GL.3}$. Suppose $\mathfrak{G}$ is a rooted Kripke frame for $\mathbf{GL.3}$ refuting $\varphi \in L$. Then clearly $\mathfrak{G}$ contains a finite subframe $\mathfrak{H}$ refuting φ. Since $\mathfrak{H}$ is a finite chain of irreflexive points, it is isomorphic to a generated subframe of $\mathfrak{F}$, contrary to $\mathfrak{F} \not\models \varphi$. Thus $\mathfrak{G} \models L$. Conversely, suppose $\mathfrak{G}$ is a Kripke frame for L. Then $\mathfrak{G}$ is irreflexive. For otherwise $\mathfrak{G}$ refutes the formula $\varphi = \Box^2(\Box p \rightarrow p) \wedge \Box(\Box p \rightarrow p) \rightarrow \Box p$, which is valid in $\mathfrak{F}$. Let us show now that $\mathfrak{G}$ is transitive. Suppose otherwise. Then $\mathfrak{G}$ refutes the formula $\Box p \rightarrow \Box(\Box p \vee (\Box q \rightarrow q))$, which is valid in $\mathfrak{F}$ because ω is a reflexive point. Finally, since $\mathfrak{G} \models \varphi$, $\mathfrak{G}$ is Noetherian and since $\mathfrak{F}$ is of width 1, we may conclude that $\mathfrak{G} \models \mathbf{GL.3}$. It follows that the subframe logic L is Kripke incomplete. Indeed, it shares the same class of Kripke frames with $\mathbf{GL.3}$ but $\Box p \rightarrow \Box\Box p \in \mathbf{GL.3} - L$.

We now follow the path described above for $\mathrm{NExt}\,\mathbf{K}_n$ and analyse the location of Kripke complete and decidable subframe logics within the lattice of subframe logics.

Subframe logics of frames with transitive relations

A number of surprisingly general decidability and completeness results have been obtained for subframe logics whose frames are based on transitive relations. The first fundamental result on the finite model property (and therefore decidability and Kripke completeness) of subframe logics is due to Fine [45]. Although the following theorem can be obtained for free using the machinery of canonical formulas, here we show its proof using the *maximal points technique* of [43].

Theorem 8.5 (Fine) *All unimodal subframe logics of transitive general frames* (*i.e., extensions of* **K4**) *have the finite model property. All finitely axiomatisable ones are decidable.*

Proof sketch. The proof is based on the following fundamental observation: call a point x in a transitive frame $\mathfrak{F} = \langle W, R, P\rangle$ *non-eliminable* if there is $X \in P$ such that $x \in X$ but no proper successor of x is in X (in other words, x is *maximal* in X); in this case we write $x \in \max_R X$. Then one can show the following: if $\mathfrak{F}$ is descriptive and $x \in X \in P$, then there exists a point $y \in \max_R X$ such that $x = y$ or xRy.

Now suppose that $L \supseteq$ **K4** is a subframe logic. To prove that L has the finite model property, suppose that $\varphi \notin L$. Take a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ refuting φ at a point x and based on a descriptive frame $\mathfrak{F} = \langle W, R, P\rangle$ for L. Say that a point $x \in W$ is *non-eliminable* relative to φ if there is a subformula ψ of φ such that $x \in \max_R\{y \in W \mid y \models \psi\}$ or $x \in \max_R\{y \in W \mid y \models \neg\psi\}$. Now select recursively a set of points W^φ as follows:

- Set $V = \{x\}$.
- If there exist $y \in V$ and $\Box\psi \in \mathsf{sub}\,\varphi$ such that $y \models \neg\Box\psi$, and there does not exist $z \in V$ with $z \models \neg\psi$ and yRz, then set $V = V \cup \{z\}$ for some non-eliminable z (relative to φ) with yRz and $z \models \neg\psi$. Otherwise, set $W^\varphi = V$ and stop.

Construct a new model $\mathfrak{M}^\varphi$ based on the frame $\mathfrak{F}^\varphi = \langle W^\varphi, R \restriction W^\varphi\rangle$ by taking $\mathfrak{V}^\varphi(p) = \mathfrak{V}(p) \cap W^\varphi$ for all variables p in φ. Clearly, the Kripke frame $\mathfrak{F}^\varphi$ is rooted, of depth $\leq l(\varphi)$, and no point has more than $\ell(\varphi)$ successors. Besides, one can easily show that $(\mathfrak{M}^\varphi, y) \models \psi$ iff $(\mathfrak{M}, y) \models \psi$, for all $\psi \in \mathsf{sub}\,\varphi$ and $y \in W^\varphi$. Finally, one can show that $\mathfrak{F}^\varphi$ is a frame for L (this is trivial if L is $\mathcal{D}$-persistent; considerably more work is required if L is not $\mathcal{D}$-persistent.) ❑

