Separating Counting from Non-Counting in Fragments of Two-Variable First-Order Logic (Extended Abstract)

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Abstract

We consider the problem of deciding whether two disjoint classes of models defined in a fragment of first-order logic (FO) with counting can be separated in the same fragment but without counting. This problem turns out to be hard. We show that separation for the two-variable fragment FO^2 extended with counting quantifiers by means of plain FO^2 is undecidable, and the same is true of the pair $\mathcal{ALCOIQ}/\mathcal{ALCOI}$ of description logics. On the other hand, we establish 2ExpTime-completeness of the separation problem for the pairs $\mathcal{ALCQ}^u/\mathcal{ALC}^u$ and $\mathcal{ALCIQ}^u/\mathcal{ALCI}^u$.

Keywords

Separation, two-variable first-order logic, counting quantifiers, bisimulation.

1. Introduction

Our concern in this paper is the following separation problems for a pair of languages L and L^s:

L/L^s**-separation:** given two mutually exclusive L-formulas φ and ψ, decide whether there exists an L^s-formula χ—a *separator* for φ and ψ—such that $\varphi \models \chi$ and $\chi \models \neg\psi$;

Craig L/L^s**-separation:** decide whether the given L-formulas φ and ψ have an L^s-separator χ that only contains common non-logical symbols (predicates and constants) of φ and ψ.

For example, φ could be an ontology $\mathcal{O}$ and ψ a concept C that is not satisfiable with respect to $\mathcal{O}$, both given in an expressive language L. Then a separator ontology $\mathcal{O}'$ in a weaker, easier to comprehend language L^s potentially explains unsatisfiability as it inherits that $\mathcal{O} \models \mathcal{O}'$ and C is not satisfiable under $\mathcal{O}'$. Similarly, if in the concept learning context, φ and ψ represent positive and negative examples for a target concept C, then any separator in an appropriately chosen language L^s could represent the concept one aims to learn.

Separation generalises definability (aka membership), which asks whether a given L-formula (say, a datalog query) is equivalent to some L^s-formula (say, a first-order query), and is regarded

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as one of the main approaches to understanding the expressive power of L relative to L^s. For instance, studying separability of regular languages by smaller language classes (e.g., a star-free language) has brought major insights into the respective formal languages, with some fundamental open problems in the area cast as separation questions [1].

Craig L/L^s-separation generalises classical Craig interpolation in L [2] because a Craig L/L-separator for φ and ψ is a Craig interpolant for $\varphi \to \neg\psi$ in L.

Our aim in this paper is to investigate the decidability and complexity of the separation problem for certain fragments L of C^2—that is, the two-variable first-order logic FO^2 extended with the counting quantifiers $\exists^{<n}x$, $\exists^{=n}x$—and the same fragments L^s but without counting.

Example 1. Consider the following C^2-formulas:

$$\varphi(x) = \exists^{=1}y\, R(x,y), \qquad \psi(x) = \exists^{=1}y\,\big(R(x,y) \wedge A(y)\big) \wedge \exists^{=1}y\,\big(R(x,y) \wedge \neg A(y)\big).$$

Then $\varphi \models \neg\psi$ and the FO^2-formula $\chi(x) = \forall y\,\big(R(x,y) \to A(y)\big) \vee \forall y\,\big(R(x,y) \to \neg A(y)\big)$ is a *separator* for $\varphi(x)$ and $\psi(x)$. For $\psi'(x) = \exists^{=2}y\, R(x,y)$, we also have $\varphi \models \neg\psi'$, but $\varphi(x)$ and $\psi'(x)$ are not separable in FO^2. On the other hand, there is no Craig FO^2-separator for $\varphi(x)$ and $\psi(x)$ as it would have to be defined using R only, and so separate $\varphi(x)$ and $\psi'(x)$ as well.

2. Logics

The logics we consider here can all be regarded as fragments of first-order logic, FO, and are defined as follows. Let σ be a *signature* containing unary and binary relation symbols and possibly constants. Fix a set *var* comprising two individual variables. Then

$\mathsf{FO}^2(\sigma)$, the *two-variable fragment of* $\mathsf{FO}(\sigma)$, is defined as the set of formulas that are built from atoms $A(x)$, $R(x,y)$, and $x = y$ with unary $A \in \sigma$, binary $R \in \sigma$, and $x, y \in$ *var*, using the Boolean connectives $\wedge$ and $\neg$ and quantifier $\exists x$ with $x \in$ *var* (other Booleans and $\forall x$ are regarded as standard abbreviations);

$\mathsf{C}^2(\sigma)$, the *two-variable fragment of* $\mathsf{FO}^2(\sigma)$ *with counting*, extends $\mathsf{FO}^2(\sigma)$ with the counting quantifiers $\exists^{<k}x$, for $k \in \mathbb{N}$ and $x \in$ *var* (other counting quantifiers $\exists^{=k}x$, $\exists^{\leq k}x$, $\exists^{\geq k}x$ can be introduced as abbreviations).

In this paper, we are only interested in formulas $\varphi(x)$ with one free variable $x \in$ *var*. The *signature* of φ is the set $\mathit{sig}(\varphi)$ of relation and constant symbols occurring in φ.

$\mathsf{FO}(\sigma)$ and its fragments are interpreted in *σ-structures* $\mathfrak{A} = (\mathrm{dom}(\mathfrak{A}), (R^{\mathfrak{A}})_{R\in\sigma}, (c^{\mathfrak{A}})_{c\in\sigma})$ with a domain $\mathrm{dom}(\mathfrak{A}) \neq \emptyset$, relations $R^{\mathfrak{A}}$ on $\mathrm{dom}(\mathfrak{A})$ of the same arity as $R \in \sigma$, and elements $c^{\mathfrak{A}} \in \mathrm{dom}(\mathfrak{A})$. A *pointed structure* is a pair $\mathfrak{A}, a$ with $a \in \mathrm{dom}(\mathfrak{A})$.

We also consider a few fragments of C^2 that correspond to some standard description logics (DLs). In the context of DLs, unary relation symbols are called *concept names*, binary ones *role names*, and constants *individual names* [3]. A *role* is a role name r or its *inverse* r^-. The *universal role* is denoted by u. A *nominal* takes the form $\{c\}$ with an individual name c.

An $\mathcal{ALCOIQ}^u(\sigma)$*-concept* is defined by the grammar

$$C \ ::= \ \top \ \mid \ A \ \mid \ \{c\} \ \mid \ \neg C \ \mid \ C \sqcap C' \ \mid \ \geq k\; r.C \ \mid \ \exists u.C,$$

where $A \in \sigma$ is a concept, $c \in \sigma$ an individual, r a role name in σ or its inverse, and $k > 0$. We consider several fragments of $\mathcal{ALCOIQ}^u$. The weakest, $\mathcal{ALC}$, is obtained by dropping the universal role (indicated by omitting $\cdot^u$ from the name), inverse roles (indicated by omitting $\mathcal{I}$), nominals (indicated by omitting $\mathcal{O}$), and only admitting qualified number restrictions of the form $\exists r.C = (\geq 1\ r.C)$ (indicated by dropping $\mathcal{Q}$). The languages between $\mathcal{ALC}$ and $\mathcal{ALCOIQ}^u$ are now defined in the obvious way.

The semantics of DLs can be defined via the *standard translation* $\cdot^\sharp$ into C^2 with constants. For any $\mathcal{ALCOIQ}^u$-concept C, we denote by $C^\sharp_x$ the C^2-formula with constants and free variable $x \in \mathit{var}$ defined inductively by taking

$$\top^\sharp_x = (x = x), \quad A^\sharp_x = A(x), \quad \{c\}^\sharp_x = (x = c), \quad (\neg C)^\sharp_x = \neg C^\sharp_x, \quad (C \sqcap D)^\sharp_x = C^\sharp_x \wedge D^\sharp_x,$$
$$(\exists u.C)^\sharp_x = \top^\sharp_x \wedge \exists \bar{x}\, C^\sharp_{\bar{x}}, \quad (\geq k\ r.C)^\sharp_x = \exists^{\geq k} \bar{x}\, \big(r(x, \bar{x}) \wedge C^\sharp_{\bar{x}}\big),$$

where $\bar{x} = y, \bar{y} = x$ and $\{x, y\} = \mathit{var}$.

The complexities of the satisfiability problems for the logics in question are as follows [3, 4]:

- FO^2, C^2, and $\mathcal{ALCOIQ}^u$ are all NExpTime-complete;
- $\mathcal{ALC}^u$, $\mathcal{ALCQ}^u$, $\mathcal{ALCIQ}^u$, and $\mathcal{ALCOI}^u$ are all ExpTime-complete.

3. Deciding Separation

Our main results are summarised in the next theorem:

Theorem 1. *The separation and Craig separation problems are*

- *undecidable for the pairs* $\mathsf{C}^2/\mathsf{FO}^2$ *and* $\mathcal{ALCOIQ}/\mathcal{ALCOI}$, *and*
- 2ExpTime-*complete for the pairs* $\mathcal{ALCIQ}^u/\mathcal{ALCI}^u$ *and* $\mathcal{ALCQ}^u/\mathcal{ALC}^u$.

The proofs of these results are based on the following model-theoretic characterisation of separation in terms of appropriate bisimulations; see [5, 6, 7] and further references therein:

Lemma 2. *Let* $\varphi(x)$ *and* $\psi(x)$ *be any* $\mathsf{C}^2(\sigma)$*-formulas,* $\varrho \subseteq \sigma$*, and let* L^s *be any of the languages introduced in Section A. Then the following conditions are equivalent*:

- $\varphi(x)$ *and* $\psi(x)$ *do not have an* $L^s(\varrho)$*-separator*;
- *there are pointed* σ*-structures* $\mathfrak{A}, a$ *and* $\mathfrak{B}, b$ *such that*

$$\mathfrak{A} \models \varphi(a), \qquad \mathfrak{B} \models \psi(b), \qquad \mathfrak{A}, a \sim_{L^s(\varrho)} \mathfrak{B}, b.$$

For Craig separation, we additionally require that $\varrho \subseteq \mathit{sig}(\varphi) \cap \mathit{sig}(\psi)$.

Here, $\mathfrak{A}, a \sim_{L^s(\varrho)} \mathfrak{B}, b$ means that there is an $L^s(\varrho)$-bisimulation $\boldsymbol{\beta}$ between $\mathfrak{A}$ and $\mathfrak{B}$ such that $(a, b) \in \boldsymbol{\beta}$, which is equivalent to $\mathfrak{A} \models \phi(a)$ iff $\mathfrak{B} \models \phi(b)$, for all $L^s(\varrho)$-formulas $\phi(x)$ [5, 8, 4]. The proof of the characterisation in Lemma 1 is similar to the characterisations of Craig interpolant nonexistence in [6, 7]. The undecidability proofs are by reduction of the halting problem for 2 register machines where the numbers in the registers are represented by the number of $L^s(\varrho)$-bisimilar nodes. The decidability proofs are based on novel adaptations of the mosaic technique for constructing $L^s(\varrho)$-bisimilar models [6, 7].

4. Related Work

While separability has so far been mainly investigated in automata theory [9, 10, 11], definability has been considered for many logics. For example, the problem of deciding whether a TBox given in a DL L can be equivalently expressed in another DL L' is considered in [12], the problem of deciding whether a GF or GNF formula is equivalent to an existential (or positive existential) GF or, respectively, GNF formula is considered in [13, 14], and there are many results on deciding when fixpoints can be dropped from a second-order extension of a fragment of FO. For instance, it is shown in [15] that it is ExpTime-complete to decide whether a modal μ-calculus formula is equivalent to a basic modal logic formula. Variants of definability explored in description logic are approximation [16] and conservative rewritability [17].

Craig separators are a generalisation of Craig interpolants where $L^s = L$. If the logic L has the Craig interpolation property (CIP), then the Craig separator existence problem for φ and ψ reduces to checking whether $\varphi \models \psi$ and is thus not harder than entailment. Only recently the problem of deciding the existence of Craig interpolants has been considered for logics without the CIP [18, 19]. In fact, the bisimulation-based method employed here makes heavy use of techniques introduced for checking Craig interpolant existence [6, 7].

5. Discussion

We have started investigating the separation problem for fragments of FO with counting by formulas in the same fragments but without counting. Many problems remain to be addressed; we mention a few of them below:

1. Our decidability proofs are non-constructive, and it would be of interest to develop algorithms that construct separators whenever they exist and to determine bounds on their size.
2. With the exception of $\mathcal{ALCQ}$, the logics with counting we considered do not have the finite model property. It would be of interest to investigate whether our results also hold on finite structures. In that case, the bisimulation criterion does not hold as formulated (because its proof uses compactness) and one has to employ a different criterion that holds on finite structures (say, bounded bisimulations).
3. Our logics have the universal role. We conjecture that without the universal role $\mathcal{ALCQ}/\mathcal{ALC}$- and $\mathcal{ALCIQ}/\mathcal{ALCI}$-separation becomes coNExpTime-complete.
4. Is definability less complex than separation for the pairs of languages considered here. For example, is $\mathsf{C}^2/\mathsf{FO}^2$-definability decidable?

References

[1] T. Place, M. Zeitoun, Separating regular languages with first-order logic, Log. Methods Comput. Sci. 12 (2016).

[2] W. Craig, Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory, J. Symb. Log. 22 (1957) 269–285. doi:10.2307/2963594.

[3] F. Baader, I. Horrocks, C. Lutz, U. Sattler, An Introduction to Description Logic, Cambridge University Press, 2017.

[4] I. Pratt-Hartmann, Fragments of First-Order Logic, Oxford Logic Guides, Oxford University Press, United Kingdom, 2023.

[5] V. Goranko, M. Otto, Model theory of modal logic, in: Handbook of Modal Logic, Elsevier, 2007, pp. 249–329.

[6] J. C. Jung, F. Wolter, Living without Beth and Craig: Definitions and interpolants in the guarded and two-variable fragments, in: Proceedings of the 36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, IEEE, 2021, pp. 1–14. URL: https://doi.org/10.1109/LICS52264.2021.9470585. doi:10.1109/LICS52264.2021.9470585.

[7] A. Artale, J. C. Jung, A. Mazzullo, A. Ozaki, F. Wolter, Living without Beth and Craig: Definitions and interpolants in description and modal logics with nominals and role inclusions, ACM Trans. Comput. Log. 24 (2023) 34:1–34:51. URL: https://doi.org/10.1145/3597301. doi:10.1145/3597301.

[8] E. Grädel, M. Otto, The freedoms of (guarded) bisimulation, in: Johan van Benthem on Logic and Information Dynamics, Springer International Publishing, 2014, pp. 3–31.

[9] T. Place, M. Zeitoun, The tale of the quantifier alternation hierarchy of first-order logic over words, ACM SIGLOG News 2 (2015) 4–17. URL: https://doi.org/10.1145/2815493.2815495. doi:10.1145/2815493.2815495.

[10] M. Bojanczyk, It is undecidable if two regular tree languages can be separated by a deterministic tree-walking automaton, Fundam. Informaticae 154 (2017) 37–46. URL: https://doi.org/10.3233/FI-2017-1551. doi:10.3233/FI-2017-1551.

[11] T. Place, Separating regular languages with two quantifier alternations, Log. Methods Comput. Sci. 14 (2018). URL: https://doi.org/10.23638/LMCS-14(4:16)2018. doi:10.23638/LMCS-14(4:16)2018.

[12] C. Lutz, R. Piro, F. Wolter, Description Logic TBoxes: Model-Theoretic Characterizations and Rewritability, in: IJCAI, 2011, pp. 983–988.

