Unique Characterisability and Learnability of Temporal Queries Mediated by an Ontology

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Abstract

Algorithms for learning database queries from examples and unique characterisations of queries by examples are prominent starting points for developing automated support for query construction and explanation. We investigate how far recent results and techniques on learning and unique characterisations of atemporal queries mediated by an ontology can be extended to temporal data and queries. Based on a systematic review of the relevant approaches in the atemporal case, we obtain general transfer results identifying conditions under which temporal queries composed of atemporal ones are (polynomially) learnable and uniquely characterisable.

1 Introduction

Providing automated support for constructing database queries from data examples has been an important research topic in database management, knowledge representation and computational logic, often subsumed under the query-by-example paradigm (Martins 2019). One prominent approach is based on exact learning using membership queries (Angluin 1987b), where one aims to identify a database query by repeatedly asking an oracle (e.g., domain expert) whether certain data examples are answers or non-answers to the query. Recently, the ability to uniquely characterise a database query by a finite set of positive and negative examples has been identified and investigated as a 'non-procedural' necessary condition for learnability via membership queries (Staworko and Wieczorek 2015; ten Cate and Dalmau 2022; Fortin et al. 2022). More precisely, a query $\boldsymbol{q}(x)$ is said to fit a pair $E = (E^+, E^-)$ of sets E^+ and E^- of pointed databases $(\mathcal{D}, a)$ if $\mathcal{D} \models \boldsymbol{q}(a)$ for all $(\mathcal{D}, a) \in E^+$, and $\mathcal{D} \not\models \boldsymbol{q}(a)$ for all $(\mathcal{D}, a) \in E^-$. The example set E uniquely characterises $\boldsymbol{q}$ within a class $\mathcal{Q}$ of queries if $\boldsymbol{q}$ is the only one (up to equivalence) in $\mathcal{Q}$ that fits E. The existence of (polynomial-size) unique characterisations is a necessary pre-condition for (polynomial) learnability via membership queries. Such characterisations can also be employed for explaining and synthesising queries.

Extending results on characterising and learning conjunctive queries (CQs) under the standard closed-world semantics (ten Cate and Dalmau 2022), there has recently been significant progress towards CQs mediated by a description logic (DL) ontology under the open-world semantics (Funk, Jung, and Lutz 2021; 2022b). The focus has been on ontologies in the tractable *DL-Lite* and $\mathcal{EL}$ families and tree-shaped CQs such as ELQs ($\mathcal{EL}$-concepts) and ELIQs ($\mathcal{ELI}$-concepts). In fact, even under the closed-world semantics, only acyclic queries can be uniquely characterised and, equivalently, learned using membership queries in polynomial time (ten Cate and Dalmau 2022).

In this paper, we aim to understand how far these characterisability and learnability results for atemporal queries mediated by an ontology can be expanded to the temporal case. Temporal ontology-mediated query answering provides a framework for accessing temporal data using a background ontology. It has been investigated for about a decade—see, e.g., (Artale et al. 2017) for a survey—resulting in different settings and a variety of query and ontology languages (Baader, Borgwardt, and Lippmann 2015; Borgwardt and Thost 2015; Artale et al. 2022; Gutiérrez-Basulto, Jung, and Kontchakov 2016; Artale et al. 2014; Wałęga et al. 2020). As a natural starting point, we assume that the background ontology holds at all times and does not admit temporal operators in its axioms. As a query language we consider a combination of ELIQs with linear temporal logic (*LTL*) operators. First observations on unique characterisability and learnability of plain *LTL* queries (Fortin et al. 2022) showed that, even without ontologies, a restriction to so-called *path queries* (defined below) is needed to obtain positive general and useful results. Our main contributions in this paper are general transfer theorems identifying abstract properties of query and ontology languages that are needed to lift unique characterisability and learnability from atemporal ontology-mediated queries and ontology-free path *LTL* queries to temporalised domain queries mediated by a DL ontology. To facilitate the transfer, we begin by revisiting the atemporal case. Below is an overview of the obtained results.

Atemporal case. We present and compare two approaches to finding unique (polysize) characterisations of atemporal queries mediated by an ontology: via frontiers and via split-partners (aka dualities). Both tools are developed under the condition that query containment in the respective atemporal DLs can be reduced to query evaluation. We call this condition *containment reduction*. It applies to all fragments of the expressive DL $\mathcal{ALCHI}$ and more general FO-ontologies without equality as well as to *DL-Lite* with functional roles.

It ensures that whenever a unique characterisation of a query mediated by an ontology exists, there is also one with a single positive example in E^+. These tools yield two essentially optimal unique characterisability results: frontiers give polynomial-size characterisations of ELIQs mediated by an ontology in the DLs $\textit{DL-Lite}_{\mathcal{H}}$ and $\textit{DL-Lite}_{\mathcal{F}}^{-}$ (Funk, Jung, and Lutz 2021; 2022b), while split-partners provide exponential-size characterisations of ELIQs mediated by an $\mathcal{ALCHI}$ ontology and polysize characterisations of ELQs mediated by an RDFS ontology.

Temporalising unique characterisations. We now assume that temporal data instances are finite sets of facts (ground unary and binary atoms) timestamped by the moments $i \in \mathbb{N}$ they happened and that queries are equipped with temporal operators. By combining the results from the atemporal case above with the techniques of (Fortin et al. 2022), we establish general transfer theorems on (polysize) unique characterisations of temporal queries mediated by a DL ontology.

We first consider the temporal operators $\bigcirc$ (at the next moment), $\Diamond$ (sometime later), and $\Diamond_r$ (now or later) and define, given a class $\mathcal{Q}$ of atemporal queries (say, ELIQs), the family $\textit{LTL}_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ of *path queries* of the form

$$\boldsymbol{q} = \boldsymbol{r}_0 \wedge \boldsymbol{o}_1(\boldsymbol{r}_1 \wedge \boldsymbol{o}_2(\boldsymbol{r}_2 \wedge \cdots \wedge \boldsymbol{o}_n \boldsymbol{r}_n)),$$

where $\boldsymbol{o}_i \in \{\bigcirc, \Diamond, \Diamond_r\}$ and $\boldsymbol{r}_i \in \mathcal{Q}$. These queries are evaluated at time 0. Even if $\mathcal{Q}$ consists of conjunctions of atoms only and no ontology is present, not all queries in $\textit{LTL}_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ can be uniquely characterised. A typical example of a non-characterisable query in this class is $\boldsymbol{q}(x) = \Diamond_r(A(x) \wedge B(x))$ (Fortin et al. 2022). We first give an effective syntactic criterion for an $\textit{LTL}_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$-query to be 'safe' in the sense of admitting a unique characterisation. Then we prove a fully general transfer theorem stating that if a DL $\mathcal{L}$ admits containment reduction and (polysize) unique characterisations for $\mathcal{Q}$-queries mediated by an $\mathcal{L}$-ontology, then so does the class of safe temporalised queries in $\textit{LTL}_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$. For example, this theorem yields polysize unique characterisations of safe queries in $\textit{LTL}_p^{\bigcirc\Diamond\Diamond_r}(\text{ELIQ})$ mediated by a $\textit{DL-Lite}_{\mathcal{F}}^{-}$ or $\textit{DL-Lite}_{\mathcal{H}}$ ontology and exponential ones for safe $\textit{LTL}_p^{\bigcirc\Diamond\Diamond_r}(\text{ELIQ})$-queries mediated by an $\mathcal{ALCHI}$ ontology.

Our second transfer result concerns temporal queries with the binary operator U (until) under the strict semantics and the family $\textit{LTL}_p^{\mathsf{U}}(\mathcal{Q})$ of path queries of the form

$$\boldsymbol{q} = \boldsymbol{r}_0 \wedge (\boldsymbol{l}_1 \,\mathsf{U}\, (\boldsymbol{r}_1 \wedge (\boldsymbol{l}_2 \,\mathsf{U}\, (\dots (\boldsymbol{l}_n \,\mathsf{U}\, \boldsymbol{r}_n) \dots)))).$$

For its subclass of 'peerless' queries, in which the $\boldsymbol{r}_i, \boldsymbol{l}_i \in \mathcal{Q}$ do not contain each other wrt $\mathcal{O}$, we prove general transfer of unique characterisations provided that unique characterisations for the atemporal class $\mathcal{Q}$ can be obtained via split-partners. For example, this result gives exponential-size unique characterisations of peerless queries in $\textit{LTL}_p^{\mathsf{U}}(\text{ELIQ})$ mediated by any $\mathcal{ALCHI}$ ontology and polysize characterisations of peerless queries in $\textit{LTL}_p^{\mathsf{U}}(\text{ELQ})$ mediated by any RDFS ontology. We also show that the general transfer fails if frontier-based characterisations of queries in $\mathcal{Q}$ are used in place of split-partners.

Temporalising learning. We apply our results on unique characterisations to learning a target query $\boldsymbol{q}_T$, known only to a teacher, wrt a given ontology $\mathcal{O}$ in Angluin's framework of exact learning. We allow the learner to use *membership queries*, which return in unit time whether a given example $(\mathcal{D}, a)$ is a positive one for $\boldsymbol{q}_T$ wrt to $\mathcal{O}$. Given that we always construct example sets effectively, it is not difficult to show that our exponential-size unique characterisations entail exponential learning algorithms. We are, however, mainly interested in efficient algorithms formalised as polynomial time or polynomial query learnability.

Obtaining such algorithms from polysize characterisations is more challenging and we currently only know how this can be done if the unique characterisation is based on polysize frontiers. Hence, we focus on queries in $\textit{LTL}_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ and show that polynomial query learnability transfers from $\mathcal{Q}$ to safe queries in $\textit{LTL}_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ and that polytime learnability transfers if natural additional conditions hold for $\mathcal{Q}$ and the DL.

Omitted details and proofs can be found in the appendix.

2 Related Work

The unique characterisation framework for temporal formulas, underpinning this paper, was originally introduced by Fortin et al. (2022). Recently, it has been generalised to finitely representable transfinite words as data examples (Sestic 2023), whose results are not directly applicable to the problems we are concerned with as the queries have no DL component and no ontology is present. It would be of interest to extend the techniques used by Sestic (2023) to the more general languages considered here.

We are not aware of any work on exact learning of temporal formulas save (Camacho and McIlraith 2019) and the related work of exact learning of finite automata starting with (Angluin 1987a). In contrast, passive learning of *LTL*-formulas has recently received significant attention (Lemieux, Park, and Beschastnikh 2015; Neider and Gavran 2018; Camacho and McIlraith 2019; Fijalkow and Lagarde 2021; Fortin et al. 2023).

The database and KR communities have been working on identifying queries and concept descriptions from data examples (Staworko and Wieczorek 2015; Konev et al. 2017; ten Cate, Dalmau, and Kolaitis 2013; Ozaki 2020; ten Cate and Dalmau 2022). In reverse engineering of queries, the goal is typically to decide whether there is a query separating given positive and negative examples. Relevant work includes (Arenas and Diaz 2016; Barceló and Romero 2017) under the closed world and (Lehmann and Hitzler 2010; Gutiérrez-Basulto, Jung, and Sabellek 2018; Funk et al. 2019; Jung et al. 2022) under the open world assumption.

The use of unique characterisations for explaining and constructing schema mappings was promoted and investigated by Kolaitis (2011) and Alexe et al. (2011).

Unique characterisability of formulas in modal logics (under the closed world assumption and without ontologies) has recently been studied by ten Cate and Koudijs (2023).

3 Atemporal Ontologies and Queries

We assume that background knowledge about the object domain is given as a standard description logic ontology. This section recaps the relevant definitions.

As usual in DL, we work with any signature of unary and binary predicate symbols, typically denoted A, B and P, R, respectively. A *data instance* is any finite set $\mathcal{A} \neq \emptyset$ of *atoms* of the form $A(a)$ and $P(a, b)$ with *individual names* a, b, and also $\top(a)$, which simply says that a exists. Denote by *ind*$(\mathcal{A})$ the set of individuals in $\mathcal{A}$ and by P^- the *inverse* of P, assuming that $P^-(a, b) \in \mathcal{A}$ iff $P(b, a) \in \mathcal{A}$. Let S range over binary predicates and their inverses. A *pointed data instance* is a pair $(\mathcal{A}, a)$ with $a \in$ *ind*$(\mathcal{A})$. The *size* $|\mathcal{A}|$ of $\mathcal{A}$ is the number of symbols in it.

In general, an *ontology*, $\mathcal{O}$, is a finite set of first-order (FO) sentences in the given signature. Ontologies and data instances are interpreted in structures $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ with domain $\Delta^\mathcal{I} \neq \emptyset$, $a^\mathcal{I} \in \Delta^\mathcal{I}$, $\top^\mathcal{I} = \Delta^\mathcal{I}$, $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, and $P^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. As usual in database theory, we assume that $a^\mathcal{I} \neq b^\mathcal{I}$ for distinct a, b; moreover, to simplify notation, we adopt the *standard name assumption* and interpret each individual name by itself, i.e., $a^\mathcal{I} = a$. Thus, $\mathcal{I}$ is a *model* of $\mathcal{A}$ if $a \in A^\mathcal{I}$ and $(a, b) \in P^\mathcal{I}$, for all $A(a) \in \mathcal{A}$ and $P(a, b) \in \mathcal{A}$. We call $\mathcal{I}$ a *model* of an ontology $\mathcal{O}$ if all sentences in $\mathcal{O}$ are true in $\mathcal{I}$, and say that $\mathcal{O}$ and $\mathcal{A}$ are *satisfiable* if they have a model.

The ontology languages we consider here are certain members of the *DL-Lite* family, $\mathcal{ALCHI}$, $\mathcal{ELHIF}$; we define them below as fragments of first-order logic:

DL-Lite$_\mathcal{F}$ (Calvanese et al. 2007b) aka *DL-Lite*$^\mathcal{F}_{core}$ (Artale et al. 2009) allows axioms of the following forms:

$$\forall x\,\big(B(x) \to B'(x)\big), \quad \forall x\,\big(B(x) \land B'(x) \to \bot\big),$$
$$\forall x, y, z\,\big(S(x, y) \land S(x, z) \to (y = z)\big), \quad (1)$$

where *basic concepts* $B(x)$ are either $A(x)$ or $\exists S(x) = \exists y\, S(x, y)$. In DL parlance, the first two axioms in (1) are written as $B \sqsubseteq B'$ and $B \sqcap B' \sqsubseteq \bot$, and the third one as $\geq 2\, S \sqsubseteq \bot$ or fun(S), a *functionality constraint* stating that relation S is *functional*.

DL-Lite$^-_\mathcal{F}$ (Funk, Jung, and Lutz 2022b) is the fragment of *DL-Lite*$_\mathcal{F}$, in which *concept inclusions* (CIs) $B \sqsubseteq B'$ cannot have $B' = \exists S$ with functional S^-.

DL-Lite$_\mathcal{H}$ (Calvanese et al. 2007b) aka *DL-Lite*$^\mathcal{H}_{core}$ (Artale et al. 2009) is obtained by disallowing the functionality constraints in *DL-Lite*$_\mathcal{F}$ and adding axioms of the form

$$\forall x, y\,(S(x, y) \to S'(x, y)) \quad (2)$$

known as *role inclusions* (RIs) and written as $S \sqsubseteq S'$.

RDFS[1] has CIs between concept names, RIs between role names, and CIs of the forms $\exists P \sqsubseteq A$ or $\exists P^- \sqsubseteq A$ saying that the domain of P and range of P are in A, respectively.

$\mathcal{ALCHI}$ (Baader et al. 2017) has the same RIs as in (2) but more expressive CIs $\forall x\,(C_1(x) \to C_2(x))$, in which the *concepts* C_i are defined inductively starting from atoms $\top(x)$ and $A(x)$ and using the constructors $C(x) \land C'(x)$, $\neg C(x)$, and $\exists y\,(S(x, y) \land C(y))$, for a fresh y—or $C \sqcap C'$, $\neg C$, and $\exists S.C$ in DL terms.

$\mathcal{ELHIF}$ (Baader et al. 2017) has RIs (2), functionality constraints, and CIs with concepts built from atoms and $\bot$ using $\land$ and $\exists y\,(S(x, y) \land C(y))$ only. $\mathcal{ELHI}$ and $\mathcal{ELIF}$ are the fragments of $\mathcal{ELHIF}$ without functionality constraints and RIs, respectively.

We reserve $\mathcal{L}$ for denoting any of these ontology languages:

$$\text{RDFS} \subset \begin{array}{ccccccc} \textit{DL-Lite}_\mathcal{H} & & \subset & & \mathcal{ELHI} & \subset & \mathcal{ALCHI} \\ & & & & & \subset & \\ \textit{DL-Lite}^-_\mathcal{F} & \subset & \textit{DL-Lite}_\mathcal{F} & \subset & \mathcal{ELIF} & \subset & \mathcal{ELHIF} \end{array}$$

The most general query language over the object domain we consider consists of *conjunctive queries* (CQs) $\boldsymbol{q}(x)$ in the signature σ with a single *answer variable* x. We often think of $\boldsymbol{q}(x)$ as the set of its atoms and denote by *var*$(\boldsymbol{q})$ and *sig*$(\boldsymbol{q})$ the sets of its individual variables and predicates symbols, respectively. We say that $\boldsymbol{q}(x)$ is *satisfiable* wrt an ontology $\mathcal{O}$ if $\mathcal{O} \cup \{\boldsymbol{q}(x)\}$ has a model.

Given a CQ $\boldsymbol{q}(x)$, an ontology $\mathcal{O}$, and a data instance $\mathcal{A}$, we say that $a \in$ *ind*$(\mathcal{A})$ is a (*certain*) *answer to* $\boldsymbol{q}$ *over* $\mathcal{A}$ *wrt* $\mathcal{O}$ and write $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}(a)$ if $\mathcal{I} \models \boldsymbol{q}(a)$ for all models $\mathcal{I}$ of $\mathcal{O}$ and $\mathcal{A}$. Recall that $\emptyset, \mathcal{A} \models \boldsymbol{q}(a)$ iff there is function $h\colon$ *var*$(\boldsymbol{q}) \to \mathcal{A}$ such that $h(x) = a$, $A(y) \in \boldsymbol{q}$ implies $A(h(y)) \in \mathcal{A}$, and $P(y, z) \in \boldsymbol{q}$ implies $P(h(y), h(z)) \in \mathcal{A}$. Such a function h is called a *homomorphism* from $\boldsymbol{q}$ to $\mathcal{A}$, written $h\colon \boldsymbol{q} \to \mathcal{A}$; h is *surjective* if $h($*var*$(\boldsymbol{q})) =$ *ind*$(\mathcal{A})$.

We say that a CQ $\boldsymbol{q}_1(x)$ is *contained* in a CQ $\boldsymbol{q}_2(x)$ wrt an ontology $\mathcal{O}$ and write $\boldsymbol{q}_1 \models_\mathcal{O} \boldsymbol{q}_2$ if $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}_1(a)$ implies $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}_2(a)$, for any data instance $\mathcal{A}$ and any $a \in$ *ind*$(\mathcal{A})$. If $\boldsymbol{q}_1 \models_\mathcal{O} \boldsymbol{q}_2$ and $\boldsymbol{q}_2 \models_\mathcal{O} \boldsymbol{q}_1$, we say that $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are *equivalent wrt* $\mathcal{O}$, writing $\boldsymbol{q}_1 \equiv_\mathcal{O} \boldsymbol{q}_2$. For $\mathcal{O} = \emptyset$, we often write $\boldsymbol{q}_1 \equiv \boldsymbol{q}_2$ instead of $\boldsymbol{q}_1 \equiv_\emptyset \boldsymbol{q}_2$.

Two smaller query languages we need are $\mathcal{ELI}$*-queries* (or ELIQs, for short) that can be defined by the grammar

$$\boldsymbol{q} \quad := \quad \top \quad | \quad A \quad | \quad \exists S.\boldsymbol{q} \quad | \quad \boldsymbol{q} \land \boldsymbol{q}'$$

and $\mathcal{EL}$*-queries* (or ELQs), which are ELIQs without inverses P^-. Semantically, an ELIQ $\boldsymbol{q}$ has the same meaning as the *tree-shaped CQ* $\boldsymbol{q}(x)$ that is defined inductively starting from atoms $\top(x)$ and $A(x)$ and using the constructors $\exists y\,(S(x, y) \land \boldsymbol{q}(y))$, for a fresh y, and $\boldsymbol{q}(x) \land \boldsymbol{q}'(x)$. The only free (i.e., answer) variable in $\boldsymbol{q}$ is x.

We reserve $\mathcal{Q}$ for denoting a class of queries with answer variable x such that whenever $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathcal{Q}$, then $\boldsymbol{q}_1 \land \boldsymbol{q}_2 \in \mathcal{Q}$. Some of our results require restricting $\mathcal{Q}$ to a *finite signature* σ: we denote by $\mathcal{Q}^\sigma$ the class of those queries in $\mathcal{Q}$ that are built from predicates in σ. The classes of all σ-ELIQs and σ-ELQs are denoted by ELIQ$^\sigma$ and ELQ$^\sigma$, respectively.

It will be convenient to include the 'inconsistency query' $\bot$ into all of our query classes. By definition, we have $\mathcal{O}, \mathcal{A} \models \bot(a)$ iff $\mathcal{O}$ and $\mathcal{A}$ are inconsistent.

4 Unique Characterisability

An *example set* is a pair $E = (E^+, E^-)$, where E^+ and E^- are finite sets of pointed data instances $(\mathcal{A}, a)$. A CQ $\boldsymbol{q}(x)$ *fits* E wrt $\mathcal{O}$ if $\mathcal{O}, \mathcal{A}^+ \models \boldsymbol{q}(a^+)$ and $\mathcal{O}, \mathcal{A}^- \not\models \boldsymbol{q}(a^-)$, for all $(\mathcal{A}^+, a^+) \in E^+$ and $(\mathcal{A}^-, a^-) \in E^-$. We say that E *uniquely characterises* $\boldsymbol{q}$ *wrt* $\mathcal{O}$ *within a given class* $\mathcal{Q}$ *of*

[1] https://www.w3.org/TR/rdf12-schema/

queries if $\boldsymbol{q}$ fits E and $\boldsymbol{q} \equiv_{\mathcal{O}} \boldsymbol{q}'$, for every $\boldsymbol{q}' \in \mathcal{Q}$ that fits E. Note that, in this case, $E^+ = \emptyset$ implies $\boldsymbol{q} \equiv_{\mathcal{O}} \bot$, and so $\boldsymbol{q}$ is not satisfiable wrt $\mathcal{O}$.

We first observe that, for a large class of ontologies $\mathcal{O}$, including all those considered here, if $\boldsymbol{q}$ is uniquely characterised by some $E = (E^+, E^-)$ wrt $\mathcal{O}$, then $\boldsymbol{q}$ has a unique characterisation of the form $E' = (\{(\hat{\boldsymbol{q}}, a)\}, E^-)$ with a single positive example $(\hat{\boldsymbol{q}}, a)$. Say that an ontology $\mathcal{O}$ *admits containment reduction* if, for any CQ $\boldsymbol{q}(x)$, there is a pointed data instance $(\hat{\boldsymbol{q}}, a)$ such that the following conditions hold:

(cr$_1$) $\boldsymbol{q}(x)$ is satisfiable wrt $\mathcal{O}$ iff $\mathcal{O}$ and $\hat{\boldsymbol{q}}$ are satisfiable;

(cr$_2$) there is a surjective $h \colon \boldsymbol{q} \to \hat{\boldsymbol{q}}$ with $h(x) = a$;

(cr$_3$) if $\boldsymbol{q}(x)$ is satisfiable wrt $\mathcal{O}$, then for every CQ $\boldsymbol{q}'(x)$, we have $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{q}'$ iff $\mathcal{O}, \hat{\boldsymbol{q}} \models \boldsymbol{q}'(a)$.

An ontology language $\mathcal{L}$ *admits containment reduction* if every $\mathcal{L}$-ontology does. If the pointed data instance $(\hat{\boldsymbol{q}}, a)$ is computable in polynomial time, for every $\mathcal{O}$ in $\mathcal{L}$, we say that $\mathcal{L}$ *admits tractable containment reduction*. The next lemma illustrates this definition by a few concrete examples.

Lemma 1. (1) *FO without equality admits tractable containment reduction*; *in particular,* $\mathcal{ALCHI}$ *admits tractable containment reduction.*

(2) $\mathcal{ELIF}$ *admits tractable containment reduction.*

(3) $\{\geq 3\, P \sqsubseteq \bot\}$ *does not admit containment reduction.*

Proof. For (1), one can define $\hat{\boldsymbol{q}}$ as $\boldsymbol{q}$, with the variables regarded as constants. To show (2), $\boldsymbol{q}$ has to be factorised first to ensure functionality; (3) is shown in appendix. □

It is readily checked that we have the following:

Lemma 2. *Suppose* $\mathcal{O}$ *admits containment reduction and* $\boldsymbol{q} \in \mathcal{Q}$ *is satisfiable wrt* $\mathcal{O}$*, having a unique characterisation* $E = (E^+, E^-)$ *wrt* $\mathcal{O}$ *within* $\mathcal{Q}$*. Then* $E' = (\{(\hat{\boldsymbol{q}}, a)\}, E^-)$ *is a unique characterisation of* $\boldsymbol{q}$ *wrt* $\mathcal{O}$ *within* $\mathcal{Q}$*, too.*

We use two ways of constructing unique characterisations: via frontiers and via split-partners. Let $\mathcal{O}$ be an ontology, $\mathcal{Q}$ a class of queries, and $\boldsymbol{q} \in \mathcal{Q}$ a satisfiable query wrt $\mathcal{O}$. A *frontier of* $\boldsymbol{q}$ *wrt* $\mathcal{O}$ *within* $\mathcal{Q}$ is a set $\mathcal{F}_{\boldsymbol{q}} \subseteq \mathcal{Q}$ such that

- for any $\boldsymbol{q}' \in \mathcal{F}_{\boldsymbol{q}}$, we have $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{q}'$ and $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}$;
- for any $\boldsymbol{q}'' \in \mathcal{Q}$, if $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{q}''$, then either $\boldsymbol{q}'' \models_{\mathcal{O}} \boldsymbol{q}$ or there is $\boldsymbol{q}' \in \mathcal{F}_{\boldsymbol{q}}$ with $\boldsymbol{q}' \models_{\mathcal{O}} \boldsymbol{q}''$.

(Note that if $\boldsymbol{q} \equiv_{\mathcal{O}} \top$, then $\mathcal{F}_{\boldsymbol{q}} = \emptyset$.) An ontology $\mathcal{O}$ is said to *admit* (*finite*) *frontiers within* $\mathcal{Q}$ if every $\boldsymbol{q} \in \mathcal{Q}$ satisfiable wrt $\mathcal{O}$ has a (finite) frontier wrt $\mathcal{O}$ within $\mathcal{Q}$. Further, if such frontiers can be computed in polynomial time, we say that $\mathcal{O}$ *admits polytime-computable frontiers.*

The next theorem follows directly from the definitions:

Theorem 1. *Suppose* $\mathcal{Q}$ *is a class of queries, an ontology* $\mathcal{O}$ *admits containment reduction,* $\boldsymbol{q} \in \mathcal{Q}$ *is satisfiable wrt* $\mathcal{O}$*, and* $\mathcal{F}_{\boldsymbol{q}}$ *is a finite frontier of* $\boldsymbol{q}$ *wrt* $\mathcal{O}$ *within* $\mathcal{Q}$*. Then* $(\{(\hat{\boldsymbol{q}}, a)\}, \{(\hat{\boldsymbol{r}}, a) \mid \boldsymbol{r} \in \mathcal{F}_{\boldsymbol{q}}\})$ *is a unique characterisation of* $\boldsymbol{q}$ *wrt* $\mathcal{O}$ *within* $\mathcal{Q}$*.*

As shown by Funk, Jung, and Lutz (2022b), the two main ontology languages that admit polytime-computable frontiers within ELIQ are *DL-Lite*$_{\mathcal{H}}$ and *DL-Lite*$^-_{\mathcal{F}}$, whereas *DL-Lite*$_{\mathcal{F}}$ itself does not admit finite ELIQ-frontiers. By Theorem 1 and Lemma 1, we then obtain:

Theorem 2. *If an ELIQ* $\boldsymbol{q}$ *is satisfiable wrt a DL-Lite*$_{\mathcal{H}}$ *or DL-Lite*$^-_{\mathcal{F}}$ *ontology* $\mathcal{O}$*, then* $\boldsymbol{q}$ *has a polysize unique characterisation wrt* $\mathcal{O}$ *within ELIQ.*

We next introduce split-partners aka dualities (McKenzie 1972; ten Cate and Dalmau 2022). Let σ be a finite signature, $\mathcal{Q}^\sigma$ a class of σ-queries, $\mathcal{O}$ a σ-ontology, and $\Theta \subseteq \mathcal{Q}^\sigma$ a finite set queries. A set $\mathcal{S}(\Theta)$ of pointed data instances $(\mathcal{A}, a)$ is called a *split-partner for* Θ *wrt* $\mathcal{O}$ *within* $\mathcal{Q}^\sigma$ if, for all $\boldsymbol{q}' \in \mathcal{Q}^\sigma$, we have

$$\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a) \text{ for some } (\mathcal{A}, a) \in \mathcal{S}(\Theta) \quad \text{iff} \quad \boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q} \text{ for all } \boldsymbol{q} \in \Theta. \qquad (3)$$

Say that an ontology language $\mathcal{L}$ *has general split-partners within* $\mathcal{Q}^\sigma$ if all finite sets of $\mathcal{Q}^\sigma$-queries have split partners wrt any $\mathcal{L}$-ontology in σ. If this holds for all singleton subsets of $\mathcal{Q}^\sigma$, we say that $\mathcal{L}$ *has split-partners within* $\mathcal{Q}^\sigma$.

We illustrate the notion of split-partner by a few examples, the last of which shows that, without the restriction to a finite signature σ, split-partners almost never exist.