We consider now what happens if we move from a unimodal subframe logic $L \in \mathrm{NExt}\,\mathbf{K4}$ to the tense logic determined by $(\mathsf{Fr}\,L)_t$. In other words, we move from unimodal logics to bimodal logics, where the second operator is interpreted by the inverse of the accessibility relation of the first operator. From a syntactic viewpoint, this semantic condition is captured by the axioms

$$p \to \Box_1\Diamond_2 p \qquad \text{and} \qquad p \to \Box_2\Diamond_1 p.$$

So, we set, for any normal unimodal logic $L = \mathbf{K} \oplus \Gamma$,

$$L_t \quad = \quad \mathbf{K}_2 \ \oplus\ \Gamma \ \oplus\ \{p \to \Box_1\Diamond_2 p, p \to \Box_2\Diamond_1 p\}$$

and call L the *minimal tense extension* of L. General questions regarding minimal tense extensions are: What is the relation between L_t and the logic determined by the class of frames $(\mathsf{Fr}\,L)_t$? Are Kripke completeness and decidability inherited from unimodal logics to minimal tense extensions? As far as Kripke completeness is concerned, the answer is negative: [161] gives an example of a logic $L \in \mathrm{NExt}\,\mathbf{K4}$ with the finite model property such that L_t is not Kripke complete. Transfer of decidability is an open problem. However, for minimal tense extensions of subframe logics in $\mathrm{NExt}\,\mathbf{K4}$ a complete answer has been given in [160,164]:

Theorem 8.6 *If $L \in \mathrm{NExt}\mathbf{K4}$ is a subframe logic then*

(i) *L_t is Kripke complete and determined by $(\mathsf{Fr}\,L)_t$;*

(ii) *L_t has the finite model property iff L is canonical iff $\mathsf{Fr}\,L$ is first-order definable;*

(iii) *L_t is decidable whenever L is finitely axiomatisable.*

So, by (i), L_t is indeed Kripke complete whenever L is a subframe logic containing **K4**. The first exciting bit of this theorem is (ii). It connects the finite model property of minimal tense extensions with first-order definability: for first-order definable subframe logics like **K4**, **S4**, **K4.3**, and **S4.3** we obtain that their minimal tense extensions still have the finite model property, while minimal tense extensions of subframe logics that are not first-order definable, say, $\mathbf{GL}_t$ and $\mathbf{Grz}_t$, do not enjoy this property. For some examples, such as $\mathbf{Grz}_t$, this can be proved easily: just observe that the Grzegorczyk axiom

$$\Box_2(\Box_2(p \to \Box_2 p) \to p) \to p$$

is refuted in $\langle \omega, \geq, \leq \rangle$ and so does not belong to $\mathbf{Grz}_t$; however, it is clearly valid in all finite partial orders (which coincide with the finite frames in $(\mathsf{Fr}\,\mathbf{Grz})_t$).

The second interesting news here is that (iii) provides us with a decidability result for a class of logics which (in general) neither have the finite model property nor the tree model property. In fact, to prove (iii) [164] introduces so-called quasi-frames, that is, frames which come together with type assignments (similar to those in Section 6.2) and with respect to which minimal tense extensions of subframe logics have the bounded finite model property.

The decidability result in Theorem 8.6 does not cover Kripke incomplete logics, and one may wonder whether there exist at all natural and interesting classes of *decidable* modal logics containing Kripke incomplete ones. The answer is 'yes:' one such class is the set of all finitely axiomatisable (possibly non-normal) subframe logics containing the bimodal provability logic

$$\mathbf{CSM}_0 \quad = \quad \mathbf{GL} \otimes \mathbf{GL} \oplus \{\Box_1 \to \Box_2 p, \Box_2 p \to \Box_1 \Box_2 p\}$$

(named so in [159] after Carlson, Smorynski and Montagna). Almost all bimodal provability logics discussed and investigated in the literature [112,137] are indeed (sometimes non-normal) subframe logics. For the proof of this result see [167].

The lattice of all subframe logics

We have seen above a lot of beautiful and useful decidability results for subframe logics over transitive relations. Unfortunately, the situation changes drastically as soon as not all accessibility relations are transitive. In this part we summarise what is known in this case. First we shall see| that at least the 'upper part' of the lattice of subframe logics is quite well-behaved. Recall that $\mathbf{K} \oplus \boldsymbol{alt}_n$ is the logic determined by the frames in which no point has more than n successors. The following result is proved by first showing that all subframe logics containing $\mathbf{K} \oplus \boldsymbol{alt}_n$ are Kripke complete [1] and then proving that if a formula φ is refuted in a frame validating $\boldsymbol{alt}_n$ then one can always select finitely many points from that frame in which φ is refuted. (It should be clear from the proof that this result still holds for multimodal subframe logics in which for each operator an axiom of the form $\boldsymbol{alt}_n$ holds.) Changed

Theorem 8.7 *All subframe logics containing some* $\mathbf{K} \oplus \boldsymbol{alt}_n$ *have the finite model property.*

Now we say that a subframe logic L is *strictly sf-complete* if there does not exist another subframe logic L' with the same Kripke frames as L. In comparison with $\mathrm{NExt}\,\mathbf{K}$, where according to Blok's dichotomy for almost every interesting L there exist uncountably many modal logic L' with the same Kripke frames as L, the situation is much more diverse in the lattice of subframe logics. Example 8.4| shows that **GL.3** is not strictly sf-complete. However, the logics **T**, **S4** and **Grz** turn out to be strictly sf-complete. As in the full lattice of normal modal logics, the notion of strict sf-completeness is closely related to the notion of splittings (now in the lattice of subframe logics) and the decidability of the axiomatisation problem for formulas axiomatising subframe logics. Say that a subframe logic $L \in \mathrm{NExt}\,\mathbf{K}$ is a *subframe union-splitting* by a set $\mathcal{F}$ of finite rooted frames, in symbols Ref?