[13] M. Benedikt, B. ten Cate, M. Vanden Boom, Effective interpolation and preservation in guarded logics, ACM Trans. Comput. Log. 17 (2016) 8. URL: https://doi.org/10.1145/2814570. doi:10.1145/2814570.

[14] V. Bárány, M. Benedikt, B. ten Cate, Some model theory of guarded negation, J. Symb. Log. 83 (2018) 1307–1344.

[15] M. Otto, Eliminating recursion in the μ-calculus, in: C. Meinel, S. Tison (Eds.), STACS 99, 16th Annual Symposium on Theoretical Aspects of Computer Science, Trier, Germany, March 4-6, 1999, Proceedings, volume 1563 of *Lecture Notes in Computer Science*, Springer, 1999, pp. 531–540. URL: https://doi.org/10.1007/3-540-49116-3_50. doi:10.1007/3-540-49116-3_50.

[16] A. Bötcher, C. Lutz, F. Wolter, Ontology approximation in Horn description logics, in: Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI 2019, Macao, China, August 10-16, 2019, 2019, pp. 1574–1580. URL: https://doi.org/10.24963/ijcai.2019/218. doi:10.24963/IJCAI.2019/218.

[17] B. Konev, C. Lutz, F. Wolter, M. Zakharyaschev, Conservative rewritability of description

logic TBoxes, in: Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI 2016, New York, NY, USA, 9-15 July 2016, 2016, pp. 1153–1159. URL: http://www.ijcai.org/Abstract/16/167.

[18] A. Kurucz, F. Wolter, M. Zakharyaschev, Definitions and (uniform) interpolants in first-order modal logic, in: Proceedings of the 20th International Conference on Principles of Knowledge Representation and Reasoning, KR 2023, Rhodes, Greece, September 2-8, 2023, 2023, pp. 417–428. URL: https://doi.org/10.24963/kr.2023/41. doi:10.24963/KR.2023/41.

[19] A. Kurucz, F. Wolter, M. Zakharyaschev, A non-uniform view of craig interpolation in modal logics with linear frames, CoRR abs/2312.05929 (2023). URL: https://doi.org/10.48550/arXiv.2312.05929. doi:10.48550/ARXIV.2312.05929. arXiv:2312.05929.

[20] A. K. Chandra, D. C. Kozen, L. J. Stockmeyer, Alternation, J. ACM 28 (1981) 114–133.

Appendix: Supplementary Material

A. Logics

In this section, we define the syntax and semantics of the fragments of first-order logic, FO, we deal with in what follows. Let σ be a *signature* containing unary and binary relation symbols and possibly constants. Fix a set *var* comprising two individual variables. Then

$\mathsf{FO}^2(\sigma)$, the *two-variable fragment of* $\mathsf{FO}(\sigma)$, is defined as the set of formulas φ that are built from atoms $A(x)$, $R(x, y)$, and $x = y$ with unary $A \in \sigma$, binary $R \in \sigma$, and $x, y \in \mathit{var}$, using the Boolean connectives $\wedge$ and $\neg$ and quantifier $\exists x$ with $x \in \mathit{var}$ (other Boolean connectives and $\forall x$ are regarded as standard abbreviations);

$\mathsf{C}^2(\sigma)$, the *two-variable fragment of* $\mathsf{FO}^2(\sigma)$ *with counting*, extends $\mathsf{FO}^2(\sigma)$ with the counting quantifiers $\exists^{<k}x$, for $k \in \mathbb{N}$ and $x \in \mathit{var}$ (other counting quantifiers $\exists^{=k}x$, $\exists^{\leq k}x$, $\exists^{\geq k}x$ can be introduced as abbreviations).

In this paper, we are only interested in formulas $\varphi(x)$ with one free variable $x \in \mathit{var}$. The *signature* of φ is the set $\mathit{sig}(\varphi)$ of relation and constant symbols occurring in φ. We denote by $\mathit{sub}(\varphi)$ the set of subformulas of φ together with their negations, setting $|\varphi| = |\mathit{sub}(\varphi)|$.

$\mathsf{FO}(\sigma)$ and its fragments are interpreted in *σ-structures* $\mathfrak{A} = (\mathrm{dom}(\mathfrak{A}), (R^{\mathfrak{A}})_{R\in\sigma}, (c^{\mathfrak{A}})_{c\in\sigma})$ with a domain $\mathrm{dom}(\mathfrak{A}) \neq \emptyset$, relations $R^{\mathfrak{A}}$ on $\mathrm{dom}(\mathfrak{A})$ of the same arity as $R \in \sigma$, and elements $c^{\mathfrak{A}} \in \mathrm{dom}(\mathfrak{A})$. A *pointed structure* is a pair $\mathfrak{A}, a$ with $a \in \mathrm{dom}(\mathfrak{A})$.

We also consider a few fragments of C^2 that correspond to some standard description (or modal) logics. In the context of DLs, unary relation symbols are called *concept names*, binary ones *role names*, and constants *individual names* [3]. A *role* is a role name r or its *inverse* r^-, with $(r^-)^- = r$. The *universal role* is denoted by u. A *nominal* takes the form $\{c\}$ with an individual name c. An $\mathcal{ALCOIQ}^u(\sigma)$-*concept* is defined by the grammar

$$C \ ::= \ \top \ \mid \ A \ \mid \ \{c\} \ \mid \ \neg C \ \mid \ C \sqcap C' \ \mid \ \geq k\ r.C \ \mid \ \exists u.C,$$

where $A \in \sigma$ is a concept name, $c \in \sigma$ an individual name, r a role name in σ or its inverse, and $k > 0$. The construct $(\geq k\ r.C)$ is known as the *qualified number restriction*. As usual, we set

$\exists r.C = (\geq 1\ r.C)$ and use $C \sqcup D$ as an abbreviation for $\neg(\neg C \sqcap \neg D)$, $C \rightarrow D$ for $\neg C \sqcup D$, $C \leftrightarrow D$ for $(C \rightarrow D) \sqcap (D \rightarrow C)$, and $\forall r.C$ for $\neg\exists r.(\neg C)$. Other counting concepts such as $\leq k\ r.C$ or $= k\ r.C$ can also be introduced as abbreviations in an obvious way.

We consider several fragments of $\mathcal{ALCOIQ}^u$. The weakest, $\mathcal{ALC}$, is obtained by dropping the universal role (indicated by omitting $\cdot^u$ from the name), inverse roles (indicated by omitting $\mathcal{I}$), nominals (indicated by omitting $\mathcal{O}$), and only admitting qualified number restrictions of the form $\exists r.C$ (indicated by dropping $\mathcal{Q}$). The languages between $\mathcal{ALC}$ and $\mathcal{ALCOIQ}^u$ are now defined in the obvious way.

The semantics of DLs can be defined via the *standard translation* $\cdot^\sharp$ into C^2 with constants. For any $\mathcal{ALCOIQ}^u$-concept C, we denote by $C^\sharp_x$ the C^2-formula with constants and free variable $x \in \mathit{var}$ defined inductively by taking

$$\top^\sharp_x = (x = x), \quad A^\sharp_x = A(x), \quad \{c\}^\sharp_x = (x = c), \quad (\neg C)^\sharp_x = \neg C^\sharp_x, \quad (C \sqcap D)^\sharp_x = C^\sharp_x \wedge D^\sharp_x,$$
$$(\exists u.C)^\sharp_x = \top^\sharp_x \wedge \exists \bar{x}\, C^\sharp_{\bar{x}}, \quad (\geq k\ r.C)^\sharp_x = \exists^{\geq k} \bar{x}\, \big(r(x, \bar{x}) \wedge C^\sharp_{\bar{x}}\big),$$

where $\bar{x} = y$, $\bar{y} = x$ and $\{x, y\} = \mathit{var}$. Then the *extension* $C^{\mathfrak{A}}$ of a concept C in $\mathfrak{A}$ is defined as

$$C^{\mathfrak{A}} = \{a \in \mathrm{dom}(\mathfrak{A}) \mid \mathfrak{A} \models C^\sharp_x(a)\}.$$

In this paper, DL concepts C are always regarded as FO-formulas $C^\sharp_x$ with one free variable x, though we often use the more succinct DL notation (denoting roles by small letters like r and s rather than R and S as in FO-formulas). A formula $\varphi(x)$ is called *satisfiable* if there is a pointed structure $\mathfrak{A}, a$ such that structure $\mathfrak{A} \models \varphi(a)$. Given σ-formulas $\varphi(x)$ and $\psi(x)$, we write $\varphi(x) \models \psi(x)$ if, for any pointed σ-structure $\mathfrak{A}, a$,

$$\mathfrak{A} \models \varphi(a) \quad \text{implies} \quad \mathfrak{A} \models \psi(a).$$

Finally, we remind the reader of the complexity of the satisfiability problem for the logics in question [3, 4]:

- FO^2, C^2, and $\mathcal{ALCOIQ}^u$ are all NExpTime-complete;
- $\mathcal{ALC}^u$, $\mathcal{ALCQ}^u$, $\mathcal{ALCIQ}^u$, and $\mathcal{ALCOI}^u$ are ExpTime-complete.

In the next section, we define the separation problems for the logics defined above and give a model-theoretic criterion of separability in terms of bisimulations.

B. Separation and Bisimulation

Let $\varphi(x)$ and $\psi(x)$ be FO-formulas. An FO-formula $\chi(x)$ is called a *separator* for $\varphi(x)$ and $\psi(x)$ if $\varphi(x) \models \chi(x)$ and $\chi(x) \models \neg\psi(x)$. If, in addition, $\mathit{sig}(\chi) \subseteq \mathit{sig}(\varphi) \cap \mathit{sig}(\psi)$, we say that $\chi(x)$ is a *Craig separator* for $\varphi(x)$ and $\psi(x)$.

Given two languages L and L^s, the *separation problem for L by L^s*—or *L/L^s-separation*, for short—is to decide whether any two given L-formulas have an L^s-separator. If we are only interested in Craig separators, we refer to the problem as *Craig L/L^s-separation*.

Lemma 3. *Let L/L^s be any of the pairs $\mathsf{C}^2/\mathsf{FO}^2$, $\mathcal{ALCOIQ}/\mathcal{ALCOI}$, $\mathcal{ALCIQ}^u/\mathcal{ALCI}^u$, $\mathcal{ALCQ}^u/\mathcal{ALC}^u$. Then L/L^s-separation is polynomial-time reducible to Craig L/L^s-separation.*

Proof. Given L-formulas $\varphi(x)$ and $\psi(x)$, consider the formulas

$$\varphi'(x) = \varphi(x) \wedge \bigwedge_{A\in sig(\psi)\setminus sig(\varphi)} \big(A(x) \rightarrow A(x)\big) \wedge \bigwedge_{R\in sig(\psi)\setminus sig(\varphi)} \big(R(x,x) \rightarrow R(x,x)\big) \wedge \bigwedge_{c\in sig(\psi)\setminus sig(\varphi)} (c = c),$$

$$\psi'(x) = \psi(x) \wedge \bigwedge_{A\in sig(\varphi)\setminus sig(\psi)} \big(A(x) \rightarrow A(x)\big) \wedge \bigwedge_{R\in sig(\varphi)\setminus sig(\psi)} \big(R(x,x) \rightarrow R(x,x)\big) \wedge \bigwedge_{c\in sig(\varphi)\setminus sig(\psi)} (c = c).$$

It is readily seen that $\varphi(x)$ and $\psi(x)$ have an L^s-separator iff $\varphi'(x)$ and $\psi'(x)$ have a Craig L^s-separator. $\dashv$

Our main tool for determining the decidability and complexity of L/L^s-separation is based on the notion of bisimulation.

Let L be any of the fragments of FO defined above and let $\varrho \subseteq \sigma$. Given pointed σ-structures $\mathfrak{A}, a$ and $\mathfrak{B}, b$, we write $\mathfrak{A}, a \equiv_{L,\varrho} \mathfrak{B}, b$ and say that $\mathfrak{A}, a$ and $\mathfrak{B}, b$ are *$L(\varrho)$-equivalent* if $\mathfrak{A} \models \varphi(a)$ iff $\mathfrak{B} \models \varphi(b)$, for all $L(\varrho)$-formulas $\varphi(x)$.

A binary relation $\boldsymbol{\beta} \subseteq \text{dom}(\mathfrak{A}) \times \text{dom}(\mathfrak{B})$ is called an *$\mathsf{FO}^2(\varrho)$-bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$* if $\boldsymbol{\beta}$ is *global* in the sense that $\text{dom}(\mathfrak{A}) \subseteq \{a \mid (a,b) \in \boldsymbol{\beta}\}$ and $\text{dom}(\mathfrak{B}) \subseteq \{b \mid (a,b) \in \boldsymbol{\beta}\}$ and, for every $(a,b) \in \boldsymbol{\beta}$, the following conditions are satisfied:

- for every $a' \in \text{dom}(\mathfrak{A})$, there is a $b' \in \text{dom}(\mathfrak{B})$ such that $(a',b') \in \boldsymbol{\beta}$ and $(a,a') \mapsto (b,b')$ is a partial ϱ-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$;
- for every $b' \in \text{dom}(\mathfrak{B})$, there is a $a' \in \text{dom}(\mathfrak{A})$ such that $(a',b') \in \boldsymbol{\beta}$ and $(a,a') \mapsto (b,b')$ is a partial ϱ-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

We write $\mathfrak{A}, a \sim_{\mathsf{FO}^2(\varrho)} \mathfrak{B}, b$ if $a \mapsto b$ is a partial ϱ-isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ and there is an $\mathsf{FO}^2(\varrho)$-bisimulation $\boldsymbol{\beta}$ between $\mathfrak{A}$ and $\mathfrak{B}$ such that $(a,b) \in \boldsymbol{\beta}$.

A non-empty binary relation $\boldsymbol{\beta} \subseteq \text{dom}(\mathfrak{A}) \times \text{dom}(\mathfrak{B})$ is called an *$\mathcal{ALC}(\varrho)$-bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$* if the following conditions are satisfied:

1. if $(a,b) \in \boldsymbol{\beta}$, then $a \in A^{\mathfrak{A}}$ iff $b \in A^{\mathfrak{B}}$ for all $A \in \varrho$;
2. if $(a,b) \in \boldsymbol{\beta}$ and $(a,a') \in R^{\mathfrak{A}}$, for $R \in \varrho$, then there exists $b' \in \text{dom}(\mathfrak{B})$ such that $(a',b') \in \boldsymbol{\beta}$ and $(b,b') \in R^{\mathfrak{B}}$;
3. if $(a,b) \in \boldsymbol{\beta}$ and $(b,b') \in R^{\mathfrak{B}}$, for $R \in \varrho$, then there exists $a' \in \text{dom}(\mathfrak{A})$ such that $(a',b') \in \boldsymbol{\beta}$ and $(a,a') \in R^{\mathfrak{A}}$.

We write $\mathfrak{A}, a \sim_{\mathcal{ALC}(\varrho)} \mathfrak{B}, b$ if there is an $\mathcal{ALC}(\varrho)$-bisimulation $\boldsymbol{\beta}$ between $\mathfrak{A}$ and $\mathfrak{B}$ such that $(a,b) \in \boldsymbol{\beta}$. We say that an $\mathcal{ALC}(\varrho)$-bisimulation $\boldsymbol{\beta}$ between $\mathfrak{A}$ and $\mathfrak{B}$ is

- an *$\mathcal{ALC}^u(\varrho)$-bisimulation* if it is global;
- an *$\mathcal{ALCI}(\varrho)$-bisimulation* if conditions 2 and 3 also hold for $r = s^-$ with $s \in \varrho$.

