Properties of products with $\mathbf{GL.3}$

David Gabelaia and Michael Zakharyaschev
Department of Computer Science, King's College London,
Strand, London WC2R 2LS, U.K.
emails: {gabelaia,mz}@dcs.kcl.ac.uk

The *Cartesian product* $L_1 \times L_2$ of two unimodal Kripke complete logics L_1 and L_2 is the bimodal logic determined by the class of all product frames $\mathfrak{F}_1 \times \mathfrak{F}_2$, where $\mathfrak{F}_i$ is a Kripke frame for L_i, $i = 1, 2$. The *commutator* $[L_1, L_2] \subseteq L_1 \times L_2$ of L_1 and L_2 is the independent join of L_1 and L_2 extended with the Church-Rosser and commutativity axioms. (For more details consult [2] and [1].) In this paper, we present some new results on products with the well-known logic $\mathbf{GL.3}$ (determined by the class of Noetherian strict linear orders).

Theorem 1. *For every Kripke complete unimodal L, the logics $L\times\mathbf{GL.3}$ and $[L, \mathbf{GL.3}]$ have the same Kripke frames (but do not necessarily coincide).*

Theorem 2. *Let $\mathcal{C}$ be a class of frames such that for at least one frame $\mathfrak{F} = (W, R)$ in $\mathcal{C}$ there is an infinite subset $Y \subset W$ and a point $x \in W$ such that xRy, for all $y \in Y$, $y \neq x$. Then the logic $\mathsf{Log}\,\mathcal{C} \times \mathbf{GL.3}$ lacks the fmp.*

Theorem 3. *Both $[\mathbf{GL.3}, \mathbf{GL.3}]$ and $[\mathbf{Grz.3}, \mathbf{GL.3}]$ are Kripke incomplete.*

Theorem 4. *Let $L \in \{\mathbf{K4}, \mathbf{K4.3}, \mathbf{S4}, \mathbf{S4.3}, \mathbf{GL}, \mathbf{Grz}\}$. Then $L \times \mathbf{GL.3}$ is undecidable.*

It should be noted that the incompleteness proof in theorem 3 is indirect. It is based on theorem 1 and the fact that the logics $[\mathbf{GL.3}, \mathbf{GL.3}]$ and $[\mathbf{Grz.3}, \mathbf{GL.3}]$ are not recursively enumerable (see theorem 7.16 in [1]).

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References

[1] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-Dimensional Modal Logics: Theory and Applications.* Elsevier, 2003.

[2] D. Gabbay and V. Shehtman. Products of modal logics. Part I. *Journal of the IGPL*, 6:73–146, 1998.