$$L \quad = \quad \mathbf{K}/^{sf}\mathcal{F},$$

if L is the smallest subframe logic such that $\mathfrak{F} \not\models L$, for at least one $\mathfrak{F} \in \mathcal{F}$. For example,

$$\mathbf{T} \quad = \quad \mathbf{K}/^{sf} \circ .$$

It is now readily checked that any Kripke complete subframe logic L which is a subframe union-splitting by finitely many rooted frames is strictly sf-complete and $\{\varphi \mid \mathbf{K} \oplus \varphi^{sf} = L\}$ is recursive whenever L is decidable. Based on these observations, the following partial results were obtained in [166]:

Theorem 8.8 (a) *A subframe logic L containing* **K4** *is strictly sf-complete iff $L \not\subseteq$* **GL.3** *iff L is a subframe union-splitting. Moreover, if $L \not\subseteq$* **GL.3** *is finitely axiomatisable, then $\{\varphi \mid \mathbf{K} \oplus \varphi^{sf} = L\}$ is recursive.*

(b) *All subframe logics $L \in \mathrm{NExt}\mathbf{K} \oplus \boldsymbol{alt}_n$ are strictly sf-complete and subframe union-splittings. Moreover, if L is finitely axiomatisable, then $\{\varphi \mid \mathbf{K} \oplus \varphi^{sf} = L\}$ is recursive.*

(c) *It is decidable whether a finitely axiomatised subframe logic is determined by a finite number of finite frames* (*is tabular*)*.*

No general *undecidability results* have been proved so far for subframe logics.

Problem 8.9 *Characterise the class of subframe logics L for which $\{\varphi \mid \mathbf{K} \oplus \varphi^{sf} = L\}$ is recursive.*

For example, although **K4** is not a subframe union-splitting it is open whether $\{\varphi \mid \mathbf{K} \oplus \varphi^{sf} = \mathbf{K4}\}$ is recursive.

9 Superintuitionistic logics

Although C.I. Lewis constructed his first modal calculus **S3** in 1918, it was Gödel's [52] two page note that attracted serious attention of mathematical logicians to modal systems. While Lewis [90] used an abstract necessity operator to avoid paradoxes of the material implication, Gödel and earlier Orlov [117] [6] treated $\Box$ as 'it is provable' to give a classical interpretation of intuitionistic propositional logic **Int** of Brouwer [12,13] and Heyting [63] by means of embedding it into a modal 'provability' system which turned out to be equivalent to Lewis' **S4**.

Approximately at the same time Gödel [51] observed that there are infinitely many logics located between **Int** and classical logic **Cl**, which—together with the creation of constructive (proper) extensions of **Int** by Kleene [74] and Rose [122] (realisability logic), Medvedev [109] (logic of finite problems), Kreisel and Putnam [78]—gave an impetus to studying the class of logics intermediate between **Int** and **Cl**, started by Umezawa [149,150]. Gödel's embedding of **Int** into **S4**, presented in an algebraic form by McKinsey and Tarski [108] and extended to all intermediate logics by Dummett and Lemmon [33], made it possible to develop theories of modal and intermediate logics in parallel ways. And the structural results of Blok [8] and Esakia [36,36] establishing an isomorphism between the lattices Ext **Int** and NExt **Grz**, along with preservation results of Maksimova and Rybakov [100] and Zakharyaschev [171], transferring various properties from modal to intermediate logics and back, showed that in many respects the theory of intermediate logics is reducible to the theory of logics in NExt **S4**.

To demonstrate this as well as some features of superintuitionistic logics is the main aim of this section. We will use the same system of notation as in the modal case. In particular, Ext **Int** is the lattice of all logics of the form **Int**$+\Gamma$ (where Γ is an arbitrary set of formulas in the language of **Int** and $+$ means taking the closure under *modus ponens* and substitution); we call them *superintuitionistic logics* or *si-logics* for short. Basic facts about the syntax and semantics of **Int** and relevant references can be found in [157,26]. A list of some 'standard' si-logics is given in Table 3.

[6] Orlov's paper remained unnoticed till the end of the 1980s. It is remarkable also for constructing the first system of relevant logic.

Cl	$=$	$\mathbf{Int} + p \vee \neg p$
SmL	$=$	$\mathbf{Int} + (\neg q \to p) \to (((p \to q) \to p) \to p)$
KC	$=$	$\mathbf{Int} + \neg p \vee \neg\neg p$
LC	$=$	$\mathbf{Int} + (p \to q) \vee (q \to p)$
SL	$=$	$\mathbf{Int} + ((\neg\neg p \to p) \to \neg p \vee p) \to \neg p \vee \neg\neg p$

Table 3
A list of standard superintuitionistic logics

9.1 Intuitionistic frames

As in the case of modal logics, the adequate relational semantics for si-logics can be constructed on the base of the Stone representation of the algebraic 'models' for **Int**, known as *Heyting* (or *pseudo-Boolean*) *algebras*. It is hard to trace now who was the first to introduce intuitionistic general frames—the earliest references we know are [35] and [119]—but in any case, having at hand [71] and [54], the construction must have been clear.