Further, we call $\boldsymbol{\beta}$ an $\mathcal{ALCI}^u(\varrho)$*-bisimulation* if it is both an $\mathcal{ALC}^u(\varrho)$- and an $\mathcal{ALCI}(\varrho)$-bisimulation; $\boldsymbol{\beta}$ is an $\mathcal{ALCOI}(\varrho)$*-bisimulation* if it is both an $\mathcal{ALCI}(\varrho)$-bisimulation and $(c^{\mathfrak{A}}, c^{\mathfrak{B}}) \in \boldsymbol{\beta}$ for all $c \in \varrho$. Finally, $\boldsymbol{\beta}$ is an $\mathcal{ALCOI}^u(\varrho)$*-bisimulation* if it is both an $\mathcal{ALCI}^u(\varrho)$ and an $\mathcal{ALCOI}(\varrho)$-bisimulation. The following characterisation is well-known; see, e.g., [**?** 8, 4]:

Lemma 4. *Let L be any of the languages introduced in Section A. For any pointed σ-structures $\mathfrak{A}, a$ and $\mathfrak{B}, b$,*

$$\mathfrak{A}, a \sim_{L(\varrho)} \mathfrak{B}, b \quad \textit{implies} \quad \mathfrak{A}, a \equiv_{L(\varrho)} \mathfrak{B}, b$$

and, conversely, if structures $\mathfrak{A}$ and $\mathfrak{B}$ are ω-saturated, then

$$\mathfrak{A}, a \equiv_{L(\varrho)} \mathfrak{B}, b \quad \textit{implies} \quad \mathfrak{A}, a \sim_{L(\varrho)} \mathfrak{B}, b.$$

Lemma 5. *Let $\varphi(x)$ and $\psi(x)$ be any $\mathsf{C}^2(\sigma)$-formulas, $\varrho \subseteq \sigma$, and let L^s be any of the languages introduced in Section A. Then the following conditions are equivalent*:

- *$\varphi(x)$ and $\psi(x)$ do not have an $L^s(\varrho)$-separator*;
- *there are pointed σ-structures $\mathfrak{A}, a$ and $\mathfrak{B}, b$ such that*

$$\mathfrak{A} \models \varphi(a), \qquad \mathfrak{B} \models \psi(b), \qquad \mathfrak{A}, a \sim_{L^s(\varrho)} \mathfrak{B}, b.$$

Proof. The proof is similar to the characterisations of Craig interpolant existence in [6, 7].⊣

C. Undecidable Separation

We show that $\mathcal{ALCOIQ}/\mathcal{ALCOI}$-(Craig) separation is undecidable with and without the universal role and that $\mathsf{C}^2/\mathsf{FO}^2$-(Craig) separation undecidable.

To this end we reduce the halting problem for two-register machines. A (deterministic) *two-register machine (2RM)* is a pair $M = (Q, P)$ with $Q = q_0, \dots, q_\ell$ a set of *states* and $P = I_0, \dots, I_{\ell-1}$ a sequence of *instructions*. By definition, q_0 is the *initial state*, and q_ℓ the *halting state*. For all $i < \ell$,

- either $I_i = +(p, q_j)$ is an *incrementation instruction* with $p \in \{0, 1\}$ a register and q_j the subsequent state;
- or $I_i = -(p, q_j, q_k)$ is a *decrementation instruction* with $p \in \{0, 1\}$ a register, q_j the subsequent state if register p contains 0, and q_k the subsequent state otherwise.

A *configuration* of M is a triple (q, m, n), with q the current state and $m, n \in \omega$ the register contents. We write $(q_i, n_1, n_2) \Rightarrow_M (q_j, m_1, m_2)$ if one of the following holds:

- $I_i = +(p, q_j)$, $m_p = n_p + 1$, and $m_{1-p} = n_{1-p}$;
- $I_i = -(p, q_j, q_k)$, $n_p = m_p = 0$, and $m_{1-p} = n_{1-p}$;
- $I_i = -(p, q_k, q_j)$, $n_p > 0$, $m_p = n_p - 1$, and $m_{1-p} = n_{1-p}$.

The *computation* of M on input $(n, m) \in \omega^2$ is the unique longest configuration sequence $(p_0, n_0, m_0) \Rightarrow_M (p_1, n_1, m_1) \Rightarrow_M \cdots$ such that $p_0 = q_0$, $n_0 = n$, and $m_0 = m$. The halting problem for 2RMs is to decide, given a 2RM M, whether its computation on input $(0, 0)$ is finite (which implies that its last state is q_ℓ).

C.1. $\mathcal{ALCOIQ}/\mathcal{ALCOI}$-separation is undecidable.

We show that $\mathcal{ALCOIQ}^u/\mathcal{ALCOI}^u$-separation is undecidable and then use spy-points to eliminate the universal role and obtain that $\mathcal{ALCOIQ}/\mathcal{ALCOI}$-separation is undecidable. We also obtain undecidability of Craig separation since the concepts used in the proof use the same signature.

We reduce the non-halting problem for 2RM to the non-existence of a separator. To this purpose, let a two-register machine $M = (Q, P)$ be given. We assume without loss of generality that the initial state q_0 only occurs initially and that I_0 is a decrementation instruction.

We introduce two concepts, C_M and D_M, both conjunctions, with the conjuncts best presented in a few groups. For C_M, the first group of conjuncts are as follows.

1. $\{a\} \sqcap Q_0$
2. $\forall u.(F \sqcup \bigsqcup_{1 \leq i \leq \ell} Q_i)$
3. $\forall u.(F \to \bigsqcup_{1 \leq i \leq \ell} \neg Q_i)$
4. $\forall u. \bigsqcap_{1 \leq i \leq \ell} (Q_i \to \bigsqcap_{i < j \leq \ell} \neg Q_j)$

These conjuncts describe the various concept names that we need: every element satisfies exactly one of F, Q_0, $\cdots$, Q_ℓ, and the unique node satisfying $\{a\}$ satisfies Q_0. We will use models of C_M to represent runs of the machine M; elements satisfying some Q_i will represent snapshots of the computations where the machine is in state q_i. The elements satisfying F are auxiliary, their purpose will become more clear once we discuss D_M.

The next group of conjuncts of C_M is as follows.

5. $\{a\} \to \forall r^{-1}.\bot$
6. $\forall u.(\neg F \to (= 1\ r.\top)$
7. $\forall u.((\neg F \land \neg\{a\}) \to (= 1\ r^{-1}.\top)$
8. $\forall u.(\neg F \to (\forall r.\neg F \sqcap \forall r^{-1}.\neg F))$
9. $\forall u.(\neg F \to \exists s.F)$

These conjuncts describe the relations r and s: every non-auxiliary element (i.e., every $\neg F$ element) has exactly one r-successor, and every such element has exactly one r-predecessor, except for the node satisfying $\{a\}$ which has no r-predecessors. Furthermore, if an element is non-auxiliary then so are its r-successor and r-predecessor. Finally, every non-auxiliary element has an s-successor that is auxiliary.

It follows from these conditions that every pointed model $\mathfrak{A}, x_0$ with $x_0 \in C_M^{\mathfrak{A}}$ contains an infinite r-chain starting at x_0 and each element of the chain satisfies exactly one Q_i. We will interpret these elements as representing a run of M.

The next group of conjuncts depends on the instructions of the machine M. If $I_i = +(p, q_j)$ we add the conjuct

10. $\forall u.(Q_i \to \forall r.Q_j)$

and if $I_i = -(p, q_j, q_k)$ we add the conjuncts

11. $\forall u.((Q_i \land E_p) \to \forall r.Q_j)$
12. $\forall u.((Q_i \land \neg E_p) \to \forall r.Q_k)$

These conjuncts express that the instructions of M are followed, where E_p (with $p \in \{0,1\}$) indicates that register p is empty. Note that we do not, at this stage, keep track of the value in each register. But *if* E_p holds if and only if register p is empty, then these conjuncts ensure that every pointed model $\mathfrak{A}, x_0$ of C_M encodes the run of M.

Finally, we add one more, very simple, conjunct,

13. $\forall u.\neg Q_\ell$

which expresses that the halting state Q_ℓ is not reached. So if C_M encodes a run of M, it must be a non-halting run.

The concept D_M can similarly be divided into groups of conjuncts. Before discussing these conjuncts, it is convenient to first define a few abbreviations:

- $U := (= 1\ s.F) \sqcup (= 3\ s.F)$
- $R_0 := (= 1\ s.F) \sqcup (= 2\ s.F)$
- $R_1 := \neg R_0$.

Note that, because U, R_0 and R_1 are mere abbreviations, they are not considered to be part of the signature. Note also that $U \sqcap R_0$ and $U \sqcap R_1$ are both consistent, while $R_0 \sqcap R_1$ obviously is not. Now, let us introduce the conjuncts of D_M.

The first group is the same as the the first group of conjuncts of C_M:

1. $\{a\} \sqcap Q_0$
2. $\forall u.(F \sqcup \bigsqcup_{1 \leq i \leq \ell} Q_i)$
3. $\forall u.(F \to \bigsqcup_{1 \leq i \leq \ell} \neg Q_i)$
4. $\forall u. \bigsqcap_{1 \leq i \leq \ell} (Q_i \to \bigsqcap_{i < j \leq \ell} \neg Q_j)$

So, as with C_M, in every pointed model $\mathfrak{B}, y_0$ of D_M, all elements satisfy exactly one of F, Q_0, $\cdots$, Q_ℓ. The remaining conjuncts are quite different, however.

5. $(= 2\; r.\top)$

6. $\exists r.(U \sqcap R_0) \sqcap \exists r.(U \sqcap R_1)$

7. $E_0 \sqcap E_1$

8. $\forall u.(R_p \to (\forall r.R_p \sqcap \forall r^{-1}.(R_p \sqcup \{a\})))$ for $p \in \{0, 1\}$

9. $\forall u.((U \sqcap \neg\{a\}) \to ((= 1\; r.U) \sqcap (= 1\; r^{-1}.U)))$

Conjunct 1 implies that in any pointed model $\mathfrak{B}, y_0$ of D_M we have that y_0 satisfies $\{a\}$, so conjuncts 5 and 6 imply that y_0 has exactly two r-successors, with one satisfying $U \sqcap R_0$ and the other $U \sqcap R_1$. We will use these branches to represent registers 0 and 1, respectively. Conjunct 7, meanwhile, says that y_0 satisfies both E_0 and E_1. Conjuncts 8 and 9 handle propagation: 8 says that R_p propagates forward and back along r (except to $\{a\}$), while 9 says that U propagates to exactly one r-successor and r-predecessor (except from $\{a\}$).

The next group of conjuncts depends on the instructions of M. If $I_i = +(p, q_j)$,

10. $\forall u.((Q_i \sqcap R_p) \to ((= 2\; r.\top) \sqcap \forall r.(= 1\; r^{-1}.\top)))$

11. $\forall u.((Q_i \sqcap \neg R_p) \to ((= 1\; r.\top) \sqcap \forall r.(= 1\; r^{-1}.\top)))$

If $i \neq 0$ and $I_i = -(p, q_j, q_k)$,

12. $\forall u((Q_i \sqcap R_p \sqcap E_p) \to ((= 1\; r.\top) \sqcap \forall r.(= 1\; r^{-1}.\top)))$

13. $\forall u((Q_i \sqcap R_p \sqcap \neg E_p) \to ((= 1\; r.\top) \sqcap \forall r.(= 2\; r^{-1}.\top)))$

14. $\forall u.((Q_i \sqcap \neg R_p) \to ((= 1\; r.\top) \sqcap \forall r.(= 1\; r^{-1}.\top)))$

Conjunct 10 expresses that if Q_i is an incrementation instruction, then an element satisfying $Q_i \sqcap R_p$ will have two r-successors that each have one r-predecessor. We will use r to represent the computation, so this means that the number of elements will double in a state with an incrementation instruction. With respect to conjuncts 12 and 13, recall that we use E_p as a marker for register p being empty. So these conjuncts say that if a decrementation instruction is given with an empty register (conjunct 12), then for every R_p element there is one r-successor that has one r-predecessor so the number of elements remains the same, and if a decrementation instruction is given with a non-empty register (conjunct 13) then for every R_p element there is a single r-successor that has two r-predecessors, so the number of elements is halved.

Conjuncts 11 and 14 state that elements in the other register (i.e., those satisfying $\neg R_p$) have exactly one successor that has exactly one predecessor, so the number of elements remains the same.

Note that we treat Q_0 separately. This is because we first need to create the two registers before we can increase or decrease the number of elements in each register. This is done by conjuncts 5 and 6. After one step of the computation, our two registers are then initialized with one element each, which we interpret as the value 0 for that register. This is why we assume that q_0 is decrementing (so the registers are empty after one step of the computation) and only occurs initially (so we do not perform the initialization again at later times).

Finally, we add one more conjunct

15. $\forall u((E_p \sqcap R_p) \to U)$

which states that the combination of E_p and R_p is possible only when U also holds.

The concepts C_M and D_M have the same signature. Hence any separator is a Craig separator. We show that a separator exists if and only if M is non-halting, from which it follows that separation is undecidable. More precisely, the reduction shows that non-separation is co-RE hard, and therefore the separation problem is RE hard.

Before proving our reduction, let us first introduce the *intended pointed models* for C_M and D_M, where M is non-halting. A pointed model $\mathfrak{A}, x_0$ of C_M will be used to represent a clock, and the state the machine is in at each time step. Recall that C_M guarantees that $\mathfrak{A}, x_0$ follows the instructions of M, as long as E_p holds when register p is empty. In our intended models, we will make sure that (i) E_p only holds if register p is empty and (ii) if register p is empty and the current instruction is to decrement register p, then E_p holds (to indicate that decrementing is impossible). If register p is empty and the current instruction is anything other than to decrement register p, we do not care whether E_p holds.

A pointed model $\mathfrak{B}, y_0$ of D_M is used to represent the content of the registers. Every point t of the computation is represented by some element e_t of $\mathfrak{A}$, and this e_t is $\mathcal{ALCOI}^u$-bisimilar to a set of elements in $\mathfrak{B}$. These elements of $\mathfrak{B}$ can be divided into those that satisfy R_0 and those that satisfy R_1. If e_t is $\mathcal{ALCOI}^u$-bisimilar to m elements that satisfy R_0 and n elements that satisfy R_1, then we say that registers 0 and 1 contains $\log_2(m)$ and $\log_2(n)$, respectively, at time t.

An example of (part of) these intended models is shown in Figure 1. In this example, $I_0 = -(0, q_1, q_1)$, $I_1 = +(0, q_4)$ and $I_4 = -(0, q_2, q_4)$. So the machine starts in state q_0 by decrementing the (already empty) register 0, and goes to state q_1. In this state, it increments register 0, and continues to state q_4. In q_4, it first decrements register 0 while staying in q_4. Then it tries to decrement register 0 again, but now that register is empty so the next state is q_2 instead of q_4.

It is straightforward (if somewhat labour intensive) to verify that the intended pointed models satisfy C_M and D_M, and that they are $\mathcal{ALCOI}^u$-bisimilar.

Proposition 6. *If M is non-halting, then there are pointed models $\mathfrak{A}, x_0$ of C_M and $\mathfrak{B}, y_0$ of D_M such that $\mathfrak{A}, x_0 \sim_{\mathcal{ALCOI}^u} \mathfrak{B}, y_0$.*

Next, we need to show the converse.