Example 1. (i) Let $\mathcal{O}$ be any ontology such that $\mathcal{O}$ and $\mathcal{A}$ are satisfiable for all data instances $\mathcal{A}$, say, $\mathcal{O} = \{A \sqsubseteq B\}$. Let $\mathcal{Q}^\sigma$ be any class of σ-CQs, for some signature σ. Then the split-partner $\mathcal{S}_\bot$ of the query $\bot$ wrt $\mathcal{O}$ within $\mathcal{Q}^\sigma$ is

$$\mathcal{S}_\bot = \{\mathcal{B}_\sigma\}, \text{for } \mathcal{B}_\sigma = \{R(a,a) \mid R \in \sigma\} \cup \{A(a) \mid A \in \sigma\}.$$

(Here and below we drop a from $(\mathcal{A}, a)$ if *ind*$(\mathcal{A}) = \{a\}$.) Clearly, $\mathcal{O}, \mathcal{B}_\sigma \models \boldsymbol{q}$, for any $\boldsymbol{q} \in \mathcal{Q}^\sigma$ different from $\bot$.

(ii) For $\mathcal{O} = \{A \sqcap B \sqsubseteq \bot\}$ and $\sigma = \{A, B\}$, we have $\mathcal{S}_\bot = \{\{A(a)\}, \{B(a)\}\}$.

(iii) There does not exist a split-partner for $\Theta = \{A\}$ wrt the empty ontology $\mathcal{O}$ within ELIQ. To show this, observe that $B \not\models_{\mathcal{O}} A$ for any unary predicate $B \neq A$. Hence, as any data instance $\mathcal{A}$ is finite, there is no finite set $\mathcal{S}(\{A\})$ satisfying (3).

In contrast, for frontiers and unique characterisations, restrictions to sets of predicates containing all symbols in the query and ontology do not make any difference. Indeed, let σ be the signature of $\mathcal{O}$ and $\boldsymbol{q}$. Then, for any class $\mathcal{Q}$ of queries, a set $\mathcal{F}_{\boldsymbol{q}}$ is a frontier for $\boldsymbol{q}$ wrt $\mathcal{O}$ within $\mathcal{Q}$ iff it is a frontier for $\boldsymbol{q}$ wrt $\mathcal{O}$ within the restriction of $\mathcal{Q}$ to σ. The same holds for unique characterisations E of $\boldsymbol{q}$ wrt $\mathcal{O}$.

The following result is proved (in the appendix) using a construction from the reduction of ontology-mediated query answering to constraint satisfaction (Bienvenu et al. 2014).

Theorem 3. $\mathcal{ALCHI}$ *has general split-partners within ELIQ*$^\sigma$ *that can be computed in exponential time.*

For ELQs, we can construct general split-partners wrt RDFS ontologies in polynomial time, provided that the number of input queries is bounded. The proof generalises the construction of split-partners for queries in ELQ wrt to the empty ontology in (Fortin et al. 2022; ten Cate et al. 2023).

Theorem 4. *Let* σ *be a signature,* $\mathcal{O}$ *a* σ*-ontology in RDFS, and* $n > 0$*. For any set* $\Theta \subseteq$ *ELQ*$^\sigma$ *with* $|\Theta| \leq n$*, one can compute in polynomial time a split-partner* $\mathcal{S}(\Theta)$ *of* Θ *wrt* $\mathcal{O}$ *within ELQ*$^\sigma$*.*

Here is our second sufficient characterisability condition:

Theorem 5. *Suppose $\mathcal{Q}$ is a class of queries, an ontology $\mathcal{O}$ admits containment reduction, $\boldsymbol{q} \in \mathcal{Q}$ is satisfiable wrt $\mathcal{O}$, and σ contains the predicate symbols in $\boldsymbol{q}$ and $\mathcal{O}$. If $\mathcal{S}_{\boldsymbol{q}}$ is a split-partner for $\{\boldsymbol{q}\}$ wrt $\mathcal{O}$ within $\mathcal{Q}^\sigma$, then $(\{(\hat{\boldsymbol{q}}, a)\}, \mathcal{S}_{\boldsymbol{q}})$ is a unique characterisation of $\boldsymbol{q}$ wrt $\mathcal{O}$ within $\mathcal{Q}$.*

As a consequence of Theorems 3, 4, 5 and Lemma 1, we obtain the following:

Theorem 6. (i) *If $\boldsymbol{q} \in ELIQ^\sigma$ is satisfiable wrt an $\mathcal{ALCHI}$-ontology $\mathcal{O}$ in a signature σ, then $\boldsymbol{q}$ has a unique characterisation wrt $\mathcal{O}$ within $ELIQ^\sigma$.*

(ii) *If $\boldsymbol{q} \in ELQ^\sigma$ and $\mathcal{O}$ is an RDFS ontology in σ, then $\boldsymbol{q}$ has a polysize unique characterisation wrt $\mathcal{O}$ within ELQ^σ.*

The sufficient conditions of Theorems 1 and 5 use the notions of frontier and split-partner, respectively. We now give examples of queries and ontologies having frontiers but no split-partners and vice versa. The query witnessing that frontiers can exist where split-partners do not exist provides a counterexample even if one admits *CQ-frontiers*, frontiers containing not only ELIQs but also CQs and defined in the obvious way in the appendix.

Theorem 7. *$\mathcal{EL}$ does not admit finite CQ-frontiers within ELIQ.*

Proof. The query $\boldsymbol{q} = A \wedge B$ does not have a finite CQ-frontier wrt the ontology $\mathcal{O} = \{A \sqsubseteq \exists R.A,\ \exists R.A \sqsubseteq A\}$ within ELIQs. □

Example 2. Observe that the following set of pointed data instances is a split-partner of $\{\boldsymbol{q}\}$ wrt $\mathcal{O}$ from the proof of Theorem 7 within ELIQ$^{\{A,B,R\}}$; here all arrows are assumed to be labelled by R:

Theorem 8. *There exist a DL-Lite$^-_\mathcal{F}$ ontology $\mathcal{O}$, a query $\boldsymbol{q}$ and a signature σ such that $\{\boldsymbol{q}\}$ does not have a finite split-partner wrt $\mathcal{O}$ within $ELIQ^\sigma$.*

Proof. Let $\mathcal{O} = \{\mathrm{fun}(P), \mathrm{fun}(P^-), B \sqcap \exists P^- \sqsubseteq \bot\}$ and $\boldsymbol{q} = A$. Then $Q = \{\boldsymbol{q}\}$ does not have a finite split-partner wrt $\mathcal{O}$ within ELIQ$^{\{A,B,P\}}$. □

Observe that $\{\top\}$ is a frontier for A wrt $\mathcal{O}$ from the proof of Theorem 8 within ELIQ and that we can combine the two proofs above to also refute the natural conjecture that frontiers and splittings together provide a 'universal tool' for constructing unique characterisations.

5 Temporal Data and Queries

We now extend the definitions of Sections 3 and 4 by adding a temporal dimension to the domain data and queries mediated by an ontology. Our definitions generalise those of (Fortin et al. 2022), where the ontology-free case was first considered.

A *temporal data instance*, denoted $\mathcal{D}$, is a finite sequence $\mathcal{A}_0, \dots, \mathcal{A}_n$ of data instances, where each $\mathcal{A}_i$ comprises the facts with timestamp i. We assume all $ind(\mathcal{A}_i)$ to be the same, adding $\top(a)$ to $\mathcal{A}_i$ if needed, and set $ind(\mathcal{D}) = ind(\mathcal{A}_0)$. The *length* of $\mathcal{D}$ is $\max(\mathcal{D}) = n$ and the *size* of $\mathcal{D}$ is $|\mathcal{D}| = \sum_{i \leq n} |\mathcal{A}_i|$. Within a temporal σ-data instance, we often denote by $\emptyset$ the instance $\{\top(a) \mid a \in ind(\mathcal{D})\}$.

Temporal queries for accessing temporal data instances we propose in this paper are built from domain queries in a given class $\mathcal{Q}$ (say, ELIQs) using $\wedge$ and the (future-time) temporal operators of the standard linear temporal logic *LTL* over the time flow $(\mathbb{N}, <)$: unary $\bigcirc$ (at the next moment), $\Diamond$ (sometime later), $\Diamond_r$ (now or later), and binary U (until); see below for the precise semantics. The class of such temporal queries that only use the operators from a set $\Phi \subseteq \{\bigcirc, \Diamond, \Diamond_r, \mathsf{U}\}$ is denoted by $LTL^\Phi(\mathcal{Q})$. The class $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ comprises *path queries* of the form

$$\boldsymbol{q} = \boldsymbol{r}_0 \wedge \boldsymbol{o}_1(\boldsymbol{r}_1 \wedge \boldsymbol{o}_2(\boldsymbol{r}_2 \wedge \dots \wedge \boldsymbol{o}_n \boldsymbol{r}_n)), \qquad (4)$$

where $\boldsymbol{o}_i \in \{\bigcirc, \Diamond, \Diamond_r\}$ and $\boldsymbol{r}_i \in \mathcal{Q}$; *path queries* in $LTL_p^{\mathsf{U}}(\mathcal{Q})$ take the form

$$\boldsymbol{q} = \boldsymbol{r}_0 \wedge (\boldsymbol{l}_1 \,\mathsf{U}\, (\boldsymbol{r}_1 \wedge (\boldsymbol{l}_2 \,\mathsf{U}\, (\dots (\boldsymbol{l}_n \,\mathsf{U}\, \boldsymbol{r}_n) \dots)))), \qquad (5)$$

where $\boldsymbol{r}_i \in \mathcal{Q}$ and either $\boldsymbol{l}_i \in \mathcal{Q}$ or $\boldsymbol{l}_i = \bot$. We use $\mathcal{C}$ to refer to classes of temporal queries. The *size* $|\boldsymbol{q}|$ of $\boldsymbol{q}$ is the number of symbols in $\boldsymbol{q}$; the *temporal depth* $tdp(\boldsymbol{q})$ of $\boldsymbol{q}$ is the maximum number of nested temporal operators in $\boldsymbol{q}$.

An (atemporal) ontology $\mathcal{O}$ and temporal data instance $\mathcal{D} = \mathcal{A}_0, \dots, \mathcal{A}_n$ are *satisfiable* if $\mathcal{O}$ and $\mathcal{A}_i$ are satisfiable for each $i \leq n$. For satisfiable $\mathcal{O}$ and $\mathcal{D}$, the *entailment relation* $\mathcal{O}, \mathcal{D}, \ell, a \models \boldsymbol{q}$ with $\ell \in \mathbb{N}$ and $a \in ind(\mathcal{D})$ is defined by induction as follows, where $\mathcal{A}_\ell = \emptyset$, for $\ell > n$:

$\mathcal{O}, \mathcal{D}, \ell, a \models \boldsymbol{q}$ iff $\mathcal{O}, \mathcal{A}_\ell \models \boldsymbol{q}(a)$, for any $\boldsymbol{q} \in \mathcal{Q}$,
$\mathcal{O}, \mathcal{D}, \ell, a \models \boldsymbol{q}_1 \wedge \boldsymbol{q}_2$ iff $\mathcal{O}, \mathcal{D}, \ell, a \models \boldsymbol{q}_i$, for $i = 1, 2$,
$\mathcal{O}, \mathcal{D}, \ell, a \models \bigcirc\boldsymbol{q}$ iff $\mathcal{O}, \mathcal{D}, \ell + 1, a \models \boldsymbol{q}$,
$\mathcal{O}, \mathcal{D}, \ell, a \models \Diamond\boldsymbol{q}$ iff $\mathcal{O}, \mathcal{D}, m, a \models \boldsymbol{q}$, for some $m > \ell$,
$\mathcal{O}, \mathcal{D}, \ell, a \models \Diamond_r\boldsymbol{q}$ iff $\mathcal{O}, \mathcal{D}, m, a \models \boldsymbol{q}$, for some $m \geq \ell$,
$\mathcal{O}, \mathcal{D}, \ell, a \models \boldsymbol{q}_1 \,\mathsf{U}\, \boldsymbol{q}_2$ iff $\mathcal{O}, \mathcal{D}, m, a \models \boldsymbol{q}_2$, for some $m > \ell$, and $\mathcal{O}, \mathcal{D}, k, a \models \boldsymbol{q}_1$, for all $k, \ell < k < m$.

If $\mathcal{O}$ and $\mathcal{D}$ are not satisfiable, we set $\mathcal{O}, \mathcal{D}, \ell, a \models \boldsymbol{q}$ to hold for all $\boldsymbol{q}$, ℓ and a. Our semantics follows the well established epistemic approach to evaluating temporal queries; see (Calvanese et al. 2007a; Artale et al. 2022) and references therein. The alternative classical Tarski semantics based on temporal interpretations is equivalent to our semantics for all *Horn ontologies* whose FO-translations belong to the Horn fragment of first-order logic (Chang and Keisler 1998), and so for all DLs we consider here except $\mathcal{ALCHI}$. A detailed discussion of the relationship between the two semantics is given in the appendix.

By an *example set* we now mean a pair $E = (E^+, E^-)$ of finite sets E^+ and E^- of pointed temporal data instances $\mathcal{D}, a$ with $a \in ind(\mathcal{D})$. We say that a query $\boldsymbol{q}$ *fits* E wrt $\mathcal{O}$ if $\mathcal{O}, \mathcal{D}^+, 0, a^+ \models \boldsymbol{q}$ and $\mathcal{O}, \mathcal{D}^-, 0, a^- \not\models \boldsymbol{q}$, for all $(\mathcal{D}^+, a^+) \in E^+$ and $(\mathcal{D}^-, a^-) \in E^-$. As before, E *uniquely characterises* $\boldsymbol{q}$ *wrt* $\mathcal{O}$ *within a class* $\mathcal{C}$ *of temporal queries* if $\boldsymbol{q}$ fits E wrt $\mathcal{O}$ and every $\boldsymbol{q}' \in \mathcal{C}$ fitting E wrt $\mathcal{O}$ is equivalent to $\boldsymbol{q}$ wrt $\mathcal{O}$. If each $\boldsymbol{q} \in \mathcal{C}$ is uniquely characterised by some E wrt $\mathcal{O}$ within $\mathcal{C}' \supseteq \mathcal{C}$, we call $\mathcal{C}$ *uniquely* *characterisable wrt $\mathcal{O}$ within $\mathcal{C}'$*. Let $\mathcal{C}^n$ be the set of queries in $\mathcal{C}$ of temporal depth $\leq n$. We say that $\mathcal{C}$ is *polysize characterisable wrt $\mathcal{O}$ for bounded temporal depth* if there is a polynomial f such that every $\boldsymbol{q} \in \mathcal{C}^n$ is characterised by some E of size $\leq f(n)$ within $\mathcal{C}^n$, $n \in \mathbb{N}$.

Note that $\Diamond \boldsymbol{r} \equiv \bigcirc\Diamond_r \boldsymbol{r}$, so $\Diamond$ does not add any expressive power to $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ and $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q}) = LTL_p^{\bigcirc\Diamond_r}(\mathcal{Q})$; however, $LTL_p^{\bigcirc\Diamond}(\mathcal{Q}) \subsetneq LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$. We also observe that our temporal query languages do not admit containment reduction as, for example, there is no temporal data instance $\hat{\boldsymbol{q}}$ for $\boldsymbol{q} = \bigcirc(A \wedge \Diamond B)$ because it will have to fix the number of steps between 0 and the moment of time where B holds.

We next prove general theorems lifting unique characterisability from domain queries considered above and ontology-free *LTL* queries of (Fortin et al. 2022) to temporal queries mediated by a DL ontology.

6 Unique Characterisations in $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$

The aim of this section is to give a criterion of (polysize) unique characterisability of temporal queries in the class $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ under certain conditions on the ontology and on the class $\mathcal{Q}$ of domain queries. It will be convenient to represent queries $\boldsymbol{q}$ of the form (4) as a sequence

$$\boldsymbol{q} = \boldsymbol{r}_0(t_0), R_1(t_0, t_1), \dots, R_m(t_{m-1}, t_m), \boldsymbol{r}_m(t_m), \quad (6)$$

where $R_i \in \{\mathit{suc}, <, \leq\}$, $\mathit{suc}(t, t')$ stands for $t' = t + 1$, and the t_i are variables over the timeline $(\mathbb{N}, <)$. We set $\mathit{var}(\boldsymbol{q}) = \{t_0, \dots, t_m\}$, ignoring the ('non-answer') variables that occur in the $\boldsymbol{r}_i$ and are different from the t_i.

Example 3. Below are a temporal query $\boldsymbol{q}$ and its representation of the form (6):

$$\boldsymbol{q} = \exists P.B \wedge \bigcirc(\exists P.A \wedge \Diamond A) \;\rightsquigarrow$$
$$\exists P.B(t_0), \mathit{suc}(t_0, t_1), \exists P.A(t_1), (t_1 < t_2), A(t_2) \quad (7)$$

with $\mathit{var}(\boldsymbol{q}) = \{t_0, t_1, t_2\}$.

We divide $\boldsymbol{q}$ of the form (6) into *blocks* $\boldsymbol{q}_i$ such that

$$\boldsymbol{q} = \boldsymbol{q}_0 \mathcal{R}_1 \boldsymbol{q}_1 \dots \mathcal{R}_n \boldsymbol{q}_n, \quad (8)$$

where $\mathcal{R}_i = R_1^i(t_0^i, t_1^i) \dots R_{n_i}^i(t_{n_i-1}^i, t_{n_i}^i)$, $R_j^i \in \{<, \leq\}$ and

$$\boldsymbol{q}_i = \boldsymbol{r}_0^i(s_0^i) \mathit{suc}(s_0^i, s_1^i) \dots \mathit{suc}(s_{k_i-1}^i, s_{k_i}^i) \boldsymbol{r}_{k_i}^i(s_{k_i}^i) \quad (9)$$

with $s_{k_i}^i = t_0^{i+1}$, $t_{n_i}^i = s_0^i$. If $k_i = 0$, the block $\boldsymbol{q}_i$ is called *primitive*.

Example 4. The query $\boldsymbol{q}$ from Example 3 has two blocks

$$\boldsymbol{q}_0 = \exists P.B(t_0), \mathit{suc}(t_0, t_1), \exists P.A(t_1) \quad \text{and} \quad \boldsymbol{q}_1 = A(t_2)$$

connected by $(t_1 < t_2)$. It contains one primitive block, $\boldsymbol{q}_1$.

Suppose we are given an ontology $\mathcal{O}$ and a class $\mathcal{Q}$ of domain queries. Then a primitive block $\boldsymbol{q}_i = \boldsymbol{r}_0^i(s_0^i)$ with $i > 0$ in $\boldsymbol{q}$ of the form (8) is called a *lone conjunct wrt $\mathcal{O}$ within $\mathcal{Q}$* if $\boldsymbol{r}_0^i$ is *meet-reducible wrt $\mathcal{O}$ within $\mathcal{Q}$* in the sense that there are queries $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathcal{Q}$ such that $\boldsymbol{r} \equiv_{\mathcal{O}} \boldsymbol{r}_1 \wedge \boldsymbol{r}_2$ and $\boldsymbol{r} \not\equiv_{\mathcal{O}} \boldsymbol{r}_i$, for $i = 1, 2$. Lone conjuncts and their impact on unique characterisability are illustrated by the next example.

Example 5. The query $\Diamond A$, which is represented by the sequence $\top(t_0), (t_0 < t_1), A(t_1)$, does not have any lone conjuncts wrt the empty ontology within ELIQ, but A is a lone conjunct of $\Diamond A$ wrt $\mathcal{O} = \{A \equiv B \wedge C\}$ within ELIQ.

The query $\boldsymbol{q} = \Diamond A$ is uniquely characterised wrt the empty ontology within $LTL_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) by the example set $E = (E^+, E^-)$, where E^+ contains two temporal data instances $\emptyset, \{A\}$ and $\emptyset, \emptyset, \{A\}$ and E^- consists of one instance $\{A\}$. However, $\boldsymbol{q} = \Diamond A$ cannot be uniquely characterised wrt $\mathcal{O} = \{A \equiv B \wedge C\}$ within $LTL_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) as it cannot be separated from queries of the form

$$\Diamond(B \wedge \Diamond_r(C \wedge \Diamond_r(B \wedge \Diamond_r(C \wedge \Diamond_r(\dots)))))$$

by a finite example set. Observe also that A is a lone conjunct in $\boldsymbol{q}' = \Diamond(A \wedge \Diamond_r D)$ wrt $\mathcal{O}' = \mathcal{O} \cup \{D \sqsubseteq A\}$ but, for the simplification $\boldsymbol{q}'' = \Diamond D$ of $\boldsymbol{q}'$, we have $\boldsymbol{q}'' \equiv_{\mathcal{O}'} \boldsymbol{q}'$ and $\boldsymbol{q}''$ does not have any lone conjuncts wrt $\mathcal{O}'$.

Example 5 shows that the notion of lone conjunct depends on the presentation of the query. To make lone conjuncts semantically meaningful, we introduce a normal form. Given an ontology $\mathcal{O}$ and a query $\boldsymbol{q}$ of the form (8), we say that $\boldsymbol{q}$ is in *normal form wrt $\mathcal{O}$* if the following conditions hold:

(n1) $\boldsymbol{r}_0^i \not\equiv_{\mathcal{O}} \top$ if $i > 0$, and $\boldsymbol{r}_{k_i}^i \not\equiv_{\mathcal{O}} \top$ if either $i > 0$ or $k_i > 0$ (thus, of all the first/last $\boldsymbol{r}$ in a block only $\boldsymbol{r}_0^0$ can be trivial);

(n2) each $\mathcal{R}_i$ is either a single $t_0^i \leq t_1^i$ or a sequence of $<$;

(n3) $\boldsymbol{r}_{k_i}^i \not\models_{\mathcal{O}} \boldsymbol{r}_0^{i+1}$ if $\boldsymbol{q}_{i+1}$ is primitive and R_{i+1} is $\leq$;

(n4) $\boldsymbol{r}_0^{i+1} \not\models_{\mathcal{O}} \boldsymbol{r}_{k_i}^i$ if $i > 0$, $\boldsymbol{q}_i$ is primitive and R_{i+1} is $\leq$;

(n5) $\boldsymbol{r}_{k_i}^i \wedge \boldsymbol{r}_0^{i+1}$ is satisfiable wrt $\mathcal{O}$ whenever R_{i+1} is $\leq$.

Lemma 3. *Let $\mathcal{O}$ be an FO-ontology (possibly with $=$). Then every query $\boldsymbol{q} \in LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ is equivalent wrt $\mathcal{O}$ to a query in normal form of size at most $|\boldsymbol{q}|$ and of temporal depth not exceeding $tdp(\boldsymbol{q})$. This query can be computed in polynomial time if containment between queries in $\mathcal{Q}$ wrt $\mathcal{O}$ is decidable in polynomial time. If $\mathcal{Q}$ = ELIQ, this is the case for DL-Lite$_{\mathcal{F}}$ but not for DL-Lite$_{\mathcal{H}}$ (unless* P = NP*).*

We call a query $\boldsymbol{q} \in LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ *safe wrt $\mathcal{O}$* if it is equivalent wrt $\mathcal{O}$ to an $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$-query in normal form that has no lone conjuncts.

We are now in a position to formulate the main result of this section.

Theorem 9. *Suppose an ontology $\mathcal{O}$ admits containment reduction and $\mathcal{Q}$ is a class of domain queries that is uniquely characterisable wrt $\mathcal{O}$. Then the following hold:*

(i) A query $\boldsymbol{q} \in LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ is uniquely characterisable within $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ wrt $\mathcal{O}$ iff $\boldsymbol{q}$ is safe wrt $\mathcal{O}$.

(ii) If $\mathcal{O}$ admits polysize characterisations within $\mathcal{Q}$, then those queries that are uniquely characterisable within $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ are actually polysize characterisable within $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$.

(iii) The class $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ is polysize characterisable for bounded temporal depth if $\mathcal{O}$ admits polysize unique characterisations within $\mathcal{Q}$.

(iv) *The class* $LTL_p^{\bigcirc\Diamond}(\mathcal{Q})$ *is uniquely characterisable. It is polysize characterisable if* $\mathcal{O}$ *admits polysize unique characterisations within* $\mathcal{Q}$.

A detailed proof of Theorem 9 is given in the appendix. To explain the intuition behind it, we show and discuss the positive and negative examples that provide the unique characterisation required for (i). Suppose $\mathcal{O}$ admits containment reduction and $\mathcal{Q}$ is a class of domain queries with a unique characterisation $(\{\hat{\boldsymbol{r}}\}, \mathcal{N}_{\boldsymbol{r}})$ of $\boldsymbol{r} \in \mathcal{Q}$ wrt $\mathcal{O}$ within $\mathcal{Q}$. Assume that $\boldsymbol{q} \in LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ in normal form wrt $\mathcal{O}$ takes the form (8) with $\boldsymbol{q}_i$ of the form (9). We define an example set $E = (E^+, E^-)$ characterising $\boldsymbol{q}$ under the assumption that $\boldsymbol{q}$ has no lone conjuncts wrt $\mathcal{O}$. Let b be the number of ocurrencies of $\bigcirc$ and $\Diamond$ in $\boldsymbol{q}$ plus 1. For every block $\boldsymbol{q}_i$ of the form (9), let $\hat{\boldsymbol{q}}_i$ be the temporal data instance

$$\hat{\boldsymbol{q}}_i = \hat{\boldsymbol{r}}_0^i \hat{\boldsymbol{r}}_1^i \dots \hat{\boldsymbol{r}}_{k_i}^i.$$

For any two blocks $\boldsymbol{q}_i, \boldsymbol{q}_{i+1}$ such that $\boldsymbol{r}_{k_i}^i \wedge \boldsymbol{r}_0^{i+1}$ is satisfiable wrt $\mathcal{O}$, we take the temporal data instance

$$\hat{\boldsymbol{q}}_i \bowtie \hat{\boldsymbol{q}}_{i+1} = \hat{\boldsymbol{r}}_0^i \dots \hat{\boldsymbol{r}}_{k_i-1}^i \widehat{\boldsymbol{r}_{k_i}^i \wedge \boldsymbol{r}_0^{i+1}} \hat{\boldsymbol{r}}_1^{i+1} \dots \hat{\boldsymbol{r}}_{k_{i+1}}^{i+1}.$$

Now, the set E^+ contains the data instances given by

– $\mathcal{D}_b = \hat{\boldsymbol{q}}_0 \emptyset^b \dots \hat{\boldsymbol{q}}_i \emptyset^b \hat{\boldsymbol{q}}_{i+1} \dots \emptyset^b \hat{\boldsymbol{q}}_n$,

– $\mathcal{D}_i = \hat{\boldsymbol{q}}_0 \emptyset^b \dots (\hat{\boldsymbol{q}}_i \bowtie \hat{\boldsymbol{q}}_{i+1}) \dots \emptyset^b \hat{\boldsymbol{q}}_n$, if $\mathcal{R}_{i+1}$ is $\leq$ and

– $\mathcal{D}_i = \hat{\boldsymbol{q}}_0 \emptyset^b \dots \hat{\boldsymbol{q}}_i \emptyset^{n_{i+1}} \hat{\boldsymbol{q}}_{i+1} \dots \emptyset^b \hat{\boldsymbol{q}}_n$, otherwise.

Here, $\emptyset^b$ is a sequence of b-many $\emptyset$ and similarly for $\emptyset^{n_{i+1}}$. By the definition of $\hat{\boldsymbol{r}}$ using containment reduction, it follows that $\mathcal{O}, \mathcal{D}, 0, a \models \boldsymbol{q}$, for all $\mathcal{D} \in E^+$. Intuitively, the data instances in E^+ force any query that is entailed to be divided into blocks in a similar way as $\boldsymbol{q}$. The set E^- contains all data instances of the form

– $\mathcal{D}_i^- = \hat{\boldsymbol{q}}_0 \emptyset^b \dots \hat{\boldsymbol{q}}_i \emptyset^{n_{i+1}-1} \hat{\boldsymbol{q}}_{i+1} \dots \emptyset^b \hat{\boldsymbol{q}}_n$, if $n_{i+1} > 1$,

– $\mathcal{D}_i^- = \hat{\boldsymbol{q}}_0 \emptyset^b \dots \hat{\boldsymbol{q}}_i \bowtie \hat{\boldsymbol{q}}_{i+1} \dots \emptyset^b \hat{\boldsymbol{q}}_n$, if $\mathcal{R}_{i+1}$ is a single $<$ and $\boldsymbol{r}_{k_i}^i \wedge \boldsymbol{r}_0^{i+1}$ is satisfiable wrt $\mathcal{O}$,

– the data instances obtained from $\mathcal{D}_b$ by applying to it exactly once each of the rules (a)–(e) defined below in all possible ways.