An *intuitionistic* (*general*) *frame* is a triple $\mathfrak{F} = \langle W, R, P\rangle$ in which R is a partial order on $W \neq \emptyset$ and P is a collection of upward closed subsets (cones) in W containing $\emptyset$ and closed under $\cap$, $\cup$, and the operation $\supset$ (for $\to$) defined by

$$X \supset Y \quad = \quad \{x \in W \mid \forall y \in x\!\uparrow (y \in X \to y \in Y)\}.$$

If P contains all upward closed subsets of W then $\mathfrak{F}$ is called an *intuitionistic Kripke frame* and denoted by $\mathfrak{F} = \langle W, R\rangle$. An important feature of intuitionistic models $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ (where $\mathfrak{V}$ maps propositional variables to sets in P) is that $\mathfrak{V}(\varphi)$, the value of a formula φ in $\mathfrak{M}$, is always upward closed.

Every intuitionistic frame $\mathfrak{F} = \langle W, R, P\rangle$ gives rise to the Heyting algebra $\mathfrak{F}^+ = \langle P, \cap, \cup, \supset, \emptyset\rangle$ called the *dual* of $\mathfrak{F}$. Conversely, given a Heyting algebra $\mathfrak{A} = \langle A, \wedge, \vee, \to, \bot\rangle$, we construct its relational representation $\mathfrak{A}_+ = \langle W, R\rangle$ by taking W to be the set of all prime filters in $\mathfrak{A}$ (a filter ∇ is *prime* if it is proper and $a \vee b \in \nabla$ implies $a \in \nabla$ or $b \in \nabla$), R to be the set-theoretic inclusion $\subseteq$ and

$$P \quad = \quad \{\{\nabla \in W \mid a \in \nabla\} \mid a \in A\}.$$

It is readily checked that $\mathfrak{A}_+$, the *dual* of $\mathfrak{A}$, is an intuitionistic frame and $\mathfrak{A} \cong (\mathfrak{A}_+)^+$. A frame $\mathfrak{F}$ is called *descriptive* if $\mathfrak{F} \cong (\mathfrak{F}^+)_+$. Duality between the basic truth-preserving operations on algebras and descriptive frames (taking p-morphic images, generated subframes and disjoint unions) is established by the same technique as in the modal case.

Since every consistent si-logic L is characterised by its Tarski–Lindenbaum algebra $\mathfrak{A}_L$, we conclude that L is also characterised by a class of intuitionistic frames, say, by the dual of $\mathfrak{A}_L$.

At the algebraic level, the connection between **Int** and **S4** discovered by Gödel is reflected by the fact, established in [107], that the algebra of open elements (i.e., elements a such that $\Box a = a$) of every modal algebra for **S4** (known as a *topological Boolean algebra*; see [118]) is a Heyting algebra and, conversely, every Heyting algebra is isomorphic to the algebra of open elements of a suitable algebra for **S4**. We explain this result in the frame-theoretic language.

Given a frame $\mathfrak{F} = \langle W, R, P\rangle$ for **S4** (which means that R is a quasi-order on W), we denote by $\boldsymbol{\rho} W$ the set of clusters in $\mathfrak{F}$—more generally, $\boldsymbol{\rho} X = \{C(x) \mid x \in X\}$—and put $C(x)\boldsymbol{\rho} C(y)$ iff xRy,

$$\boldsymbol{\rho} P \quad = \quad \{\boldsymbol{\rho} X \mid \; X \in P \;\&\; X = \Box X\} \quad = \quad \{\boldsymbol{\rho} X \mid X \in P \wedge X \;\&\; X\!\uparrow\}.$$

It is readily checked that $\boldsymbol{\rho}\mathfrak{F} = \langle \boldsymbol{\rho} W, \boldsymbol{\rho} R, \boldsymbol{\rho} P\rangle$ is an intuitionistic frame; we call it the *skeleton* of $\mathfrak{F}$. The *skeleton* of a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V}\rangle$ for **S4** is the model $\boldsymbol{\rho}\mathfrak{M} = \langle \boldsymbol{\rho}\mathfrak{F}, \boldsymbol{\rho}\mathfrak{V}\rangle$, where $\boldsymbol{\rho}\mathfrak{V}(p) = \mathfrak{V}(\Box p)$.

Denote by T the *Gödel translation* prefixing $\Box$ to all subformulas of a given intuitionistic formula.[7] Then for every model $\mathfrak{M}$ for **S4**, every intuitionistic formula φ and every point x in $\mathfrak{M}$,

$$(\boldsymbol{\rho}\mathfrak{M}, C(x)) \models \varphi \qquad \text{iff} \qquad (\mathfrak{M}, x) \models T(\varphi).$$

It follows that $\varphi \in \mathbf{Int}$ implies $T(\varphi) \in \mathbf{S4}$. To prove the converse, we should be able to convert intuitionistic frames $\mathfrak{F}$ into modal ones with the skeleton (isomorphic to) $\mathfrak{F}$. This is trivial if $\mathfrak{F}$ is a Kripke frame—we can just regard it to be a frame for **S4**, which in view of the Kripke completeness of both **Int** and **S4**, shows that T really embeds the former into the latter, i.e.,

$$\varphi \in \mathbf{Int} \qquad \text{iff} \qquad T(\varphi) \in \mathbf{S4}.$$