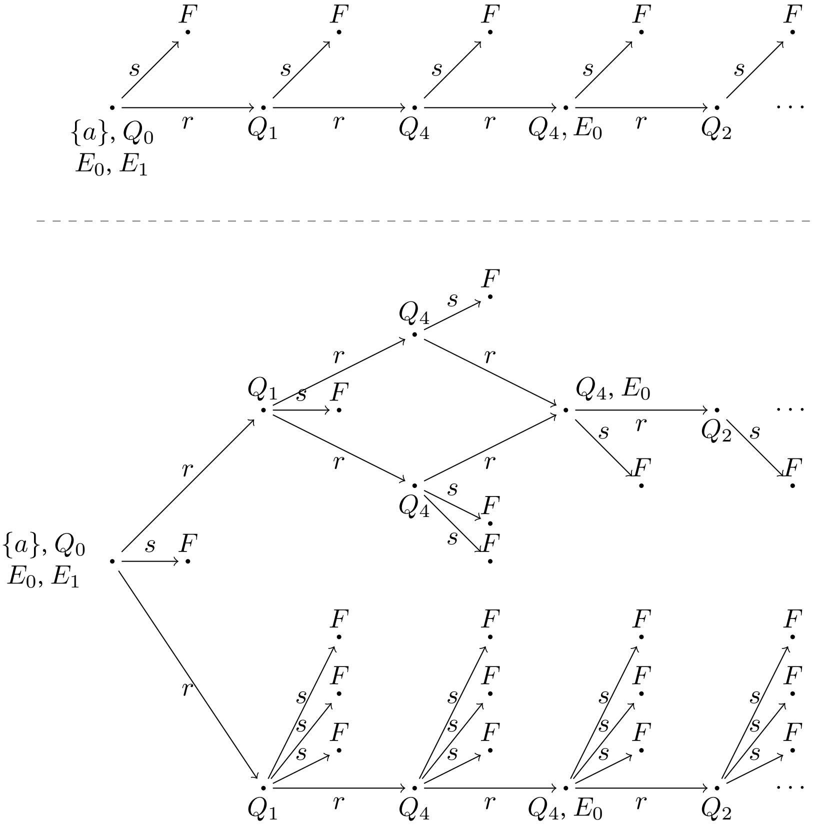
Proposition 7. *If there are there are pointed models $\mathfrak{A}, x_0$ of C_M and $\mathfrak{B}, y_0$ of D_M such that $\mathfrak{A}, x_0 \sim_{\mathcal{ALCOI}^u} \mathfrak{B}, y_0$, then M is non-halting.*

Proof. We will show that $\mathfrak{A}$ and $\mathfrak{B}$ contain intended models as sub-models. $\mathfrak{A}$ contains a sequence of elements $x_0, x_1, \cdots$ where $x_0 = a^{\mathfrak{A}}$, x_{t+1} is the unique r-successor of x_t and x_t is the unique r-predecessor of x_{t+1}. Furthermore, on each x_t some Q_i (with $0 \leq i \leq \ell$) holds.

In $\mathfrak{B}$ we also have $y_0 = a^{\mathfrak{B}}$. Let $Y_0 = \{y_0\}$, and for every t let Y_{t+1} be the r-successors of the elements of Y_t. Let $\boldsymbol{\beta} \subseteq \mathsf{dom}(\mathfrak{A}) \times \mathsf{dom}(\mathfrak{B})$ be the $\mathcal{ALCOI}^u$-bisimulation between $\mathfrak{A}, x_0$ and $\mathfrak{B}, y_0$.

Claim 1: For every $t \in \mathbb{N}$, we have (i) if $(x_t, y) \in \boldsymbol{\beta}$ then $y \in Y_t$ and (ii) if $y \in Y_t$, then $(x_t, y) \in \boldsymbol{\beta}$.

Figure 1: Example pointed models of C_M (above the dashed line) and D_M (below the dashed line). Elements that are drawn below each other are $\mathcal{ALCOI}^u$-bisimilar. Recall that R_0, R_1 and U are abbreviations that depend on the number of s-successors satisfying F. All elements in the upper branch of the pointed model of D_M have 1 or 2 s-successors, to they satisfy R_0. In the lower branch they have more than 2 s-successors, so they satisfy R_1. All but one elements have 1 or 3 s-successors, and therefore satisfy U. The one exception is where the upper branch splits; there can be at most one R_0-element that satisfies U, so the lower of the two R_0 elements has 2 s-successors, which is a $\neg U$ amount.



Proof of claim 1: By induction on t. As base case, note that x_0 and y_0 are the unique elements satisfying $\{a\}$ in their respective models, so x_0 and y_0 can only be $\mathcal{ALCOI}^u$-bisimilar to each other. Since $\boldsymbol{\beta}$ is, by assumption, a $\mathcal{ALCOI}^u$-bisimulation, we must have $(x_0, y_0) \in \boldsymbol{\beta}$. Assume then as induction hypothesis that $t > 0$ and that the claim holds for all $t' < t$.

Take any y such that $(x_t, y) \in \boldsymbol{\beta}$. Because $\boldsymbol{\beta}$ is an $\mathcal{ALCOI}^u$-bisimulation and x_t has x_{t-1} as an r-predecessor, y must have some y' as r-predecessor such that $(t_{x-1}, y') \in \boldsymbol{\beta}$. By the induction hypothesis, $y' \in Y_{t-1}$. This implies that y is an r-successor of an element from Y_{t-1} so, by definition, $y \in Y_t$.

Now, take any $y \in Y_t$. By the definition of Y_t, there is some $y' \in Y_{t-1}$ that is an r-predecessor of y. by the induction hypothesis, $(x_{t-1}, y) \in \boldsymbol{\beta}$. Because $\boldsymbol{\beta}$ is a bisimulation and y is an r-successor of y', there must be some x' that is an r-successor of x_{t-1} such that $(x', y) \in \boldsymbol{\beta}$. The only r-successor of x_{t-1} is x_t, so we have $x' = x_t$, and hence $(x_t, y) \in \boldsymbol{\beta}$. This completes the induction step, and thereby the proof of Claim 1.

Claim 2: For every $y \in Y_t$, if y' is an r-predecessor of y, then $y' \in Y_{t-1}$.

Proof of claim 2: By claim 1, $(x_t, y) \in \boldsymbol{\beta}$. Since y' is an r-predecessor of y, we must have $(y', x') \in \boldsymbol{\beta}$ for some r-predecessor x' of x_t. The only r-predecessor of x_t is x_{t-1}, so we have $(x_{t-1}, y') \in \boldsymbol{\beta}$. By claim 1, this implies that $y' \in Y_{t-1}$. This completes the proof of claim 2.

Claim 3: For every t, there is exactly one i_t such that Q_{i_t} holds on any of x_t and Y_t. This Q_{i_t} holds on all of x_t and Y_t. Furthermore, E_p either holds on all of x_t and Y_t, or on none.

Proof of claim 3: Because $\mathfrak{A}$ is a model of C_M, every x_t satisfies exactly one Q_{i_t}. The rest of the claim follows immediately from $\boldsymbol{\beta}$ being a bisimulation.

Before continuing with further claims, let use define Y_t^p, for $t > 0$ and $p \in \{0, 1\}$, by $Y_t^p = \{y \in Y_t \mid \mathfrak{B}, y \models R_p\}$. If $p \in \{0, 1\}$, we write $\overline{p}$ for the other element of $\{0, 1\}$, i.e., $\overline{0} = 1$ and $\overline{1} = 0$.

Claim 4: For $p \in \{0, 1\}$ and $t > 0$, the r-successors of Y_t^p are in Y_{t+1}^p and the r-predecessors of Y_{t+1}^p are in Y_t^p.

Proof of claim 4: Conjuncts 8 says that R_p propagates forward and back trough r.

Claim 5: For every $t > 0$,

- if $I_{i_t} = +(p, q_j)$ then $i_{t+1} = j$, $|Y_{t+1}^p| = 2 \times |Y_t^p|$,
- if $I_{i_t} = -(p, q_j, q_k)$ and E_p holds on x_t, then $I_{t+1} = j$, $|Y_{t+1}^p| = |Y_t^p|$,
- if $I_{i_t} = -(p, q_j, q_k)$ and E_p does not hold on x_t, then $i_{t+1} = k$, $|Y_{t+1}^p| = \frac{1}{2} \times |Y_t^p|$,

and, in each case, $|Y_{t+1}^{\overline{p}}| = |Y_t^{\overline{p}}|$.

Proof of claim 5: Conjuncts 10, 11 and 12 of C_M guarantee that i_{t+1} has the appropriate value.

Consider, then the sets Y_t^p and Y_{t+1}^p. By claim 4, all the r-successor of Y_t^p are in Y_{t+1}^p, and all the r-predecessors of Y_{t+1}^p are in Y_t^p. Furthermore, conjunct 10 of D_M implies that, if $I_{i+t} = +(p, q_j)$ then every $y \in Y_t^p$ has exactly two r-successors that each have exactly one r-predecessor. It follows that $|Y_{t+1}^p| = 2 \times |Y_t^p|$. Similarly, if $I_{i_t} = -(p, q_j, q_k)$ and E_p holds, then conjunct 12 says that every $y \in Y_t^p$ has exactly one r-successor that has exactly one r-predecessor, so $|Y_{t+1}^p| = |Y_t^p|$. If $I_{i+t} = -(p, q_j, q_k)$ and E_p does not hold, then conjunct 13 makes sure that every $y \in Y_t^p$ has exactly one r-successor that has exactly two r-predecessors, so $|Y_{t+1}^p| = \frac{1}{2} \times |Y_t^p|$.

Finally, the conjuncts 11 and 14 guarantee that every $y \in Y_t^{\overline{p}}$ has exactly one r-successor that has exactly one r-predecessor, so $|Y_{t+1}^{\overline{p}}| = |Y_t^{\overline{p}}|$.

Claim 6: $|Y_1^0| = |Y_1^1| = 1$.

Proof of claim 6: By conjuncts 5 and 6 of D_M, $\{a\}$ has exactly two successors, one of which satisfies $U \sqcap R_0$ while the other satisfies $U \sqcap R_1$.

Claim 7: For every $t > 0$ and $p \in \{0, 1\}$, there is exactly one $y \in Y_t^p$ that satisfies U.

Proof of claim 7: For $t = 1$, the claim holds because, by conjunct 6 of D_M, there is at least one $y \in Y_1^p$ that satisfies U. As Y_1^p is a singleton (by claim 6), there must be exactly one such y. Conjunct 9 implies that every U element (other than $\{a\}$) has exactly one r-predecessor and one r-successor that satisfies U, so if Y_t^p has exactly one element satisfying U then so does Y_{t+1}^p.

Claim 8: for every $t > 0$ and $p \in \{0, 1\}$,

- if E_p holds on x_t and Y_t, then Y_t^p is a singleton and
- if $I_t = -(p, q_j, q_k)$ and Y_t^p is a singleton, then E_p holds on x_t and Y_t.

Proof of claim 8: Suppose that E_p holds on x_t and Y_t. Then $E_p \sqcap R_p$ is true for every element of Y_t^p. By conjunct 15, this implies that U is true for all of Y_t^p. Claim 7 says that only one element of Y_t^p can satisfy U, so Y_t^p must be a singleton.

Suppose then that $I_t = -(p, q_j, q_k)$ and Y_t^p is a singleton. Assume towards a contradiction that E_p does not hold on x_t. Then, by claim 5, $|Y_{t+1}^p| = \frac{1}{2} \times |Y_t^p|$. But Y_{t+1}^p cannot have half as many elements as Y_t^p, since Y_t^p is a singleton. From this contradiction, we conclude that E_p holds on x_t (and therefore also on Y_t).

Claim 9: for every t and $p \in \{0, 1\}$, let v_t^p be the value in register p at time t in the run of M and q_{i_t} the state M is in at time t. The following hold:

- Q_{i_t} holds on x_t and Y_t,
- if $t > 0$, then $|Y_t^p| = 2^{v_t^p}$.

Proof of claim 9: As we remarked when we introduced them, conjuncts 10–12 of C_M guarantee that the state transition instruction of M are obeyed, if E_p holds in the appropriate places. Claim 5 shows that $\mathfrak{B}$ follows the incrementation/decrementation instructions of M, again under the assumption that E_p holds when appropriate. Finally, claim 8 shows that E_p holds where appropriate.

Claim 10: M is non halting.

Proof of claim 10: Follows from conjunct 13 of C_M together with claim 9. ⊣

We have shown the following.

Theorem 8. *$\mathcal{ALCOIQ}^u/\mathcal{ALCOI}^u$)-(Craig) separation is RE-complete.*

We observe that the proof above can be adapted so that the universal role u is not used. Let C_M and D_M be as defined above, except that we treat u not as the universal role but instead as a role name. Now, add the following additional conjuncts to both C_M and D_M:

- $\exists u.\{a\}$
- $\forall u.(\forall r.\exists u^-.\{a\} \sqcap \forall r^-.\exists u^-.\{a\})$
- $\forall u.(\forall s.\exists u^-.\{a\} \sqcap \forall s^-.\exists u^-.\{a\})$

These additional conjuncts guarantee that while u is not necessarily interpreted as the universal relation, all elements that are relevant in the proof are u-successors of $a^{\mathfrak{A}}$ and $a^{\mathfrak{B}}$, respectively, and hence all conjuncts that start with $\forall u$ have the desired effect. Undecidability of $\mathcal{ALCOIQ}/\mathcal{ALCOI}$-(Craig) separation follows immediately.

Theorem 9. *$\mathcal{ALCOIQ}/\mathcal{ALCOI}$-(Craig) separation is RE-complete.*

C.2. $\mathsf{C}^2/\mathsf{FO}^2$-separation is undecidable

We show that $\mathsf{C}^2/\mathsf{FO}^2$-(Craig) separation is undecidable. Our reduction is very similar to the one presented in the previous Section for $\mathcal{ALCOIQ}/\mathcal{ALCIO}$-separation. The reduction is given below, but before we consider the details we discuss the modifications that are required.

Since we are now working in C^2, instead of $\mathcal{ALCOIQ}^u$, we will use different notation, but the main idea remains the same. We use two formulas φ_M and ψ_M, where the pointed model $\mathfrak{A}, x_0$ of ϕ_M is essentially an r-chain encoding the the run of M, and the model $\mathfrak{B}, y_0$ of ψ_M represents the contents of the registers. Synchronisation between $\mathfrak{A}$ and $\mathfrak{B}$ is done through the bisimulation between them. All of this is exactly as in the $\mathcal{ALCOIQ}^u/\mathcal{ALCOI}^u$ case.

Where things get more complicated is that now the bisimulation has to be a FO^2-bisimulation, which is more constraining than a $\mathcal{ALOIQ}^u$-bisimulation. Recall that a FO^2-bisimulation requires that if $(x, y) \in \boldsymbol{\beta}$, then for every x' there is an y' such that $(x', y') \in \boldsymbol{\beta}$ and $(x, x') \mapsto (y, y')$ is a partial isomorphism, and similarly for every y' there must be an x' with the same properties. Here $(x, x') \mapsto (y, y')$ being a partial isomorphism means that (i) for every unary predicate P in the relevant signature, $P(x)$ iff $P(y)$ and $P(x')$ iff $P(y')$, (ii) for every binary predicate R in the relevant signature, $R(x, x')$ iff $R(y, y')$ and $R(x', x)$ iff $R(y', y')$ and (iii) $x = x'$ iff $y = y'$.

Perhaps surprisingly, it is only condition (iii) that will give us trouble, and that requires us to make the current proof slighltly more complicated than the $\mathcal{ALCOIQ}^u/\mathcal{ALCOI}^u$ one. On the intended pointed models of C_M and D_M, we can already choose an y' (or x') that satisfies properties (i) and (ii). Condition (iii), however, is not satisfied for those models.