It follows from the assumption that $\boldsymbol{q}$ is in normal form and the reduced 'gaps' between blocks in $\mathcal{D}_i^-$ that we have $\mathcal{O}, \mathcal{D}_i^-, 0, a \not\models \boldsymbol{q}$ for all $\mathcal{D}_i^-$. To obtain a unique charcaterisation, the additional data instances obtained by applying rules (a)–(e) to $\mathcal{D}_b$ are crucial. They 'weaken' $\mathcal{D}_b$ by replacing some $\hat{\boldsymbol{r}}$ by negative examples in $\mathcal{N}_{\boldsymbol{r}}$ or by introducing big 'gaps' between some $\hat{\boldsymbol{r}}$s. To make our notation more uniform, we think of the pointed data instances in $\mathcal{N}_{\boldsymbol{r}}$ as having the form $\hat{\boldsymbol{r}}'$, for a suitable CQ $\boldsymbol{r}'$ (which is not necessarily in $\mathcal{Q}$). The rules are as follows:

(a) replace some $\hat{\boldsymbol{r}}_j^i$ with $\boldsymbol{r}_j^i \not\equiv_{\mathcal{O}} \top$ by an $\hat{\boldsymbol{r}} \in \mathcal{N}_{\boldsymbol{r}_j^i}$, for i, j such that $(i, j) \neq (0, 0)$—that is, the rule is not applied to $\boldsymbol{r}_0^0$;

(b) replace some pair $\hat{\boldsymbol{r}}_j^i \hat{\boldsymbol{r}}_{j+1}^i$ within block i by $\hat{\boldsymbol{r}}_j^i \emptyset^b \hat{\boldsymbol{r}}_{j+1}^i$;

(c) replace some $\hat{\boldsymbol{r}}_j^i$ such that $\boldsymbol{r}_j^i \not\equiv_{\mathcal{O}} \top$ by $\hat{\boldsymbol{r}}_j^i \emptyset^b \hat{\boldsymbol{r}}_j^i$, where $k_i > j > 0$—that is, the rule is not applied to $\boldsymbol{r}_j^i$ if it is on the border of its block;

(d) replace $\hat{\boldsymbol{r}}_{k_i}^i$ $(k_i > 0)$ by $\hat{\boldsymbol{r}} \emptyset^b \hat{\boldsymbol{r}}_{k_i}^i$, for some $\hat{\boldsymbol{r}} \in \mathcal{N}_{\boldsymbol{r}_{k_i}^i}$, or replace $\hat{\boldsymbol{r}}_0^i$ $(k_i > 0)$ by $\hat{\boldsymbol{r}}_0^i \emptyset^b \hat{\boldsymbol{r}}$, for some $\hat{\boldsymbol{r}} \in \mathcal{N}_{\boldsymbol{r}_0^i}$;

(e) replace $\hat{\boldsymbol{r}}_0^0$ with $\boldsymbol{r}_0^0 \not\equiv_{\mathcal{O}} \top$ by $\hat{\boldsymbol{r}} \emptyset^b \hat{\boldsymbol{r}}_0^0$, for $\hat{\boldsymbol{r}} \in \mathcal{N}_{\boldsymbol{r}_0^0}$, if $k_0 = 0$, and by $\hat{\boldsymbol{r}}_0^0 \emptyset^b \hat{\boldsymbol{r}}_0^0$ if $k_0 > 0$.

The proof that (E^+, E^-) as defined above uniquely characterises $\boldsymbol{q}$ wrt $\mathcal{O}$ if $\boldsymbol{q}$ contains no lone conjuncts is nontrivial and extends ideas from the ontology-free case investigated in (Fortin et al. 2022). Claim (ii) follows from the observation that the unique characterisation constructed in (i) is polynomial in the size of the characterisations $(\{\hat{\boldsymbol{r}}\}, \mathcal{N}_{\boldsymbol{r}})$ of the domain queries used in $\boldsymbol{q}$. For (iii), assume that $tdp(\boldsymbol{q}) \leq n$. Then we add to rules (a)–(e) the following rule: if $\hat{\boldsymbol{r}}$ is a lone conjunct in $\boldsymbol{q}$, then replace $\hat{\boldsymbol{r}}$ by $(\hat{\boldsymbol{r}}_1 \emptyset^b \cdots \emptyset^b \hat{\boldsymbol{r}}_k)^n$ in $\mathcal{D}_b$ for $\mathcal{N}_{\boldsymbol{r}} = \{\hat{\boldsymbol{r}}_1, \dots, \hat{\boldsymbol{r}}_k\}$ with $\boldsymbol{r}_i \not\equiv_{\mathcal{O}} \boldsymbol{r}_j$, for $i \neq j$. As $\boldsymbol{r}$ is meet-reducible wrt $\mathcal{O}$, one can first show that $|\mathcal{N}_{\boldsymbol{r}}| \geq 2$ and then that we obtain a unique characterisation of $\boldsymbol{q}$ wrt $\mathcal{O}$ within the class of queries in $\mathcal{Q}$ of temporal depth $\leq n$. To show (iv), one can follow the proof of (i) without $\Diamond_r$ in $\boldsymbol{q}$ but possibly with lone conjuncts. Now, rules (c), (d), and (e) are not needed in the construction of E^-.

As an immediate consequence of Lemma 1 and Theorems 2, 6 and 9 we obtain:

Theorem 10. (i) *For any DL-Lite*$_{\mathcal{H}}$ *or DL-Lite*$_{\mathcal{F}}^-$ *ontology* $\mathcal{O}$*, the following hold*:

(i_1) *any query* $\boldsymbol{q} \in LTL_p^{\bigcirc\Diamond\Diamond_r}(ELIQ)$ *is uniquely characterisable—in fact, polysize characterisable—wrt* $\mathcal{O}$ *within* $LTL_p^{\bigcirc\Diamond\Diamond_r}(ELIQ)$ *iff* $\boldsymbol{q}$ *is safe wrt* $\mathcal{O}$;

(i_2) $LTL_p^{\bigcirc\Diamond\Diamond_r}(ELIQ)$ *is polysize characterisable wrt* $\mathcal{O}$ *for bounded temporal depth*;

(i_3) $LTL_p^{\bigcirc\Diamond}(ELIQ)$ *is polysize characterisable wrt* $\mathcal{O}$.

(ii) *Let* σ *be a signature. Then claims* (i_1)–(i_3) *also hold for* $\mathcal{ALCHI}$ *ontologies provided that 'polysize' is replaced by 'exponential-size' and ELIQ by ELIQ*$^\sigma$.

7 Unique Characterisations in $LTL_p^{\mathsf{U}}(\mathcal{Q}^\sigma)$

We next consider temporalisations by means of the binary operator U (until), which is more expressive than $\bigcirc$ and $\Diamond$ as $\bigcirc\boldsymbol{q} \equiv \bot \,\mathsf{U}\, \boldsymbol{q}$ and $\Diamond\boldsymbol{q} \equiv \top \,\mathsf{U}\, \boldsymbol{q}$ under the strict semantics. Compared to the previous section, we now have to restrict queries to a finite signature because otherwise the implicit universal quantification in U makes queries such as $\bot \,\mathsf{U}\, A$ not uniquely characterisable wrt the empty ontology (Fortin et al. 2022). For the same reason, we also have to disallow nesting of U on the left-hand side of U in queries. Finally, in the ontology-free case, polysize unique characterisations for propositional *LTL*-queries with U are only available for the so-called peerless queries (Fortin et al. 2022). These observations lead to the following classes of temporal queries, for which we are going to obtain our transfer results.

Let $\mathcal{Q}$ be a domain query language and σ a finite signature of unary and binary predicate symbols. Then $\mathcal{Q}^\sigma$ denotes the set of queries in $\mathcal{Q}$ that only use symbols in σ. The class $LTL_p^{\mathsf{U}}(\mathcal{Q}^\sigma)$ comprises temporal path queries of the form (5) where each $\boldsymbol{r}_i \in \mathcal{Q}^\sigma$ and each $\boldsymbol{l}_i$ is either in $\mathcal{Q}^\sigma$ or $\bot$ (recall that $\boldsymbol{q}$, $\boldsymbol{r}_i$, $\boldsymbol{l}_i$ have a single answer domain variable x and that we evaluate $\boldsymbol{q}$ at time point 0). Given an ontology $\mathcal{O}$, we consider the class $\mathit{LTL}^{\mathsf{U}}_{pp}(\mathcal{Q}^\sigma)$ of *$\mathcal{O}$-peerless queries* in $\mathit{LTL}^{\mathsf{U}}_{p}(\mathcal{Q}^\sigma)$ of the form (5), in which $\boldsymbol{r}_i \not\models_{\mathcal{O}} \boldsymbol{l}_i$ and $\boldsymbol{l}_i \not\models_{\mathcal{O}} \boldsymbol{r}_i$, for all $i \leq n$. In what follows we write $\mathcal{O}, \mathcal{D} \models \boldsymbol{q}$ instead of $\mathcal{O}, \mathcal{D}, 0, a \models \boldsymbol{q}$ when a is clear from context. We also write $\mathcal{D} \models \boldsymbol{q}$ instead of $\emptyset, \mathcal{D} \models \boldsymbol{q}$ (that is, for the empty ontology).

A fundamental difference to the previous section and Theorem 9 is that now containment reduction and unique characterisability of domain queries are not sufficient to guarantee transfer to the temporal case. Recall that *DL-Lite*$^-_{\mathcal{F}}$ admits polytime computable frontiers but no split-partners.

Theorem 11. *There exist a DL-Lite*$^-_{\mathcal{F}}$ *ontology $\mathcal{O}$, a signature σ and a query $\boldsymbol{q} \in \mathit{LTL}^{\mathsf{U}}_{pp}(\mathit{ELIQ}^\sigma)$ such that $\boldsymbol{q}$ is not uniquely characterisable wrt $\mathcal{O}$ within $\mathit{LTL}^{\mathsf{U}}_{p}(\mathit{ELIQ}^\sigma)$.*

In fact, one can take $\mathcal{O}$ and σ from the proof of Theorem 8 and set $\boldsymbol{q} = \bot \mathsf{U} A \equiv \bigcirc A$. Observe that to separate $\bigcirc A$ from $\boldsymbol{q}' \mathsf{U} A$ with a σ-ELIQ $\boldsymbol{q}'$ such that $\boldsymbol{q}' \not\models_{\mathcal{O}} A$, one has to add to E^- a temporal σ-data instance $\mathcal{D} = \{\top(a)\}, \mathcal{A}, \{A(a)\}$ such that $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a)$ but $\mathcal{O}, \mathcal{A} \not\models A(a)$. Such $\mathcal{A}$ could be provided by a finite split-partner for $\{A\}$ wrt $\mathcal{O}$ within ELIQ$^\sigma$ had it existed, but not from a frontier.

We establish the following general transfer theorem, assuming containment reduction and split-partners:

Theorem 12. *Suppose $\mathcal{Q}$ is a class of domain queries, σ a signature, an ontology language $\mathcal{L}$ has general split-partners within $\mathcal{Q}^\sigma$, and $\mathcal{O}$ is a σ-ontology in $\mathcal{L}$ admitting containment reduction. Then the following hold*:

(i) *Every query $\boldsymbol{q} \in \mathit{LTL}^{\mathsf{U}}_{pp}(\mathcal{Q}^\sigma)$ is uniquely characterisable wrt $\mathcal{O}$ within $\mathit{LTL}^{\mathsf{U}}_{p}(\mathcal{Q}^\sigma)$.*

(ii) *If a split-partner for any set Θ, $|\Theta| \leq 2$, of $\mathcal{Q}^\sigma$ queries wrt $\mathcal{O}$ within $\mathcal{Q}^\sigma$ is exponential, then there is an exponential-size unique characterisation of $\boldsymbol{q}$ wrt $\mathcal{O}$.*

(iii) *If a split-partner of any set Θ as above is polynomial and a split-partner $\mathcal{S}_\bot$ of $\bot(x)$ within $\mathcal{Q}^\sigma$ wrt $\mathcal{O}$ is a singleton, then there is a polynomial-size unique characterisation of $\boldsymbol{q}$ wrt $\mathcal{O}$.*

The detailed proof of Theorem 12 given in the appendix is by reduction to the ontology-free *LTL* case, using a characterisation of (Fortin et al. 2022). Here, we define the example set that provides the characterisation for (i). Suppose a signature σ, a σ-ontology $\mathcal{O}$, and a query $\boldsymbol{q} \in \mathit{LTL}^{\mathsf{U}}_{pp}(\mathcal{Q}^\sigma)$ of the form (5) are given. We may assume that $\boldsymbol{r}_n \not\equiv_{\mathcal{O}} \top$. We obtain the set E^+ of positive examples by taking

$(\mathfrak{p}'_0)$ $\hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_n$;

$(\mathfrak{p}'_1)$ $\hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1} \hat{\boldsymbol{l}}_i \hat{r}_i \dots \hat{\boldsymbol{r}}_n$;

$(\mathfrak{p}'_2)$ $\hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1} \hat{\boldsymbol{l}}^k_i \hat{\boldsymbol{r}}_i \dots \hat{\boldsymbol{r}}_{j-1} \hat{\boldsymbol{l}}_j \hat{\boldsymbol{r}}_j \dots \hat{\boldsymbol{r}}_n,\ i < j,\ k = 1, 2.$

Here, $\hat{\boldsymbol{l}}^k_i$ is a sequence of k-many $\hat{\boldsymbol{l}}_i$. The negative examples E^- comprise the following instances $\mathcal{D}$ whenever $\mathcal{D} \not\models \boldsymbol{q}$:

$(\mathfrak{n}'_0)$ $\mathcal{A}_1, \dots, \mathcal{A}_n$ and $\mathcal{A}_1, \dots, \mathcal{A}_{n-i}, \mathcal{A}, \mathcal{A}_{n-i+1}, \dots, \mathcal{A}_n$, for $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{r}_i\})$ and $(\mathcal{A}_1, a), \dots, (\mathcal{A}_n, a) \in \mathcal{S}_\bot$;

$(\mathfrak{n}'_1)$ $\mathcal{D} = \hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1} \mathcal{A} \hat{\boldsymbol{r}}_i \dots \hat{\boldsymbol{r}}_n$, where $(\mathcal{A}, a)$ is an element of $\mathcal{S}(\{\boldsymbol{l}_i, \boldsymbol{r}_i\}) \cup \mathcal{S}(\{\boldsymbol{l}_i\}) \cup \mathcal{S}_\bot$;

$(\mathfrak{n}'_2)$ for all i and $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{l}_i, \boldsymbol{r}_i\}) \cup \mathcal{S}(\{\boldsymbol{l}_i\}) \cup \mathcal{S}_\bot$, *some* data instance

$$\mathcal{D}^i_{\mathcal{A}} = \hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1} \mathcal{A} \hat{\boldsymbol{r}}_i \hat{\boldsymbol{l}}^{k_{i+1}}_{i+1} \hat{\boldsymbol{r}}_{i+1} \dots \hat{\boldsymbol{l}}^{k_n}_n \boldsymbol{r}_n,$$

if any, such that $\max(\mathcal{D}^i_{\mathcal{A}}) \leq (n+1)^2$ and $\mathcal{D}^i_{\mathcal{A}} \not\models \boldsymbol{q}^\dagger$ for $\boldsymbol{q}^\dagger$ obtained from $\boldsymbol{q}$ by replacing all $\boldsymbol{l}_j$, for $j \leq i$, with $\bot$.

We have (ii) since (E^+, E^-) is at most exponential in the size of split-partners of sets with at most two queries. For (iii), observe that $(\mathfrak{n}'_1)$ is exponential in $|\mathcal{S}_\bot|$ iff $|\mathcal{S}_\bot| \geq 2$.

As a consequence of Lemma 1, Theorem 12 (ii) and (iii), and Theorems 3 and 4 we obtain the following (note that, for every RDFS ontology, the split partner $\mathcal{S}_\bot$ of $\bot$ is a singleton by Example 1 (i)):

Theorem 13. (i) *Each $\boldsymbol{q} \in \mathit{LTL}^{\mathsf{U}}_{pp}(\mathit{ELIQ}^\sigma)$ is exponential-size uniquely characterisable wrt any $\mathcal{ALCHI}$ ontology in σ within $\mathit{LTL}^{\mathsf{U}}_{p}(\mathit{ELIQ}^\sigma)$.*

(ii) *Each $\boldsymbol{q} \in \mathit{LTL}^{\mathsf{U}}_{pp}(\mathit{ELQ}^\sigma)$ is polysize uniquely characterisable wrt any RDFS ontology in σ within $\mathit{LTL}^{\mathsf{U}}_{p}(\mathit{ELQ}^\sigma)$.*

8 Exact Learnability

We apply the results on unique characterisability obtained in Section 6 to exact learnability of queries wrt ontologies. Given a query class $\mathcal{C}$ and an ontology $\mathcal{O}$, the *learner* aims to identify a *target query* $\boldsymbol{q}_T \in \mathcal{C}$ by means of membership queries of the form 'does $\mathcal{O}, \mathcal{D}, 0, a \models \boldsymbol{q}_T$ hold?' to the *teacher*. We call $\mathcal{C}$ *polynomial time learnable wrt $\mathcal{L}$ ontologies using membership queries* if there is a learning algorithm that given $\mathcal{O}$ constructs $\boldsymbol{q}_T$ (up to equivalence wrt $\mathcal{O}$) in time polynomial in the sizes of $\boldsymbol{q}_T$ and $\mathcal{O}$. For the weaker requirement of *polynomial query learnability*, it suffices that the total size of the examples given to the oracle be bounded by a polynomial. We start with making the following observation, where exponential query learnability is defined in the expected way.

Theorem 14. *Let $\mathcal{L}$ be an ontology language and $\mathcal{C}$ be a class of queries which admits effective exponential size unique characterizations wrt $\mathcal{L}$ ontologies. Then, $\mathcal{C}$ is exponential query learnable wrt $\mathcal{L}$ ontologies.*

Proof. Let $\boldsymbol{q}_T \in \mathcal{C}$ be the target query and $\mathcal{O}$ be an $\mathcal{L}$ ontology. We enumerate all queries in $\mathcal{C}$ in increasing size (this is possible assuming that $\mathcal{C}$ has an effective syntax). For every enumerated $\boldsymbol{q}$, we compute its unique characterisation (E^+, E^-) wrt $\mathcal{O}$ and use membership queries to check whether all examples in E^+ are positive examples and all examples in E^- are negative examples. If so, output $\boldsymbol{q}$. $\square$

Our main focus, however, is polynomial time and query learnability. As the presence of $\sqcap$ and $\bot$ in the ontology language precludes polynomial query learnability already in the atemporal case, c.f. Theorem 6 in (Funk, Jung, and Lutz 2022b), we follow their approach and assume that the learner also receives an initial positive example $\mathcal{D}, a$ with $\mathcal{D}$ and $\mathcal{O}$ satisfiable. In order to state our main result, we introduce one further natural condition. An ontology language $\mathcal{L}$ *admits polynomial time instance checking* if given an $\mathcal{L}$ ontology $\mathcal{O}$, a pointed instance $(\mathcal{A}, a)$, and a concept name A, it is decidable in polynomial time whether $\mathcal{O}, \mathcal{A} \models A(a)$.

Theorem 15. *Let $\mathcal{L}$ be an ontology language that contains only $\mathcal{ELHI}$ or only $\mathcal{ELIF}$ ontologies and that admits poly-size frontiers within ELIQ that can be computed. Then:*

(i) *The safe LTL$_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) queries are polynomial query learnable wrt $\mathcal{L}$ ontologies using membership queries.*

(ii) *The class LTL$_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) is polynomial query learnable wrt $\mathcal{L}$ ontologies using membership queries if the learner knows the temporal depth of the target query.*

(iii) *The class LTL$_p^{\bigcirc\Diamond}$(ELIQ) is polynomial query learnable wrt $\mathcal{L}$ ontologies using membership queries.*

If $\mathcal{L}$ further admits polynomial time instance checking and polynomial time computable frontiers within ELIQ, then in (ii) and (iii), polynomial query learnability can be replaced by polynomial time learnability. If, in addition, meet-reducibility wrt $\mathcal{L}$ ontologies can be decided in polynomial time, then also in (i) polynomial query learnability can be replaced by polynomial time learnability.

To achieve the generality of the results independently of the exact languages, in the proof of Theorem 15 we rely on the results and techniques from Section 6 and general results proved in the context of exact learning of (atemporal) ELIQs wrt ontologies (Funk, Jung, and Lutz 2022a).

Let $\boldsymbol{q}_T$ be a target query, $\mathcal{O}$ be an ontology, and $\mathcal{D}, a$ be a positive example with $\mathcal{D} = \mathcal{A}_0 \dots \mathcal{A}_n$ and $\mathcal{D}$ and $\mathcal{O}$ satisfiable. The idea is to modify $\mathcal{D}$ in a number of steps such that, in the end, $\mathcal{D}$ viewed as temporal query is equivalent to $\boldsymbol{q}_T$.

We describe how to show (i); (ii) and (iii) are slight modifications thereof. In **Step 1**, the goal is to find a temporal data instance $\mathcal{D}$ where each $\mathcal{A}_i$ is *tree-shaped* and hence can be viewed as an ELIQ. This can be done separately for each time point using membership queries and standard unraveling techniques from the atemporal setting (Funk, Jung, and Lutz 2022a). In **Step 2**, we exhaustively apply Rules (a)-(e) from the proof of Theorem 9 to $\mathcal{D}$, as long as $\mathcal{D}, a$ remains a positive example. In **Step 3**, we take care of lone conjuncts in $\mathcal{D}$ (when viewed as a temporal query) – recall that $\boldsymbol{q}_T$ is safe and thus does not have any. For this step, we rely on a characterisation of meet-reducibility in terms of *minimal* frontiers. For computing those, we exploit the fact that containment of ELIQs wrt $\mathcal{ELHI}$ and $\mathcal{ELIF}$ ontologies is decidable (Bienvenu et al. 2016). After Step 3, $\mathcal{D}$ (viewed as query) is already very similar to $\boldsymbol{q}_T$. More precisely, when representing $\boldsymbol{q}_T$ in shape (8) as a sequence of blocks $\boldsymbol{q}_0\mathcal{R}_1\boldsymbol{q}_1 \dots \mathcal{R}_m\boldsymbol{q}_m$, then $\mathcal{D}$ has the shape $\mathcal{D}_0\emptyset^b \dots \emptyset^b\mathcal{D}_m$, for sufficiently large b, and each $\boldsymbol{q}_i$ is isomorphic to $\mathcal{D}_i$. So in **Step 4**, it remains to identify the precise separators $\mathcal{R}_i$. They can be a single $\leq$ or a sequence of $<$, and the two cases can be distinguished using suitable membership queries.

In order to show that this entire process terminates after asking polynomially many membership queries, we lift the notion of *generalisation sequences* from (Funk, Jung, and Lutz 2022a) to the temporal setting. For the sake of convenience, we treat the data instances in the time points as CQs. A sequence $\mathcal{D}_1, \dots$ of temporal data instances is a *generalisation sequence towards $\boldsymbol{q}_T$ wrt $\mathcal{O}$* if for all $i \geq 1$:

- $\mathcal{D}_{i+1}$ is obtained from $\mathcal{D}_i$ by modifying one non-temporal CQ $\boldsymbol{r}_j$ in $\mathcal{D}_i$ to $\boldsymbol{r}'_j$ such that $\boldsymbol{r}_j \models_\mathcal{O} \boldsymbol{r}'_j$ and $\boldsymbol{r}'_j \not\models_\mathcal{O} \boldsymbol{r}_j$;
- $\mathcal{O}, \mathcal{D}_i, 0, a \models \boldsymbol{q}_T$ for all $i \geq 1$.

Intuitively, data instances in generalisation sequences become weaker and weaker, and based on this, we show that the length of generalisation sequences towards $\boldsymbol{q}_T$ wrt $\mathcal{O}$ is bounded by a polynomial in $\max(\mathcal{D}_1)$ and the sizes of $\boldsymbol{q}_T, \mathcal{O}$. The crucial observation is that the sequences of temporal data instances obtained by rule application are mostly generalisation sequences towards $\boldsymbol{q}_T$ wrt $\mathcal{O}$; thus the steps terminate in polynomial time. If they are not, we use a different (but usually easier) termination argument.

It remains to note that the sketched algorithm runs in polynomial time when $\mathcal{L}$ satisfies all the required criteria. □

We finally apply Theorem 15 to concrete ontology languages, namely *DL-Lite*$_\mathcal{F}^-$ and *DL-Lite*$_\mathcal{H}$.

Theorem 16. *The following learnability results hold:*

(i) *The class of safe queries in LTL$_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) is polynomial query learnable wrt DL-Lite$_\mathcal{H}$ ontologies using membership queries and polynomial time learnable wrt DL-Lite$_\mathcal{F}^-$ ontologies using membership queries.*

(ii) *The class LTL$_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) is polynomial time learnable wrt both DL-Lite$_\mathcal{F}^-$ and DL-Lite$_\mathcal{H}$ ontologies using membership queries if the learner knows the temporal depth of the target query in advance.*

(iii) *The class LTL$_p^{\bigcirc\Diamond}$(ELIQ) is polynomial time learnable wrt both DL-Lite$_\mathcal{F}^-$ and DL-Lite$_\mathcal{H}$ ontologies using membership queries.*

Theorem 16 is a direct consequence of Theorem 15 and the fact that the considered ontology languages satisfy all conditions mentioned there. In particular, we show in the appendix that meet-reducibility of ELIQs wrt *DL-Lite*$_\mathcal{F}^-$ ontologies Turing reduces to ELIQ containment wrt *DL-Lite*$_\mathcal{F}^-$ ontologies which is tractable (Bienvenu et al. 2013). The latter is not true for *DL-Lite*$_\mathcal{H}$ which explains the difference in (i). We leave it for future work whether *LTL*$_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) is polynomial *time* learnable wrt *DL-Lite*$_\mathcal{H}$ ontologies.

9 Outlook

Many interesting and challenging problems remain to be addressed. We discuss a few of them below.

(1) Is it possible to overcome our 'negative' unique characterisability results by admitting some form of infinite (but finitely presentable) examples? Some results in this direction without ontologies are obtained in (Sestic 2023).

(2) We have not considered learnability using membership queries of temporal queries with U. In fact, it remains completely open how far our characterisability results for these queries can be exploited to obtain polynomial query (or time) learnability.

(3) We only considered path queries with no temporal operator occurring in the scope of a DL operator. This is motivated by the negative results of (Fortin et al. 2022), which showed that (i) applying $\exists P$ to $\bigcirc\Diamond$-queries quickly leads to non-characterisability and that (ii) even without DL-operators and without ontology, branching $\Diamond$-queries are often not uniquely characterisable. We still believe there is some scope for useful positive characterisability results.

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A Proofs and detailed definitions for Section 4

Lemma 1. (1) *FO without equality admits tractable containment reduction*; *in particular,* $\mathcal{ALCHI}$ *admits tractable containment reduction.*

(2) $\mathcal{ELIF}$ *admits tractable containment reduction.*

(3) $\{\geq 3\,P \sqsubseteq \bot\}$ *does not admit containment reduction.*

Proof. (1) By (Bienvenu et al. 2014, Proposition 5.9), for any FO-ontology $\mathcal{O}$ without =, any CQ $\boldsymbol{q}$, and any pointed instances $\mathcal{A}_1, a_1$ and $\mathcal{A}_2, a_2$, if there is $h\colon \mathcal{A}_1 \to \mathcal{A}_2$ with $h(a_1) = a_2$, then $\mathcal{O}, \mathcal{A}_1 \models \boldsymbol{q}(a_1)$ implies $\mathcal{O}, \mathcal{A}_2 \models \boldsymbol{q}(a_2)$. Let $(\hat{\boldsymbol{q}}(x), a)$ be *induced* by $\boldsymbol{q}(x)$, i.e., obtained by replacing the variables in $\boldsymbol{q}$ by distinct constants, with x replaced by a. Suppose $\mathcal{O}, \hat{\boldsymbol{q}} \models \boldsymbol{q}'$ and $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}(a)$ but $\mathcal{O}, \mathcal{A} \not\models \boldsymbol{q}'(a)$. Take a model $\mathcal{I}$ witnessing $\mathcal{O}, \mathcal{A} \not\models \boldsymbol{q}_2(a)$. Then $\mathcal{I} \models \boldsymbol{q}_1(a)$ and this is witnessed by a homomorphism $h : \boldsymbol{q}_1 \to \mathcal{I}$. Take the image $h(\boldsymbol{q}_1)$. Then $\mathcal{O}, \widehat{h(\boldsymbol{q}_1)} \not\models \boldsymbol{q}_2(x)$ is witnessed by $\mathcal{I}$, and so $\mathcal{O}, \hat{\boldsymbol{q}}_1 \not\models \boldsymbol{q}_2(x)$, which is a contradiction.

(2) Let $\mathcal{O}$ be a $\mathcal{ELIF}$ ontology. Given a CQ $\boldsymbol{q}(x)$, define an equivalence relation $\sim$ on $\mathit{var}(\boldsymbol{q})$ as the transitive closure of the following relation: $y \sim' z$ iff there is $u \in \mathit{var}(\boldsymbol{q})$ such that $S(u, y), S(u, z) \in \boldsymbol{q}$, for a functional S in $\mathcal{O}$. Let $\boldsymbol{q}/_\sim$ be obtained by identifying (glueing together) all of the variables in each equivalence class $y/_\sim$. Clearly, $\boldsymbol{q}/_\sim(x/_\sim)$ is a homomorphic image of $\boldsymbol{q}(x)$ and $\boldsymbol{q}(x) \equiv_{\mathcal{O}} \boldsymbol{q}/_\sim(x/_\sim)$. We define $(\hat{\boldsymbol{q}}, a)$ as the pointed data instance induced by $\boldsymbol{q}/_\sim(x/_\sim)$. Conditions ($\mathrm{cr}_1$) and ($\mathrm{cr}_2$) are obvious, and ($\mathrm{cr}_3$) follows from the fact that $\boldsymbol{q}/_\sim \models_{\mathcal{O}} \boldsymbol{q}'/_\sim$ iff $\boldsymbol{q}/_\sim \models_{\mathcal{O}'} \boldsymbol{q}'/_\sim$, where $\mathcal{O}'$ is obtained from $\mathcal{O}$ by omitting all of its functionality constraints, which is in the scope of part (1).