In general, one can construct a modal frame from an intuitionistic frame $\mathfrak{F} = \langle W, R, P\rangle$ by taking the closure $\boldsymbol{\sigma}P$ of P under the Boolean operations. Then $\langle W, R, \boldsymbol{\sigma}P\rangle$ is a partially ordered modal frame; we denote it by $\boldsymbol{\sigma}\mathfrak{F}$. Moreover, we clearly have $\mathfrak{F} \cong \boldsymbol{\rho\sigma}\mathfrak{F}$. It is worth noting that if $\mathfrak{F} = \langle W, R\rangle$ is a finite intuitionistic Kripke frame then $\boldsymbol{\sigma}\mathfrak{F}$ is also a Kripke frame. However, for an infinite $\mathfrak{F}$, $\boldsymbol{\sigma}\mathfrak{F}$ is not in general a Kripke frame—witness $\langle \omega, \leq\rangle$.

9.2 Canonical formulas

The language of canonical formulas, axiomatising all si-logics and characterising the structure of their frames, can be easily developed following the scheme of constructing the canonical formulas for **K4** and using the connection between modal and intuitionistic frames established above. We confine ourselves here only to pointing out the differences from the modal case and some interesting peculiarities; details can be found in [169,170,26]. Actually, there are two important differences. First, in the definition of subreduction of $\mathfrak{F} = \langle W, R, P\rangle$ to $\mathfrak{G}$ the third condition does not correspond to the fact that all sets in P are upward closed. We replace it by the following condition

- $\forall X \in \overline{Q}\ f^{-1}(X){\downarrow} \in \overline{P}$, where $\overline{Q} = \{V - X \mid X \in Q\}$ and $\overline{P} = \{W - X \mid X \in P\}$.

$\mathfrak{G}$ is a *subframe* of $\mathfrak{F}$ if there is an injective subreduction of $\mathfrak{F}$ to $\mathfrak{G}$. It is of interest to note that in the intuitionistic case (cofinal) subreductions are dual to IC(N)-subalgebras of Heyting algebras which only preserve implication, conjunction (and negation) but do not necessarily preserve disjunction.

Second, we have to change the definition of open domains. Now we say an antichain $\mathfrak{a}$ (of at least two points) is an *open domain* in an intuitionistic model $\mathfrak{N}$ relative to a formula φ if there ia a pair $t_{\mathfrak{a}} = (\Gamma_{\mathfrak{a}}, \Delta_{\mathfrak{a}})$ such that $\Gamma_{\mathfrak{a}} \cup \Delta_{\mathfrak{a}} = \mathsf{sub}\,\varphi$, $\bigwedge \Gamma_{\mathfrak{a}} \to \bigvee \Delta_{\mathfrak{a}} \notin \mathbf{Int}$ and

- $\psi \in \Gamma_{\mathfrak{a}}$ iff $a \models \psi$ for all $a \in \mathfrak{a}$.

It is worth noting that in any intuitionistic model every antichain $\mathfrak{a}$ is open relative to every disjunction free formula φ. Indeed, let $\Gamma_{\mathfrak{a}}$ be defined by condition above and $\Delta_{\mathfrak{a}} = \mathsf{sub}\,\varphi - \Gamma_{\mathfrak{a}}$. It should be clear that $\psi \wedge \chi \in \Gamma_{\mathfrak{a}}$ iff $\psi \in \Gamma_{\mathfrak{a}}$ and $\chi \in \Gamma_{\mathfrak{a}}$. And if $\psi \to \chi \in \Gamma_{\mathfrak{a}}$, $\psi \in \Gamma_{\mathfrak{a}}$ but $\chi \in \Delta_{\mathfrak{a}}$ then $a \models \psi$ for every $a \in \mathfrak{a}$ and $b \not\models \chi$ for some $b \in \mathfrak{a}$, whence $b \not\models \psi \to \chi$, which is a contradiction. It follows that $\bigwedge \Gamma_{\mathfrak{a}} \to \bigvee \Delta_{\mathfrak{a}} \notin \mathbf{Int}$.

Now, as in the modal case, with every finite rooted intuitionistic frame $\mathfrak{F} = \langle W, R\rangle$ and a set $\mathfrak{D}$ of antichains in it we can associate two formulas $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\beta(\mathfrak{F}, \mathfrak{D})$, called the *canonical* and *negation free canonical formulas*, respectively, so that $\mathfrak{G} \not\models \beta(\mathfrak{F}, \mathfrak{D}, \bot)$ ($\mathfrak{G} \not\models \beta(\mathfrak{F}, \mathfrak{D})$) iff there is a (cofinal) subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CDC) for $\mathfrak{D}$.

Theorem 9.1 *There is an algorithm which, given an intuitionistic φ, returns canonical formulas $\beta(\mathfrak{F}_1, \mathfrak{D}_1, \bot), \ldots, \beta(\mathfrak{F}_n, \mathfrak{D}_n, \bot)$ such that*

$$\mathbf{Int} + \varphi \quad = \quad \mathbf{Int} + \beta(\mathfrak{F}_1, \mathfrak{D}_1, \bot) + \cdots + \beta(\mathfrak{F}_n, \mathfrak{D}_n, \bot).$$

[7] The translation defined in [52] does not prefix $\Box$ to conjunctions and disjunctions. However, this difference is of no importance as far as embeddings into logics in NExt**S4** are concerned.