This is because condition (iii) implies that a FO^2-bisimulation can never relate a singleton set to a multi-element set (i.e., if $(x, y) \in \boldsymbol{\beta}$ and $(x, y') \in \boldsymbol{\beta}$ with $y \neq y'$ then there must be a $x' \neq x$ such that $(x', y) \in \boldsymbol{\beta}$ and $(x', y') \in \boldsymbol{\beta}$). After all, if $(x, y), (x, y') \in \boldsymbol{\beta}$ with $y \neq y'$ then there must be some x' such that $(x', y') \in \boldsymbol{\beta}$ and $(x, x') \mapsto (y, y')$ is a partial isomorphism. In particular, we must then have $x = x'$ iff $y = y'$, so from $y \neq y'$ it follows that $x \neq x'$.

In order to solve this issue, we make a small modification to the intended models for C_M and D_M. Specifically, we create two copies of everything: the intended models of ϕ_M and ψ_M are simply two disjoint copies of C_M and D_M, respectively.

Now, let us consider the precise formulas φ_M and φ_M that achieve this effect. We begin with φ_M, which is the conjunction of the following formulas:

1. $\exists^{=2} x\, Q_0(x)$
2. $\forall x\, (F(x) \vee \bigvee_{1 \leq i \leq \ell} Q_i(x))$
3. $\forall x\, (F(x) \to \bigwedge_{1 \leq i \leq \ell} \neg Q_i(x))$
4. $\forall x\, \bigwedge_{1 \leq i \leq \ell} (Q_i(x) \to \bigwedge_{i < j \leq \ell} \neg Q_j(x))$
5. $\forall x\, (Q_0(x) \to \forall y \neg R(y, x))$
6. $\forall x\, (\neg F(x) \to \exists^{=1} y R(x, y))$
7. $\forall x\, ((\neg F(x) \wedge \neg Q_0(x)) \to \exists^{=1} y R(y, x))$

8. $\forall x\,(\neg F(x) \to \forall y\,((R(x,y) \vee R(y,x)) \to \neg F(y)))$

9. $\forall x\,(\neg F \to \exists^{\geq 2} y(S(x,y) \wedge F(y)))$

if $I_i = +(p, q_j)$ then

10. $\forall x\,(Q_i(x) \to \forall y\,(R(x,y) \to Q_j(y)))$

and if $I_i = -(p, q_j, q_k)$ then

11. $\forall x\,((Q_i(x) \wedge E_p(x)) \to \forall y\,(R(x,y) \to Q_j(y)))$

12. $\forall x\,((Q_i(x) \wedge \neg E_p(x)) \to \forall y\,(R(x,y) \to Q_k(y)))$

And, finally,

13. $\forall x\,\neg Q_\ell(x)$

The reader may notice that most conjuncts of φ_M are simply translations of the conjunct of C_M with the same number to C^2. The only exceptions, and these are very minor exceptions, are conjuncts 1 and 9. The corresponding C_M conjuncts guarantee the existence of one Q_0 element and one s-successor, while here we guarantee the existence of two Q_0 elements and two S-successors.

The formula ψ_M uses abbreviations U, R_0 and R_1, similar to how we used them in D_M. We need to scale the numbers up a bit, however, due to the aforementioned issue regarding bisimulations and singletons.

- $U(x) := \exists^{=2} y\,(S(x,y) \wedge F(y)) \vee \exists^{=4} y\,(S(x,y) \wedge F(y))$
- $R_0(x) := \exists^{=2} y\,(S(x,y) \wedge F(y)) \vee \exists^{=3} y\,(S(x,y) \wedge F(y))$
- $R_1(x) := \neg R_0(x)$.

Using these abbreviations, we can define the conjuncts of ψ_M.

1. $\exists^{=2} x\, Q_0(x)$
2. $\forall x\,(F(x) \vee \bigvee_{1 \leq i \leq \ell} Q_i(x))$
3. $\forall x\,(F(x) \to \bigwedge_{1 \leq i \leq \ell} \neg Q_i(x))$
4. $\forall x\, \bigwedge_{1 \leq i \leq \ell}(Q_i(x) \to \bigwedge_{i < j \leq \ell} \neg Q_j(x))$
5. $\forall x\,(Q_0(x) \to \exists^{=2} y\, R(x,y))$
6. $\forall x\,(Q_0(x) \to \bigwedge_{p \in \{0,1\}} \exists^{=1} y\,(R(x,y) \wedge U(y) \wedge R_p(y)))$
7. $\forall x\,(Q_0(x) \to (E_0(x) \wedge E_1(x)))$
8. $\forall x\,(R_p(x) \to (\forall y\,((R(x,y) \vee R(y,x)) \to (R_p(y) \vee Q_0(y)))))$
9. $\forall x\,((U(x) \wedge \neg Q_0(x)) \to (\exists^{=1} y\,(R(x,y) \wedge U(y)) \wedge \exists^{=1} y\,(R(y,x) \wedge U(y))))$

If $I_i = +(p, q_j)$ then

10. $\forall x\, ((Q_i(x) \wedge R_p(x)) \to (\exists^{=2} y\, R(x,y) \wedge \forall y\, (R(x,y) \to \exists^{=1} x\, R(x,y))))$

11. $\forall x\, ((Q_i(x) \wedge \neg R_p(x)) \to (\exists^{=1} y\, R(x,y) \wedge \forall y\, (R(x,y) \to \exists^{=1} x\, R(x,y))))$

If $I_i = -(p, q_j, q_j)$ then

12. $\forall x\, ((Q_i(x) \wedge R_p(x) \wedge E_p) \to (\exists^{=1} y\, R(x,y) \wedge \forall y\, (R(x,y) \to \exists^{=1} x\, R(x,y))))$

13. $\forall x\, ((Q_i(x) \wedge R_p(x) \wedge \neg E_p) \to (\exists^{=1} y\, R(x,y) \wedge \forall y\, (R(x,y) \to \exists^{=2} x\, R(x,y))))$

14. $\forall x\, ((Q_i(x) \wedge \neg R_p(x)) \to (\exists^{=1} y\, R(x,y) \wedge \forall y\, (R(x,y) \to \exists^{=1} x\, R(x,y))))$

And finally

15. $\forall x\, ((E_p(x) \wedge R_p(x)) \to U(x))$

Note that φ_M and ψ_M use the same signature ϱ.

Proposition 10. *If M is non-halting, then there are pointed models $\mathfrak{A}, x_0$ and $\mathfrak{B}, y_0$ with $\mathfrak{A} \models \varphi_M(x_0)$ and $\mathfrak{B} \models \psi_M(y_0$ such that $\mathfrak{A}, x_0 \sim_{\mathsf{FO}^2(\varrho)} \mathfrak{B}, y_0$.*

Proof. The models in question are slight variations on the intended models for C_M and D_M that we discussed previously. The required modifications are as follows:

- For notation reasons, we call the relations R and S, instead of r and s.
- In the intended model of C_M, we gave every $\neg F$ element a single s-successor that satisfies F, in the intended model of φ_M we give every $\neg F$ element *two* S-successors that satisfy F.
- Similarly, in the intended model of D_M, we gave every $\neg F$ element between 1 and 4 s-successors satisfying F, with the exact amount determined by whether U, R_0 and R_1 should hold. In the intended model for ψ_M we increase this amount by one, so we add between 2 and 5 S-successors satisfying F for every $\neg F$ element.
- Finally, we take two disjoint copies of the structure obtained so far.

Apart from the renaming of r and s to R and S, these changes are necessary because, as discussed above, a singleton set can never be FO^2-bisimilar to a multi-element set.

Verifying that these structures are indeed models of $\mathfrak{A}$ and $\mathfrak{B}$, and that they are FO^2-bisimilar, is straightforward.

$\dashv$

Next, the other direction.

Proposition 11. *If there are pointed models $\mathfrak{A}, x_0$ and $\mathfrak{B}, y_0$ with $\mathfrak{A} \models \varphi_M(x_0)$ and $\mathfrak{B} \models \psi_M(y_0$ such that $\mathfrak{A}, x_0 \sim_{\mathsf{FO}^2(\varrho)} \mathfrak{B}, y_0$, then M is non-halting.*

Proof. This proof is mostly similar to that of Proposition 7, except for the first few claims, in which we show certain elements to be bisimilar. We therefore prove the first few claims in detail, and refer to the proof of Proposition 7 for details on the remainder.

Let $\boldsymbol{\beta}$ be the bisimulation betweem $\mathfrak{A}$ and $\mathfrak{B}$, and let α_0 be the elements in $\mathfrak{A}$ that satisfy Q_0 and β_0 the ones in $\mathfrak{B}$. Then, for $n \in \mathbb{N}$, let α_{n+1} be the set of R-successors of elements of α_n and β_{n+1} the set of R-successors of β_n. The remained of this proof is organized by several numbered claims.

Claim 1: $\boldsymbol{\beta}$ can only relate an element of α_n to elements of β_n, and vice versa.

Proof of claim 1: By induction. We show the base case for one direction, the other direction can be shown similarly. Suppose, towards a contradiction, that $(a, b) \in \boldsymbol{\beta}$ with $a \in \alpha_0$ and $b \notin \beta_0$. Then, for any x, $(a, a) \mapsto (b, x)$ is not a partial isomorphism, since $Q_0(a)$ but $\neg Q_0(b)$. This contradicts $\boldsymbol{\beta}$ being a bisimulation, so such a and b cannot exist.

Assume then, as induction hypothesis, that the claim holds for all $n' < n$. Again, we show one direction; that $(a, b) \in \boldsymbol{\beta}$ and $a \in \alpha_n$. The element a has an a-predecessor $a' \in \alpha_{n-1}$. Because $\boldsymbol{\beta}$ is a bisimulation, there must be a b' such that $(b, b') \in \boldsymbol{\beta}$ and $(a, a') \mapsto (b, b')$ is a partial isomorphism. We have $R(a', a)$, so we must also have $R(b', b)$. Furthermore, by the induction hypothesis, $(a', b') \in \boldsymbol{\beta}$, together with $a' \in \alpha_{n-1}$, implies that $b' \in \beta_{n-1}$. Hence b is the R-successor of an element in β_{n-1}, which, by definition, means $b \in \beta_n$. This completes the induction step, thereby proving the claim.

Claim 2: If $a \in \alpha_{n+1}$ and $R(a', a)$ then $a' \in \alpha_n$, and if $b \in \beta_{n+1}$ and $R(b', b)$, then $b' \in \beta_n$.

Proof of claim 2: In $\mathfrak{A}$, every element, other than the two that satisfy Q_0, has exactly one R-predecessor. So it immediately follows that every R-predecessor of $a \in \alpha_{n+1}$ is in α_n.

Now, take any $b \in \beta_{n+1}$ and b' such that $R(b', b)$. There must be some a such that $(a, b) \in \boldsymbol{\beta}$. By Claim 1, this implies that $a \in \alpha_{n+1}$. Furthermore, there is some a' such that $(a', b') \in \boldsymbol{\beta}$ and $(a', a) \mapsto (b', b)$ is a partial isomorphism. As $R(b', b)$, this implies that $R(a', a)$ as well, so we have $a' \in \alpha_n$. By Claim 1, this implies that $b' \in \beta_n$.

Claim 3: If $a_0 \in \alpha_0$ and $b_0 \in \beta_0$, then $(a_0, b_0) \in \boldsymbol{\beta}$.

Proof of claim 3: Let b_0 be one of the two elements of β_0. Then there is at least one $a_0 \in \alpha_0$ such that $(a_0, b_0) \in \boldsymbol{\beta}$. Let a_0' be the other element of α_0. This b_0 has exactly two R-successors $b_1, b_1' \in \beta_1$ (with $b_1 \neq b_1'$). There must be a_1, a_1' such that $(a_1, b_1) \in \boldsymbol{\beta}$ and $(a_1', b_1') \in \boldsymbol{\beta}$, and $(a_0, a_1) \mapsto (b_0, b_1)$ and $(a_0, a_1') \mapsto (b_0, b_1')$ are partial isomorphisms. From $R(b_0, b_1)$ and $R(b_0, b_1')$ it therefore follows that $R(a_0, a_1)$ and $R(a_0, a_0')$. As a_0 has exactly one R-successor, this implies that $a_0 = a_0'$.

Now, consider the pair (b_1, b_1'). We have $(a_1, b_1) \in \boldsymbol{\beta}$, so there must be some a_1'' such that $(a_1'', b_1') \in \boldsymbol{\beta}$ and $(a_1, a_1'') \mapsto (b_1, b_1')$ is a partial isomorphism. This implies that, in particular, $a_1 = a_1''$ if and only if $b_1 = b_1'$. We have $b_1 \neq b_1'$, so $a_1 \neq a_1''$. From $(a_1'', b_1') \in \boldsymbol{\beta}$ and Claim 1, it follows that $a_1'' \in \alpha_1$. Because $a_1 \neq a_1''$, this a_1'' must be the other element of α_1, which is the R-successor of a_0'.

Because $(a_1'', b_1') \in \boldsymbol{\beta}$ and $R(a_0', a_1'')$, there must be some b_0' such that $(a_0', b_0') \in \boldsymbol{\beta}$ and $R(b_0', b_1')$. But b_1' is an R-successor of b_0 and has only one R-predecessor, so $b_0' = b_0'$. Hence

$(a'_0, b'_0) \in \boldsymbol{\beta}$ implies $(a'_0, b_0) \in \boldsymbol{\beta}$. We have now shown that, for arbitrary $b_0 \in \beta_0$, we have $(a_0, b_0) \in \boldsymbol{\beta}$ and $(a'_0, b_0) \in \boldsymbol{\beta}$, with $a_0 \neq a'_0$. The claim follows immediately.

Claim 4: If $a_n \in \alpha_n$ and $b_n \in \beta_n$, then $(a_n, b_n) \in \boldsymbol{\beta}$.

Proof of claim 4: By induction. The base case is Claim 3. So assume as induction hypothesis that $n > 0$ and that Claim 4 holds for all $n' < n$. Take any $a_n \in \alpha_n$ and $b_n \in \beta_n$. There are $a_{n-1} \in \alpha_{n-1}$ and $\beta_{n-1} \in \beta_{n-1}$ such that $R(a_{n-1}, a_n)$ and $R(b_{n-1}, b_n)$. By the induction hypothesis, $(a_{n-1}, b_{n-1}) \in \boldsymbol{\beta}$. Now, there must be some a'_n such that $(a'_n, b_n) \in \boldsymbol{\beta}$ and $(a_{n-1}, a'_n) \mapsto (b_{n-1}, b_n)$ is a partial isomorphism. In particular, because $R(b_{n-1}, b_n)$, we have $R(a_{n-1}, a'_n)$.

We already had $R(a_{n-1}, a_n)$, and a_{n-1} has only one R-successor, so $a_n = a'_n$. As we have $(a'_n, b_n) \in \boldsymbol{\beta}$, this shows that $(a_n, b_n) \in \boldsymbol{\beta}$, thereby completing the induction step and therefore proving the claim.

The remainder of the proof proceeds in the same way as the proof of Proposition 7, so we omit it here. ⊣

Corollary 12. $\mathsf{C}^2/\mathsf{FO}^2$*-separability is RE-complete.*

D. $\mathcal{ALCQ}^u/\mathcal{ALC}^u$-separation is 2ExpTime-complete

In this section, we prove the following:

Theorem 13. $\mathcal{ALCQ}^u/\mathcal{ALC}^u$*-(Craig) separation is* 2ExpTime*-complete.*

We begin with the upper bound proof. Let L be one of the fragments of $\mathcal{ALCIQ}^u$ defined in Section A, ϱ a signature, and let C_1, C_2 be any L-concepts. To simplify presentation, we assume that C_1, C_2 contain only one role name $r \in \varrho$. It is straightforward to extend the upper bound proofs given in this and the next section to arbitrarily-many role names in and out of ϱ.