(3) Suppose otherwise. Let $\mathcal{O} = \{\geq 3\,P \sqsubseteq \bot\}$ and let $\boldsymbol{q}(x) = \{P(x, y_i), A_i(y_i) \mid i = 1, 2, 3\}$ with a suitable $(\hat{\boldsymbol{q}}, a)$. As $\mathcal{O}$ and the instance induced by $\boldsymbol{q}$ are not satisfiable and in view of (cr_2), $\hat{\boldsymbol{q}}$ contains at most three individuals, say, $\hat{\boldsymbol{q}} = \{P(a, b), A_1(b), A_2(b), P(a, c), A_3(c)\}$. But then, by ($\mathrm{cr}_3$), $\boldsymbol{q}'(x) = \{P(x, y), A_1(y), A_2(y), P(x, z), A_3(z)\}$ should satisfy $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{q}'$, which is not the case as witnessed by $\mathcal{A} = \{P(a, b), A_1(b), P(a, c), A_2(c), A_3(c)\}$ because $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}(a)$ but $\mathcal{O}, \mathcal{A} \not\models \boldsymbol{q}'(a)$. □

Lemma 2. *Suppose* $\mathcal{O}$ *admits containment reduction and* $\boldsymbol{q} \in \mathcal{Q}$ *is satisfiable wrt* $\mathcal{O}$*, having a unique characterisation* $E = (E^+, E^-)$ *wrt* $\mathcal{O}$ *within* $\mathcal{Q}$*. Then* $E' = (\{(\hat{\boldsymbol{q}}, a)\}, E^-)$ *is a unique characterisation of* $\boldsymbol{q}$ *wrt* $\mathcal{O}$ *within* $\mathcal{Q}$*, too.*

Proof. To show that E' is as required, we first observe that $\boldsymbol{q}$ fits E' by (**cr**$_1$) and (**cr**$_3$). Suppose $\boldsymbol{q}' \not\equiv_{\mathcal{O}} \boldsymbol{q}$ for some $\boldsymbol{q}' \in \mathcal{Q}$. We show that then either $\mathcal{O}, \hat{\boldsymbol{q}} \not\models \boldsymbol{q}'$ or $\mathcal{O}, \mathcal{D} \models \boldsymbol{q}'$ for some $\mathcal{D} \in E^-$. Let $\boldsymbol{q} \not\models_{\mathcal{O}} \boldsymbol{q}'$. Then $\mathcal{O}, \hat{\boldsymbol{q}} \not\models \boldsymbol{q}'$ by (**cr**$_3$). Let $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{q}'$ and $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}$. Then $\mathcal{O}, \mathcal{D} \models \boldsymbol{q}'$ for all $\mathcal{D} \in E^+$, and so $\mathcal{O}, \mathcal{D} \models \boldsymbol{q}'$, for some $\mathcal{D} \in E^-$, because E is a unique characterisation of $\boldsymbol{q}$ wrt $\mathcal{O}$. □

Theorem 1. *Suppose* $\mathcal{Q}$ *is a class of queries, an ontology* $\mathcal{O}$ *admits containment reduction,* $\boldsymbol{q} \in \mathcal{Q}$ *is satisfiable wrt* $\mathcal{O}$*, and* $\mathcal{F}_{\boldsymbol{q}}$ *is a finite frontier of* $\boldsymbol{q}$ *wrt* $\mathcal{O}$ *within* $\mathcal{Q}$*. Then* $(\{(\hat{\boldsymbol{q}}, a)\}, \{(\hat{\boldsymbol{r}}, a) \mid \boldsymbol{r} \in \mathcal{F}_{\boldsymbol{q}}\})$ *is a unique characterisation of* $\boldsymbol{q}$ *wrt* $\mathcal{O}$ *within* $\mathcal{Q}$*.*

Proof. By ($\mathbf{cr}_2$), $\mathcal{O}, \hat{\boldsymbol{q}} \models \boldsymbol{q}(a)$. To show $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{q}(a)$ for all $\boldsymbol{r} \in \mathcal{F}_{\boldsymbol{q}}$, we observe that $\boldsymbol{r} \not\models_{\mathcal{O}} \boldsymbol{q}$ by the definition of $\mathcal{F}_{\boldsymbol{q}}$, so $\boldsymbol{r}(x)$ is consistent with $\mathcal{O}$ and by ($\mathbf{cr}_3$) for $\boldsymbol{r}$, from which $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{q}(a)$. Thus, $\boldsymbol{q}$ fits E.

Let $\boldsymbol{q}' \in \mathcal{Q}$ and $\boldsymbol{q} \not\equiv_{\mathcal{O}} \boldsymbol{q}'$. We show that either $\mathcal{O}, \hat{\boldsymbol{q}} \not\models \boldsymbol{q}'$ or $\mathcal{O}, \hat{\boldsymbol{r}} \models \boldsymbol{q}'$ for some $\boldsymbol{r} \in \mathcal{F}_{\boldsymbol{q}}$. If $\boldsymbol{q} \not\models_{\mathcal{O}} \boldsymbol{q}'$, then, since $\mathcal{O}$ admits containment reduction and $\boldsymbol{q}(x)$ is satisfiable wrt $\mathcal{O}$, we obtain $\mathcal{O}, \hat{\boldsymbol{q}} \not\models \boldsymbol{q}'$ by ($\mathbf{cr}_3$). So suppose $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{q}'$ and $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}$. As $\mathcal{F}_{\boldsymbol{q}}$ is a frontier of $\boldsymbol{q}$ wrt $\mathcal{O}$, there is $\boldsymbol{r} \in \mathcal{F}_{\boldsymbol{q}}$ with $\boldsymbol{r} \models_{\mathcal{O}} \boldsymbol{q}'$. If $\boldsymbol{r}(x)$ is unsatisfiable wrt $\mathcal{O}$, then $\mathcal{O}$ and $\hat{\boldsymbol{r}}$ are unsatisfiable by ($\mathbf{cr}_1$), and so $\mathcal{O}, \hat{\boldsymbol{r}} \models \boldsymbol{q}'$. And if $\boldsymbol{r}(x)$ is satisfiable wrt $\mathcal{O}$, we obtain $\mathcal{O}, \hat{\boldsymbol{r}} \models \boldsymbol{q}'(a)$ by ($\mathbf{cr}_3$). □

Theorem 3. *$\mathcal{ALCHI}$ has general split-partners within ELIQ$^\sigma$ that can be computed in exponential time.*

Proof. Suppose a finite set $Q \subseteq$ ELIQ$^\sigma$ and an $\mathcal{ALCHI}$-ontology $\mathcal{O}$ in the signature σ are given. Let $sub_{\mathcal{O},Q}$ be the closure under single negation of the set of subconcepts of concepts in Q and $\mathcal{O}$. A *type for* $\mathcal{O}$ is any maximal subset $tp \subseteq sub_{\mathcal{O},Q}$ consistent with $\mathcal{O}$. Let $\boldsymbol{T}$ be the set of all types for $\mathcal{O}$. Define a σ-data instance $\mathcal{A}$ with $ind(\mathcal{A}) = \boldsymbol{T}$, $A(tp) \in \mathcal{A}$ for all concept names $A \in \sigma$ and tp such that $A \in tp$, and $P(tp, tp') \in \mathcal{A}$ for all role names $P \in \sigma$, tp and tp' such that tp and tp' can be satisfied by the domain elements of a model of $\mathcal{O}$ that are related via P. We consider an interpretation $\mathcal{I}_{\mathcal{A}}$ with $\Delta^{\mathcal{I}_{\mathcal{A}}} = \{ind(\mathcal{A})\}$, $A^{\mathcal{I}_{\mathcal{A}}} = \{tp \mid A(tp) \in \mathcal{A}\}$ for concept names $A \in \sigma$, $A^{\mathcal{I}_{\mathcal{A}}} = \emptyset$ for $A \notin \sigma$, $P^{\mathcal{I}_{\mathcal{A}}} = \{(tp, tp') \mid P(tp, tp') \in \mathcal{A}\}$ for role names $P \in \sigma$, $P^{\mathcal{I}_{\mathcal{A}}} = \emptyset$ for $P \notin \sigma$. It can be readily checked that, for any $\boldsymbol{q} \in \mathcal{Q}^\sigma$,

$$\mathcal{I}_{\mathcal{A}} \models \mathcal{O}, \mathcal{A}, \tag{10}$$

$$\boldsymbol{q} \in tp \quad \text{iff} \quad \mathcal{I}_{\mathcal{A}}, tp \models \boldsymbol{q} \quad \text{iff} \quad \mathcal{O}, \mathcal{A} \models \boldsymbol{q}(tp). \tag{11}$$

Let $Q = \{\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n\}$ and let $\mathcal{A}^n$ be the n-times direct product of $\mathcal{A}$. Set

$$\mathcal{S}(Q) = \{(\mathcal{A}^n, (tp_1, \ldots, tp_n)) \mid \neg\boldsymbol{q}_i \in tp_i,\ i = 1, \ldots n\}.$$

We prove (3) for an arbitrary $\boldsymbol{q}' \in \mathcal{Q}^\sigma$. For the ($\Rightarrow$) direction, suppose $\mathcal{O}, \mathcal{A}^n \models \boldsymbol{q}'(\vec{tp})$ for some $(\mathcal{A}^n, \vec{tp}) \in \mathcal{S}(Q)$ and $\vec{tp} = (tp_1, \ldots, tp_n)$. Fix $i \in \{1, \ldots, n\}$. Observe that the *projection* map $h((tp'_1, \ldots, tp'_n)) = tp'_i$ for $(tp'_1, \ldots, tp'_n) \in \boldsymbol{T}^n$ is a homomorphism from $\mathcal{A}^n$ to $\mathcal{A}$ such that $h(\vec{tp}) = tp_i$. As in the proof of Lemma 1 (1), we obtain $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(tp_i)$. Recall that $\neg\boldsymbol{q}_i \in tp_i$. Then, by (11), we have $\mathcal{I}_{\mathcal{A}}, tp_i \models \boldsymbol{q}'$, $\mathcal{I}_{\mathcal{A}}, tp_i \not\models \boldsymbol{q}_i$, and so using (10) we obtain $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}_i$. For the opposite direction, suppose $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}_i$ for all $1 \leq i \leq n$. It follows that, for each i, there exists $tp_i \in \boldsymbol{T}$ such that $\boldsymbol{q}', \neg\boldsymbol{q}_i \in tp_i$. Let $\vec{tp} = (tp_1, \ldots, tp_n)$. Clearly, $(\mathcal{A}^n, \vec{tp}) \in \mathcal{S}(Q)$ and it remains to show that $\mathcal{O}, \mathcal{A}^n \models \boldsymbol{q}'(\vec{tp})$. Observe that, for each tp_i, by (11), there exists a homomorphism h_i that maps $\boldsymbol{q}'$ into $\mathcal{I}_{\mathcal{A}}$ with its root mapped to tp_i. By the construction of $\mathcal{I}_{\mathcal{A}}$, the same holds for $\mathcal{A}$ in place of $\mathcal{I}_{\mathcal{A}}$. Because $\mathcal{A}^n$ is a direct product, there exists a homomorphism that maps $\boldsymbol{q}'$ into $\mathcal{A}^n$ with its root mapped to $\vec{tp}$. Thus, $\mathcal{A}^n \models \boldsymbol{q}'(\vec{tp})$. □

We observe that split partners for conjunctions can be obtained from split partners for the conjuncts.

Lemma 4. *Let σ be a signature, $\mathcal{Q}^\sigma$ be a subset of CQ and $\mathcal{L}$ an arbitrary logic. Let $\boldsymbol{q} = \boldsymbol{q}_1 \wedge \boldsymbol{q}_2$ be any CQ and $\mathcal{O}$ be an $\mathcal{L}$-ontology. Then, if $\mathcal{S}_1, \mathcal{S}_2$ are split partners for $\boldsymbol{q}_1, \boldsymbol{q}_2$ wrt $\mathcal{O}$ within $\mathcal{Q}^\sigma$, then $\mathcal{S}_1 \cup \mathcal{S}_2$ is a split partner for $\boldsymbol{q}$ wrt $\mathcal{O}$ within $\mathcal{Q}^\sigma$.*

Proof. Let $\boldsymbol{q}' \in \mathcal{Q}^\sigma$ arbitrary.

Suppose first $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a)$ for some $(\mathcal{A}, a) \in \mathcal{S}$. Then $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a)$ for some $(\mathcal{A}, a) \in \mathcal{S}_i$, for some $i \in \{1, 2\}$. Since $\mathcal{S}_i$ is a split-partner for $\boldsymbol{q}_i$ wrt $\mathcal{O}$ within $\mathcal{Q}^\sigma$, we have $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}_i$, and thus $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}$.

Suppose now that $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}$. Thus $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}_i$, for some $i \in \{1, 2\}$. Since $\mathcal{S}_i$ is a split-partner for $\boldsymbol{q}_i$ wrt $\mathcal{O}$ within $\mathcal{Q}^\sigma$, we have $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a)$ for some $(\mathcal{A}, a) \in \mathcal{S}_i$. Hence, $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a)$ for some $(\mathcal{A}, a) \in \mathcal{S}$. □

Theorem 4. *Let σ be a signature, $\mathcal{O}$ a σ-ontology in RDFS, and $n > 0$. For any set $\Theta \subseteq$ ELQ$^\sigma$ with $|\Theta| \leq n$, one can compute in polynomial time a split-partner $\mathcal{S}(\Theta)$ of Θ wrt $\mathcal{O}$ within ELQ$^\sigma$.*

Proof. We prove the statement for $n = 1$, the generalisation is straightforward. Let $Q = \{\boldsymbol{q}\}$. The construction is by induction over the depth of $\boldsymbol{q}$. Assume depth$(\boldsymbol{q}) = 0$. Due to Lemma 4, it suffices to consider $\boldsymbol{q} = A$ with A a concept name. Define a data instance $\mathcal{A}$ by taking

$$\begin{aligned} \mathcal{A} \quad = \quad & \{B(a) \mid \mathcal{O} \not\models B \sqsubseteq A, B \in \sigma\} \cup \\ & \{R(a, b) \mid \mathcal{O} \not\models \exists R \sqsubseteq A, R \in \sigma\} \cup \\ & \{B(b), R(b, b) \mid B, R \in \sigma\} \end{aligned}$$

and set $\mathcal{S}(\boldsymbol{q}) = \{(\mathcal{A}, a)\}$. We show that $\mathcal{S}(\boldsymbol{q})$ is as required. Assume

$$\boldsymbol{q}' = \bigwedge_{i=1}^{m_1} B_i \wedge \bigwedge_{i=1}^{m_2} \exists R_i.\boldsymbol{q}_i.$$

If $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}$, then

- $\mathcal{O} \not\models B_j \sqsubseteq A$ for all B_j;
- $\mathcal{O} \not\models \exists R_j \sqsubseteq A$ for all R_j.

Then $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a)$, as required.

Conversely, if $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a)$, then

- for all B_j there exists $B(a) \in \mathcal{A}$ with $\mathcal{O} \models B \sqsubseteq B_j$. Then $\mathcal{O} \not\models B \sqsubseteq A$, and so $\mathcal{O} \not\models B_j \sqsubseteq A$;
- for all $\exists R_j.\boldsymbol{q}_j$ there exists $R(a, b) \in \mathcal{A}$ with $\mathcal{O} \models R \sqsubseteq R_j$. Then $\mathcal{O} \not\models \exists R \sqsubseteq A$, and so $\mathcal{O} \not\models \exists R_j \sqsubseteq A$.

Hence, $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}$, as required.

Assume now that depth$(\boldsymbol{q}) = n + 1$ and that split partners $\mathcal{S}(\boldsymbol{q}')$ have been defined for queries of depth $\leq n$. In view of Lemma 4 it suffices to consider the case

$$\boldsymbol{q} = \exists S_1.\boldsymbol{q}_1,$$

for an ELIQ $\boldsymbol{q}_1$ of depth $\leq n$ with split partner $\mathcal{S}(\boldsymbol{q}_1) = \{(\mathcal{A}_1, a_1), \ldots, (\mathcal{A}_k, a_k)\}$.

Let for all R with $\mathcal{O} \models R \sqsubseteq S_1$, I_R be the set of $j \leq k$ with $\mathcal{O}, \mathcal{A}_j \models A(a_j)$ whenever $\mathcal{O} \models \exists R^- \sqsubseteq A$. Define a data instance $\mathcal{A}$ by taking

$$\begin{aligned}\mathcal{A} \quad = \quad & \{B(a) \mid B \in \sigma\} \cup \\ & \{R(a,b), S(b,b), B(b) \mid \mathcal{O} \not\models R \sqsubseteq S_1, R, B, S \in \sigma\} \cup \\ & \{R(a, a_j) \mid j \in I_R, R \in \sigma\} \cup \\ & \mathcal{A}_1(a_1) \cup \cdots \cup \mathcal{A}_k(a_k).\end{aligned}$$

and set $\mathcal{S}(\boldsymbol{q}) = \{(\mathcal{A}, a)\}$. We show that $\mathcal{S}(\boldsymbol{q})$ is as required. We show the following claim for arbitrary $\boldsymbol{q}'$ of the form

$$\boldsymbol{q}' = \bigwedge_{i=1}^{m_1} B_i \wedge \bigwedge_{i=1}^{m_2} \exists R_i.\boldsymbol{q}_i'.$$

Claim 1. $\boldsymbol{q}' \not\models_{\mathcal{O}} \exists S_1.\boldsymbol{q}_1$ iff $\mathcal{O}, \mathcal{A}_i \models \boldsymbol{q}'(a)$.

Proof of Caim 1. If $\boldsymbol{q}' \not\models_{\mathcal{O}} \exists S_1.\boldsymbol{q}_1$, then for all $\exists R_j.\boldsymbol{q}_j'$ with $\mathcal{O} \models R_j \sqsubseteq S_1$ we have

$$\exists R_j^- \sqcap \boldsymbol{q}_j' \not\models_{\mathcal{O}} \boldsymbol{q}_1.$$

Take any j. Let C_{R_j} be the conjunction of all A with $\mathcal{O} \models \exists R_j^- \sqsubseteq A$. Then

$$C_{R_j} \sqcap \boldsymbol{q}_j' \not\models_{\mathcal{O}} \boldsymbol{q}_1.$$

By the definition of split-partners, there is ℓ with

$$\mathcal{O}, \mathcal{A}_\ell \models C_{R_j} \sqcap \boldsymbol{q}_j'(a_\ell).$$

It now follows immediately that $\mathcal{O}, \mathcal{A} \models \exists R_j.\boldsymbol{q}_j'(a)$. Hence $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a)$ follows, as required.

Conversely, assume $\boldsymbol{q}' \models_{\mathcal{O}} \exists S_1.\boldsymbol{q}_1$. Then there exists $\exists R_j.\boldsymbol{q}_j'$ with $\mathcal{O} \models R_j \sqsubseteq S_1$ and

$$\exists R_j^- \sqcap \boldsymbol{q}_j' \models_{\mathcal{O}} \boldsymbol{q}_1.$$

Let again C_{R_j} be the conjunction of all A with $\mathcal{O} \models \exists R_j^- \sqsubseteq A$. Then

$$C_{R_j} \sqcap \boldsymbol{q}_j' \models_{\mathcal{O}} \boldsymbol{q}_i.$$

By the definition of split-partners,

$$\mathcal{O}, \mathcal{A}_\ell \not\models C_{R_j} \sqcap \boldsymbol{q}_j'(a_\ell).$$

for all $\ell \leq k$. But then $\mathcal{O}, \mathcal{A} \not\models \exists R_j.\boldsymbol{q}_j'(a)$ and so $\mathcal{O}, \mathcal{A} \not\models \boldsymbol{q}'(a)$, as required.

This finishes the proof of Claim 1 and, in fact, of the Theorem. □

Theorem 5. *Suppose $\mathcal{Q}$ is a class of queries, an ontology $\mathcal{O}$ admits containment reduction, $\boldsymbol{q} \in \mathcal{Q}$ is satisfiable wrt $\mathcal{O}$, and σ contains the predicate symbols in $\boldsymbol{q}$ and $\mathcal{O}$. If $\mathcal{S}_{\boldsymbol{q}}$ is a split-partner for $\{\boldsymbol{q}\}$ wrt $\mathcal{O}$ within $\mathcal{Q}^\sigma$, then $(\{(\hat{\boldsymbol{q}}, a)\}, \mathcal{S}_{\boldsymbol{q}})$ is a unique characterisation of $\boldsymbol{q}$ wrt $\mathcal{O}$ within $\mathcal{Q}$.*

Proof. Clearly, $\boldsymbol{q}$ fits E as $\mathcal{O}, \hat{\boldsymbol{q}} \models \boldsymbol{q}(a)$ and $\mathcal{O}, \mathcal{A} \not\models \boldsymbol{q}(a)$ for any $(\mathcal{A}, a) \in \mathcal{S}_{\boldsymbol{q}}$ as otherwise $\boldsymbol{q} \not\models_{\mathcal{O}} \boldsymbol{q}$. Let $\boldsymbol{q}' \not\equiv_{\mathcal{O}} \boldsymbol{q}$. If $\boldsymbol{q}' \models_{\mathcal{O}} \boldsymbol{q}$, then $\boldsymbol{q} \not\models_{\mathcal{O}} \boldsymbol{q}'$, and so $\mathcal{O}, \hat{\boldsymbol{q}} \not\models \boldsymbol{q}'(a)$. Hence $\boldsymbol{q}'$ does not fit E. If $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}$, then there exists $(\mathcal{A}, a) \in \mathcal{S}_{\boldsymbol{q}}$ with $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a)$, and so again $\boldsymbol{q}'$ does not fit E. □

A *CQ-frontier* for an ELIQ $\boldsymbol{q}$ wrt to an ontology $\mathcal{O}$ is a set $\mathcal{F}_{\boldsymbol{q}}$ of CQs such that

- if $\boldsymbol{q}' \models_{\mathcal{O}} \boldsymbol{q}''$, for a CQ $\boldsymbol{q}' \in \mathcal{F}_{\boldsymbol{q}}$ and an ELIQ $\boldsymbol{q}''$, then $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{q}''$ and $\boldsymbol{q}'' \not\models_{\mathcal{O}} \boldsymbol{q}$;
- if $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{q}''$ and $\boldsymbol{q}'' \not\models_{\mathcal{O}} \boldsymbol{q}$, for an ELIQ $\boldsymbol{q}''$, then there exists $\boldsymbol{q}' \in \mathcal{F}_{\boldsymbol{q}}$ such that $\boldsymbol{q}' \models_{\mathcal{O}} \boldsymbol{q}''$.

Clearly standard ELIQ frontiers defined above are also CQ-frontiers.

Theorem 7. *$\mathcal{EL}$ does not admit finite CQ-frontiers within ELIQ.*

Proof. We show that the query $\boldsymbol{q} = A \wedge B$ does not have a finite CQ-frontier wrt the ontology

$$\mathcal{O} = \{A \sqsubseteq \exists R.A,\ \exists R.A \sqsubseteq A\}$$

within ELIQs. Suppose otherwise. Let $\mathcal{F}_{\boldsymbol{q}}$ be such a CQ-frontier. Consider the ELIQs $\boldsymbol{r}_{n,m} = \exists R^n \exists R^{-m}.B$ with $n > m > 0$. Clearly, $\boldsymbol{q} \not\models_{\mathcal{O}} \boldsymbol{r}_{n,m}$, and so $\boldsymbol{r}_{n,m}$ cannot be entailed wrt $\mathcal{O}$ by any CQ in $\mathcal{F}_{\boldsymbol{q}}$. Thus, if $\boldsymbol{q}' \in \mathcal{F}_{\boldsymbol{q}}$ and $B(x)$ is in $\boldsymbol{q}'(x)$, then we cannot have an R-cycle in $\boldsymbol{q}'$ reachable from x along an R-path as otherwise we would have $\boldsymbol{q}' \models_{\mathcal{O}} \boldsymbol{r}_{n,m}$ for suitable n, m.

Consider now the ELIQs $\boldsymbol{q}_n = B \wedge \exists R^n.\top$, for all $n \geq 1$. Clearly, $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{q}_n$. As $\boldsymbol{q}_n \not\models_{\mathcal{O}} \boldsymbol{q}$, infinitely many $\boldsymbol{q}_n$ are entailed by some $\boldsymbol{q}' \in \mathcal{F}_{\boldsymbol{q}}$ wrt $\mathcal{O}$. Take such a $\boldsymbol{q}'$. Since $B(x) \in \boldsymbol{q}'$ because of $\boldsymbol{q}' \models_{\mathcal{O}} \boldsymbol{q}_n$, no R-cycle is reachable from x via an R-path in $\boldsymbol{q}'$. Note also that no y with $A(y) \in \boldsymbol{q}'$ can be reached from x along an R-path as otherwise $\boldsymbol{q}' \models_{\mathcal{O}} B \wedge \exists R^k.A$ for some $k \geq 0$ and, since $B \wedge \exists R^k.A \models_{\mathcal{O}} \boldsymbol{q}$ by the second axiom in $\mathcal{O}$, we would have $\boldsymbol{q}' \models_{\mathcal{O}} \boldsymbol{q}$.

To derive a contradiction, we show now that there is an R-path of any length n starting at x in $\boldsymbol{q}'$. Suppose this is not the case. Let n be the length of a longest R-path starting at x in $\boldsymbol{q}'$. We construct a model $\mathcal{I}$ of $\mathcal{O}$ and $\boldsymbol{q}'$ refuting $\boldsymbol{q}_{n+l}$, for any $l \geq 1$. Define $\mathcal{I}$ by taking

- $\Delta^{\mathcal{I}} = \mathit{var}(\boldsymbol{q}') \cup \{d_1, d_2, \dots\}$, for fresh d_i;
- $a \in B^{\mathcal{I}}$ if $B(a) \in \boldsymbol{q}'$;
- $a \in A^{\mathcal{I}}$ if there is an R-path in $\boldsymbol{q}'$ from a to some y with $A(y) \in \boldsymbol{q}'$ or $a = d_i$ for some i;
- $(a, b) \in R^{\mathcal{I}}$ if $R(a, b) \in \boldsymbol{q}'$ or there is an R-path in $\boldsymbol{q}'$ from a to some y with $A(y) \in \boldsymbol{q}'$ and $b = d_1$, or $a = d_i$ and $b = d_{i+1}$.

By the construction and the fact that no y with $A(y) \in \boldsymbol{q}'$ can be reached from x along an r-path, $\mathcal{I}$ is a model of $\mathcal{O}$ and $\boldsymbol{q}'$ refuting $\boldsymbol{q}_{n+l}$. □

Theorem 8. *There exist a DL-Lite$_{\mathcal{F}}^-$ ontology $\mathcal{O}$, a query $\boldsymbol{q}$ and a signature σ such that $\{\boldsymbol{q}\}$ does not have a finite split-partner wrt $\mathcal{O}$ within ELIQ$^\sigma$.*

Proof. Let $\mathcal{O} = \{\text{fun}(P), \text{fun}(P^-), B \sqcap \exists P^- \sqsubseteq \bot\}$ and $\boldsymbol{q} = A$. We show that $Q = \{\boldsymbol{q}\}$ does not have a finite split partner wrt $\mathcal{O}$ within ELIQ$^{\{A,B,P\}}$. For suppose $\mathcal{S}(Q)$ is such a split-partner. Then there exists $(\mathcal{A}, a) \in \mathcal{S}(Q)$ with $\mathcal{O}, \mathcal{A} \models B \sqcap \exists P^n.\top(a)$ for all sufficiently large n because

$B \sqcap \exists P^n.\top \not\models_{\mathcal{O}} A$. Then $\mathcal{A}$ must contain n nodes if $\mathcal{O}$ and $\mathcal{A}$ are satisfiable, so $\mathcal{S}(Q)$ is infinite.

On the other hand, $\{\top\}$ is a frontier for A wrt $\mathcal{O}$ within ELIQ. □

The following example shows that even by taking frontiers and splittings together we do not obtain a universal method for constructing unique characterisations with a single positive example.

Example 6. Consider the ELIQ $\boldsymbol{q} = A \wedge B$ and the $\mathcal{ELIF}$-ontology $\mathcal{O}$ with the following axioms:

$$A \sqsubseteq \exists R.A,\ \exists R.A \sqsubseteq A,\ \text{fun}(P),\ \text{fun}(P^-),\ E \sqcap \exists P^- \sqsubseteq \bot.$$

It can be shown in the same way as above that $\boldsymbol{q}$ has no frontier wrt $\mathcal{O}$ within ELIQ and that $\boldsymbol{q}$ does not have any split-partner wrt $\mathcal{O}$ within $\text{ELIQ}^{\{A,B,R,P,E\}}$. However, a unique characterisation of $\boldsymbol{q}$ wrt $\mathcal{O}$ within ELIQ is obtained by taking $E^+ = \{\hat{\boldsymbol{q}}\}$ and E^- the same as in Example 2. (To show the latter one only has to observe that E^- is a split-partner of $\boldsymbol{q}$ wrt $\mathcal{O}$ within $\text{ELIQ}^{\{A,B,R\}}$ and that $\mathcal{O}, \hat{\boldsymbol{q}} \not\models \boldsymbol{r}$ for any $\boldsymbol{r}$ containing any of the symbols P or E.)

B Results on Meet-Reducibility

Recall that a query $\boldsymbol{r} \in \mathcal{Q}$ is called *meet-reducible* (McKenzie 1972) wrt $\mathcal{O}$ within $\mathcal{Q}$ if there are queries $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathcal{Q}$ such that $\boldsymbol{r} \equiv_{\mathcal{O}} \boldsymbol{r}_1 \wedge \boldsymbol{r}_2$ and $\boldsymbol{r} \not\equiv_{\mathcal{O}} \boldsymbol{r}_i$, $i = 1, 2$.

Lemma 5. (i) *If an ontology $\mathcal{O}$ admits frontiers within $\mathcal{Q}$, then $\boldsymbol{q} \in \mathcal{Q}$ is meet-reducible wrt $\mathcal{O}$ within $\mathcal{Q}$ iff $|\mathcal{F}_{\boldsymbol{q}}| \geq 2$ provided that $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}''$, for any distinct $\boldsymbol{q}', \boldsymbol{q}'' \in \mathcal{F}_{\boldsymbol{q}}$.*

(ii) *If an ontology $\mathcal{O}$ admits containment reduction within $\mathcal{Q}$, then, for any meet-reducible $\boldsymbol{q} \in \mathcal{Q}$ wrt $\mathcal{O}$ within $\mathcal{Q}$, we have $|\mathcal{N}_{\boldsymbol{q}}| \geq 2$ for every characterisation of $\boldsymbol{q}$ wrt $\mathcal{O}$ within $\mathcal{Q}$ of the form $(\{\hat{\boldsymbol{q}}\}, \mathcal{N}_{\boldsymbol{q}})$.*

Proof. $(i, \Leftarrow)$ Let $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathcal{F}_{\boldsymbol{q}}$ be distinct and $\boldsymbol{r} = \boldsymbol{r}_1 \wedge \boldsymbol{r}_2$. If $\boldsymbol{q} \not\equiv_{\mathcal{O}} \boldsymbol{r}$, then there is $\boldsymbol{r}' \in \mathcal{F}_{\boldsymbol{q}}$ with $\boldsymbol{r}' \models_{\mathcal{O}} \boldsymbol{r}$, and so $\boldsymbol{r}' \models_{\mathcal{O}} \boldsymbol{r}_i$, which is impossible.