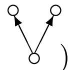
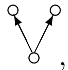
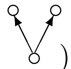
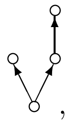
Cl	=	**Int** + $\beta($  $)$
SmL	=	**Int** + $\beta($  $) + \beta($  $)$
KC	=	**Int** + $\beta($  $, \bot)$
LC	=	**Int** + $\beta($  $)$
SL	=	**Int** + $\beta^\sharp($  $, \bot)$

Table 4
Canonical axioms of standard superintuitionistic logics

If φ is negation free then one can use only negation free canonical formulas. And if φ is disjunction free then all the $\mathfrak{D}_i$ are empty.

In the intuitionistic case we have the following syntactical characterisation of subframe and cofinal subframe si-logics:

Theorem 9.2 (1) *A si-logic is axiomatisable by implicative formulas iff it is determined by a class of* (*finite*) *frames closed under subframes.*

(2) *A si-logic is axiomatisable by disjunction-free formulas iff it is determined by a class of* (*finite*) *frames closed under cofinal subframes.*

It follows from these two theorems (and the refutability criterion for the canonical formulas) that all cofinal subframe si-logics have the finite model property and are decidable if finitely axiomatisable. In fact, there are a continuum of subframe logics [173]. That all si-logics with disjunction-free axioms have the fmp was first proved by McKay [104] with the help of Diego's [31] theorem according to which there are only finitely many pairwise nonequivalent in **Int** disjunction free formulas in variables $p_1, \ldots, p_n$ (see also [151]). An algebraic approach to superintuitionistic (cofinal) subframe logics has been recently developed in [2].

Problem 9.3 *Characterise the computational complexity of cofinal subframe si-logics.*

Table 4 shows canonical axiomatisations of the si-logics in Table 3. Using this 'geometrical' representation it is not hard to see, for instance, that **SmL**, known as the *Smetanich logic*, is the greatest consistent extension of **Int** different from **Cl**; it is the logic of the two-point rooted frame. **KC**, the logic of the *Weak Law of the Excluded Middle*, is characterised by the class of directed frames. It is the greatest si-logic containing the same negation free formulas as **Int** [68]. **LC**, the *Dummett* or *chain logic*, is characterised by the class of linear frames [32].

Jankov [69] proved that logics of the form **Int**+$\beta(\mathfrak{F}, \mathfrak{D}^\sharp, \bot)$ and only them are splittings of Ext **Int**. However, not every si-logic is a union-splitting of Ext **Int**.

Problem 9.4 *Characterise the degree of Kripke incompleteness of si-logics.*

9.3 Modal companions and preservation theorems

The fact that the Gödel translation T embeds **Int** into **S4** and the relationship between intuitionistic and modal frames shown above can be used to reduce various problems concerning **Int** (e.g., proving decidability, Kripke completeness or the fmp) to those for **S4** and vice versa. Moreover, it turns out that each logic in Ext **Int** is embedded by T into some logics in NExt **S4**, and for each logic in NExt **S4** there is a logic in Ext **Int** embeddable in it.

We say a modal logic $M \in$ NExt **S4** is a *modal companion* of a si-logic L if L is embedded in M by T, i.e., if for every intuitionistic formula φ, we have $\varphi \in L$ iff $T(\varphi) \in M$. If M is a modal companion of L then L is called the *si-fragment* of M and denoted by $\boldsymbol{\rho} M$. The reason for denoting the operator 'modal logic $\mapsto$ its si-fragment' by the same symbol we used for the skeleton operator is explained by the simple fact that $\boldsymbol{\rho} M = \{\varphi \mid T(\varphi) \in M\}$ and, moreover, if M is characterised by a class $\mathcal{C}$ of modal frames then $\boldsymbol{\rho} M$ is characterised by the class $\boldsymbol{\rho}\mathcal{C} = \{\boldsymbol{\rho}\mathfrak{F} \mid \mathfrak{F} \in \mathcal{C}\}$ of intuitionistic frames.

Thus, $\boldsymbol{\rho}$ maps NExt**S4** into Ext**Int**. The following observation of [33] shows that actually $\boldsymbol{\rho}$ is a surjection. Given a logic $L \in$ Ext **Int**, let

$$\boldsymbol{\tau} L \quad = \quad \mathbf{S4} \oplus \{T(\varphi) \mid \varphi \in L\}.$$

Then, for every si-logic L, $\boldsymbol{\tau} L$ is a modal companion of L.

Now we use the language of canonical formulas to show a general characterisation of all modal companions of a given si-logic L obtained in [170,171]. Notice first that for every modal frame $\mathfrak{G}$ and every intuitionistic canonical formula $\beta(\mathfrak{F}, \mathfrak{D}, \bot)$, $\mathfrak{G} \models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ iff $\boldsymbol{\rho}\mathfrak{G} \models \beta(\mathfrak{F}, \mathfrak{D}, \bot)$, and so $\mathbf{S4} \oplus T(\beta(\mathfrak{F}, \mathfrak{D}, \bot)) = \mathbf{S4} \oplus \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. The same holds for the negation free canonical formulas.