Denote by $\mathit{sub}(C_1, C_2)$ the closure under single negation of the set of subconcepts in C_1 and C_2. By a *type*, t, we mean any subset of $\mathit{sub}(C_1, C_2)$ for which there exists a structure $\mathfrak{A}$ with some $d \in \mathrm{dom}(\mathfrak{A})$ such that $t = \mathit{tp}(\mathfrak{A}, d)$, where

$$\mathit{tp}(\mathfrak{A}, d) = \{C \in \mathit{sub}(C_1, C_2) \mid d \in C^{\mathfrak{A}}\}$$

is the *type realised by* d *in* $\mathfrak{A}$. Denote by Tp the set of all types. Clearly, $|\mathit{Tp}| \leq 2^{|C_1|+|C_2|}$ and, moreover, Tp can be computed in time exponential in $|C_1| + |C_2|$ for all considered logics (fragments of $\mathcal{ALCIQ}^u$).

A *mosaic* is a set M of types. For structures $\mathfrak{A}_1, \mathfrak{A}_2$ and $i \in \{1, 2\}$, the mosaic defined by $d \in \mathrm{dom}(\mathfrak{A}_i)$ in $\mathfrak{A}_1, \mathfrak{A}_2$ is the set $T(d)$, where

$$T(d) = \{\mathit{tp}(\mathfrak{A}_j, e) \mid e \in \mathrm{dom}(\mathfrak{A}_j),\ \mathfrak{A}_i, d \sim_{\mathcal{ALC}^u(\varrho)} \mathfrak{A}_j, e, j = 1, 2\}.$$

We then say that M is a *mosaic defined by* $\mathfrak{A}_1, \mathfrak{A}_2$ if there exists $d \in \mathrm{dom}(\mathfrak{A}_1) \cup \mathrm{dom}(\mathfrak{A}_2)$ such that $M = T(d)$. Clearly, there are at most doubly exponentially many mosaics.

By Lemmas 5 and 3, we can prove the 2ExpTime upper bound of Theorem 13 by checking in double-exponential time whether there exist pointed structures $\mathfrak{A}_1, d_1$ and $\mathfrak{A}_2, d_2$ such that

$d_1 \in C_1^{\mathfrak{A}_1}$, $d_2 \in C_2^{\mathfrak{A}_2}$ and $\mathfrak{A}_1, d_1 \sim_{\mathcal{ALC}^u(\varrho)} \mathfrak{A}_2, d_2$, for $\varrho = sig(C_1) \cap sig(C_2)$. We do this by means of a mosaic-elimination procedure whose aim is to determine all mosaics M such that every $t \in M$ can be realised by mutually $\mathcal{ALC}^u(\varrho)$-bisimilar elements of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. To formulate the elimination conditions, we define several compatibility and existential saturation conditions between mosaics, which are similar to those used in standard type-elimination procedures.

We say that types t_1 and t_2 are *u-equivalent* when $\exists u.C \in t_1$ iff $\exists u.C \in t_2$, for every $\exists u.C \in \textit{sub}(C_1, C_2)$.

Let M be a mosaic and $\mathcal{M}$ a set of mosaics. We call $\mathcal{M}$ an *existential witness for M* if there is a *copying function f* and a binary *satisfying relation $\mathcal{R}$* such that f associates with every (t', M') with $t' \in M'$ a positive natural number $f(t', M')$ and, for

- $\Delta = \{(t, M) \mid t \in M\}$ and
- $\Gamma = \{((t', M'), j) \mid t' \in M',\ j < f(t', M'),\ M' \in \mathcal{M}\}$,

we have $\mathcal{R} \subseteq \Delta \times \Gamma$ and the following conditions are satisfied:

- if $(t, M)\mathcal{R}((t', M'), j)$, then t and t' are u-equivalent;
- for any $(\geq n\ r.C) \in \textit{sub}(C_1, C_2)$ and $(t, M) \in \Delta$, we have $(\geq n\ r.C) \in t$ iff (t, M) has at least n-many $\mathcal{R}$-successors $((t', M'), j)$ with $C \in t'$;
- for any $(\leq n\ r.C) \in \textit{sub}(C_1, C_2)$ and $(t, M) \in \Delta$, we have $(\leq n\ r.C) \in t$ iff (t, M) has at most n-many $\mathcal{R}$-successors $((t', M'), j)$ with $C \in t'$;
- for any $(t, M) \in \Delta$ and any $M' \in \mathcal{M}$, there exists an $\mathcal{R}$-successor of (t, M) of the form $((t', M'), j)$.

We can now define the mosaic elimination procedure. Let $\mathcal{S}$ be a set of mosaics. We call $M \in \mathcal{S}$ *bad* in $\mathcal{S}$ if it violates at least one of the following conditions:

1. $A \in t$ iff $A \in t'$, for all $A \in \varrho$ and all $t, t' \in M$;
2. there exists a set $\mathcal{M}$ of mosaics in $\mathcal{S}$ such that $\mathcal{M}$ is an existential witness for M;
3. if $\exists u.C \in t \in M$, then there exists $M' \in \mathcal{S}$ such that $C \in t' \in M'$ and t, t' are u-equivalent.

Lemma 14. *Given a set $\mathcal{S}$ of mosaics and $M \in \mathcal{S}$, it can be decided in double-exponential time in $|C_1| + |C_2|$ whether M is bad in $\mathcal{S}$.*

Let $\mathcal{S}_0$ be the set of all mosaics. Then we obtain the set $\mathcal{S}_{i+1}$ from $\mathcal{S}_i$, for $i \geq 0$, by eliminating mosaics that are bad in $\mathcal{S}_i$. Denote by $\mathcal{S}^*$ the resulting set without bad mosaics, which can be constructed in double-exponential time.

Lemma 15. *The following conditions are equivalent:*

1. *C_1 and C_2 are satisfied in $\mathcal{ALC}^u(\varrho)$-bisimilar pointed structures;*

2. *there is $M \in \mathcal{S}^*$ with $t_1, t_2 \in M$ such that $C_1 \in t_1$ and $C_2 \in T_2$.*

Proof. $(1 \Rightarrow 2)$ Suppose $\mathfrak{A}_1, d_1 \sim_{\mathcal{ALC}^u(\varrho)} \mathfrak{A}_2, d_2$, $d_1 \in C_1^{\mathfrak{A}_1}$ and $d_2 \in C_2^{\mathfrak{A}_2}$. Let $\mathcal{S}$ be the set of all mosaics defined by $\mathfrak{A}_1, \mathfrak{A}_2$. It is easy to see that $\mathcal{S}$ does not contain any bad M in $\mathcal{S}$, and so $\mathcal{S} \subseteq \mathcal{S}^*$.

$(2 \Rightarrow 1)$ We construct structures $\mathfrak{A}_1, \mathfrak{A}_2$ from $\mathcal{S}^*$ and a mosaic $M^* \in \mathcal{S}^*$ with $t_1^*, t_2^* \in M^*$ such that $C_1 \in t_1^*$ and $C_2 \in t_2^*$. For every $M \in \mathcal{S}^*$, take an existential witness $\mathcal{M}_M$ for M with a copying function F_M and a witnessing relation $\mathcal{R}_M \subseteq \Delta_M \times \Gamma_M$. The domain of $\mathfrak{A}_i$ contains all words

$$s = (t_0, M_0)(t_1, M_1, i_1) \dots (t_n, M_n, i_n) \tag{1}$$

such that

- $M_0 \in \mathcal{S}^*$;
- $t_0, t_1, \dots, t_n, t_i^*$ are u-equivalent;
- $t_j \in M_j$ for all $j \leq n$;
- $M_{j+1} \in \mathcal{M}_{M_j}$ and $i_{j+1} < f_{M_j}(t_{j+1}, M_{j+1})$.

We next interpret the concept names A by setting, for s defined by (1), $s \in A^{\mathfrak{A}_i}$ iff $A \in t_n$. We let $(s, s') \in r^{\mathfrak{A}_i}$, for s defined by (1) and

$$s' = (t'_0, M'_0)(t'_1, M'_1, i'_1) \cdots (t'_m, M'_m, i'_m) \in \text{dom}(\mathfrak{A}_i), \tag{2}$$

iff s is an initial part of s' with $m = n + 1$ and $(t_n, M_n)\mathcal{R}_{M_n}(t'_{n+1}, M'_{n+1}, i'_{n+1})$. It is not hard to check that, for all $C \in \mathit{sub}(C_1, C_2)$ and all s given by (1), we have $s \in C^{\mathfrak{A}_i}$ iff $C \in t_n$.

Finally, we define an $\mathcal{ALC}^u(\varrho)$-bisimulation $\boldsymbol{\beta}$ between $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by setting, for s given by (1) and s' by (2), $s\boldsymbol{\beta}s'$ iff $n = m$ and $M_i = M'_i$ for all $i \leq n$. One can show that $\boldsymbol{\beta}$ is a $\mathcal{ALC}^u(\varrho)$-bisimulation between $\mathfrak{A}_1$ and $\mathfrak{A}_2$, as required.

We now prove the 2ExpTime-lower bound, first for Craig-separation and then for separation. The proof is by reduction of acceptance for alternating Turing machines and is very similar to the proof that interpolant existence is 2ExpTime-hard for concept inclusions under $\mathcal{ALCH}$ ontologies given in [7].

An *alternating Turing machine (ATM)* is a tuple $M = (Q, \Theta, \Gamma, q_0, \Delta)$ where $Q = Q_\exists \uplus Q_\forall$ is a finite set of states partitioned into *existential states* $Q_\exists$ and *universal states* $Q_\forall$. Further, Θ is the input alphabet and Γ is the tape alphabet that contains a *blank symbol* $\Box \notin \Theta$, $q_0 \in Q_\forall$ is the *initial state*, and $\Delta \subseteq Q \times \Gamma \times Q \times \Gamma \times \{L, R\}$ is the *transition relation*. We assume without loss of generality that the set $\Delta(q, a) := \{(q', a', M) \mid (q, a, q', a', M) \in \Delta\}$ contains exactly two or zero elements for every $q \in Q$ and $a \in \Gamma$. Moreover, the state q' must be from $Q_\forall$ if $q \in Q_\exists$ and from $Q_\exists$ otherwise, that is, existential and universal states alternate. Acceptance of ATMs is defined in a slightly unusual way, without using accepting states. Intuitively, an ATM accepts if it runs forever on all branches and rejects otherwise. More formally, a *configuration* of an ATM is a word wqw' with $w, w' \in \Gamma^*$ and $q \in Q$. We say that wqw' is *existential* if q is, and likewise for *universal*. *Successor configurations* are defined in the usual way. Note that every configuration has exactly zero or two successor configurations. A *computation tree* of an ATM M on input w is a (possibly infinite) tree whose nodes are labeled with configurations of M such that

- the root is labeled with the initial configuration q_0w;
- if a node is labeled with an existential configuration wqw', then it has a single successor which is labeled with a successor configuration of wqw';
- if a node is labeled with a universal configuration wqw', then it has two successors which are labeled with the two successor configurations of wqw'.

An ATM M *accepts* an input w if there is a computation tree of M on w. Note that we can convert any ATM M in which acceptance is based on accepting states to our model by assuming that M terminates on any input and then modifying it to enter an infinite loop from the accepting states. It is well-known that there are 2^n-space bounded ATMs which recognize a 2ExpTime-hard language [20], where n is the length of the input w.

Let us fix such an ATM $M = (Q, \Theta, \Gamma, q_0, \Delta)$ and an input $w = a_0 \dots a_{n-1}$ of length n. We aim to construct C_1, C_2 such that M accepts w iff C_1, C_2 are satisfied in $\mathcal{ALC}^u(\varrho)$-bisimilar pointed models, where

$$\varrho = \{r, s, Z, B_\forall, B^1_\exists, B^2_\exists\} \cup \{A_\sigma \mid \sigma \in \Gamma \cup (Q \times \Gamma)\}.$$

C_1 generates an r-chain of length n as follows:

$$C_1 = \exists r^n.\top \sqcap \forall r^n.\forall r.\bot \sqcap \bigwedge_{0 \leq i < n} \forall r^i(= 1\ r.\top)$$

C_2 first generates a binary r-tree of depth n with leafs satisfying counter values from 0 to $2^n - 1$ represented by concepts names $A_0, \dots, A_{n-1}$. Hence C_1 contains the following conjuncts:

1. $\forall r^i.(= 2\ r.\top)$ for $0 \leq i < n$;
2. $\forall r^i.(\exists r.A_i \sqcap \exists r.\neg A_i)$ for $0 \leq i < n$;
3. $\forall r^i.\big((A_j \to \forall r.A_j) \sqcap (\neg A_j \to \forall r.\neg A_j)\big)$, for $0 \leq j < i < n$.

Next, we make a concept name L_R true in all leafs of the tree

$$\forall r^n.L_R$$

and start from the leafs s-trees with two counters, realised using concept names U_i and V_i, $0 \leq i < n$, and initialised to 0 and the value of the $A_0, \dots, A_{n-1}$ counter, respectively:

1. $\forall u.(L_R \to (U = 0))$;
2. $\forall u.(L_R \to (A_j \leftrightarrow V_j))$ for $0 \leq j < n$;
3. $\forall u.(L_R \to \exists s.\top)$.

Observe that if $d \in C_1^{\mathfrak{A}_1}$, $e \in {C'_2}^{\mathfrak{A}_2}$, and $\mathfrak{A}_1, d \sim_{\mathcal{ALC}^u(\varrho)} \mathfrak{A}_2, e$ for C'_2 the conjuncts of C_2 constructed up to now, then there is a node d' in $\mathfrak{A}_1$ which is reachable via an r-chain of length n from d and there are leafs $e_0, \ldots, e_{2^n-1} \in L_R^{\mathfrak{A}_2}$ of the binary tree in $\mathfrak{A}_2$ with root e such that

$$\mathfrak{A}_1, d' \sim_{\mathcal{ALC}^u(\varrho)} \mathfrak{A}_2, e_0 \sim_{\mathcal{ALC}^u(\varrho)} \cdots \sim_{\mathcal{ALC}^u(\varrho)} \mathfrak{A}_2, e_{2^n-1}.$$

Now we add further conjuncts to C_2 that start from L_R s-trees encoding the computation of M on input w using the counters U and V in exactly the same way as in [7] such that the following conditions are equivalent:

1. M accepts w;
2. there exist structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that $\mathfrak{A}_1, d \sim_{\mathcal{ALC}^u(\varrho)} \mathfrak{A}_2, e$, for some $d \in C_1^{\mathfrak{A}_1}$ and $e \in C_2^{\mathfrak{A}_2}$.

This completes the proof of 2ExpTime-hardness of Craig separation. To prove 2ExpTime-hardness of separation, we replace all concept names A in C_2 that are not in ϱ by $(= 2 r_A.\top)$ with R_A a fresh role name and denote the resulting concept by D_2. Let σ be the set of all concept and role names in C_1, D_2. Then one can show that the following conditions are equivalent:

1. there exist structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that $\mathfrak{A}_1, d \sim_{\mathcal{ALC}^u(\varrho)} \mathfrak{A}_2, e$, for some $d \in C_1^{\mathfrak{A}_1}$ and $e \in C_2^{\mathfrak{A}_2}$.
2. there exist structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ such that $\mathfrak{A}_1, d \sim_{\mathcal{ALC}^u(\sigma)} \mathfrak{A}_2, e$, for some $d \in C_1^{\mathfrak{A}_1}$ and $e \in D_2^{\mathfrak{A}_2}$.