$(i, \Rightarrow)$ Suppose $\boldsymbol{q} \equiv_{\mathcal{O}} \boldsymbol{q}_1 \wedge \boldsymbol{q}_2$ and $\boldsymbol{q} \not\equiv_{\mathcal{O}} \boldsymbol{q}_i$, for $i = 1, 2$. Then there are $\boldsymbol{r}_i \in \mathcal{F}_{\boldsymbol{q}}$ with $\boldsymbol{r}_i \not\equiv_{\mathcal{O}} \boldsymbol{q}_i$. Clearly, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are distinct because otherwise $\boldsymbol{r}_i \models_{\mathcal{O}} \boldsymbol{q}$.

(ii) Suppose $\boldsymbol{q} \equiv_{\mathcal{O}} \boldsymbol{q}_1 \wedge \boldsymbol{q}_2$ and $\boldsymbol{q} \not\equiv_{\mathcal{O}} \boldsymbol{q}_i$, for $i = 1, 2$. If $\mathcal{N}_{\boldsymbol{q}} = \{\hat{\boldsymbol{r}}\}$, then $\mathcal{O}, \hat{\boldsymbol{r}} \models \boldsymbol{q}_i$, for $i = 1, 2$, because if $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{q}_i$, then $\boldsymbol{q}_i$ would fit $(\{\hat{\boldsymbol{q}}\}, \mathcal{N}_{\boldsymbol{q}})$, and so would be equivalent to $\boldsymbol{q}$ wrt $\mathcal{O}$, which is not the case. □

Lemma 6. (i) *Deciding whether an ELIQ $\boldsymbol{q}$ is meet-reducible wrt to a DL-Lite$^-_{\mathcal{F}}$-ontology is in* PTIME.

(ii) *Deciding whether an ELIQ $\boldsymbol{q}$ is meet-reducible wrt to a DL-Lite$_{\mathcal{H}}$-ontology is* coNP*-complete.*

Proof. (i) We first compute a frontier $\mathcal{F}_{\boldsymbol{q}}$ of $\boldsymbol{q}$ in polynomial time (Funk, Jung, and Lutz 2022b). Then we remove from $\mathcal{F}_{\boldsymbol{q}}$ every $\boldsymbol{q}''$ for which there is a different $\boldsymbol{q}' \in \mathcal{F}_{\boldsymbol{q}}$ with $\boldsymbol{q}' \models_{\mathcal{O}} \boldsymbol{q}''$. This can also be done in polynomial time because ELIQ containment in *DL-Lite*$_{\mathcal{F}}$ is tractable (Bienvenu et al. 2013). It remains to use Lemma 5 (i) to check if the resulting set is a singleton.

(ii) Assume $\mathcal{O}$ and $\boldsymbol{q}$ are given. For the upper bound, first compute a frontier $\mathcal{F}_{\boldsymbol{q}}$ of $\boldsymbol{q}$ in polynomial time (Funk, Jung, and Lutz 2022b). To check that $\boldsymbol{q}$ is meet-reducible guess queries $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathcal{F}_{\boldsymbol{q}}$ and witness models showing that $\boldsymbol{q}_1 \not\models_{\mathcal{O}} \boldsymbol{q}_2$ and $\boldsymbol{q}_2 \not\models_{\mathcal{O}} \boldsymbol{q}_1$. For the lower bound, consider the ontologies $\mathcal{O}$, ABox $\{A_0(a)\}$, and ELIQ $\boldsymbol{q}$ constructed in the proof of (Kikot, Kontchakov, and Zakharyaschev 2011, Theorem 1) for Boolean CNFs. Recall that $\boldsymbol{q}$ has a single answer variable x, an atom $A_0(x)$, and A_0 does not occur elsewhere in $\boldsymbol{q}$. As shown in that proof, the problem $\mathcal{O}, A_0(a) \models \boldsymbol{q}(a)$ is NP-hard. Take a copy $\mathcal{O}'$ of $\mathcal{O}$ obtained by replacing each predicate symbol S in $\mathcal{O}$ except A_0 by a fresh S'. Similarly, take a copy $\boldsymbol{q}'$ of $\boldsymbol{q}$. We show that

$$\boldsymbol{q} \wedge \boldsymbol{q}' \equiv_{\mathcal{O} \cup \mathcal{O}'} \boldsymbol{q} \text{ implies } \mathcal{O}, A_0(a) \models \boldsymbol{q}(a) \qquad (12)$$

$(\Rightarrow)$ Suppose $\mathcal{O}, A_0(a) \not\models \boldsymbol{q}(a)$, then there exists a model $\mathcal{I}'$ of $\mathcal{O}'$ with $a^{\mathcal{I}'} \in A_0^{\mathcal{I}'}$ not satisfying $\boldsymbol{q}'$ at $a^{\mathcal{I}'}$. Let $a^{\mathcal{I}'} = d$. Take the interpretation $\mathcal{J}$ that looks like $\hat{\boldsymbol{q}}$, let its root be d. From (Kikot, Kontchakov, and Zakharyaschev 2011, Theorem 1) it follows that there exists $\mathcal{I} \supseteq \mathcal{J}$ such that $\mathcal{I} \models \mathcal{O}$. Clearly, $\mathcal{I} \models \mathcal{O}'$ (because all $S'^{\mathcal{I}} = \emptyset$). By taking the union of $\mathcal{I}$ and $\mathcal{I}'$, we obtain an interpretation that satisfies $\mathcal{O} \cup \mathcal{O}'$, $A_0(a)$, satisfies $\boldsymbol{q}$ at d and does not satisfy $\boldsymbol{q}'$ at d. It follows $\boldsymbol{q} \not\models_{\mathcal{O} \cup \mathcal{O}'} \boldsymbol{q} \wedge \boldsymbol{q}'$.

Using(12), we now show that $\boldsymbol{q} \wedge \boldsymbol{q}'$ is not meet-reducible w.r.t. $\mathcal{O} \cup \mathcal{O}'$ iff $\mathcal{O}, A_0(a) \models \boldsymbol{q}(a)$. If $\boldsymbol{q} \wedge \boldsymbol{q}'$ is not meet-reducible, then $\boldsymbol{q} \wedge \boldsymbol{q}' \equiv_{\mathcal{O} \cup \mathcal{O}'} \boldsymbol{q}$ and we are done. For the opposite direction, suppose $\mathcal{O}, A_0(a) \models \boldsymbol{q}(a)$. We immediately observe that $\boldsymbol{q} \equiv_{\mathcal{O}} A_0(x)$. It follows then that $\boldsymbol{q} \wedge \boldsymbol{q}' \equiv_{\mathcal{O} \cup \mathcal{O}'} A_0(x)$. Suppose $\boldsymbol{q}_1 \wedge \boldsymbol{q}_2 \equiv_{\mathcal{O} \cup \mathcal{O}'} \boldsymbol{q} \wedge \boldsymbol{q}'$, for some $\boldsymbol{q}_1, \boldsymbol{q}_2$. It follows that $\boldsymbol{q}_1 \wedge \boldsymbol{q}_2 \equiv_{\mathcal{O} \cup \mathcal{O}'} A_0(x)$. Clearly, some $\boldsymbol{q}_i$ contains $A_0(x)$ for otherwise we easily get $\boldsymbol{q}_1 \wedge \boldsymbol{q}_2 \not\models_{\mathcal{O} \cup \mathcal{O}'} A_0(x)$ contrary to our assumption. But then it follows that $\boldsymbol{q}_i \equiv_{\mathcal{O} \cup \mathcal{O}'} \boldsymbol{q} \wedge \boldsymbol{q}'$ and $\boldsymbol{q} \wedge \boldsymbol{q}'$ is not meet-reducible. □

C Comments for Section 5

We discuss the relationship between the epistemic semantics used in this article for temporal queries and Tarski semantics based on *temporal structures* $\mathcal{I}$, which are a sequences $\mathcal{I}_0, \mathcal{I}_1, \dots$ of domain structures $\mathcal{I}_i$ as introduced above such that $a^{\mathcal{I}_n} = a^{\mathcal{I}_m}$, for any individual a and $n, m \in \mathbb{N}$. $\mathcal{I}$ is a *model* of $\mathcal{D} = \mathcal{A}_0, \dots, \mathcal{A}_n$ if each $\mathcal{I}_i$ is a model of $\mathcal{A}_i$, for $i \leq n$; and $\mathcal{I}$ is a model of $\mathcal{O}$ if each $\mathcal{I}_i$ is a model of $\mathcal{O}$, for $i \in \mathbb{N}$. The truth relation $\mathcal{I}, \ell, a \models \boldsymbol{q}$ is then defined in the obvious way. We write $\mathcal{O}, \mathcal{D}, \ell, a \models_T \boldsymbol{q}$ if $\mathcal{I}, \ell, a \models \boldsymbol{q}$, for every model $\mathcal{I}$ of $\mathcal{O}$ and $\mathcal{D}$. It is easy to see that $\models$ coincides with $\models_T$ for any Horn ontology $\mathcal{O}$, in particular, all *DL-Lite* logics considered here and $\mathcal{ELHIF}$. Thus, the results presented in this paper also hold under $\models_T$ if one considers such ontologies. In general, however, the two entailment relations do not coincide: consider the ontology $\mathcal{O} = \{\top \sqsubseteq A \sqcup B\}$ and the data instance $\mathcal{D} = \emptyset, \mathcal{A}_1, \emptyset, \mathcal{A}_3$ with $\mathcal{A}_1 = \{A(a)\}$ and $\mathcal{A}_3 = \{B(a)\}$. Then $\mathcal{O}, \mathcal{D}, 0, a \models_T \Diamond(A \wedge \bigcirc B)$ but $\mathcal{O}, \mathcal{D}, 0, a \not\models \Diamond(A \wedge \bigcirc B)$. We leave an investigation of $\models_T$ in the non-Horn case for future work.

D Proof for Section 6

Lemma 3. *Let $\mathcal{O}$ be an FO-ontology* (*possibly with* $=$)*. Then every query $\boldsymbol{q} \in LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ is equivalent wrt $\mathcal{O}$*

to a query in normal form of size at most $|\boldsymbol{q}|$ and of temporal depth not exceeding $tdp(\boldsymbol{q})$. This query can be computed in polynomial time if containment between queries in $\mathcal{Q}$ wrt $\mathcal{O}$ is decidable in polynomial time. If $\mathcal{Q}$ = ELIQ, this is the case for DL-Lite$_\mathcal{F}$ but not for DL-Lite$_\mathcal{H}$ (unless P = NP).

Proof. The transformation is straightforward: to ensure **(n1)**, drop any $\boldsymbol{r}_0^i$ and $\boldsymbol{r}_{k_i}^i$ for which **(n1)** fails and add one $<$ to the relevant $\mathcal{R}_i$. To ensure **(n2)**, replace any $\mathcal{R}_i$ containing at least one occurrence of $<$ with the sequence obtained from $\mathcal{R}_i$ by dropping all occurrences of $\leq$ and replace any $\mathcal{R}_i$ not containing any occurrence of $<$ by a single $\leq$. To ensure **(n3)**, drop any $\boldsymbol{r}_0^{i+1}$ with $\boldsymbol{r}_{k_i}^i \models_\mathcal{O} \boldsymbol{r}_0^{i+1}$ if $\boldsymbol{q}_{i+1}$ is primitive and $\mathcal{R}_{i+1}$ is $\leq$. To ensure **(n4)** drop any $\boldsymbol{r}_{k_i}^i$ with $\boldsymbol{r}_0^{i+1} \models_\mathcal{O} \boldsymbol{r}_{k_i}^i$ if $i > 0$, $\boldsymbol{q}_i$ is primitive and $\mathcal{R}_{i+1}$ is $\leq$. Finally, to ensure **(n5)**, replace $\mathcal{R}_{i+1}$ by $<$ if $\boldsymbol{r}_{k_i}^i \wedge \boldsymbol{r}_0^{i+1}$ is not satisfiable wrt $\mathcal{O}$ and $\mathcal{R}_{i+1}$ is $\leq$.

The second part follows from the fact that query containment in *DL-Lite*$_\mathcal{F}$ is in P and NP-hard in *DL-Lite*$_\mathcal{H}$ (Kikot, Kontchakov, and Zakharyaschev 2011). □

Theorem 9. *Suppose an ontology $\mathcal{O}$ admits containment reduction and $\mathcal{Q}$ is a class of domain queries that is uniquely characterisable wrt $\mathcal{O}$. Then the following hold:*

(i) *A query $\boldsymbol{q} \in LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ is uniquely characterisable within $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ wrt $\mathcal{O}$ iff $\boldsymbol{q}$ is safe wrt $\mathcal{O}$.*

(ii) *If $\mathcal{O}$ admits polysize characterisations within $\mathcal{Q}$, then those queries that are uniquely characterisable within $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ are actually polysize characterisable within $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$.*

(iii) *The class $LTL_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ is polysize characterisable for bounded temporal depth if $\mathcal{O}$ admits polysize unique characterisations within $\mathcal{Q}$.*

(iv) *The class $LTL_p^{\bigcirc\Diamond}(\mathcal{Q})$ is uniquely characterisable. It is polysize characterisable if $\mathcal{O}$ admits polysize unique characterisations within $\mathcal{Q}$.*

To prove Theorem 9 we first provide some notation for talking about the entailment relation $\mathcal{O}, \mathcal{D} \models \boldsymbol{q}$. Let $\mathcal{D} = \mathcal{A}_0, \dots, \mathcal{A}_n$, $a \in ind(\mathcal{D})$, and let $\boldsymbol{q}$ take the form (6). A map $h\colon var(\boldsymbol{q}) \to [0, \max(\mathcal{D})]$ is called a *root $\mathcal{O}$-homomorphism* from $\boldsymbol{q}$ to $(\mathcal{D}, a)$ if $h(t_0) = 0$, $\mathcal{O}, \mathcal{A}_{h(t)} \models \boldsymbol{r}(a)$ if $\boldsymbol{r}(t) \in \boldsymbol{q}$, $h(t') = h(t) + 1$ if $suc(t, t') \in \boldsymbol{q}$, and $h(t)\, R\, h(t')$ if $R(t, t') \in \boldsymbol{q}$ for $R \in \{<, \leq\}$. It is readily seen that $\mathcal{O}, \mathcal{D}, a, 0 \models \boldsymbol{q}$ iff there exists a root $\mathcal{O}$-homomorphism from $\boldsymbol{q}$ to $(\mathcal{D}, a)$.

Let $b \geq 1$. The instance $\mathcal{D}$ is said to be *b-normal wrt $\mathcal{O}$* if it takes the form

$$\mathcal{D} = \mathcal{D}_0 \emptyset^b \mathcal{D}_1 \dots \emptyset^b \mathcal{D}_n, \text{ where } \mathcal{D}_i = \mathcal{A}_0^i \dots \mathcal{A}_{k_i}^i, \qquad (13)$$

with $b > k_i \geq 0$ and $\mathcal{A}_0^i \not\equiv_\mathcal{O} \emptyset$ if $i > 0$, and $\mathcal{A}_{k_i}^i \not\equiv_\mathcal{O} \emptyset$ if $i > 0$ or $k_i > 0$ (thus, of all the first/last $\mathcal{A}$ in a $\mathcal{D}_i$ only $\mathcal{A}_0^0$ can be trivial). Following the terminology for queries, we call each $\mathcal{D}_i$ *a block of* $\mathcal{D}$. For a block $\mathcal{D}_i$ in $\mathcal{D}$, we denote by $I(\mathcal{D}_i)$ the subset of $[0, \max(\mathcal{D})]$ occupied by $\mathcal{D}_i$. Then we call a root $\mathcal{O}$-homomorphism $h\colon \boldsymbol{q} \to \mathcal{D}$ *block surjective* if every $j \in I(\mathcal{D}_i)$ with a block $\mathcal{D}_i$ is in the range $ran(h)$ of h. We next aim to state that after 'weakening' the non-temporal data instances in blocks, no root $\mathcal{O}$-homomorphism from $\boldsymbol{q}$ to $\mathcal{D}$ exists. This is only required if the non-temporal data instances are obtained from queries in $\mathcal{Q}$, and so we express 'weakening' for characterisations $(\{\hat{\boldsymbol{s}}\}, \mathcal{N}_{\boldsymbol{s}})$ of queries $\boldsymbol{s}$ in $\mathcal{Q}$ wrt $\mathcal{O}$.

In detail, suppose $\mathcal{A}_j^i = \hat{\boldsymbol{s}}_j^i$, for queries $\boldsymbol{s}_j^i \in \mathcal{Q}$. Given $\boldsymbol{s} \in \mathcal{Q}$, take the characterisation $(\{\hat{\boldsymbol{s}}\}, \mathcal{N}_{\boldsymbol{s}})$ of $\boldsymbol{s}$ wrt $\mathcal{O}$ within $\mathcal{Q}$, where $\{\hat{\boldsymbol{s}}\}$ is the only positive data instance, and $\mathcal{N}_{\boldsymbol{s}}$ is the set of negative data instances. To make our notation more uniform, we think of the pointed data instances in $\mathcal{N}_{\boldsymbol{s}}$ as having the form $\hat{\boldsymbol{s}}'$, for a suitable CQ $\boldsymbol{s}'$ (which is not necessarily in $\mathcal{Q}$).

For $\ell \in ran(h)$, let

$$\boldsymbol{r}_\ell = \bigwedge_{\boldsymbol{r}(t) \in \boldsymbol{q}, h(t) = \ell} \boldsymbol{r}.$$

Then h is called *data surjective* if $\mathcal{O}, \hat{\boldsymbol{s}} \not\models \boldsymbol{r}_\ell(a)$, for any $\boldsymbol{s} \in \mathcal{N}_{\boldsymbol{s}_j^i}$ and any $\ell \in ran(h)$ such that $\boldsymbol{s}_j^i \not\equiv \top$, where $\hat{\boldsymbol{s}}_j^i$ is the data instance placed at ℓ in $\mathcal{D}$.

We call the root $\mathcal{O}$-homomorphism $h : \boldsymbol{q} \to \mathcal{D}$ a *root $\mathcal{O}$-isomorphism* if it is data surjective and, for the blocks $\boldsymbol{q}_0, \dots, \boldsymbol{q}_m$ of $\boldsymbol{q}$, we have $n = m$ and h restricted to $var(\boldsymbol{q}_i)$ is a bijection onto $I(\mathcal{D}_i)$ for all $i \leq n$ (in particular, h is block surjective). Intuitively, if we have a root $\mathcal{O}$-isomorphism $h\colon \boldsymbol{q} \to \mathcal{D}$, then $\boldsymbol{q}$ is almost the same as $\mathcal{D}$ except for differences between the sequences $\mathcal{R}_i$ in $\boldsymbol{q}$ and the gaps between blocks in $\mathcal{D}$.

Let $\mathcal{Q}, \mathcal{O}, \boldsymbol{q}$, and $\mathcal{D}$ be as before with $\mathcal{D}$ of the form (13), where $\mathcal{A}_j^i = \hat{\boldsymbol{s}}_j^i$, for $\boldsymbol{s}_j^i \in \mathcal{Q}$. The following rules will be used to define the negative examples in the unique characterisation of $\boldsymbol{q}$ and as steps in the learning algorithm. They are applied to $\mathcal{D}$:

(a) replace some $\hat{\boldsymbol{s}}_j^i$ with $\boldsymbol{s}_j^i \not\equiv_\mathcal{O} \top$ by an $\hat{\boldsymbol{s}} \in \mathcal{N}_{\boldsymbol{s}_j^i}$, for i, j such that $(i, j) \neq (0, 0)$—that is, the rule is not applied to $\boldsymbol{s}_0^0$;

(b) replace some pair $\hat{\boldsymbol{s}}_j^i \hat{\boldsymbol{s}}_{j+1}^i$ within block i by $\hat{\boldsymbol{s}}_j^i \emptyset^b \hat{\boldsymbol{s}}_{j+1}^i$;

(c) replace some $\hat{\boldsymbol{s}}_j^i$ with $\boldsymbol{s}_j^i \not\equiv_\mathcal{O} \top$ by $\hat{\boldsymbol{s}}_j^i \emptyset^b \hat{\boldsymbol{s}}_j^i$, where $k_i > j > 0$—that is, the rule is not applied to $\boldsymbol{s}_j^i$ if it is on the border of its block;

(d) replace $\hat{\boldsymbol{s}}_{k_i}^i$ $(k_i > 0)$ by $\hat{\boldsymbol{s}} \emptyset^b \hat{\boldsymbol{s}}_{k_i}^i$, for some $\hat{\boldsymbol{s}} \in \mathcal{N}_{\boldsymbol{s}_{k_i}^i}$, or replace $\hat{\boldsymbol{s}}_0^i$ $(k_i > 0)$ by $\hat{\boldsymbol{s}}_0^i \emptyset^b \hat{\boldsymbol{s}}$, for some $\hat{\boldsymbol{s}} \in \mathcal{N}_{\boldsymbol{s}_0^i}$;

(e) replace $\hat{\boldsymbol{s}}_0^0$ with $\boldsymbol{s}_0^0 \not\equiv_\mathcal{O} \top$ by $\hat{\boldsymbol{s}} \emptyset^b \hat{\boldsymbol{s}}_0^0$, for $\hat{\boldsymbol{s}} \in \mathcal{N}_{\boldsymbol{s}_0^0}$, if $k_0 = 0$, and by $\hat{\boldsymbol{s}}_0^0 \emptyset^b \hat{\boldsymbol{s}}_0^0$ if $k_0 > 0$.

If $k_i = 0$, $i > 0$, and $\boldsymbol{s}_0^i$ is meet-reducible wrt $\mathcal{O}$ within $\mathcal{Q}$, then we say that $\boldsymbol{s}_0^i$ is a *lone conjunct wrt $\mathcal{O}$ within $\mathcal{Q}$* in $\mathcal{D}$.

Lemma 7. *Assume $\mathcal{Q}, \mathcal{O}$, and $\boldsymbol{q}$ are as above. Let b exceed the number of $\Diamond$ and $\bigcirc$ in $\boldsymbol{q}$, let $\boldsymbol{q}$ be in normal form, and let $\mathcal{D}$ be b-normal without lone conjuncts wrt $\mathcal{O}$ within $\mathcal{Q}$. If $\mathcal{O}, \mathcal{D} \models \boldsymbol{q}$ but $\mathcal{O}, \mathcal{D}' \not\models \boldsymbol{q}$, for any $\mathcal{D}'$ obtained from $\mathcal{D}$ by applying any of the rules (a)–(e), then any root $\mathcal{O}$-homomorphim $h\colon \boldsymbol{q} \to \mathcal{D}$ is a root $\mathcal{O}$-isomorphism.*

Proof. We assume that $\boldsymbol{q}$ is constructed using $\boldsymbol{r}_j^i \in \mathcal{Q}$ and $\mathcal{D}$ is constructed using $\boldsymbol{s}_j^i \in \mathcal{Q}$. Let $\mathcal{O}, \mathcal{D} \models \boldsymbol{q}$. Take a root $\mathcal{O}$-homomorphism $h\colon \boldsymbol{q} \to \mathcal{D}$.

Suppose first that h is not block surjective. Since $h(0) = 0$, we find i, j with $(i, j) \neq (0, 0)$ such that the time point of $\hat{\boldsymbol{s}}^i_j$ is not in the range of h. If $\hat{\boldsymbol{s}}^i_j$ is not on the border of its block, that is $0 < j < k_i$, then we obtain from h a root $\mathcal{O}$-homomorphism into the data instance $\mathcal{D}'$ obtained from $\mathcal{D}$ by rule (b) and have derived a contradiction. If $0 = j$ or $k_i = j$, then we obtain from h a root $\mathcal{O}$-homomorphism into the data instance $\mathcal{D}'$ obtained from $\mathcal{D}$ by rule (a) applied to $\boldsymbol{s}^i_j$ and have derived a contradiction.

Assume next that h is block surjective but not data surjective. Then we find $\ell \in \mathit{ran}(h)$ with $\hat{\boldsymbol{s}}^i_j$ such that $\boldsymbol{s}^i_j \not\equiv_{\mathcal{O}} \top$ placed at ℓ such that $\mathcal{O}, \hat{\boldsymbol{s}} \models \boldsymbol{r}_\ell(a)$ for some $\boldsymbol{s} \in \mathcal{N}_{\boldsymbol{s}^i_j}$. But then h is a root $\mathcal{O}$-homomorphism into the data instance $\mathcal{D}'$ obtained from $\mathcal{D}$ by applying rule (a) to $\boldsymbol{s}^i_j$, that is, replacing $\hat{\boldsymbol{s}}^i_j$ by $\hat{\boldsymbol{s}}$ with $\boldsymbol{s} \in \mathcal{N}_{\boldsymbol{s}^i_j}$.

Suppose now that $h\colon \boldsymbol{q} \to \mathcal{D}$ is a block and data surjective, $(t \leq t') \in \boldsymbol{q}$ and $h(t) = h(t') = \ell$ lies in the ith block of $\mathcal{D}$. Then $h^{-1}(\ell) = \{t_1, \dots, t_k\}$ with $k \geq 2$ and $(t_j \leq t_{j+1}) \in \boldsymbol{q}$, $1 \leq j < k$. Let $\boldsymbol{r}_1, \dots, \boldsymbol{r}_k$ be the queries with $\boldsymbol{r}_j(t_j)$ in $\boldsymbol{q}$. As $\boldsymbol{q}$ satisfies **(n1)** and **(n2)**, there is j with $\boldsymbol{r}_j \not\equiv_{\mathcal{O}} \top$. Hence, $\boldsymbol{s}^i_{j_0} \not\equiv_{\mathcal{O}} \top$ for the query $\boldsymbol{s}^i_{j_0}$ with $\hat{\boldsymbol{s}}^i_{j_0}$ at ℓ in $\mathcal{D}$. Moreover, by data surjectivity, $\boldsymbol{r}_1 \wedge_{\mathcal{O}} \dots \wedge_{\mathcal{O}} \boldsymbol{r}_k \equiv_{\mathcal{O}} \boldsymbol{s}^i_{j_0}$. Consider possible locations of j_0 in its block.

Case 1: j_0 has both a left and a right neighbour in its block. Then there is $\mathcal{D}'$ obtained by (c)—i.e., by replacing $\hat{\boldsymbol{s}}^i_{j_0}$ with $\hat{\boldsymbol{s}}^i_{j_0}\emptyset^b\hat{\boldsymbol{s}}^i_{j_0}$—and a root $\mathcal{O}$-homomorphism $h'\colon \boldsymbol{q} \to \mathcal{D}'$, which 'coincides' with h except that $h'(t_1)$ is the point with the first $\hat{\boldsymbol{s}}^i_{j_0}$ and $h'(t_j)$, for $j = 2, \dots, k$, is the point with the second $\hat{\boldsymbol{s}}^i_{j_0}$.

Case 2: j_0 has no neighbours in its block and $i \neq 0$, so this block is primitive and $\boldsymbol{s}^i_{j_0}$ is not equivalent to a conjunction of queries as $\mathcal{D}$ has no lone conjuncts by our assumption. Observe that the blocks of $t_1, \dots, t_k$ are all different and primitive. As $\boldsymbol{s}^i_{j_0}$ is not equivalent to a conjunction of queries, we have

$$\boldsymbol{r}_1 \equiv_{\mathcal{O}} \dots \equiv_{\mathcal{O}} \boldsymbol{r}_k \equiv_{\mathcal{O}} \boldsymbol{s}^i_{j_0}.$$

However, $\boldsymbol{r}_j \not\equiv_{\mathcal{O}} \boldsymbol{r}_{j+1}$ by **(n3)** and **(n4)**. Thus, Case 2 cannot happen.

Case 3: j_0 has a left neighbour in its block but no right neighbour. Then $\boldsymbol{r}_1 \not\models_{\mathcal{O}} \boldsymbol{r}_2$ in view of **(n3)**, and so $\boldsymbol{r}_1 \not\models_{\mathcal{O}} \boldsymbol{s}^i_{j_0}$. As $\boldsymbol{s}^i_{j_0} \models_{\mathcal{O}} \boldsymbol{r}_1$, there is $\hat{\boldsymbol{s}} \in \mathcal{N}_{\boldsymbol{s}^i_{j_0}}$ with $\mathcal{O}, \hat{\boldsymbol{s}} \not\models_{\mathcal{O}} \boldsymbol{r}_1$. Let $\mathcal{D}'$ be obtained by the first part of (d) by replacing $\hat{\boldsymbol{s}}^i_{j_0}$ with $\hat{\boldsymbol{s}}\emptyset^b\hat{\boldsymbol{s}}^i_{j_0}$. Then there is a root $\mathcal{O}$-homomorphism $h'\colon \boldsymbol{q} \to \mathcal{D}'$ that sends t_1 to the point of $\boldsymbol{s}$ and the remaining t_j to the point of $\boldsymbol{s}^i_{j_0}$.

Case 4: j_0 has a right neighbour in its block, $i \neq 0$, and it has no left neighbour. This case is dual to Case 3 and we use the second part of (d).

Case 5: $i = 0$ and $j_0 = 0$. If block 0 is primitive, then all of the $\boldsymbol{r}_i(t_i)$ are primitive blocks in $\boldsymbol{q}$. By **(n3)**, $\boldsymbol{r}_1 \not\models_{\mathcal{O}} \boldsymbol{r}_2$, and so $\boldsymbol{r}_1 \not\models_{\mathcal{O}} \boldsymbol{s}^i_{j_0}$. Take $\hat{\boldsymbol{s}} \in \mathcal{N}_{\boldsymbol{s}^i_{j_0}}$. By the first part of (e), we have $\mathcal{D}'$ obtained by replacing $\hat{\boldsymbol{s}}^0_0$ with $\hat{\boldsymbol{s}}\emptyset^b\hat{\boldsymbol{s}}^0_0$. Then there is a root $\mathcal{O}$-homomorphism $h'\colon \boldsymbol{q} \to \mathcal{D}'$ that sends t_1 to the point of $\boldsymbol{s}$ and the remaining t_j to the point of $\boldsymbol{s}^0_0$.