Theorem 9.5 *$M \in$ NExt **S4** is a modal companion of $L = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \mathfrak{D}_i, \bot) \mid i \in I\}$ iff M can be represented in the form*

$$M \quad = \quad \mathbf{S4} \quad \oplus \quad \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) \mid i \in I\} \quad \oplus \quad \{\alpha(\mathfrak{F}_j, \mathfrak{D}_j, \bot) \mid j \in J\},$$

where every frame $\mathfrak{F}_j$, for $j \in J$, contains a proper cluster.

Thus, we have:

$$\boldsymbol{\rho}\mathbf{S4} = \boldsymbol{\rho}\mathbf{S4.1} = \boldsymbol{\rho}\mathbf{Dum} = \boldsymbol{\rho}\mathbf{Grz} = \mathbf{Int},$$

$$\boldsymbol{\rho}\mathbf{S4.2} = \boldsymbol{\rho}(\mathbf{S4.2} \oplus \mathbf{Grz}) = \mathbf{KC},$$

$$\boldsymbol{\rho}\mathbf{S4.3} = \boldsymbol{\rho}(\mathbf{S4.3} \oplus \mathbf{Grz}) = \mathbf{LC},$$

$$\boldsymbol{\rho}\mathbf{S5} = \boldsymbol{\rho}(\mathbf{S5} \oplus \mathbf{Grz}) = \mathbf{Cl}.$$

Corollary 9.6 *The set of modal companions of every consistent si-logic L forms the interval*

$$\boldsymbol{\rho}^{-1}(L) \quad = \quad [\boldsymbol{\tau} L, \boldsymbol{\tau} L \oplus \alpha(②)] \quad = \quad \{M \in \text{NExt}\,\mathbf{S4} \mid \boldsymbol{\tau} L \subseteq M \subseteq \boldsymbol{\tau} L \oplus \mathbf{Grz}\}$$

and contains an infinite descending chain of logics.

Proof Notice first that $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha(\mathfrak{F}, \mathfrak{D})$ are in **Grz** iff $\mathfrak{F}$ contains a proper cluster. So $\boldsymbol{\rho}^{-1}(L) \subseteq [\boldsymbol{\tau} L,\, \boldsymbol{\tau} L \oplus \alpha(②)]$. On the other hand, the si-fragments of all logics in the interval are the same, namely L. Therefore, $\boldsymbol{\rho}^{-1}(L) = [\boldsymbol{\tau} L, \boldsymbol{\tau} L \oplus \alpha(②)]$. Now, if L is consistent then $\beta(\circ) \notin L$ and so

$$\boldsymbol{\tau} L \subset \ldots \subset \boldsymbol{\tau} L \oplus \alpha(ⓚ) \subset \cdots \subset \boldsymbol{\tau} L \oplus \alpha(②) \subset \boldsymbol{\tau} L \oplus \alpha(①) = \mathbf{For},$$

where **For** is the set of all intuitionistic formulas. ❑

This result is due to Maksimova and Rybakov [100], Blok [8] and Esakia [37]. Thus, all modal companions of every si-logic L are contained between the least companion $\boldsymbol{\tau} L$ and the greatest one, viz., $\boldsymbol{\tau} L \oplus \alpha(②)$, which will be denoted by $\boldsymbol{\sigma} L$.

The following theorem, which is also a consequence of Theorem 9.5, describes lattice-theoretic properties of the maps $\boldsymbol{\rho}$, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$. Items (i), (ii) and (iv) in it were first proved in [100]; (iii) is known as the Blok–Esakia theorem [8,37].

Property of logics	Preserved under		
	$\boldsymbol{\rho}$	$\boldsymbol{\tau}$	$\boldsymbol{\sigma}$
Decidability	Yes	Yes	Yes
Kripke completeness	Yes	Yes	No
Finite model property	Yes	Yes	Yes
Tabularity	Yes	No	Yes
Interpolation property	Yes	No	No
First-order definability	Yes	Yes	No

Table 5
Preservation theorems

Theorem 9.7 (i) *The map* $\boldsymbol{\rho}$ *is a homomorphism of the lattice* NExt **S4** *onto the lattice* Ext **Int**.

(ii) *The map* $\boldsymbol{\tau}$ *is an isomorphism from* Ext **Int** *into* NExt **S4**.

(iii) *The map* $\boldsymbol{\sigma}$ *is an isomorphism from* Ext **Int** *onto* NExt **Grz**.

(iv) *All these maps preserve infinite sums and intersections of logics.*

The following theorem provides a deductive characterisation of the maps $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$.

Theorem 9.8 *For every si-logic L and every modal canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ built on a quasi-ordered frame $\mathfrak{F}$,*

(i) $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \boldsymbol{\tau}L$ *iff* $\beta(\boldsymbol{\rho}\mathfrak{F}, \boldsymbol{\rho}\mathfrak{D}, \bot) \in L$;

(ii) $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \boldsymbol{\sigma}L$ *iff either $\mathfrak{F}$ is partially ordered and $\beta(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ or $\mathfrak{F}$ contains a proper cluster.*

The theorems above can be used for transferring various properties of modal logics to their si-fragments and back. Some results of that sort are collected in Table 5; for proofs see [26].