This complete the hardness proof. ⊣

E. $\mathcal{ALCIQ}^u/\mathcal{ALCI}^u$ separation is 2ExpTime-complete

In this section, we prove the following:

Theorem 16. *$\mathcal{ALCIQ}^u/\mathcal{ALCI}^u$-(Craig) separation is 2ExpTime-complete.*

The lower bound proof is essentially the same as in the previous section. Hence we focus on the upper bound proof. Assume $\mathcal{ALCIQ}^u$-concepts C_1, C_2 and a signature ϱ are given. As before, we assume that C_1, C_2 contain only one role name $r \in \varrho$. It is straightforward to extend the upper bound proofs given in this section to arbitrarily many role names in and out of ϱ. We use the notion of types introduced in the previous section. Other notions such as that of a mosaic are a bit different. Let m_{C_1,C_2} be the maximal parameter in C_1, C_2.

We first introduce the notion of an *extended type* as a pair (t, P) with t a type and P a set of expressions containing, for each type t', exactly one expression of the form $(= n\ r.t')$ with $0 \leq n \leq m_{C_1,C_2}$ or $(> m_{C_1,C_2}\ r.t')$ and exactly one expression of the form $(= n\ r^-.t')$ with $0 \leq n \leq m_{C_1,C_2}$ or $(> m_{C_1,C_2}\ r^-.t')$. The semantics of these expressions is defined in the obvious way. Note that t and $(= n\ s.t')$ with $n > 0$ can only be satisfied if t and t' are u-equivalent.

For $s \in \{r, r^{-1}\}$, an extended type specifies exactly how many s-successors there are that satisfy type t'. In some places we only care about a coarser distinction, however, namely whether the number of s-successors that have type t' is greater than 0. We therefore write $(> 0\ s.t') \in P$ as a shorthand for '$(= n\ s.t') \in P$ for some $1 \leq n \leq m_{C_1,C_2}$ or $(> m_{C_1,C_2}\ s.t') \in P$'.

The *extended type* $(tp(\mathfrak{A}, d), e(\mathfrak{A}, d, D))$ *realised in* $\mathfrak{A}$ *at a pair* (d, D) with $d \in \text{dom}(\mathfrak{A})$ and $D \subseteq \text{dom}(\mathfrak{A})$ is defined by setting

- $(= n\ s.t') \in \text{e}(\mathfrak{A}, d, D)$ if $n \leq m_{C_1,C_2}$ is the number of $d' \in D$ with $(d, d') \in s^{\mathfrak{A}}$ such that $t' = \text{tp}(\mathfrak{A}, d')$;
- $(> m_{C_1,C_2}\ s.t') \in \text{e}(\mathfrak{A}, d, D)$ if the number of $d' \in D$ with $(d, d') \in s^{\mathfrak{A}}$ such that $t' = \text{tp}(\mathfrak{A}, d')$ is larger than m_{C_1,C_2}.

An extended type (t, P) is called a *root* if $n = 0$ for all $(= n\ s.t') \in P$ and P contains no $(> m_{C_1,C_2} s.t')$.

We are going to construct bisimilar models that are (almost) tree-shaped. When constructing these models, we distinguish between types that are realised in a parent node and types that are realised in child nodes. Hence we define a *two-way extended type* as a tuple of the form (t, E) with $E = (E_{up}, E_{down})$, where (t, E_{up}) and (t, E_{down}) are extended types.

Given $d \in \mathfrak{A}$ and $D \subseteq \text{dom}(\mathfrak{A})$, the *two-way extended type* realised in $\mathfrak{A}$ at (d, D), is defined as

$$\text{tw}(\mathfrak{A}, d, D) = (\text{tp}(\mathfrak{A}, d), \text{e}(\mathfrak{A}, d, D), \text{e}(\mathfrak{A}, d, \text{dom}(\mathfrak{A}) \setminus D))$$

We next aim at spelling out when adding up the number of witnesses for the number restrictions in E_{up} and E_{down} ensures that we satisfy the number restrictions in t. Assume $P \in \{E_{up}, E_{down}\}$ is given. The *s-profile of* P, $Pr_s(P)$ contains for every $C \in \text{sub}(C_1, C_2)$

- $(= n\ s.C)$ if n is the sum over all k with $(= k\ s.t') \in P$ and $C \in t'$;
- $(> m_{C_1,C_2}\ s.C)$ if the sum above exceeds m_{C_1,C_2} or $(> m_{C_1,C_2}\ s.t') \in P$ for some t' with $C \in t'$.

The *joint s-profile of* E_{up}, E_{down}, $Pr_s(E_{up}, E_{down})$, contains for $C \in \text{sub}(C_1, C_2)$

- $(= n\ s.C)$ if $n = n_1 + n_2 \leq m_{C_1,C_2}$ for $(= n_1\ s.C)$ in the s-profile of E_{up} and $(= n_2\ s.C)$ in the s-profile of E_{down};
- $(> m_{C_1,C_2}\ s.C)$ otherwise

We call (t, E_{up}, E_{down}) *consistent* if for $s \in \{r, r^-\}$

- if $(\geq n\ s.C) \in t$ and $(= n'\ s.C) \in Pr_s(E_{up}, E_{down})$, then $n' \geq n$,
- if $(\leq n\ s.C) \in t$ and $(= n'\ s.C) \in Pr_s(E_{up}, E_{down})$, then $n' \leq n$,
- if $(\leq n\ s.C) \in t$, then $(> m_{C_1,C_2}\ s.C) \notin Pr_s(E_{up}, E_{down})$.
- if $(> 0\ s.t')$ occurs in $E_{up} \cup E_{down}$, then t and t' are u-equivalent.

Clearly any realised two-way extended type $\mathrm{tw}(\mathfrak{A}, d, D)$ is consistent.

A *mosaic* M is a set of consistent two-way extended types. A mosaic is a root if (t, E_{up}) is a root for all $(t, E_{up}, E_{down}) \in m$.

For structures $\mathfrak{A}_1, \mathfrak{A}_2$ and $i \in \{1, 2\}$, the mosaic defined by (d, d') with $d, d' \in \mathrm{dom}(\mathfrak{A}_i)$ for some $i \in \{1, 2\}$ in $\mathfrak{A}_1, \mathfrak{A}_2$, is defined as

$$T(d, d') = \{\mathrm{tw}(\mathfrak{A}_j, e, D) \mid e \in \mathrm{dom}(\mathfrak{A}_j), \mathfrak{A}_i, d \sim_{\mathcal{ALCI}^u,\varrho} \mathfrak{A}_j, e, j \in \{1, 2\}\}\},$$

for $D = \{e' \in \mathrm{dom}(\mathfrak{A}_j) \mid \mathfrak{A}_i, e' \sim_{\mathcal{ALCI}^u,\varrho} \mathfrak{A}_j, d'\}$, for $j = 1, 2$.

We say that a mosaic M of two-way extended types is a *mosaic defined by* $\mathfrak{A}_1, \mathfrak{A}_2$ if there exist $d, d' \in \mathrm{dom}(\mathfrak{A}_1) \cup \mathrm{dom}(\mathfrak{A})$ such that $m = (T(d, d')$.

Observe that such a mosaic is a root iff there do not exist d, d' defining m with $(d, d') \in s^{\mathfrak{A}_1}$ for some $s \in \{r, r^-\}$.

Given a mosaic M, we now provide criteria for when a set $\mathcal{M}$ of mosaics provides appropriate witnesses for the number restrictions in E_{down} for all two-way extended types $(t, E_{up}, E_{down}) \in M$. For extended types (t, P) and (t', P'), we write $(t, P) \to_s (t', P')$ and say that (t, P) and (t', P') are *s-coherent* if $(> 0 \ s.t') \in P$ and $(> 0 \ s^{-1}.t) \in P'$ and t, t' are u-equivalent. Intuitively, s-coherence captures that we can draw an s-edge between nodes representing (t, P) and (t', P'). For two-way extended types (t, E) and (t', E') we write $(t, E) \to_s (t', E')$ and say that they are *s-coherent* if $(t, E_{down}) \to_s (t', E'_{up})$.

Let M be a mosaic and $\mathcal{M}$ a set of mosaics, in what follows we assume the two-way extended types they contain are consistent. Then $\mathcal{M}$ is a *syntactic existential witness* for M if there are relations $r, r^{-1} \subseteq M \times \{((t', E'), M') \mid (t', E') \in M'\}$ such that for every $s \in \{r, r^{-1}\}$

1. if $((t, E), ((t', E'), M')) \in s$ then $(t, E) \to_s (t', E')$,
2. for every $M' \in \mathcal{M}$, either
   - for every $(t, E) \in M$ there is some $(t', E') \in M'$ such that $((t, E), ((t', E'), M')) \in s$ and
   - for every $(t', E') \in M'$ there is some $(t, E) \in M$ such that $((t, E), ((t', E'), M')) \in s$,

   in which case we say that M' is an s-successor of M, or
   - for every $(t, E) \in M$ and every $(t', E') \in M'$, $((t, E), ((t', E'), M')) \notin s$

   in which case M' is not an s-successor of M,
3. for every $(t, E) \in M$, if $(= n \ s.t') \in E_{down}$ or $(> m_{C_1,C_2} \ s.t') \in E_{down}$ then there are E' and M' such that $((t, E), ((t', E'), M')) \in s$,
4. for every $M' \in \mathcal{M}$ and every $(t', E') \in M'$, if $(= n \ s^{-1}.t) \in E'_{up}$ or $(> m_{C_1,C_2} \ s^{-1}.t) \in E_{up}$, then there is some $(t, E) \in M$ such that $((t, E), ((t', E'), M')) \in s$.
5. for every $((t, E), ((t', E'), M')) \in s$, there is a function f that assigns to each s-successor M'' of M an extended type $(t'', E'') \in M''$ such that
   - $f(M') = (t', E')$,

- if $(= n\ s.t'') \in E_{down}$ then $|\{M'' \mid \exists E'' : f(M'') = (t'', E'')\}| \leq n$.

We next provide a definition of existential witnesses that is closer to the intended semantics than syntactic existential witnesses. We call $\mathcal{M}$ an *existential witness* for M if there are relations $\boldsymbol{r}, \boldsymbol{r}^{-1} \subseteq \Delta \times \Gamma$, where $\Delta = \{((t, E), j) \mid j < \omega, (t, E) \in M\}$ and $\Gamma = \{((t', E'), j, M') \mid j < \omega, (t', E') \in M', M' \in \mathcal{M}\}$ such that for every $\boldsymbol{s} \in \{\boldsymbol{r}, \boldsymbol{r}^{-1}\}$,

- if $((t, E), j) \in \Delta$ and $(= n\ s.t') \in E_{down}$ then there are exactly n different E', k and M' such that $((t', E'), k, M')$ is an $\boldsymbol{s}$-successor of $((t, E), j)$,
- if $((t, E), j) \in \Delta$ and $(> m_{C_1,C_2}\ s.t') \in E_{down}$ then there are more than m_{C_1,C_2} different E', k and M' such that $((t', E'), k, M')$ is an $\boldsymbol{s}$-successor of $((t, E), j)$,
- if $((t', E'), k, M') \in \Gamma$ and $(= n\ s^{-1}.t) \in E'_{up}$, then there are exactly n different E and j such that $((t', E'), k, M')$ is an $\boldsymbol{s}$-successor of $((t, E), j)$,
- if $((t', E'), k, M') \in \Gamma$ and $(> m_{C_1,C_2}\ s^{-1}.t) \in E'_{up}$, then there are more than m different E and j such that $((t', E'), k, M')$ is an $\boldsymbol{s}$-successor of $((t, E), j)$,
- if $((t, E), j) \in \Delta$ has an s-successor $((t', E'), k, M') \in \Gamma$ then for every $(t^*, E^*), j^*) \in \Delta$ there are t'^*, E'^* and k^* such that $((t'^*, E'^*), k^*, M')$ is an $\boldsymbol{s}$-successor of $((t^*, E^*), j^*)$,
- if $((t', E'), k, M') \in \Gamma$ has an s-predecessor $((t, E), j) \in \Delta$ then for every $((t'^*, E'^*), k^*, M') \in \Gamma$ there are t^* and j^* such that $((t'^*, E'^*), k^*, M')$ is an $\boldsymbol{s}$-successor of $((t^*, E^*), j^*)$.

Lemma 17. *If $\mathcal{M}$ is a syntactic existential witness for M then $\mathcal{M}$ is an existential witness for M.*

Proof. Let r, r^{-1} be relations witnessing the fact that $\mathcal{M}$ is a syntactic existential witness for M, and enumerate all elements of $\Gamma \cup \Delta$. We will add $\boldsymbol{r}$ and $\boldsymbol{r}^{-1}$ edges for each $x \in \Gamma \cup \Delta$ in enumeration order. Throughout the process, we will keep invariant the property that, by the time we reach $x \in \Gamma \cup \Delta$, there will be at most one $\boldsymbol{r}$ or $\boldsymbol{r}^{-1}$ edge to or from x already. We first treat the two cases where x does not have any edges yet, then the two cases where one edge has already been added.

Case 1: Suppose $x = ((t, E), j) \in \Delta$ and no edges from x have been added yet. For every $s \in \{r, r^{-1}\}$, do the following.

Let $\mathcal{M}_s \subseteq \mathcal{M}$ be the set of s-successors of M. Take any $M' \in \mathcal{M}_s$. Then there is some $(t', E') \in M'$ such that $((t, E), ((t', E'), M')) \in s$. Let f be the assignment function for $((t, E), ((t', E'), M')) \in s$, and let $\mathcal{E} = \{((t'', E''), M'') \mid f(M'') = (t'', E'')\}$.

Consider any t'' such that $(= ns.t'') \in E_{down}$. It is a property of f that there are at most n elements in $\mathcal{E}$ that have a t'' component. If there are less than n such elements, take any $((t'', E''), M'')$ such that $((t, E), ((t'', E''), M'')) \in s$, and add enough copies of this $((t'', E''), M'')$ to $\mathcal{E}$ to make the total number of elements with a t'' component exactly n. We turn $\mathcal{E}$ into a multi-set by doing this. Note that t'', E'', M'' with the required property exist by condition 3 of syntactic existential witness.

Similarly, if $(> m_{C_1,C_2}\ s.t'') \in E_{down}$ and there are $\leq M_{C_1,C_2}$ elements in $\mathcal{E}$ with t'' component, add copies of $((t'', E''), M'')$ until we have more than m_{C_1,C_2} such elements.

Now, for every $((t', E'), M') \in \mathcal{E}$, add an $\boldsymbol{s}$-edge from $x = ((t, E), j)$ to the lowest (in the enumeration) element $((t', E'), k, M') \in \Gamma$ that does not yet have any edges to it.

Case 2: Suppose $x = ((t', E'), k, M') \in \Gamma$ and no edges to x have been added yet. For every $s \in \{r, r^{-1}\}$, do the following.

If $(= \ n \ \ s^{-1}.t) \in E'_{up}$ or $(> \ m_{C_1,C_2} \ \ s^{-1}.t) \in E'_{up}$, let (t, E) be such that $((t, E), ((t'E'), M')) \in s$. Create $\boldsymbol{s}$ edges to x from the first n (if $(= \ n \ \ s^{-1}.t) \in E'_{up}$) or the first $m_{C_1,C_2} + 1$ (if $(> m_{C_1,C_2} \ s^{-1}.t) \in E'_{up}$) elements of the form $((t, E), j) \in \Delta$ that do not yet have any edges going from them.

Case 3: Suppose $x = ((t, E), j) \in \Delta$ and there is already an $\boldsymbol{s}$ edge from x to $((t', E'), k, M') \in \Gamma$. Add edges for $\boldsymbol{s}^{-1}$ as in case 1. With regard to $\boldsymbol{s}$, add edges like in case 1 except that instead of taking any s-successor we now take the successor $((t', E'), M')$ as a starting point, and take the already existing $\boldsymbol{s}$ edge from $((t, E), j)$ to $((t', E'), k, M')$ instead of one of the edges that we would otherwise add.