Finally, if block 0 is not primitive, the second part of (e) gives $\mathcal{D}'$ by replacing $\hat{\boldsymbol{s}}^0_0$ in $\mathcal{D}$ with $\hat{\boldsymbol{s}}^0_0\emptyset^b\hat{\boldsymbol{s}}^0_0$. We obtain a root $\mathcal{O}$-homomorphism from $\boldsymbol{q}$ to $\mathcal{D}'$ by sending t_1 to the first $\hat{\boldsymbol{s}}^0_0$ and the remaining t_j to the second $\hat{\boldsymbol{s}}^0_0$. ☐

We can now prove Theorem 9. Suppose an ontology $\mathcal{O}$ admits characterisations $(\{\hat{\boldsymbol{s}}\}, \mathcal{N}_{\boldsymbol{s}})$ of queries $\boldsymbol{s}$ wrt $\mathcal{O}$ within a class of domain queries $\mathcal{Q}$. Let $\boldsymbol{q} \in \mathit{LTL}_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ in normal form wrt $\mathcal{O}$ take the form (8) with $\boldsymbol{q}_i$ of the form (9). We define an example set $E = (E^+, E^-)$ characterising $\boldsymbol{q}$ under the assumption that $\boldsymbol{q}$ has no lone conjuncts wrt $\mathcal{O}$. Let b be the number of $\bigcirc$ and $\Diamond$ in $\boldsymbol{q}$ plus 1. For every block $\boldsymbol{q}_i$ of the form (9), let $\hat{\boldsymbol{q}}_i$ be the temporal data instance

$$\hat{\boldsymbol{q}}_i = \hat{\boldsymbol{r}}^i_0\hat{\boldsymbol{r}}^i_1 \dots \hat{\boldsymbol{r}}^i_{k_i}.$$

For any two blocks $\boldsymbol{q}_i, \boldsymbol{q}_{i+1}$ such that $\boldsymbol{r}^i_{k_i} \wedge \boldsymbol{r}^{i+1}_0$ is satisfiable wrt $\mathcal{O}$, we take the temporal data instance

$$\hat{\boldsymbol{q}}_i \bowtie \hat{\boldsymbol{q}}_{i+1} = \hat{\boldsymbol{r}}^i_0 \dots \hat{\boldsymbol{r}}^i_{k_{i-1}} \widehat{\boldsymbol{r}^i_{k_i} \wedge \boldsymbol{r}^{i+1}_0} \hat{\boldsymbol{r}}^{i+1}_1 \dots \hat{\boldsymbol{r}}^{i+1}_{k_{i+1}}.$$

Now, the set E^+ contains the data instances given by

- $\mathcal{D}_b = \hat{\boldsymbol{q}}_0\emptyset^b \dots \hat{\boldsymbol{q}}_i\emptyset^b\hat{\boldsymbol{q}}_{i+1} \dots \emptyset^b\hat{\boldsymbol{q}}_n$,
- $\mathcal{D}_i = \hat{\boldsymbol{q}}_0\emptyset^b \dots (\hat{\boldsymbol{q}}_i \bowtie \hat{\boldsymbol{q}}_{i+1}) \dots \emptyset^b\hat{\boldsymbol{q}}_n$ if $\mathcal{R}_{i+1}$ is $\leq$,
- $\mathcal{D}_i = \hat{\boldsymbol{q}}_0\emptyset^b \dots \hat{\boldsymbol{q}}_i\emptyset^{n_{i+1}}\hat{\boldsymbol{q}}_{i+1} \dots \emptyset^b\hat{\boldsymbol{q}}_n$ otherwise.

Here, $\emptyset^b$ is a sequence of b-many $\emptyset$ and similarly for $\emptyset^{n_{i+1}}$. The set E^- contains all data instances of the form

- $\mathcal{D}^-_i = \hat{\boldsymbol{q}}_0\emptyset^b \dots \hat{\boldsymbol{q}}_i\emptyset^{n_{i+1}-1}\hat{\boldsymbol{q}}_{i+1} \dots \emptyset^b\hat{\boldsymbol{q}}_n$ if $n_{i+1} > 1$,
- $\mathcal{D}^-_i = \hat{\boldsymbol{q}}_0\emptyset^b \dots \hat{\boldsymbol{q}}_i \bowtie \hat{\boldsymbol{q}}_{i+1} \dots \emptyset^b\hat{\boldsymbol{q}}_n$ if $\mathcal{R}_{i+1}$ is a single $<$ and $\boldsymbol{r}^i_{k_i} \wedge \boldsymbol{r}^{i+1}_0$ is satisfiable wrt $\mathcal{O}$,
- the data instances obtained from $\mathcal{D}_b$ by applying to it each of the rules (a)–(e) in all possible ways exactly once.

We show that E characterises $\boldsymbol{q}$. Clearly, $\mathcal{O}, \mathcal{D} \models \boldsymbol{q}$ for all $\mathcal{D} \in E^+$. To establish $\mathcal{O}, \mathcal{D} \not\models \boldsymbol{q}$ for $\mathcal{D} \in E^-$, we need the following:

Claim 1. (i) *There is only one root $\mathcal{O}$-homomorphism $h\colon \boldsymbol{q} \to \mathcal{D}_b$, and it maps isomorphically each $\mathit{var}(\boldsymbol{q}_i)$ onto $I(\hat{\boldsymbol{q}}_i)$.*

(ii) *$\mathcal{O}, \mathcal{D}^-_i \not\models \boldsymbol{q}$, for any $\mathcal{R}_i$ different from $\leq$.*

(iii) *If $\mathcal{D}'_b$ is obtained from $\mathcal{D}_b$ by replacing some $\hat{\boldsymbol{q}}_i$ with $\hat{\boldsymbol{q}}'_i$ such that $\mathcal{O}, \hat{\boldsymbol{q}}'_i, \ell \not\models \boldsymbol{q}_i$ for any $\ell \leq \max(\hat{\boldsymbol{q}}'_i)$, then $\mathcal{O}, \mathcal{D}'_b \not\models \boldsymbol{q}$. In particular, $\mathcal{O}, \mathcal{D} \not\models \boldsymbol{q}$, for all $\mathcal{D} \in E^-$.*

Proof of claim. (i) Let h be a root $\mathcal{O}$-homomorphism. As $\boldsymbol{q}$ is in normal form and the gaps between $\hat{\boldsymbol{q}}_i$ and $\hat{\boldsymbol{q}}_{i+1}$ are not shorter than any block in $\boldsymbol{q}$, every $\mathit{var}(\boldsymbol{q}_i)$, where $\boldsymbol{q}_i$ is a block in $\boldsymbol{q}$, is mapped by h to a single $I(\hat{\boldsymbol{q}}_j)$, where $\hat{\boldsymbol{q}}_j$ is a block of D_b. Hence we can define a function $f\colon [0, n] \to [0, n]$ by setting $f(i) = j$ if $f(\mathit{var}(\boldsymbol{q}_i)) \subseteq I(\hat{\boldsymbol{q}}_j)$. Observe that $f(0) = 0$ and $i < j$ implies $f(i) \leq f(j)$. It also follows from the definition of the normal form that if $f(i) = i$, then h isomorphically maps $\mathit{var}(\boldsymbol{q}_i)$ onto $I(\hat{\boldsymbol{q}}_i)$ and $f(i-1) < i$ and $f(i+1) > i$ (observe that here (**n3**) and (**n4**) are required as they prohibit that $\mathit{var}(\boldsymbol{q}_i)$ and $\mathit{var}(\boldsymbol{q}_{i+1})$ are merged if $R_{i+1} =\leq$ and $\mathit{var}(\boldsymbol{q}_i)$ or $\mathit{var}(\boldsymbol{q}_{i+1})$ are a singleton). It remains to show that $f(i) = i$ for all i.

We first observe that $f(1) \geq 1$ and $f(j) = j$, for $j = \max\{i \mid f(i) \geq i\}$, from which again $f(j-1) < j$

and $f(j+1) > j$. Then we can proceed in the same way inductively by considering h and f restricted to the smaller intervals $[j,n]$ and $[0,j]$.

(ii) Suppose $\mathcal{R}_i$ is not $\leq$ but there is a root $\mathcal{O}$-homomorphism $h\colon \boldsymbol{q} \to \mathcal{D}_i^-$. Consider the location of $h(s_0^i) = \ell$. One can show similarly to (i) that $\ell \in I(\hat{\boldsymbol{q}}_j)$ for some $j \geq i$. Since $\boldsymbol{r}_{k_{i+1}}^{i+1} \not\equiv_{\mathcal{O}} \top$ and by the construction of $\mathcal{D}_i^-$, $h(s_0^{i+1})$ lies in some $I(\hat{\boldsymbol{q}}_j)$ with $j > i+1$. But then there is a root $\mathcal{O}$-homomorphism $h'\colon \boldsymbol{q} \to \mathcal{D}_b$ different from the one in (i), which is impossible.

(iii) is proved analogously. This completes the proof of the claim.

Now assume that $\boldsymbol{q}' \in \mathcal{Q}$ in normal form is given and $\boldsymbol{q}' \not\equiv_{\mathcal{O}} \boldsymbol{q}$. We have to show that $\boldsymbol{q}'$ does not fit E. If $\mathcal{O}, \mathcal{D}_b \not\models \boldsymbol{q}'$, we are done as $\mathcal{D}_b \in E^+$. Otherwise, let h be a root $\mathcal{O}$-homomorphism witnessing $\mathcal{O}, \mathcal{D}_b \models \boldsymbol{q}'$. If h is not a root $\mathcal{O}$-isomorphism, then by Lemma 7, there exists a data instance $\mathcal{D}$ obtained from $\mathcal{D}_b$ by applying one of the rules (a)–(e) such that $\mathcal{O}, \mathcal{D} \models \boldsymbol{q}'$. As $\mathcal{D} \in E^-$, we are done.

So suppose $h\colon \boldsymbol{q}' \to \mathcal{D}_b$ is a root $\mathcal{O}$-isomorphism. Then the difference between $\boldsymbol{q}'$ and $\boldsymbol{q}$ can only be in the sequences of $\Diamond$ and $\Diamond_r$ between blocks. To be more precise, $\boldsymbol{q}$ is of the form (8),

$$\boldsymbol{q}' = \boldsymbol{q}_0 \mathcal{R}_1' \boldsymbol{q}_1 \dots \mathcal{R}_n' \boldsymbol{q}_n \tag{14}$$

and $\mathcal{R}_i \neq \mathcal{R}_i'$ for some i. Four cases are possible:

– $\mathcal{R}_i = (r_0 \leq r_1)$ and $\mathcal{R}_i' = (s_0 < s_1) \dots (s_{l-1} < s_l)$, for $l \geq 1$. In this case, $\mathcal{O}, \mathcal{D}_i \not\models \boldsymbol{q}'$, for $\mathcal{D}_i \in E^+$.

– $\mathcal{R}_i = (r_0 < r_1) \dots (r_{k-1} < r_k)$, $\mathcal{R}_i' = (s_0 < s_1) \dots (s_{l-1} < s_l)$, for $l > k$. Then again $\mathcal{O}, \mathcal{D}_i \not\models \boldsymbol{q}'$.

– $\mathcal{R}_i = (r_0 < r_1) \dots (r_{k-1} < r_k)$, $\mathcal{R}_i' = (s_0 \leq s_1)$, for $k \geq 1$. In this case $\mathcal{O}, \mathcal{D}_i^- \models \boldsymbol{q}'$, for $\mathcal{D}_i^- \in E^-$. (Note that the compatibility condition is satisfied as $\boldsymbol{q}'$ is in normal form.)

– $\mathcal{R}_i = (r_0 < r_1) \dots (r_{k-1} < r_k)$ and $\mathcal{R}_i' = (s_0 < s_1) \dots (s_{l-1} < s_l)$, for $l < k$. Then again $\mathcal{O}, \mathcal{D}_i^- \models \boldsymbol{q}'$.

We now show the converse direction in Theorem 9 (i). Suppose $\boldsymbol{q}$ in normal form (8) does contain a lone conjunct $\boldsymbol{q}_i = \boldsymbol{r}$ wrt $\mathcal{O}$ within $\mathcal{Q}$. Let $\boldsymbol{r}^-$ be the last query of the block $\boldsymbol{q}_{i-1}$ and let $\boldsymbol{r}^+$ be the first query of the block $\boldsymbol{q}_{i+1}$.

Now let $\boldsymbol{r} \equiv_{\mathcal{O}} \boldsymbol{r}_1 \wedge \boldsymbol{r}_2$ and $\boldsymbol{r}_i \not\models_{\mathcal{O}} \boldsymbol{r}$, $i = 1,2$. Observe that, for $\boldsymbol{s} \in \{\boldsymbol{r}^-, \boldsymbol{r}^+\}$,

- $\boldsymbol{s} \wedge \boldsymbol{r}_i$ is satifiable wrt $\mathcal{O}$ if $\boldsymbol{r}^- \wedge \boldsymbol{r}$ is satisfiable wrt $\mathcal{O}$;
- if $\boldsymbol{s} \not\models_{\mathcal{O}} \boldsymbol{r}$, then $\boldsymbol{s} \not\models_{\mathcal{O}} \boldsymbol{r}_1$ or $\boldsymbol{s} \not\models_{\mathcal{O}} \boldsymbol{r}_2$.

Hence one of the queries $\boldsymbol{s}_1'$ or $\boldsymbol{s}_1''$ below is in normal form:

$$\boldsymbol{s}_1' = \boldsymbol{q}_0 \mathcal{R}_1 \dots \mathcal{R}_i \boldsymbol{s}_1 (\leq) \boldsymbol{s}_2 \mathcal{R}_{i+1} \dots \mathcal{R}_n \boldsymbol{q}_n,$$
$$\boldsymbol{s}_2' = \boldsymbol{q}_0 \mathcal{R}_1 \dots \mathcal{R}_i \boldsymbol{s}_1 (\leq) \boldsymbol{s}_2 (\leq) \boldsymbol{s}_1 \mathcal{R}_{i+1} \dots \mathcal{R}_n \boldsymbol{q}_n,$$

where $\{\boldsymbol{s}_1, \boldsymbol{s}_2\} = \{\boldsymbol{r}_1, \boldsymbol{r}_2\}$. Pick one of $\boldsymbol{s}_1'$ and $\boldsymbol{s}_2'$, which is in normal form, and denote it by $\boldsymbol{s}_1'$. For $n \geq 2$, let $\boldsymbol{s}_n'$ be the query obtained from $\boldsymbol{s}_1'$ by duplicating n times the part $\boldsymbol{s}_1(\leq)\boldsymbol{s}_2$ in $\boldsymbol{s}_1'$ and inserting $\leq$ between the copies. It is readily seen that $\boldsymbol{s}_n'$ is in normal form. Clearly, $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{s}_n'$ and, similarly to the proof of Claim 1, one can show that $\boldsymbol{s}_n' \not\models_{\mathcal{O}} \boldsymbol{q}$, for any $n \geq 1$.

Suppose $E = (E^+, E^-)$ characterises $\boldsymbol{q}$ and $n = \max\{\max(\mathcal{D}) \mid \mathcal{D} \in E^-\} + 1$. Then there exists $\mathcal{D} \in E^-$ with $\mathcal{O}, \mathcal{D} \models \boldsymbol{s}_n'$, so we have a root $\mathcal{O}$-homomorphism $h\colon \boldsymbol{s}_n' \to \mathcal{D}$. By the pigeonhole principle, h maps some variables of the queries $\boldsymbol{s}_1, \boldsymbol{s}_2$ in $\boldsymbol{s}_n'$ to the same point in $\mathcal{D}$. But then h can be readily modified to obtain a root $\mathcal{O}$-homomorphism $h'\colon \boldsymbol{q} \to \mathcal{D}$, which is a contradiction. This finishes the proof of (i).

(ii) follows from the proof of (i) as (E^+, E^-) is of polynomial size if the characterisations $(\{\hat{s}\}, \mathcal{N}_{\boldsymbol{s}})$ of the domain queries $\boldsymbol{s}$ wrt $\mathcal{O}$ within $\mathcal{Q}$ are of polynomial size.

(iii) We aim to characterise $\boldsymbol{q}$ in normal form (8), which may contain lone conjuncts wrt $\mathcal{O}$ within $\mathcal{Q}$ in the class of queries from $\mathit{LTL}_p^{\bigcirc\Diamond\Diamond_r}(\mathcal{Q})$ of temporal depth at most $n = \mathit{tdp}(\boldsymbol{q})$. We first observe a variation of Lemma 7. Extend the rules (a)–(e) by the following rule: if $\hat{s}$ is a block in $\mathcal{D}$ with $\boldsymbol{s}$ a lone conjunct in $\mathcal{D}$, then let $\mathcal{N}_{\boldsymbol{q}} = \{\boldsymbol{s}_1, \dots, \boldsymbol{s}_k\}$ with $\boldsymbol{s}_i \not\equiv_{\mathcal{O}} \boldsymbol{s}_j$, for $i \neq j$, and

(f$_n$) replace $\boldsymbol{s}$ with $(\boldsymbol{s}_1 \emptyset^b \cdots \emptyset^b \boldsymbol{s}_k)^n$.

By Lemma 5 (ii), $|\mathcal{N}_{\boldsymbol{q}}| \geq 2$. Now Lemma 7 still holds if we admit lone conjuncts in $\mathcal{D}$ but only consider $\boldsymbol{q}$ with at most n blocks and add rule (f$_n$) to (a)–(e). To see this, one only has to modify the argument for Case 2 in a straightforward way. With the above modification of Lemma 7, we continue as follows. The set E^+ of positive examples is defined as before. The set E^- of negative examples is defined by adding to the set E^- defined under (i) the results of applying (f$_n$) to $\mathcal{D}_b$ in all possible ways exactly once.

For the proof that (E^+, E^-) characterises $\boldsymbol{q}$ within the class of queries of temporal depth at most n, observe that $\mathcal{O}, \mathcal{D}' \not\models \boldsymbol{q}$ for the data instance $\mathcal{D}'$ obtained from $\mathcal{D}_b$ by applying (f$_n$).

(iv) Assume $\boldsymbol{q} \in \mathit{LTL}_p^{\bigcirc\Diamond}(\mathcal{Q})$ is given. The proof of (i) shows that (E^+, E^-), defined in the same way as in (i), characterises $\boldsymbol{q}$ wrt $\mathcal{O}$ within $\mathit{LTL}_p^{\bigcirc\Diamond}(\mathcal{Q})$ even if $\boldsymbol{q}$ contains lone conjuncts: the proof of Lemma 7 becomes much simpler as any block and type surjective root $\mathcal{O}$-homomorphism h is now a root $\mathcal{O}$-isomorphism. Note that therefore rules (c), (d), and (e) are not needed. This completes the proof of Theorem 9.

E Proofs for Section 7

Theorem 12. *Suppose $\mathcal{Q}$ is a class of domain queries, σ a signature, an ontology language $\mathcal{L}$ has general split-partners within $\mathcal{Q}^\sigma$, and $\mathcal{O}$ is a σ-ontology in $\mathcal{L}$ admitting containment reduction. Then the following hold*:

(i) *Every query $\boldsymbol{q} \in \mathit{LTL}_{pp}^{\mathsf{U}}(\mathcal{Q}^\sigma)$ is uniquely characterisable wrt $\mathcal{O}$ within $\mathit{LTL}_p^{\mathsf{U}}(\mathcal{Q}^\sigma)$.*

(ii) *If a split-partner for any set Θ, $|\Theta| \leq 2$, of $\mathcal{Q}^\sigma$ queries wrt $\mathcal{O}$ within $\mathcal{Q}^\sigma$ is exponential, then there is an exponential-size unique characterisation of $\boldsymbol{q}$ wrt $\mathcal{O}$.*

(iii) *If a split-partner of any set Θ as above is polynomial and a split-partner $\mathcal{S}_\bot$ of $\bot(x)$ within $\mathcal{Q}^\sigma$ wrt $\mathcal{O}$ is a single-*

ton, then there is a polynomial-size unique characterisation of $\boldsymbol{q}$ wrt $\mathcal{O}$.

The proof is by reduction to the ontology-free propositional *LTL* case. Namely, we require the following result proved in (Fortin et al. 2022), where $\mathcal{P}$ is the class of propositional queries (conjunctions of unary atoms:

Theorem 17 (Fortin et al. 2022)**.** *$LTL^{\mathsf{U}}_{pp}(\mathcal{P}^\sigma)$ is polysize characterisable within $LTL^{\mathsf{U}}_{p}(\mathcal{P}^\sigma)$ wrt the empty ontology, with the characterisation defined below*:

Let $\boldsymbol{s} \in \mathcal{P}^\sigma \cup \{\bot\}$. We treat each such $\boldsymbol{s} \neq \bot$ as a set of its conjuncts and define $\bar{\boldsymbol{s}} = \{A(a) \mid A(x) \in \boldsymbol{s}\}$. For $\boldsymbol{s} = \bot$, we set $\bar{\boldsymbol{s}} = \varepsilon$, where ε is the empty word in the sense that $\varepsilon\mathcal{D} = \mathcal{D}$, for any data instance $\mathcal{D}$, and $\varepsilon\varepsilon = \varepsilon$. Consider $\boldsymbol{q} \in LTL^{\mathsf{U}}_{pp}(\mathcal{P}^\sigma)$ of the form (5)*. Then $\boldsymbol{q}$ is uniquely characterised within $LTL^{\mathsf{U}}_{p}(\mathcal{P}^\sigma)$ by the example set $E = (E^+, E^-)$, where E^+ contains all data instances of the following forms*:

($\mathfrak{p}_0$) $\bar{\boldsymbol{r}}_0 \dots \bar{\boldsymbol{r}}_n$,

($\mathfrak{p}_1$) $\bar{\boldsymbol{r}}_0 \dots \bar{\boldsymbol{r}}_{i-1}\bar{\boldsymbol{l}}_i\bar{\boldsymbol{r}}_i \dots \bar{\boldsymbol{r}}_n$,

($\mathfrak{p}_2$) $\bar{\boldsymbol{r}}_0 \dots \bar{\boldsymbol{r}}_{i-1}\bar{\boldsymbol{l}}_i^k\bar{\boldsymbol{r}}_i \dots \bar{\boldsymbol{r}}_{j-1}\bar{\boldsymbol{l}}_j\bar{\boldsymbol{r}}_j \dots \bar{\boldsymbol{r}}_n$, *for $i < j$, and $k = 1, 2$ (where $\bar{\boldsymbol{l}}_i^k$ is a sequence of k-many $\bar{\boldsymbol{l}}_i$)*;

and E^- contains all instances $\mathcal{D}$ with $\mathcal{D} \not\models \boldsymbol{q}$ of the forms:

($\mathfrak{n}_0$) *$\bar{\sigma}^n$ and $\bar{\sigma}^{n-i}\overline{\sigma \setminus \{A\}}\bar{\sigma}^i$, for $A(x) \in \boldsymbol{r}_i$ (here, the whole σ is regarded as a query),*

($\mathfrak{n}_1$) *$\bar{\boldsymbol{r}}_0 \dots \bar{\boldsymbol{r}}_{i-1}\overline{\sigma \setminus X}\bar{\boldsymbol{r}}_i \dots \bar{\boldsymbol{r}}_n$, for $X = \{A, B\}$ with $A(x) \in \boldsymbol{l}_i$, $B(x) \in \boldsymbol{r}_i$, $X = \emptyset$, and $X = \{A\}$ with $A(x) \in \boldsymbol{l}_i$,*

($\mathfrak{n}_2$) *for all i and $A(x) \in \boldsymbol{l}_i \cup \{\bot(x)\}$,* some *data instance*

$$\mathcal{D}^i_A = \bar{\boldsymbol{r}}_0 \dots \bar{\boldsymbol{r}}_{i-1}(\overline{\sigma \setminus \{A\}})\bar{\boldsymbol{r}}_i\bar{\boldsymbol{l}}_{i+1}^{k_{i+1}} \dots \bar{\boldsymbol{l}}_n^{k_n}\bar{\boldsymbol{r}}_n, \tag{15}$$

if any, such that $\max(\mathcal{D}^i_A) \leq (n+1)^2$ and $\mathcal{D}^i_A \not\models \boldsymbol{q}^\dagger$ for $\boldsymbol{q}^\dagger$ obtained from $\boldsymbol{q}$ by replacing $\boldsymbol{l}_j$, for all $j \leq i$, with $\bot$. (Note that $\mathcal{D}^i_A \not\models \boldsymbol{q}$ for peerless $\boldsymbol{q}$.)

Returning to the proof of Theorem 12, assume a signature σ, an ontology $\mathcal{O}$ in σ admitting containment reduction and general split-partners within $\mathcal{Q}^\sigma$, and a $\boldsymbol{q} \in LTL^{\mathsf{U}}_{pp}(\mathcal{Q}^\sigma)$ of the form (5) are given. We may assume that $\boldsymbol{r}_n \not\equiv_{\mathcal{O}} \top$. We obtain the set E^+ of positive examples by taking the following data instances:

($\mathfrak{p}'_0$) $\hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_n$,

($\mathfrak{p}'_1$) $\hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1}\hat{\boldsymbol{l}}_i\hat{\boldsymbol{r}}_i \dots \hat{\boldsymbol{r}}_n = \mathcal{D}^i_{\boldsymbol{q}}$,

($\mathfrak{p}_2$)′ $\hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1}\hat{\boldsymbol{l}}_i^k\hat{\boldsymbol{r}}_i \dots \hat{\boldsymbol{r}}_{j-1}\hat{\boldsymbol{l}}_j\hat{\boldsymbol{r}}_j \dots \hat{\boldsymbol{r}}_n = \mathcal{D}^j_{i,k}$, for $i < j$ and $k = 1, 2$.

We obtain the set E^- of negative examples by taking the following data instances $\mathcal{D}$ whenever $\mathcal{D} \not\models \boldsymbol{q}$:

($\mathfrak{n}'_0$) $\mathcal{A}_1, \dots, \mathcal{A}_n$ and $\mathcal{A}_1, \dots, \mathcal{A}_{n-i}, \mathcal{A}, \mathcal{A}_{n-i+1}, \dots, \mathcal{A}_n$, for $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{r}_i\})$ and $(\mathcal{A}_1, a), \dots, (\mathcal{A}_n, a) \in \mathcal{S}_\bot$;

($\mathfrak{n}'_1$) $\hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1}\mathcal{A}\hat{\boldsymbol{r}}_i \dots \hat{\boldsymbol{r}}_n$, where $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{l}_i, \boldsymbol{r}_i\}) \cup \mathcal{S}(\{\boldsymbol{l}_i\}) \cup \mathcal{S}_\bot$;

($\mathfrak{n}'_2$) for all i and $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{l}_i, \boldsymbol{r}_i\}) \cup \mathcal{S}(\{\boldsymbol{l}_i\}) \cup \mathcal{S}_\bot$, *some* data instance

$$\mathcal{D}^i_{\mathcal{A}} = \hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1}\mathcal{A}\hat{\boldsymbol{r}}_i\hat{\boldsymbol{l}}_{i+1}^{k_{i+1}}\hat{\boldsymbol{r}}_{i+1} \dots \hat{\boldsymbol{l}}_n^{k_n}\boldsymbol{r}_n,$$

if any, such that $\max(\mathcal{D}^i_{\mathcal{A}}) \leq (n+1)^2$ and $\mathcal{D}^i_{\mathcal{A}} \not\models \boldsymbol{q}^\dagger$ for $\boldsymbol{q}^\dagger$ obtained from $\boldsymbol{q}$ by replacing all $\boldsymbol{l}_j$, for $j \leq i$, with $\bot$.

We show now that $\boldsymbol{q}$ is uniquely characterised by the constructed example set $E = (E^+, E^-)$ wrt $\mathcal{O}$ within $LTL^{\mathsf{U}}_{p}(\mathcal{Q}^\sigma)$. Consider any query

$$\boldsymbol{q}' = \boldsymbol{r}'_0 \land (\boldsymbol{l}'_1 \mathbin{\mathsf{U}} (\boldsymbol{r}'_1 \land (\boldsymbol{l}'_2 \mathbin{\mathsf{U}} (\dots (\boldsymbol{l}'_m \mathbin{\mathsf{U}} \boldsymbol{r}'_m) \dots))))$$

in $LTL^{\mathsf{U}}_{p}(\mathcal{Q}^\sigma)$ such that $\boldsymbol{q}' \not\equiv_{\mathcal{O}} \boldsymbol{q}$. We can again assume that $\boldsymbol{r}'_m \not\equiv_{\mathcal{O}} \top$. Thus, in what follows we can safely ignore what the ontology $\mathcal{O}$ entails after the timepoint $\max\mathcal{D}$, for any database $\mathcal{D}$, as these points do not contribute to entailment of $\boldsymbol{q}$ or $\boldsymbol{q}'$. In order to show that $\boldsymbol{q}$ fits E and $\boldsymbol{q}'$ does not fit E, we need a few definitions.