9.4 Completeness

In this section we briefly discuss the most important general results concerning completeness of si-logics with respect to various classes of Kripke frames. (As in the modal case, the fmp and decidability of a good many concrete si-logics was proved using various forms of filtration; see, e.g., [46,116,136,47,39].)

That not all si-logics are complete with respect to Kripke frames was discovered by Shehtman [132], who found a way to adjust Fine's [43] idea to the intuitionistic case. As to general positive results, notice first that the preservation theorems yield the following translation of Theorems 5.5–5.6 (si-logics of finite width were studied by Sobolev [138]).

Theorem 9.9 (1) *Every si-logic of finite depth has the fmp* (*in fact, is locally tabular*).

(2) *Every si-logic of width n is characterised by a class of Noetherian Kripke frames of width $\leq n$.*

Problem 9.10 *Are finitely axiomatisable si-logics of width $n > 1$ decidable? What is their computational complexity?*

An intuitionistic formula is said to be *essentially negative* if every occurrence of a variable in it is in the scope of some $\neg$. If φ is essentially negative then $T(\varphi)$ is a $\Box\Diamond$-formula, which—together with Theorem 4.6—yields the following result of McKay [105]:

Theorem 9.11 (McKay) *If a si-logic L is decidable* (*or has the fmp*) *and φ is an essentially negative formula then $L + \varphi$ is decidable* (*has the fmp*).

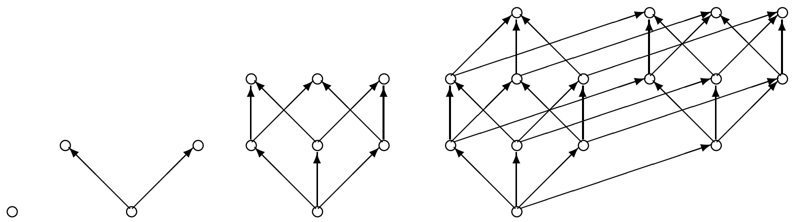


Fig. 5.

Say that an occurrence of a variable in a formula is *essential* if it is not in the scope of any $\neg$. A formula φ is *mild* if every two essential occurrences of the same variable in φ are either both positive or both negative. Kuznetsov [83] claimed (we have not seen the proof) that all si-logics whose extra axioms do not contain negative occurrences of essential variables have the fmp. And Wroński [168] announced that if L is a decidable si-logic and φ a mild formula then $L + \varphi$ is also decidable.

Since frames for **Int** contain no clusters, Theorem 8.1 and its analogue for cofinal subframe logics reduce in the intuitionistic case to the following result which is due to [27,121,134,174]:

Theorem 9.12 *All si-logics with disjunction free axioms are first-order definable* (*definable by* $\forall\exists$-*sentences*) *and* $\mathcal{D}$*-persistent.*

Nishimura [115] described all si-logics axiomatisable by one-variable formulas. As a consequence of his result and Theorem 4.14 we obtain the following theorem due to Sobolev [138]:

Theorem 9.13 (Sobolev) *All si-logics with extra axioms in one variable have the fmp and are decidable.*

In fact, Sobolev proved a more general (but rather complicated) syntactical sufficient condition of the fmp and constructed a formula in two variables axiomatising a si-logic without the fmp (Shehtman's [132] incomplete si-logic has also axioms in two variables).

By the Blok–Esakia and preservation theorems, the situation with tabular logics in Ext**Int** is the same as in NExt **Grz**. In particular, there are only three pretabular logics in Ext**Int** [95], and the tabularity problem is decidable in Ext**Int**.

9.5 Medvedev's logic

Although this chapter's main concern is *classes* of logics rather than individual systems, we conclude it with a brief discussion of a very elegant and interesting si-logic introduced by Medvedev [109,110,111] and known as the *logic of finite problems* or the *Medvedev logic* **ML**. Semantically it can be defined as the set of intuitionistic formulas that are valid in all 'topless' n-ary Boolean cubes depicted in Fig. 5 for $n = 1, 2, 3, 4$. More precisely, let W_n be the family of nonempty subsets of a set with $n > 0$ elements and xR_ny means $y \subseteq x$, for $x, y \in W_n$. Then

$$\mathbf{ML} \quad = \quad \{\varphi \mid \langle W_n, R_n\rangle \models \varphi \ \text{ for all } n \geq 1\}.$$

It turns out that **ML** is a *constructive* si-logic in the sense that it enjoys the following *disjunction property*

$$\varphi \vee \psi \in \mathbf{ML} \quad \text{implies} \quad \varphi \in \mathbf{ML} \ \text{ or } \ \psi \in \mathbf{ML}$$

and, moreover, no proper extension of **ML** is constructive in this sense [89,99]. In fact, there are a continuum of maximal constructive si-logics [73,98,20,39], and not a single one of them is known to be finitely axiomatisable or decidable. In particular, [101] shows that **ML** is not finitely axiomatisable.

Problem 9.14 *Does there exist a decidable maximal si-logic with the disjunction property? In particular, is* **ML** *decidable?*

Problem 9.15 *Does there exist a finitely axiomatisable maximal si-logic with the disjunction property?*

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