Case 4: Suppose $x = ((t', E'), k, M') \in \Gamma$ and there is already an $\boldsymbol{s}$ edge from $((t, E), j)$ to x. Add edges for $\boldsymbol{s}^{-1}$ as in case 2. For $\boldsymbol{s}$, do the same except we take the edge from $((t, E), j)$ to x instead of one edge that would otherwise be added.

It is immediate from the construction of $\boldsymbol{r}$ and $\boldsymbol{r}^{-1}$ that the number restriction conditions of an existential witness are satisfied. The bisimilarity conditions follow from $\mathcal{M}$ being a syntactic existential witness. $\dashv$

The converse direction should be obvious.

Lemma 18. *If $\mathcal{M}$ is a existential witness for M then $\mathcal{M}$ is a syntactic existential witness for M.*

We are going to construct bisimilar models starting with a root mosaic, then satisfying number restrictions in the mosaic by adding existential witnesses, then satisfying number restrictions in the E_{down}-part of the fresh mosaics in existential witnesses, and so on.

Extended types (t, P) and (t', P') are called *profile-equivalent* if $t = t'$ and $Pr_s(P) = Pr_s(P')$ for $s \in \{r, r^-\}$. Two-way extended types (t, E_{up}, E_{down}) and (t', E'_{up}, E'_{down}) are

- *up profile-equivalent* if (t, E_{up}) and (t', E'_{up}) are profile-equivalent;
- *down profile-equivalent* if (t, E_{down}) and (t', E'_{down}) are profile-equivalent;

We say that mosaics M, M' are

- *down profile-equivalent* if for any $(t, E_{up}, E_{down}) \in M$ there exists $(t', E'_{up}, E'_{down}) \in M'$ which is down profile-equivalent and vice versa;
- *up profile-equivalent* if for any $(t, E_{up}, E_{down}) \in M$ there exists $(t', E'_{up}, E'_{down}) \in M'$ which is up profile-equivalent and vice versa.
- *down/up profile-equivalent* if for any $(t, E_{up}, E_{down}) \in M$ there exists $(t', E'_{up}, E'_{down}) \in M'$ such that (t, E_{down}) and (t', E'_{up}) are profile-equivalent and vice versa.

Lemma 19. *Assume that $\mathcal{M}$ is a syntactic existential witness for M, and let $\{r, r^{-1}\}$ be the witnessing relations. Let $\mathcal{M}' \subseteq \mathcal{M}$ be such that for every $s \in \{r, r^{-1}\}$, every $M' \in \mathcal{M} \setminus \mathcal{M}'$, every $(t, E) \in M$ and every $(t', E') \in M'$ such that M' is an s-successor of M and $(t, E) \to_s$*

(t', E') there are some $M'' \in \mathcal{M}'$ and $(t', E'') \in M''$ such that M'' is an s-successor of M and $(t, E) \to_s (t', E'')$.

Then $\mathcal{M}'$ is also a syntactic existential witness for M.

Proof. We obtain witnessing relations $\{r', r'^{-1}\}$ for M and $\mathcal{M}'$ be replacing any edge $((t, E), ((t', E'), M')) \in s$ where $M' \in \mathcal{M} \setminus \mathcal{M}'$ by the edge $((t, E), ((t', E''), M''))$. $\dashv$

Lemma 20. *Assume that $\mathcal{M}$ is a syntactic existential witness for M, and let $\{r, r^{-1}\}$ be the witnessing relations. Let $\mathcal{M}'$ be obtainable from $\mathcal{M}$ be replacing some $M' \in \mathcal{M}$ by any M'' with the properties that (i) $M'' \subseteq M'$ and (ii) for every $s \in \{r, r^{-1}\}$, every $(t, E) \in M$ and every $(t', E') \in M'$, if $(t, E) \to_s (t', E')$ then there is a $(t', E'') \in M''$ such that $(t, E) \to_s (t', E'')$.*

Then $\mathcal{M}'$ is also a syntactic existential witness for M.

Proof. We obtain witnessing relations $\{r', r'^{-1}\}$ for M and $\mathcal{M}'$ be replacing any edge $((t, E), ((t', E'), M')) \in s$ by the edge $((t, E), ((t', E''), M'))$. $\dashv$

Lemma 21. *Assume that $\mathcal{M}$ is a syntactic existential witness for M, and let $\{r, r^{-1}\}$ be the witnessing relations. Let $M^* \subseteq M$ be such that for every $s \in \{r, r^{-1}\}$, every $M' \in \mathcal{M}$, every $(t, E) \in M$ and $(t', E') \in M'$, if $((t, E), ((t', E'), M')) \in s$ then there is some $(t, E^*) \in M^*$ such that $((t, E^*), ((t', E'), M')) \in s$.*

Then $\mathcal{M}$ is a syntactic existential witness for M^*.

Proof. The witnessing relations r^* and r^{*-1} are simply the restrictions of r and r^{-1} to M^*.$\dashv$

Lemma 22. *Assume that $\mathcal{M}$ is a syntactic existential witness for M. Then there are $\mathcal{M}'$ and $M^* \subseteq M$ such that for any $N' \in \mathcal{M}'$ there exists $M' \in \mathcal{M}$ with $N' \subseteq M'$ such that*

- *$\mathcal{M}'$ is a syntactic existential witness for M';*
- *$|\mathcal{M}'| \leq K_1$, where K_1 is exponential in $|C_1| + |C_2|$;*
- *$|M^*| \leq K_2$, where K_2 is exponential in $|C_1| + |C_2|$;*
- *for all $N' \in \mathcal{M}'$, $|N'| \leq K_3$, where K_3 is exponential in $|C_1| + |C_2|$;*
- *M and M^* are up profile-equivalent;*
- *for every $N' \in \mathcal{M}'$ there exists a down profile-equivalent $M' \in \mathcal{M}$.*

Proof. For any $M' \in \mathcal{M}$, let $N' \subseteq M'$ be such that (i) for every $(t', E') \in M'$ there is a down profile-equivalent $(t', E'') \in N'$, (ii) for every t, if there is a $(t', E') \in M'$ such that $(> 0s.t') \in E'_{up}$ then there is a $(t', E'') \in N'$ such that $(> 0s.t') \in E''_{up}$ and (iii) N' is at most exponential in size. Such N' exists because there are at most exponentially many profiles and exponentially many types, and we need at most one witness for each.

Let $\mathcal{N}$ be the result of replacing every $M' \in \mathcal{M}$ by the corresponding N'. By Lemma 20, $\mathcal{N}$ is a syntactic existential witness for M. Let r, r^{-1} be the witnessing relations.

Now, let $\mathcal{M}'$ be a subset of $\mathcal{N}$ such that (i) for every every $s \in \{r, r^{-1}\}$, every $M' \in \mathcal{N} \setminus \mathcal{M}'$, every $(t, E) \in M$ and every $(t', E') \in M'$ such that M' is an s-successor of M and $(t, E) \to_s$

(t', E') there are some $M'' \in \mathcal{M}'$ and $(t', E'') \in M''$ such that M'' is an s-successor of M and $(t, E) \to_s (t', E'')$ and (ii) $\mathcal{M}'$ is at most exponential in size. Such $\mathcal{M}'$ exists because for every $s \in \{r, r^{-1}\}$ and every t, t' we need only one witnessing mosaic. Furthermore, by Lemma 19, $\mathcal{M}'$ is a syntactic existential witness for M.

Now, let M^* be a subset of M such that (i) for every $(t, E) \in M$ there is an up profile-equivalent $(t, E^*) \in M^*$, (ii) for every $s \in \{r, r^{-1}\}$ and every $((t, E), ((t', E'), M')) \in s$, there is a $((t, E^*), ((t', E'), M')) \in s$ with $(t, E^*) \in M^*$ and (iii) M^* is at most exponential in size. such M^* exists because there are at most exponentially-many profiles and exponentially many t and $((t', E'), M')$, and hence we need at most exponentially-many E^* such that $((t, E^*), ((t', E'), M')) \in s$. By Lemma 21, $\mathcal{M}'$ is a syntactic existential witness for M^*. $\dashv$

Next we define how existential witnesses are extracted from bisimilar models. We now return to the view of mosaics as tuples (T_1, T_2). Assume $\mathfrak{A}_1, d_1 \sim_{\mathcal{ALCI}^u(\varrho)} \mathfrak{A}_2, d_2$. Consider (d, d') with $d, d' \in \text{dom}(\mathfrak{A}_i)$ realising a mosaic M. Then let $F_{d,d'}$ be the set of pairs (f, d) such that

- $(d, f) \in s^{\mathfrak{A}_i}$ for some $s \in \{r, r^{-}\}$;
- $\mathfrak{A}_i, f \not\sim_{\mathcal{ALCI}^u(\varrho)} \mathfrak{A}_i, d'$.

Let $\mathcal{M}$ be the set of mosaics realised by pairs in $F_{d,d'}$. It is straightforward to show that $\mathcal{M}$ is an existential witness for M. We say that $\mathcal{M}$ is a existential witness defined by $\mathfrak{A}_1, \mathfrak{A}_2$ for M. (Note that there can be many existential witnesses for one and the same M defined by $\mathfrak{A}_1, \mathfrak{A}_2$ as M could be realised in many different ways.)

We are now in a position to give the mosaic elimination procedure. Let $\mathcal{S}_0$ be the set of all mosaics M containing only consistent two-way extended types and with $|M| \leq (K_2 + K_3)^2$. Clearly $\mathcal{S}_0$ can be computed in double exponential time in $|C_1| + |C_2|$.

Let $\mathcal{S} \subseteq \mathcal{S}_0$. We call $M \in \mathcal{S}$ *bad* in $\mathcal{S}$ if it violates at least one of the following conditions:

1. $A \in t$ iff $A \in t'$ for all $A \in \varrho$ and all $(t, E), (t', E') \in M$;
2. there exists a set $\mathcal{M} \subseteq \mathcal{S}$ with $|\mathcal{M}| \leq K_1$ such that $\mathcal{M}$ is a syntactic existential witness for M.
3. $\exists u.C \in t$ for some $(t, E) \in M$ then there exists a root mosaic $M' \in \mathcal{S}$ such that $C \in t'$ for some $(t', E') \in M$ such that t and t' are u-equivalent.

We compute a sequence $\mathcal{S}_0, \mathcal{S}_1, \ldots$, where we obtain, for $i \geq 0$, the set $\mathcal{S}_{i+1}$ from $\mathcal{S}_i$ by eliminating all mosaics that are bad in $\mathcal{S}_i$. Let $\mathcal{S}^*$ be the set of mosaics where the sequence stabilises. Clearly $\mathcal{S}^*$ is obtained from $\mathcal{S}_0$ in at most double exponential time.

Lemma 23. *The following conditions are equivalent:*

1. *C_1, C_2 are satisfied in $\mathcal{ALCI}^u(\varrho)$-bisimilar pointed structures.*
2. *There is a root mosaic $M \in \mathcal{S}^*$ and $(t_1, E_1), (t_2, E_2) \in M$ with $C_1 \in t_1$ and $C_2 \in T_2$.*

Proof. Assume Point 1 holds. Assume $\mathfrak{A}_1, d_1 \sim_{\mathcal{ALCI}^u(\varrho)} \mathfrak{A}_2, d_2$ and $d_1 \in C_1^{\mathfrak{A}_1}$ and $d_2 \in C_2^{\mathfrak{A}_2}$.

Consider the set of all pairs $p = (M, \mathcal{M}_M)$ such that $\mathcal{M}_M$ is an existential witness for M defined by $\mathfrak{A}_1, \mathfrak{A}_2$. Next apply Lemma 22 to these pairs and obtain the pairs $(M^p, \mathcal{M}_M^p)$. We call any M^p *down-good* (as we have found the relevant witnesses for number restrictions in children) and all mosaics in any $\mathcal{M}_M^p$ *up-good* (as we have found the relevant witnesses for number restrictions in parents). Now define for any up-good M_1 and down-good M_2 which are up/down profile equivalent a fresh mosaic M_{M_1,M_2} by setting $(t, E_{up}, E_{down}) \in M_{M_1,M_2}$ iff there are $(t, E_{up}, E'_{down}) \in M_1$ and $(t, E'_{up}, E_{down}) \in M_2$ such that (t, E'_{down}) and (t, E'_{up}) are profile equivalent. Let $\mathcal{S}$ be the set of all these mosaics. Then by construction $\mathcal{S} \subseteq \mathcal{S}_0$ and $\mathcal{S}$ does not contain any mosaics that are bad in $\mathcal{S}$. Hence $\mathcal{S} \subseteq \mathcal{S}^*$ and Point 2 follows.

Conversely, assume that Point 2 holds. We construct structures $\mathfrak{A}_1, \mathfrak{A}_2$ from $\mathcal{S}^*$ and the root mosaic $M^* \in \mathcal{S}^*$ with $(t_1^*, E_1^*), (t_2^*, E_2^*) \in M^*$ with $C_1 \in t_1^*$ and $C_2 \in t_2^*$.

Take for every $M \in \mathcal{S}^*$ an existential witness $\mathcal{M}_M \subseteq \mathcal{S}^*$ for M and witnessing relations $\boldsymbol{r}_M, \boldsymbol{r}_M^{-1} \subseteq \Delta_M \times \Gamma_M$.

The domain of $\mathfrak{A}_i$ contains all words

$$s = ((t_0, E_0), j_0, M_0)((t_1, E_1), j_1, M_1) \cdots ((t_n, E_n), j_n, M_n) \tag{3}$$

such that

- $M_0 \in \mathcal{S}^*$ is a root mosaic;
- $t_0, t_1, \ldots, t_n, t_i^*$ are all u-equivalent;
- $j_\ell < \omega$, for all $\ell \leq n$;
- $(t_\ell, E_\ell) \in M_\ell$, for all $\ell \leq n$;
- $M_{\ell+1} \in \mathcal{M}_{M_\ell}$, for all $\ell < n$.

We next interpret the concept names A by setting for s of the form (3), $s \in A^{\mathfrak{A}_i}$ if $A \in t_n$. We let $(s, s') \in r^{\mathfrak{A}_i}$ for s of the form (3) and

$$s' = ((t'_0, E'_0), j'_0, M'_0)((t'_1, E'_1), j'_1, M'_1) \cdots ((t'_m, E'_m), j'_m, M'_m) \in \text{dom}(\mathfrak{A}_i) \tag{4}$$

if s is an initial part of s' with $m = n + 1$ and

- $((t_n, E_n), j_n, M_n)\boldsymbol{r}_{M_n}((t'_{n+1}, E'_{n+1}), j'_{n+1}, M'_{n+1})$ or
- $((t'_{n+1}, E'_{n+1}), j'_{n+1}, M'_{n+1})\boldsymbol{r}_{M_n}^{-1}((t_n, E_n), j_n, M_n)$.

One can now prove by induction on the construction of C the following

Claim. For all $C \in \text{sub}(C_1, C_2)$ and all s given by (3), $s \in C^{\mathfrak{A}_i}$ iff $C \in t_n$.

We define the $\mathcal{ALCI}^u(\varrho)$-bisimulation $\boldsymbol{\beta}$ by setting for s of the form (3) and s' of the form (4), $s\boldsymbol{\beta}s'$ iff $n = m$ and $M_i = M'_i$ for all $i \leq n$.

Claim. $\boldsymbol{\beta}$ is a $\mathcal{ALCI}^u(\varrho)$-bisimulation between $\mathfrak{A}_1$ and $\mathfrak{A}_2$

This finishes the proof of Point 1. $\dashv$