We define a map f that reduces the 2D case to the 1D case. Consider the alphabet

$$\Gamma = \{\boldsymbol{r}_0, \dots, \boldsymbol{r}_n, \boldsymbol{l}_1, \dots, \boldsymbol{l}_n, \boldsymbol{r}'_0, \dots, \boldsymbol{r}_m, \boldsymbol{l}'_1, \dots, \boldsymbol{l}'_m, \} \setminus \{\bot\},$$

in which we regard the CQs $\boldsymbol{r}_i, \boldsymbol{l}_i, \boldsymbol{r}'_j, \boldsymbol{l}'_j$ as symbols. Let $\hat{\Gamma} = \{(\hat{\boldsymbol{a}}, a) \mid \boldsymbol{a} \in \Gamma\}$, that is, $\hat{\Gamma}$ consists of the pointed databases corresponding to the CQs $\boldsymbol{a} \in \Gamma$. For any CQ $\boldsymbol{a}$, we set

$$f(\boldsymbol{a}) = \{\boldsymbol{b}(x) \mid \boldsymbol{b} \in \Gamma \text{ and } \mathcal{O}, \hat{\boldsymbol{a}} \models \boldsymbol{b}(a)\}.$$

Similarly, for any pointed data instance $(\mathcal{A}, a)$, we set

$$f(\mathcal{A}, a) = \{\boldsymbol{b}(x) \mid \boldsymbol{b} \in \Gamma \text{ and } \mathcal{O}, \mathcal{A} \models \boldsymbol{b}(a)\}$$

and, for any temporal data instance $\mathcal{D} = \mathcal{A}_0, \dots, \mathcal{A}_k$ with a point a, set

$$f(\mathcal{D}, a) = (f(\mathcal{A}_0, a), \dots, f(\mathcal{A}_k, a)),$$

which is a temporal data instance over the signature Γ. Finally, we define an $LTL^{\mathsf{U}}_{p}(\mathcal{P}^\Gamma)$-query

$$f(\boldsymbol{q}) = \rho_0 \land (\lambda_1 \mathbin{\mathsf{U}} (\rho_1 \land (\lambda_2 \mathbin{\mathsf{U}} (\dots (\lambda_n \mathbin{\mathsf{U}} \rho_n) \dots))))$$

by taking $\rho_i = f(\boldsymbol{r}_i)$ and $\lambda_i = f(\boldsymbol{l}_i)$, and similarly for $\boldsymbol{q}'$. It follows immediately from the definition that, for any data instance $\mathcal{D}$, we have $\mathcal{O}, \mathcal{D} \models \boldsymbol{q}$ iff $f(\mathcal{D}, a) \models f(\boldsymbol{q})$ and $\mathcal{O}, \mathcal{D} \models f(\boldsymbol{q}')$ iff $f(\mathcal{D}, a) \models f(\boldsymbol{q}')$.

We first observe that $f(\boldsymbol{q})$ is a peerless $LTL^{\mathsf{U}}_{p}(\mathcal{P}^\Gamma)$-query: indeed, since $\mathcal{O}, \hat{\boldsymbol{r}}_i \not\models \boldsymbol{l}_i(a)$, we have $\boldsymbol{l}_i \in f(\boldsymbol{l}_i) \setminus f(\boldsymbol{r}_i)$, and since $\mathcal{O}, \hat{\boldsymbol{l}}_i \not\models \boldsymbol{r}_i(a)$, we have $\boldsymbol{r}_i \in f(\boldsymbol{r}_i) \setminus f(\boldsymbol{l}_i)$. It follows that $f(\boldsymbol{q}) \not\equiv f(\boldsymbol{q}')$

Let $E_{\text{prop}} = (E^+_{\text{prop}}, E^-_{\text{prop}})$ be the example set defined for $f(\boldsymbol{q})$ using ($\mathfrak{p}_0$)–($\mathfrak{p}_2$) and ($\mathfrak{n}_0$)–($\mathfrak{n}_2$). By Theorem 17, $f(\boldsymbol{q})$ fits E_{prop} and $f(\boldsymbol{q}')$ does not fit E_{prop}.

A *satisfying root $\mathcal{O}$-homomorphism* for any query

$$\boldsymbol{r}_0 \land (\boldsymbol{l}_1 \mathbin{\mathsf{U}} (\boldsymbol{r}_1 \land (\boldsymbol{l}_2 \mathbin{\mathsf{U}} (\dots (\boldsymbol{l}_n \mathbin{\mathsf{U}} \boldsymbol{r}_n) \dots))))$$

in $\mathcal{D}, a = (\mathcal{A}_0, a) \dots, (\mathcal{A}_k, a)$ is a map h from $\{0, \dots, n\}$ to $\mathbb{N}$ such that $h(0) = 0$ and $h(i) < h(i+1)$ for $i < n$ and

- $\mathcal{O}, \mathcal{A}_{f(i)} \models \boldsymbol{r}_i(a)$;

- $\mathcal{O}, \mathcal{A}_{i'} \models \boldsymbol{l}_i(a)$ for all $i' \in (f(i), f(i+1))$.

Clearly, such a root $\mathcal{O}$-homomorphism exists iff the query is satisfied in $\mathcal{D}, a$. If the query is in $LTL_p^{\mathsf{U}}(\mathcal{P}^\sigma)$ and $\mathcal{O}$ is empty, then we call the homomorphism above a *satisfying homomorphism*.

We are now in a position to show that $\boldsymbol{q}$ fits E but $\boldsymbol{q}'$ does not fit E. It is immediate from the definitions that $\boldsymbol{q}$ fits E. So we show that $\boldsymbol{q}'$ does not fit E.

Assume first that $f(\boldsymbol{q}')$ is not entailed by some example in E_p^+. Then $\boldsymbol{q}'$ is not entailed by some example in E^+ as the examples from $(\mathfrak{p}_0)$–$(\mathfrak{p}_2)$ are exactly the f-images of the examples $(\mathfrak{p}_0')$–$(\mathfrak{p}_2')$.

Assume now that $f(\boldsymbol{q}')$ is entailed by all data instances in E^+_{prop} and is also entailed by some $\mathcal{D}$ from E^-_{prop}. We show that then there is a data instance in E^- that entails $\boldsymbol{q}'$ under $\mathcal{O}$.

If $\mathcal{D} = \Gamma^n$, then it follows that the temporal depth of $f(\boldsymbol{q}')$ is less than the temporal depth of $f(\boldsymbol{q})$. Then $m < n$ and the query $\boldsymbol{q}'$ is entailed by some $\mathcal{A}_1, \dots, \mathcal{A}_n \in E^-$ with $(\mathcal{A}_i, a) \in \mathcal{S}_\perp$: we obtain $\mathcal{A}_i$ by taking $(\mathcal{A}_i, a) \in \mathcal{S}_\perp$ such that $\mathcal{O}, \mathcal{A}_i \models \boldsymbol{r}_i'(a)$.

Suppose $\mathcal{D} = \Gamma^{n-i}(\Gamma \setminus \{\boldsymbol{a}\})\Gamma^i \models f(\boldsymbol{q}')$. Observe that the only satisfying homomorphim that witnesses this is the identity mapping. So we have $f(\boldsymbol{r}_{n-i}) \not\subseteq \Gamma \setminus \{\boldsymbol{a}\}$ and therefore $\mathcal{O}, \hat{\boldsymbol{r}}_{n-i} \models \boldsymbol{a}(a)$ but $f(\boldsymbol{r}_{n-i}') \subseteq \Gamma \setminus \{\boldsymbol{a}\}$. Then $\mathcal{O}, \hat{\boldsymbol{r}}_{n-i}' \not\models \boldsymbol{a}(a)$, and so $\boldsymbol{r}_{n-i}' \not\models_{\mathcal{O}} \boldsymbol{r}_i$. Therefore, there is $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{r}_{n-i}\})$ such that $\mathcal{O}, \mathcal{A} \models \boldsymbol{r}_{n-i}'(a)$. Observe that also $\mathcal{O}, \mathcal{A} \not\models \boldsymbol{r}_{n-i}(a)$.

Now take, for any $j \neq n-i$, some $(\mathcal{A}_j, a) \in \mathcal{S}_\perp$ with $\mathcal{O}, \mathcal{A}_j \models \boldsymbol{r}_j'$. Then

$$\mathcal{O}, \mathcal{A}_0 \cdots \mathcal{A}_{n-i-1}\mathcal{A}\mathcal{A}_{n-i+1} \cdots \mathcal{A}_n \not\models \boldsymbol{q},$$
$$\mathcal{O}, \mathcal{A}_0 \cdots \mathcal{A}_{n-i-1}\mathcal{A}\mathcal{A}_{n-i+1} \cdots \mathcal{A}_n \not\models \boldsymbol{q}'.$$

Assume next that $\mathcal{D}$ is from $(\mathfrak{n}_1)$. We have $\mathcal{D} \models f(\boldsymbol{q}')$ and $\mathcal{D} \not\models f(\boldsymbol{q})$. As $\mathcal{D} \models f(\boldsymbol{q}')$, we have a satisfying homomorphism h for $f(\boldsymbol{q}')$ in $\mathcal{D}$.

If there is j such that $h(j) = i$, then let $\boldsymbol{r} = \boldsymbol{r}_j$. Otherwise, there is j such that $h(j) < i < h(j+1)$. Then let $\boldsymbol{r} = \boldsymbol{l}_j$. In both cases $f(\boldsymbol{r}) \subseteq Y$, where Y depends on $\mathcal{D}$ and is either:

1. $\Gamma \setminus \{\boldsymbol{a}, \boldsymbol{b}\}$ with $\mathcal{O}, \hat{\boldsymbol{l}}_i \models \boldsymbol{a}(a)$ and $\mathcal{O}, \hat{\boldsymbol{r}}_i \models \boldsymbol{b}(a)$ or
2. $\Gamma \setminus \{\boldsymbol{a}\}$ with $\mathcal{O}, \hat{\boldsymbol{l}}_i \models \boldsymbol{a}(a)$ or
3. Γ (only if $\boldsymbol{l}_i = \perp$) or
4. $\Gamma \setminus \{\boldsymbol{b}\}$.

Case 1. We have $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{a}(a)$ and $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{b}(a)$. Hence $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{l}_i(a)$ and $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{r}_i(a)$. By the definition of split-partners, there exists $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{l}_i, \boldsymbol{r}_i\})$ such that $\mathcal{O}, \mathcal{A} \models \boldsymbol{r}(a)$. But then h is also a satisfying root $\mathcal{O}$-homomorphism in $\mathcal{D}', a$ witnessing that $\boldsymbol{q}'$ is entailed by $\mathcal{D}' = \hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1}\mathcal{A}\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_{i+1} \dots \boldsymbol{r}_n$ wrt $\mathcal{O}$.

It remains to show that $\mathcal{O}, \mathcal{D}' \not\models \boldsymbol{q}$. Assume otherwise. Take a satisfying root $\mathcal{O}$-homomorphism h^* witnessing $\mathcal{O}, \mathcal{D}' \models \boldsymbol{q}$. By peerlessness of $\boldsymbol{q}$, $h^*(j) = j$ for all $j < i$. But then $\mathcal{O}, \mathcal{A} \models \boldsymbol{l}_i$ or $\mathcal{O}, \mathcal{A} \models \boldsymbol{r}_i$ which both contradict to $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{l}_i, \boldsymbol{r}_i\})$.

Case 2. We have $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{a}(a)$. Hence $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{l}_i(a)$. We now distinguish two cases. If also $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{r}_i(a)$, then we proceed as in the previous case and choose a split-partner $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{l}_i, \boldsymbol{r}_i\})$ such that $\mathcal{O}, \mathcal{A} \models \boldsymbol{r}(a)$. We proceed as in Case 1.

If $\mathcal{O}, \hat{\boldsymbol{r}} \models \boldsymbol{r}_i(a)$, then we proceed as follows. Choose a split-partner $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{l}_i\})$ such that $\mathcal{O}, \mathcal{A} \models \boldsymbol{r}(a)$. Then h is also a satisfying root $\mathcal{O}$-homomorphism in $\mathcal{D}', a$ witnessing that $\boldsymbol{q}'$ is entailed by $\mathcal{D}' = \hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1}\mathcal{A}\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_{i+1} \dots \boldsymbol{r}_n$ wrt $\mathcal{O}$.

It remains to show that $\mathcal{O}, \mathcal{D}' \not\models \boldsymbol{q}$. Assume otherwise. Take a satisfying root $\mathcal{O}$-homomorphism h^* in $\mathcal{D}', a$ witnessing $\mathcal{D}' \models \boldsymbol{q}$. By peerlessness of $\boldsymbol{q}$, $h^*(j) = j$ for all $j < i$. Then $h^*(i) = i$ as $\mathcal{O}, \mathcal{A} \models \boldsymbol{l}_i$ would contradict $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{l}_i\})$. But then h^* is a satisfying homomorphism in $\mathcal{D}, a$ witnessing $\mathcal{D} \models f(\boldsymbol{q})$ and we have derived a contradiction.

Case 3. We set $\mathcal{D}' = \hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1}\mathcal{A}\hat{\boldsymbol{r}}_i\hat{\boldsymbol{l}}_{i+1}^{k_{i+1}}\hat{\boldsymbol{r}}_{i+1} \dots \hat{\boldsymbol{l}}_n^{k_n}\boldsymbol{r}_n$ for some $(\mathcal{A}, a) \in \mathcal{S}_\perp$ with $\mathcal{O}, \mathcal{A} \models \boldsymbol{r}_i'$. It directly follows from $\mathcal{D} \models f(\boldsymbol{q}')$ that $\mathcal{O}, \mathcal{D}' \models \boldsymbol{q}'$ and also from $\mathcal{D} \not\models f(\boldsymbol{q})$ that $\mathcal{O}, \mathcal{D}' \not\models \boldsymbol{q}$.

Case 4. We have $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{b}(a)$. Hence $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{r}_i(a)$. We distinguish two cases. If also $\mathcal{O}, \hat{\boldsymbol{r}} \not\models \boldsymbol{l}_i(a)$, then we proceed as in Case 1 and choose split-partner $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{l}_i, \boldsymbol{r}_i\})$ such that $\mathcal{O}, \mathcal{A} \models \boldsymbol{r}(a)$.

If $\mathcal{O}, \hat{\boldsymbol{r}} \models \boldsymbol{l}_i(a)$, then we proceed as follows. Choose a split-partner $(\mathcal{A}, a) \in \mathcal{S}(\{\boldsymbol{r}_i\})$ such that $\mathcal{O}, \mathcal{A} \models \boldsymbol{r}(a)$. Then h is also a satisfying root $\mathcal{O}$-homomorphism in $\mathcal{D}', a$ witnessing that $\boldsymbol{q}'$ is entailed by $\mathcal{D}' = \hat{\boldsymbol{r}}_0 \dots \hat{\boldsymbol{r}}_{i-1}\mathcal{A}\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_{i+1} \dots \boldsymbol{r}_n$ wrt $\mathcal{O}$.

It remains to show that $\mathcal{O}, \mathcal{D}' \not\models \boldsymbol{q}$. Assume otherwise. Take a satisfying root $\mathcal{O}$-homomorphism h^* in $\mathcal{D}', a$ witnessing $\mathcal{O}, \mathcal{D}' \models \boldsymbol{q}$. By peerlessness of $\boldsymbol{q}$, $h^*(j) = j$ for all $j < i$. Then $h^*(i) > i$ as $\mathcal{O}, \mathcal{A} \models \boldsymbol{r}_i(a)$ would contradict the definition of $\mathcal{A}$. But then, as $\Gamma \setminus \{\boldsymbol{b}\} \models f(\boldsymbol{l}_i)$, h^* is also a satisfying homomorphism in $\mathcal{D}, a$ witnessing $\mathcal{D} \models f(\boldsymbol{q})$ and we have derived a contradiction.

The case when $\mathcal{D}$ is from $(\mathfrak{n}_2)$ is considered similarly to the case of $(\mathfrak{n}_1)$.

Theorem 11. *There exist a DL-Lite$_{\mathcal{F}}^-$ ontology $\mathcal{O}$, a signature σ and a query $\boldsymbol{q} \in LTL_{pp}^{\mathsf{U}}(ELIQ^\sigma)$ such that $\boldsymbol{q}$ is not uniquely characterisable wrt $\mathcal{O}$ within $LTL_p^{\mathsf{U}}(ELIQ^\sigma)$.*

Proof. Consider the ontology

$$\mathcal{O} = \{\text{fun}(P), \text{fun}(P^-), B \sqcap \exists P^- \sqsubseteq \perp\}$$

from the proof of Theorem 8. We know from (Funk, Jung, and Lutz 2022b) and that proof that $\mathcal{O}$ admits frontiers within $\text{ELIQ}^{\{A,B,P\}}$ but not split-partners. We show that the query $\bigcirc A$ is not uniquely characterisable wrt $\mathcal{O}$ within $LTL_p^{\mathsf{U}}(\text{ELIQ}^{\{A,B,P\}})$. Indeed, suppose $E = (E^+, E^-)$ is such a unique characterisation.

Consider the following set of pointed data instances:

$$\mathcal{S}(\{A\}) = \{(\mathcal{A}_i, a) \mid i > 0,\ \exists \mathcal{D} = \mathcal{A}_0, \dots, \mathcal{A}_n \in E^-,$$
$$\mathcal{O}, \mathcal{A}_j \not\models A(a) \text{ for } 0 < j \leq i\}.$$

We claim that the defined $\mathcal{S}(\{A\})$ is a split-partner for $\{A\}$ within $\text{ELIQ}^{\{A,B,P\}}$, which is a contradiction.

Take any $\boldsymbol{q}' \in \text{ELIQ}^{\{A,B,P\}}$. If $\mathcal{O}, \mathcal{A} \models \boldsymbol{q}'(a)$, for some $(\mathcal{A}, a) \in \mathcal{S}(\{A\})$, then $\boldsymbol{q}' \not\models_{\mathcal{O}} \bigcirc A$ because otherwise $\mathcal{O}, \mathcal{A} \models A(a)$ which is not the case by definition of $\mathcal{S}(\{A\})$.

Now suppose $\mathcal{O}, \mathcal{A} \not\models \boldsymbol{q}'(a)$ for all $(\mathcal{A}, a) \in \mathcal{S}(\{A\})$. Then $\mathcal{D} \not\models \boldsymbol{q}' \cup A$ for all $\mathcal{D}$ of the form $\mathcal{A}_0, \dots, \mathcal{A}_n$ in E^- with $\mathcal{O}, \mathcal{A}_1 \not\models A(a)$. Hence $\mathcal{D} \not\models \boldsymbol{q}' \cup A$ for all $\mathcal{D} \in E^-$. On the other hand, from $\bigcirc A \models \boldsymbol{q}' \cup A$ we obtain $\mathcal{D} \models \boldsymbol{q}' \cup A$ for all $\mathcal{D} \in E^+$, and so $\boldsymbol{q}' \cup A$ is equivalent to $\bigcirc A$ wrt $\mathcal{O}$. By the shape of $\mathcal{O}$, this implies that $\boldsymbol{q}'$ is equivalent to $\bot$, and so $\boldsymbol{q}' \models_{\mathcal{O}} A$, as required by the definition of split-partners. □

F Proofs for Section 8

This section is mainly devoted to give a full proof of Theorem 15, but we need some preparation.

F.1 Normal Form

In order to lift some results obtained in the atemporal case (Funk, Jung, and Lutz 2022a) to the temporal setting, we have to rely on the same normal form for ontologies. An $\mathcal{ELHIF}$ ontology is in *normal form* if every concept inclusion takes one of the following forms:

$$A \sqsubseteq \exists R.A', \qquad \exists R.A \sqsubseteq A', \qquad A \sqcap A' \sqsubseteq B,$$

where A, A' are concept names or $\top$, B is a concept name or $\bot$, and R is a role.

We describe next how to convert an $\mathcal{ELHIF}$ ontology $\mathcal{O}$ into an $\mathcal{ELHIF}$ ontology $\mathcal{O}'$ in normal form. Let us use $\mathfrak{C}(\mathcal{O})$ to denote the set of all concepts that occur in a concept inclusion in $\mathcal{O}$. Note that $\mathfrak{C}(\mathcal{O})$ is closed under taking subconcepts. We introduce a fresh concept name X_C for every complex concept $C \in \mathfrak{C}(\mathcal{O})$, and set $X_\bot = \bot$ and $X_A = A$ for concept names $A \in \mathfrak{C}(\mathcal{O})$. The ontology $\mathcal{O}'$ consists of all functionality assertions and all role inclusions in $\mathcal{O}$ and the following concept inclusions:

- $X_C \sqsubseteq X_D$ for every $C \sqsubseteq D \in \mathcal{O}$;
- $X_{D_1 \sqcap D_2} \sqsubseteq X_{D_i}$ and $X_{D_1} \sqcap X_{D_2} \sqsubseteq X_{D_1 \sqcap D_2}$, for every $D_1 \sqcap D_2 \in \mathfrak{C}(\mathcal{O})$ and $i \in \{1, 2\}$;
- $X_{\exists R.C} \sqsubseteq \exists R.X_C$ and $\exists R.X_C \sqsubseteq X_{\exists R.C}$, for every $\exists R.C \in \mathfrak{C}(\mathcal{O})$.

Clearly, $\mathcal{O}'$ can be computed in polynomial time. Regarding the relationship between $\mathcal{O}$ and $\mathcal{O}'$, we observe the following consequences of the definition of $\mathcal{O}'$.

Lemma 8.

1. *$\mathcal{O}'$ is a conservative extension of $\mathcal{O}$;*
2. *$sig(\mathcal{O}') = sig(\mathcal{O}) \cup \{X_C \mid C \in \mathfrak{C}(\mathcal{O})\}$;*
3. *$\mathcal{O}' \models X_C \equiv C$, for all $C \in \mathfrak{C}(\mathcal{O})$.*

Lemma 8 essentially says that $\mathcal{O}'$ is a conservative extension of $\mathcal{O}$, but is slightly stronger in also making precise how exactly a model of $\mathcal{O}$ can be extended to a model of $\mathcal{O}'$.

We next show that it suffices to provide learning algorithms wrt ontologies in normal form.

Lemma 9. *Let $\mathcal{L}$ be an ontology language contained in $\mathcal{ELHI}$ or $\mathcal{ELIF}$. If a class $\mathcal{Q} \subseteq$ LTL$_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) of queries is polynomial query learnable wrt $\mathcal{ELHIF}$ ontologies in normal form using membership queries, then the same is true for $\mathcal{L}$ ontologies. If, additonally, $\mathcal{L}$ admits polynomial time instance checking, then even polynomial time learnability is preserved.*

Proof. Let L' be a polynomial time learning algorithm for $\mathcal{Q}$ wrt ontologies in normal form. We transform it into a polynomial time learning algorithm L for $\mathcal{Q}$ wrt unrestricted $\mathcal{ELIF}$ ontologies, relying on the normal form provided by Lemma 8. The construction for $\mathcal{ELHI}$ is similar, and we strongly conjecture that it is possible to lift it to full $\mathcal{ELHIF}$ but it is beyond the scope of the paper.

Given an $\mathcal{ELIF}$ ontology $\mathcal{O}$ and a signature $\Sigma = \text{sig}(\mathcal{O})$ with $\text{sig}(q_T) \subseteq \Sigma$, algorithm L first computes the ontology $\mathcal{O}'$ in normal form as per Lemma 8, choosing the fresh concept names so that they are not from Σ. It then runs L' on $\mathcal{O}'$ and $\Sigma' = \Sigma \cup \text{sig}(\mathcal{O}')$. In contrast to L', the oracle still works with the original ontology $\mathcal{O}$. To ensure that the answers to the queries posed to the oracle are correct, L modifies L' as follows.

Whenever L' asks a membership query $\mathcal{D}', a$ with $\mathcal{D}' = \mathcal{A}'_0, \dots, \mathcal{A}'_n$, we may assume that each $\mathcal{A}'_i$ satisfies the functionality assertions from $\mathcal{O}$, since otherwise the answer is trivially "yes". Then, L asks the membership query $\mathcal{D}, a$, where $\mathcal{D}$ is obtained from $\mathcal{D}'$. Note that the $\mathcal{D}$ we are going to construct contains concept assertions $C(d)$ for complex concepts C, but these can be removed at the cost of introducing more fresh individuals and using standard concept assertions.

We start with setting $\mathcal{A}_i = \mathcal{A}'_i \cup \{C(d) \mid X_C(d) \in \mathcal{A}'_i\}$, for all i and then extending the $\mathcal{A}_i$, for every role R and every individual $b \in \text{ind}(\mathcal{A}'_i)$ as follows:

(†) Let $C_{R,b}$ be the set of all concepts $\exists R.D \in \mathfrak{C}(\mathcal{O})$ such that $\mathcal{O}', \mathcal{A}'_i \models \exists R.D(b)$ but $\mathcal{O}', \mathcal{A}'_i \not\models D(b')$ for any $R(b, b') \in \mathcal{A}'_i$. Then
  - if $\text{fun}(R) \notin \mathcal{O}$, then add for each $\exists R.D \in C_{R,b}$ one fresh individual c together with assertions $R(b, c), D(c)$;
  - otherwise, add one fresh individual c and add assertions $R(b, c)$ and $D(c)$, for all $\exists R.D \in C_{R,b}$.

By the following claim, the answer to the modified membership query coincides with that to the original query.

Claim 1. $\mathcal{O}', \mathcal{D}', 0, a \models \boldsymbol{q}$ iff $\mathcal{O}, \mathcal{D}, 0, a \models \boldsymbol{q}$ for all $\boldsymbol{q} \in$ *LTL*$_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) that only use symbols from Σ, and all $a \in \text{ind}(\mathcal{D}')$.

Proof of Claim 1. For "if", suppose that $\mathcal{O}, \mathcal{D}, 0, a \models \boldsymbol{q}$ and let $\mathcal{I}'$ be a model of $\mathcal{D}'$ and $\mathcal{O}'$. We can assume that $\Delta^{\mathcal{I}'}$ does not mention any of the individuals that were introduced in the construction of $\mathcal{D}$. We will construct a model $\mathcal{I}$ of $\mathcal{D}$ and $\mathcal{O}$ such that $(\mathcal{I}_i, a) \to (\mathcal{I}'_i, a)$, for every $0 \le i \le n$. This clearly suffices since $\mathcal{I}, 0, a \models \boldsymbol{q}$.

Fix some i with $0 \le i \le n$ and start with setting $\mathcal{I}_i$ to the restriction of $\mathcal{I}'_i$ to $\text{ind}(\mathcal{A}'_i)$. Then process every individual $b \in \text{ind}(\mathcal{A}'_i)$ and every role R.

Let $C_{R,b}$ be the set of concepts in (†). We distinguish cases:

- If $\text{fun}(R) \notin \mathcal{O}$, process each $\exists R.D \in C_{R,b}$ as follows. By definition of $C_{R,b}$, we have $\mathcal{O}', \mathcal{A}'_i \models \exists R.D(b)$. As $\mathcal{I}'_i$ is a model of $\mathcal{O}'$ and $\mathcal{A}'_i$, there is an element c with $(b, c) \in R^{\mathcal{I}'_i}$ and $c \in D^{\mathcal{I}'_i}$. Take the unraveling $\mathcal{J}_c$ of $\mathcal{I}'_i$ at c, omit the R^--successor of c if $\text{fun}(R^-) \in \mathcal{O}$, and add the root of $\mathcal{J}_c$ as an R-successor of b.
- If $\text{fun}(R) \in \mathcal{O}$, we proceed as follows. By definition of $C_{R,b}$, we have $\mathcal{O}', \mathcal{A}'_i \models \exists R.D(b)$, for all $\exists R.D \in C_{R,b}$. As $\mathcal{I}'$ is a model of $\mathcal{O}'$ and $\mathcal{A}'_i$, there is an element c with $(b, c) \in R^{\mathcal{I}'_i}$ and $c \in D^{\mathcal{I}'_i}$, for all $\exists r.D \in C_{R,b}$. By definition of $C_{R,b}$ and since $\text{fun}(R) \in \mathcal{O}$, we know that there is no $b' \in \text{ind}(\mathcal{A}'_i)$ with $(b, b') \in \mathcal{A}'_i$. Take the unraveling $\mathcal{J}_c$ of $\mathcal{I}'_i$ at c, omit the R^--successor of c if $\text{fun}(R^-) \in \mathcal{O}$, and add the root of $\mathcal{J}_c$ as an R-successor of b.

For the sake of completeness, we provide a formal definition of $\mathcal{J}_c$. Its domain $\Delta^{\mathcal{J}_c}$ consists of all sequences $a_0 R_1 a_1 \dots R_n a_n$ such that

- $a_0 = c$;
- $a_i \in \Delta^{\mathcal{I}'}$, for all i with $0 \leq i \leq n$;
- $(a_i, a_{i+1}) \in R^{\mathcal{I}'}_{i+1}$, for all i with $0 \leq i < n$;
- if $\text{fun}(R^-_i) \in \mathcal{O}$, then $R_{i+1} \neq R^-_i$, for all i with $0 \leq i < n$;
- if $R_1 = R^-$ then $\text{fun}(R^-) \notin \mathcal{O}$.

The interpretation of concept names $A \in \mathsf{N_C}$ and role names $r \in \mathsf{N_R}$ is then as expected:

$$A^{\mathcal{J}_c} = \{a_0 R_1 a_1 \dots R_n a_n \in \Delta^{\mathcal{J}_c} \mid a_n \in A^{\mathcal{I}'}\}$$
$$r^{\mathcal{J}_c} = \{(\pi, \pi r a) \mid \pi r a \in \Delta^{\mathcal{J}_c}\} \cup \{(\pi r^- a, \pi) \mid \pi r^- a \in \Delta^{\mathcal{J}_c}\}.$$

Note that each $\mathcal{J}_c$ has a homomorphism into $\mathcal{I}'$: just map every sequence $a_0 R_1 \dots a_n$ to a_n.

Let $\mathcal{I}$ be the result of the above process. Due to the initialization, we have $\mathcal{I} \models \mathcal{A}$. It is routine to verify that $\mathcal{I}$ is also a model of $\mathcal{O}$ and that there is a homomorphism $(\mathcal{I}, a) \to (\mathcal{I}', a)$.

For "only if", suppose that $\mathcal{O}', \mathcal{D}', 0, a \models \boldsymbol{q}$ and let $\mathcal{I}$ be a model of $\mathcal{D}$ and $\mathcal{O}$. Since $\mathcal{O}'$ is a conservative extension of $\mathcal{O}$, there is a model $\mathcal{I}'$ of $\mathcal{O}'$ that coincides (in every time point) with $\mathcal{I}$ on Σ. Moreover, by Point 3 of Lemma 8, it is also a model of $\mathcal{D}'$. It follows that $\mathcal{I}, 0, a \models \boldsymbol{q}$ as required. This finishes the proof of Claim 1.

For $\mathcal{ELHI}$, we use the following variant of (†):

(‡) Let $C_{R,b}$ be the set of all concepts $\exists R.D \in \mathfrak{C}(\mathcal{O})$ such that $\mathcal{O}', \mathcal{A}'_i \models \exists R.D(b)$. Then add for each $\exists R.D \in C_{R,b}$ one fresh individual c together with assertions $R(b, c), D(c)$.

Now, polynomial query learnability is preserved simply due to the fact that the construction of $\mathcal{D}$ from $\mathcal{D}'$ is computable because instance checking wrt $\mathcal{ELHIF}$ ontologies is decidable. If the ontology language admits polynomial time instance checking, then the construction can actually be computed in polynomial time; thus polynomial time learnability is preserved. □

F.2 Generalisation Sequences

Generalisation sequences have been introduced as a generic tool to show that exact learning algorithms in the atemporal case need only polynomially many steps (Funk, Jung, and Lutz 2022a). We recall the definition.

A *generalisation sequence for a CQ $\boldsymbol{q}$ wrt $\mathcal{O}$* is a sequence $\boldsymbol{q}_1, \boldsymbol{q}_2, \dots$ of CQs that satisfies the following conditions, for all $i \geq 1$:

- $\boldsymbol{q}_i \models_{\mathcal{O}} \boldsymbol{q}_{i+1}$ and $\boldsymbol{q}_{i+1} \not\models_{\mathcal{O}} \boldsymbol{q}_i$, and
- $\boldsymbol{q}_i \models_{\mathcal{O}} \boldsymbol{q}$.

Intuitively, a generalisation sequence is a sequence of weaker and weaker CQs which, however, still entail $\boldsymbol{q}$ wrt $\mathcal{O}$. We recall next that suitable generalisation sequences have bounded length.

Let us fix CQs $\boldsymbol{q}, \boldsymbol{q}_T$. We say that $\boldsymbol{q}$ is *$(\boldsymbol{q}_T, \mathcal{O})$-minimal* if $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}_T$, for every restriction $\boldsymbol{q}'$ of $\boldsymbol{q}$ to a strict subset of the variables in $\boldsymbol{q}$. For a variable $y \in \text{var}(\boldsymbol{q})$, we denote with $\boldsymbol{q}(y)$, the variant of $\boldsymbol{q}$ where the unique free variable is y. We then say that $\boldsymbol{q}$ is *$\mathcal{O}$-saturated* if $\boldsymbol{q}(y) \models_{\mathcal{O}} A(y)$ implies that $A(y)$ is a conjunct in $\boldsymbol{q}$, for every variable y in $\boldsymbol{q}$ and every concept name A that occurs in $\mathcal{O}$. As usual, a CQ is *rooted* if the graph $(\text{var}(\boldsymbol{q}), \{\{x, y\} \mid r(x, y) \in \boldsymbol{q}\})$ is connected. Clearly, all ELIQs are rooted.

We recall Theorem 13 from (Funk, Jung, and Lutz 2022a), adapted to our notation.

Theorem 18. *Let $\mathcal{O}$ be an $\mathcal{ELIF}$ ontology in normal form, $\boldsymbol{q}_T$ be a rooted CQ, and $\boldsymbol{q}_1, \boldsymbol{q}_2, \dots$ be a generalization sequence towards $\boldsymbol{q}$ wrt $\mathcal{O}$ such that $\boldsymbol{q}_1$ is satisfiable wrt $\mathcal{O}$. If all $\boldsymbol{q}_i$ are $(\boldsymbol{q}_T, \mathcal{O})$-minimal and $\mathcal{O}$-saturated, then the length of the sequence is bounded by a polynomial in the sizes of $\mathcal{O}$ and $\boldsymbol{q}_T$.*

Using the same techniques it can be proved that Theorem 18 is remains true for $\mathcal{ELHIF}$ ontologies.

We lift the notion of generalisation sequences to temporal data instances as discussed in the main part of the paper, and show an analogue of Theorem 18. We repeat the definition here for the sake of convenience.

Let $\boldsymbol{q}_T \in LTL_p^{\bigcirc\Diamond\Diamond_r}(\text{ELIQ})$ be a temporal query, and let us fix throughout the rest of the subsection an individual name a. A sequence $\mathcal{D}_1, \dots$ of temporal data instances is a *generalisation sequence towards $\boldsymbol{q}_T$ wrt $\mathcal{O}$* if for all $i \geq 1$:

- $\mathcal{D}_{i+1}$ is obtained from $\mathcal{D}_i$ by modifying one non-temporal CQ $\boldsymbol{r}_j$ in $\mathcal{D}_i$ to $\boldsymbol{r}'_j$ such that $\boldsymbol{r}_j \models_{\mathcal{O}} \boldsymbol{r}'_j$ and $\boldsymbol{r}'_j \not\models_{\mathcal{O}} \boldsymbol{r}_j$;
- $\mathcal{O}, \mathcal{D}_i, 0, a \models \boldsymbol{q}_T$ for all $i \geq 1$.

The notion of $\mathcal{O}$-saturatedness lifts from CQs to temporal data instances $\mathcal{D} = \boldsymbol{q}_0 \dots \boldsymbol{q}_n$ as expected: $\mathcal{D}$ is $\mathcal{O}$-saturated if every $\boldsymbol{q}_i$ is. We further say that $\mathcal{D}$ is *$(\boldsymbol{q}_T, \mathcal{O})$-minimal* if the result $\mathcal{D}'$ of dropping any atom from any $\boldsymbol{q}_i$ satisfies $\mathcal{O}, \mathcal{D}', 0, a \not\models q_T$. The *support* $\text{supp}(\mathcal{D})$ of a temporal data instance $\mathcal{D} = \mathcal{A}_0 \dots \mathcal{A}_n$ the set of all i such that $\mathcal{A}_i \neq \emptyset$

Lemma 10. *Let $\boldsymbol{q}_T \in LTL_p^{\bigcirc\Diamond\Diamond_r}(ELIQ)$. The length of a generalisation sequence $\mathcal{D}_1, \dots, \mathcal{D}_n$ towards $\boldsymbol{q}_T$ wrt $\mathcal{O}$ such that all $\mathcal{D}_i$ are satisfiable wrt $\mathcal{O}$, $\mathcal{O}$-saturated, and $(\boldsymbol{q}_T, \mathcal{O})$-minimal is bounded by a polynomial in the sizes of $\boldsymbol{q}_T$, $\mathcal{O}$, and $|supp(\mathcal{D}_1)|$.*

Proof. Consider a time point i and let $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots$ be the sequence of different queries at time point i that occur in the generalisation sequence, that is, $\boldsymbol{r}_j \models_{\mathcal{O}} \boldsymbol{r}_{j+1}$ and $\boldsymbol{r}_{j+1} \not\models_{\mathcal{O}} \boldsymbol{r}_j$, for each j. Let h be a root homomorphism from $\boldsymbol{q}_T$ to $\mathcal{D}_n$ and let I be the set of all t with $h(t) = i$. (By construction, h is a root homomorphism from $\boldsymbol{q}_T$ to all $\mathcal{D}_j$.) Consider $\boldsymbol{q}' = \bigwedge_{i \in I} \boldsymbol{q}_i$. Clearly, $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots,$ is a generalisation sequence towards $\boldsymbol{q}'$ wrt $\mathcal{O}$. Since all $\mathcal{D}_j$ are satisfiable wrt $\mathcal{O}$, $\mathcal{O}$-saturated and $(\boldsymbol{q}_T, a, \mathcal{O})$-minimal, it follows that in particular, all $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots$ are satisfiable wrt $\mathcal{O}$, $\mathcal{O}$-saturated, and $(\boldsymbol{q}_T, \mathcal{O})$-minimal. By Theorem 18, the length of $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots$ is bounded by a polynomial in the sizes of $\boldsymbol{q}_T$ and $\mathcal{O}$. Since there are only $|\text{supp}(\mathcal{D}_1)|$ time points to consider, the overall sequence $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n$ is bounded by a polynomial in $\boldsymbol{q}_T$, $\mathcal{O}$, and $|\text{supp}(\mathcal{D}_1)|$. □

F.3 Proof of Theorem 15

We restate Theorem 15 for convenience.

Theorem 15. *Let $\mathcal{L}$ be an ontology language that contains only $\mathcal{ELHI}$ or only $\mathcal{ELIF}$ ontologies and that admits polysize frontiers within ELIQ that can be computed. Then:*

(i) *The safe $\text{LTL}_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) queries are polynomial query learnable wrt $\mathcal{L}$ ontologies using membership queries.*

(ii) *The class $\text{LTL}_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) is polynomial query learnable wrt $\mathcal{L}$ ontologies using membership queries if the learner knows the temporal depth of the target query.*

(iii) *The class $\text{LTL}_p^{\bigcirc\Diamond}$(ELIQ) is polynomial query learnable wrt $\mathcal{L}$ ontologies using membership queries.*

If $\mathcal{L}$ further admits polynomial time instance checking and polynomial time computable frontiers within ELIQ, then in (ii) and (iii), polynomial query learnability can be replaced by polynomial time learnability. If, in addition, meet-reducibility wrt $\mathcal{L}$ ontologies can be decided in polynomial time, then also in (i) polynomial query learnability can be replaced by polynomial time learnability.

Proof. Let $\mathcal{L}$ be as in the theorem. Let $\boldsymbol{q}_T$ be the target query, $\mathcal{O}$ an $\mathcal{L}$ ontology, and $\mathcal{D}, a$ be a positive example with $\mathcal{D} = \mathcal{A}_0 \dots \mathcal{A}_n$ and such that $\mathcal{D}$ is satisfiable wrt $\mathcal{O}$. By Lemma 9, we can assume that $\mathcal{O}$ is actually in normal form. Moreover, by Lemma 3, we can assume $\boldsymbol{q}_T$ to be in normal form as well. We further assume $\boldsymbol{q}_T$ to be of shape (8):

$$\boldsymbol{q}_T = \boldsymbol{q}_0 \mathcal{R}_1 \boldsymbol{q}_1 \dots \mathcal{R}_n \boldsymbol{q}_n.$$

As $\boldsymbol{q}_T$ is safe, it does not have lone conjuncts.

We start with showing (i) and then describe the necessary modifications for (ii) and (iii). The idea of the proof is to modify $\mathcal{D}$ in a number of steps such that in the end $\mathcal{D}$ viewed as a temporal query is equivalent to $\boldsymbol{q}_T$. As a general proviso we assume that at all times: each $\mathcal{A}_i$ (viewed as CQ) is $\mathcal{O}$-saturated; this is without loss of generality since instance checking wrt $\mathcal{ELHIF}$ ontologies is decidable (Tobies 2001),

We call a temporal data instance $\mathcal{D}$ *temporally minimal* if there is no time point i such that $\mathcal{D}', a$ is a positive example where $\mathcal{D}'$ is obtained from $\mathcal{D}$ by dropping $\mathcal{A}_i$ from $\mathcal{D}$. Clearly, temporal minimality can be established using at most $\max(\mathcal{D})$ membership queries, and a temporally minimal data instance $\mathcal{D}$ satisfies that $\max(\mathcal{D})$ is at least the number of occurrences of suc and $<$ in $\boldsymbol{q}_T$ and at most as large as the size of $\boldsymbol{q}_T$.

Thus, we can assume without loss of generality that the initial data example $\mathcal{D}$ is temporally minimal. Thus, every root $\mathcal{O}$-homomorphism $h : \boldsymbol{q}_T \to \mathcal{D}$ is block surjective[2] for the block size

$$b := \max(\mathcal{D}) + 1,$$

as $\mathcal{D}$ has only one block for this b. In fact, during all modifications, we maintain the invariant that every root $\mathcal{O}$-homomorphism is block surjective for this number b. We use this initial constant b in steps 2, 3, and 4 below.

Step 1. We first aim to find a temporal data instance which is *tree-shaped*, meaning that in $\mathcal{D} = \mathcal{A}_0 \dots \mathcal{A}_n$ each $\mathcal{A}_i$ is tree-shaped. To achieve this, we exhaustively apply the following rules Unwind and Minimise with a preference given to Minimise. A *cycle* in a data instance is a sequence $R_1(a_1, a_2), \dots, R_n(a_n, a_1)$ of distinct atoms such that $a_1, \dots, a_n$ are distinct.

Minimise. If there is some i and some individual $b \in ind(\mathcal{A}_i)$ such that $\mathcal{D}', a$ is a positive example where $\mathcal{D}'$ is obtained from $\mathcal{D}$ by dropping from $\mathcal{A}_i$ all atoms that mention b, then replace $\mathcal{D}$ with $\mathcal{D}'$.

Unwind. Choose an atom $R(a_1, a_2) \in \mathcal{A}_i$ that is part of a cycle. Obtain $\mathcal{A}_i'$ by first adding a disjoint copy $\mathcal{A}_i'$ of $\mathcal{A}_i$ to $\mathcal{A}_i$ and let a_1', a_2' be the copies of a_1, a_2 in $\mathcal{A}_i'$. Then replace all atoms $S(a_1, a_2)$ (respectively, $S(a_1', a_2')$) by $S(a_1, a_2')$ (respectively, $S(a_1', a_2)$), for all roles S.

It is clear that the resulting temporal data instance is tree-shaped as required. It is still temporally minimal and the invariant that every root $\mathcal{O}$-homomorphism is block surjective is preserved.

Step 2. In this step, we 'close' $\mathcal{D}$ under applications of the Rules (a)–(e) used in Lemma 7. Formally, consider the following Rule 2(x), for x $\in$ {a,b,c,d,e}.

2(x) Let $\mathcal{D}'$ be a data instance obtained from $\mathcal{D}$ by applying Rule (x) from the proof of Lemma 7. If $\mathcal{D}', a$ is a positive example, replace $\mathcal{D}$ with the result of the exhaustive application of Minimise to $\mathcal{D}'$.

We first apply 2(b) and 2(c) until $\mathcal{D}$ stabilises. Then, we exhaustively apply 2(a), 2(d), and 2(e) giving preference to 2(a).

After *Step 2*, $\mathcal{D}$ satisfies that, if $\mathcal{D}'$ is the result of an application of Rules (a)–(e), then $\mathcal{D}', a$ is not a positive example.

Step 3. In this step, we take care of lone conjuncts in $\mathcal{D}$ by applying $(*)$ below as long as $\mathcal{D}$ contains one. Recall that $\boldsymbol{q}_T$ does not, so we can simplify $\mathcal{D}$.

[2]Recall the notion of *(block surjective) root $\mathcal{O}$-homomorphisms* from the proof of Theorem 9.

($*$) Choose a primitive block $\emptyset^b\mathcal{A}\emptyset^b$ in $\mathcal{D}$ such that $\mathcal{A}$, viewed as CQ $\boldsymbol{q}$ is meet-reducible wrt $\mathcal{O}$ within ELIQ. Let $\mathcal{F}_{\boldsymbol{q}} = \{\boldsymbol{q}_1, \dots, \boldsymbol{q}_\ell\}$ and $w = \hat{\boldsymbol{q}}_1\emptyset^b\hat{\boldsymbol{q}}_2\emptyset^b \dots \hat{\boldsymbol{q}}_\ell\emptyset^b$. Denote with $\mathcal{D}_k$ the result of replacing $\emptyset^b\mathcal{A}\emptyset^b$ in $\mathcal{D}$ with $\emptyset^b(w)^k$. Then identify some $i \geq 1$ such that $\mathcal{D}_i, a$ is a positive example, by using membership queries for $i = 1, 2, \dots$. Notice that this requires only polynomially many membership queries as $\mathcal{D}_k, a$ is a positive example for $k = |\boldsymbol{q}_T|$, and that all queries are of polynomial size since $\mathcal{F}_{\boldsymbol{q}}$ is of polynomial size. Replace $\mathcal{D}$ with the result of exhaustively applying Rule 2(a) to $\mathcal{D}_i$ and subsequently shortening blocks $\emptyset^d$ for $d > b$ to $\emptyset^b$.

Let $\mathcal{D}$ be the result of *Step 3*. It is routine to verify that 2(a)–2(e) are not applicable, that $\mathcal{D}$ is b-normal, and that $\mathcal{D}$ is without lone conjuncts wrt $\mathcal{O}$ within $LTL_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ). By Lemma 7, any root $\mathcal{O}$-homomorphism is a root $\mathcal{O}$-isomorphism. Thus, the algorithm has identified all blocks in the following sense. Suppose that $\boldsymbol{q}_T = \boldsymbol{q}_0\mathcal{R}_1\boldsymbol{q}_1 \dots \mathcal{R}_m\boldsymbol{q}_m$ is a sequence of blocks $\boldsymbol{q}_i = \boldsymbol{r}_0^i \dots \boldsymbol{r}_{\ell_i}^i$ and

$$\mathcal{D} = \mathcal{D}_0\emptyset^b\mathcal{D}_1 \dots \emptyset^b\mathcal{D}_n \text{ where } \mathcal{D}_i = \mathcal{A}_0^i \dots \mathcal{A}_{k_i}^i.$$

Then $m = n$ and each block $\mathcal{D}_i$ in $\mathcal{D}$ is isomorphic to $\boldsymbol{q}_i$, that is, $\ell_i = k_i$ and $\hat{\boldsymbol{r}}_j^i = \mathcal{A}_j^i$, for all i, j with $0 \leq i \leq n$ and $0 \leq j \leq k_i$. It is unclear, however, whether the $\mathcal{R}_i$ are (a single) $\leq$ or a sequence of $<$. This is resolved in the final step.

Step 4. We determine $\mathcal{R}_{i+1}$, for each i with $0 \leq i < n$, as follows:

- If $\boldsymbol{r}_{k_i}^i \wedge \boldsymbol{r}_0^{i+1}$ is satisfiable wrt $\mathcal{O}$ and $\mathcal{D}_i, a$ with $\mathcal{D}_i = \mathcal{D}_0\emptyset^b \dots \emptyset^b\mathcal{D}_i \bowtie \mathcal{D}_{i+1}\emptyset^b \dots \emptyset^b\mathcal{D}_n$ ($\bowtie$ defined as in the proof of Theorem 9) is a positive example, then $\mathcal{R}_{i+1}$ is $\leq$. Otherwise, let s be minimal such that $\mathcal{D}_i', a$ is a positive example for $\mathcal{D}_i' = \mathcal{D}_0\emptyset^b \dots \emptyset^b\mathcal{D}_i\emptyset^s\mathcal{D}_{i+1}\emptyset^b \dots \emptyset^b\mathcal{D}_n$. Then, $\mathcal{R}_{i+1}$ is a sequence of s times $<$.

We have thus shown that indeed the returned query is equivalent to $\boldsymbol{q}_T$. It remains to argue that the algorithm issues only polynomially many membership queries. We analyse *Steps 1–4* separately.

For *Step 1*, let $\mathcal{D}_1, \mathcal{D}_2, \dots$ be the sequence of temporal data instances that Unwind is applied to during *Step 1*. Clearly, all these queries are $(\boldsymbol{q}_T, \mathcal{O})$-minimal (recall that we give preference to Minimise) and $\mathcal{O}$-saturated. Since an application of Minimise decreases the overall number of individuals in the instance, there are only polynomially many applications of Minimise between $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$. In the proof of Lemma 14 in (Funk, Jung, and Lutz 2022a), it is shown that the operation Unwind[3] applied to a $(\boldsymbol{q}_T, \mathcal{O})$-minimal CQ $\boldsymbol{q}$ leads to a strictly weaker CQ $\boldsymbol{q}'$, that is, $\boldsymbol{q} \models_{\mathcal{O}} \boldsymbol{q}'$, but not vice versa. This applies here as well, and implies that $\mathcal{D}_1, \mathcal{D}_2, \dots$ is a generalisation sequence towards $\boldsymbol{q}_T$ wrt $\mathcal{O}$. Applying Lemma 10 yields that *Step 1* terminates in time polynomial in the size of $\boldsymbol{q}_T$, $\mathcal{O}$, and $|\text{supp}(\mathcal{D})|$ which in turn is bounded by the size of $\boldsymbol{q}_T$ (recall that $\mathcal{D}$ is temporally minimal).

We next analyse *Step 2*, starting with Rules 2(b) and 2(c). First note that the number of applications of Rules 2(b) and 2(c) is bounded by the number of $<$ and $\leq$ in $\boldsymbol{q}_T$. To see this, we inductively show that Rules 2(b) and 2(c) preserve the fact that every root $\mathcal{O}$-homomorphism $h : \boldsymbol{q}_T \to \mathcal{D}$ is block surjective. As $\mathcal{D}$ is temporally minimal, this certainly holds before *Step 2*. Suppose now that $\mathcal{D}'$ is obtained by a single application of 2(b) or 2(c) to $\mathcal{D}$, and that there is a root $\mathcal{O}$-homomorphism that is not block surjective. Then we can easily construct a non-block surjective homomorphism from $\boldsymbol{q}_T$ to $\mathcal{D}$, a contradiction. Applications of Minimise also preserve the claim. Also note that the block-surjectivity implies that the support of the resulting $\mathcal{D}$ is bounded by the size of $\boldsymbol{q}_T$.

We next analyse Rules 2(a), 2(d), and 2(e). Let $\mathcal{D}_1, \mathcal{D}_2, \dots$ be a sequence of temporal data instances obtained by a sequence of applications of 2(a). Clearly, $\mathcal{D}_1, \mathcal{D}_2, \dots$ is a generalisation sequence towards $\boldsymbol{q}_T$ wrt $\mathcal{O}$ such that all $\mathcal{D}_i$ are satisfiable wrt $\mathcal{O}$, $\mathcal{O}$-saturated, and $(\boldsymbol{q}_T, \mathcal{O})$-minimal. By Lemma 10, the length of the sequence is bounded by a polynomial in the sizes of $\boldsymbol{q}_T$ and $\mathcal{O}$, and in $|\text{supp}(\mathcal{D})| \leq |\boldsymbol{q}_T|$. Further note that applications of 2(a) preserve that every root $\mathcal{O}$-homomorphism is block surjective and that 2(b) and 2(c) remain not applicable.

Next consider an application of 2(d) to a temporal data instance $\mathcal{D}$ where 2(a) is not applicable and such that every root $\mathcal{O}$-homomorphism is block surjective. Let $\mathcal{D}'$ be the result. Since 2(a) is not applicable, every root $\mathcal{O}$-homomorphism to $\mathcal{D}'$ must also be block surjective. Thus, the number of applications of 2(d) is bounded by the number of $<$ and $\leq$ in $\boldsymbol{q}_T$. The same argument works for 2(e). It is readily seen that 2(b) and 2(c) are still not applicable, thus none of the rules is applicable to $\mathcal{D}$. Overall, we obtain a polynomial number of rule applications. This finishes the analysis of *Step 2*.

Consider now *Step 3*. Recall that, by Lemma 5, a CQ $\boldsymbol{q}$ is meet-reducible wrt $\mathcal{O}$ in ELIQ iff $|\mathcal{F}_{\boldsymbol{q}}| \geq 2$ provided that $\boldsymbol{q}' \not\models_{\mathcal{O}} \boldsymbol{q}''$, for all distinct $\boldsymbol{q}', \boldsymbol{q}'' \in \mathcal{F}_{\boldsymbol{q}}$. Thus, to find a lone conjunct $\emptyset^b\mathcal{A}\emptyset^b$ in $\mathcal{D}$, we can compute such a minimal frontier $\mathcal{F}_{\boldsymbol{q}}$ of $\boldsymbol{q}$ wrt $\mathcal{O}$ by first computing any frontier F (which is possible by assumption) and then exhaustively removing from F queries $\boldsymbol{q}''$ such that $\boldsymbol{q}' \models_{\mathcal{O}} \boldsymbol{q}''$ for some $\boldsymbol{q}' \in F$ with $\boldsymbol{q}' \neq \boldsymbol{q}''$. Note that the test $\boldsymbol{q}' \models_{\mathcal{O}} \boldsymbol{q}''$ is decidable for $\mathcal{ELHIF}$ ontologies (Bienvenu et al. 2016).

As noted in ($*$), identifying the right $\mathcal{D}_i$ needs only polynomially many membership queries (despite the fact that deciding meet-reducibility might require more time). Since exhaustive application of 2(a) requires only polynomial time, a single application of ($*$) requires only polynomially many membership queries. Moreover, using the fact that 2(a) is not applicable before application of ($*$) one can show that the number of 'gaps' is increased and that the rule preserves that every root $\mathcal{O}$-homomorphism is block surjective. Hence, ($*$) is applied at most once for each $\leq$ in $\boldsymbol{q}_T$.

It remains to analyse the running time of *Step 4*. Clearly, only linearly many (in the size of $\boldsymbol{q}_T$) membership queries are asked. To finish the argument, it remains to note that

[3] Unwind is called *Double Cycle* in (Funk, Jung, and Lutz 2022a).

$\mathcal{ELHIF}$ admits tractable containment reduction and that satisfiability wrt $\mathcal{ELHIF}$ ontologies is decidable.

We argue next that the above algorithm runs in polynomial time if $\mathcal{L}$ additionally admits polynomial time instance checking, polynomial time computable frontiers, and meet-reducibility of ELIQs wrt $\mathcal{L}$ ontologies can be decided in polynomial time. First note that $\mathcal{O}$-saturation of $\mathcal{D}$ (which is assumed throughout the algorithm) can be established in polynomial time via instance checking. Then observe that, in *Step 3*, a (not necessarily minimal) frontier $\mathcal{F}_{\boldsymbol{q}}$ can be computed in polynomial time and meet-reducibility can also be decided in polynomial time, by assumption. Together with the analysis of *Step 3* above, this yields that *Step 3* needs only polynomial time. Finally observe that also *Step 4* runs in polynomial time since the tests for satisfiability can be reduced to (polynomial time) instance checking.

It remains to prove Points (ii) and (iii) from Theorem 15. The learning algorithm for Point (ii) is similar to the algorithm provided above, but with a modified *Step 3* since in this case $\boldsymbol{q}_T$ might have lone conjuncts and possibly more than one variable from $var(\boldsymbol{q}_T)$ is mapped to the same time point in $\mathcal{D}$. Let T be the temporal depth of the target query.

Step 3′. In this step, we apply $(*')$ until $\mathcal{D}$ stabilises.

$(*')$ Choose a primitive block $\emptyset^b\mathcal{A}\emptyset^b$ and an ELIQ $\boldsymbol{q}$ with $\mathcal{A} = \hat{\boldsymbol{q}}$. Let $\mathcal{F}_{\boldsymbol{q}} = \{\boldsymbol{q}_1, \dots, \boldsymbol{q}_\ell\}$ and $w = \hat{\boldsymbol{q}}_1\emptyset^b\hat{\boldsymbol{q}}_2\emptyset^b \dots \hat{\boldsymbol{q}}_\ell\emptyset^b$. Let $\mathcal{D}'$ be the result of replacing $\emptyset^b\mathcal{A}\emptyset^b$ in $\mathcal{D}$ with $\emptyset^b(w)^T$. If $\mathcal{D}', a$ is a positive example, then replace $\mathcal{D}$ with the result of exhaustively applying Rule 2(a) to $\mathcal{D}'$ and subsequently shortening blocks $\emptyset^d$ for $d > b$ to $\emptyset^b$.

Let $\mathcal{D}$ be the result of *Step 3*. It is routine to verify that 2(a)–2(e) are not applicable. The following can be proven similar to Lemma 7.

Lemma 11. *Let $\mathcal{O}$ and $\boldsymbol{q}_T$ be as above. Let b exceed the number of $\Diamond$ and $\bigcirc$ in $\boldsymbol{q}_T$, and let $\mathcal{D}$ be b-normal. If Rules 2(a)–(e) and $(*')$ are not applicable, then any root $\mathcal{O}$-homomorphim $h\colon \boldsymbol{q} \to \mathcal{D}$ is a root $\mathcal{O}$-isomorphism.*

As an immediate consequence, after *Step 3′*, the modified algorithm has identified all blocks in $\boldsymbol{q}_T$ as described above and it remains to apply *Step 4*.

The learning algorithm for Point (iii) is a similar modification. Note that for the query class $LTL_p^{\bigcirc\Diamond}$(ELIQ), we know that the temporal depth of $\boldsymbol{q}_T$ is exactly $T_0 = \max(\mathcal{D})$ for the temporally minimal input example $\mathcal{D}$. This T_0 can then be used in place of T in $(*')$ above. This finishes the proof of Theorem 15. □

F.4 Proof of Theorem 16

Theorem 16. *The following learnability results hold:*

(i) *The class of safe queries in LTL$_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) is polynomial query learnable wrt DL-Lite$_\mathcal{H}$ ontologies using membership queries and polynomial time learnable wrt DL-Lite$_\mathcal{F}^-$ ontologies using membership queries.*

(ii) *The class LTL$_p^{\bigcirc\Diamond\Diamond_r}$(ELIQ) is polynomial time learnable wrt both DL-Lite$_\mathcal{F}^-$ and DL-Lite$_\mathcal{H}$ ontologies using membership queries if the learner knows the temporal depth of the target query in advance.*

(iii) *The class LTL$_p^{\bigcirc\Diamond}$(ELIQ) is polynomial time learnable wrt both DL-Lite$_\mathcal{F}^-$ and DL-Lite$_\mathcal{H}$ ontologies using membership queries.*

Proof. The theorem is a direct consequence of Theorem 15 and the fact that the considered ontology languages satisfy all conditions mentioned in that theorem. Most importantly:

- *DL-Lite$_\mathcal{F}$* admits polynomial time instance checking (Calvanese et al. 2007b) and *DL-Lite$_\mathcal{F}^-$* admits polynomial time computable frontiers (Funk, Jung, and Lutz 2022b). Meet-reducibility in *DL-Lite$_\mathcal{F}^-$* is decidable in polynomial time, by Lemma 6.
- *DL-Lite$_\mathcal{H}$* admits polynomial time instance checking (Calvanese et al. 2007b) and admits polynomial time computable frontiers (Funk, Jung, and Lutz 2022b).

This completes the proof. □