Automated Reasoning for Static Program Analysis

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https://www.dcs.bbk.ac.uk/~carsten/satsmtar2024/

Quality Assurance for Software by Program Analysis

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- Static analysis:
  Analyse the program text without actually running the program.
  - \+ can prove (verify) correctness of the program
    - → important for safety-critical applications
    - → motivating example: first flight of Ariane 5 rocket in 1996
      - https://www.youtube.com/watch?v=PK_yguLapgA
      - https://en.wikipedia.org/wiki/Ariane_5_Flight_501
  - — manual static analysis requires high effort and expertise
  - ⇒ for broad applicability:

Use automatic reasoning for static analysis!

Static Analysis: the User's Perspective (1/2)

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Ask me in the coffee break!

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Note: All these properties are **undecidable**!
⇒ use automatable sufficient criteria in practice

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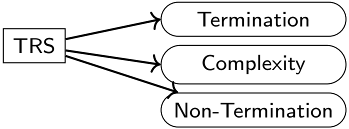

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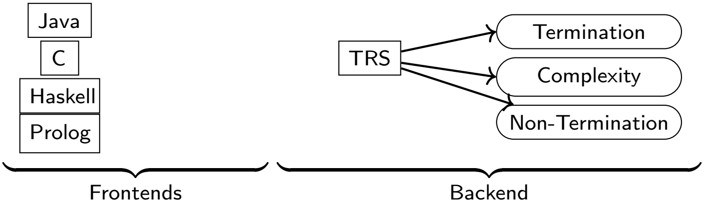

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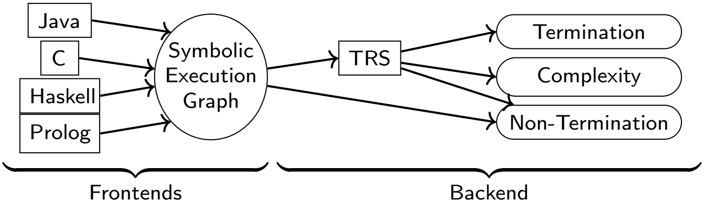

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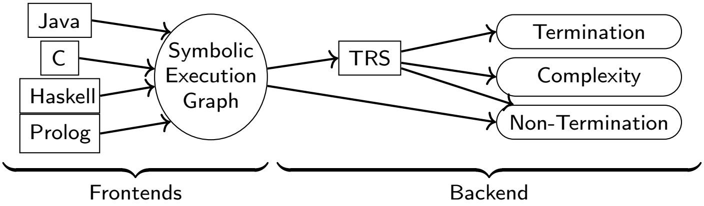

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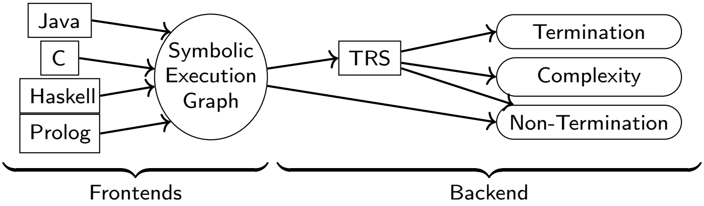

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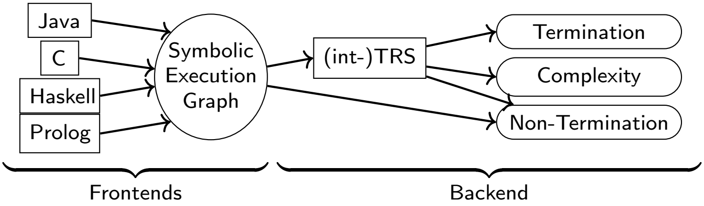

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- Performs the analysis of the desired property

$\Rightarrow$ Result carries over to original program

I. Termination Analysis

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2011: PHP and Java issues with floating-point number parser

- http://www.exploringbinary.com/php-hangs-on-numeric-value-2-2250738585072011e-308/
- http://www.exploringbinary.com/java-hangs-when-converting-2-2250738585072012e-308/

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- That's not even semi-decidable!
- But, fear not . . .

Termination Analysis, Classically

Turing 1949

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Example (Does this program terminate for all $x \in \mathbb{Z}$?)

while $x > 0$:

$x = x - 1$

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In practice:

- Encode only one proof **step** at a time
  $\longrightarrow$ try to prove only **part** of the program terminating
- **Repeat** until the whole program is proved terminating

Termination Proving in the Back-End and in the Front-End

Back-End:

1. Term Rewrite Systems (TRSs)
2. Imperative Programs (as Integer Transition Systems, ITSs)
3. Both together! Logically Constrained Term Rewrite Systems

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Front-End: processing practical programming languages
Example: Java

I.1 Termination Analysis of Term Rewrite Systems

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Syntactic approach for reasoning in equational first-order logic

Core functional programming language without many restrictions (and features) of "real" FP:

- first-order (usually)
- no fixed evaluation strategy $\longrightarrow$ non-determinism!
- no fixed order of rules to apply (Haskell: top to bottom) $\longrightarrow$ non-determinism!
- untyped (unless you really want types)
- no pre-defined data structures (integers, arrays, ...)

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$$\mathcal{R} = \begin{cases} \mathsf{minus}(x, 0) & \to & x \\ \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{minus}(x, y) \\ \mathsf{quot}(0, \mathsf{s}(y)) & \to & 0 \\ \mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \to & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y))) \end{cases}$$

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Calculation:

$$\mathsf{minus}(\mathsf{s}(\mathsf{s}(0)), \mathsf{s}(0)) \quad \to_{\mathcal{R}} \quad \mathsf{minus}(\mathsf{s}(0), 0) \quad \to_{\mathcal{R}} \quad \mathsf{s}(0)$$

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  - Logic programming: Prolog
    [van Raamsdonk, *ICLP '97*; Schneider-Kamp et al, *TOCL '09*; Giesl et al, *PPDP '12*]

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- Termination needed by theorem provers
- Translate program P with inductive data structures (trees) to TRS, represent data structures as terms

  $\Rightarrow$ Termination of TRS implies termination of P

  - Logic programming: Prolog
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  - (Lazy) functional programming: Haskell [Giesl et al, *TOPLAS '11*]

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  - Object-oriented programming: Java [Otto et al, *RTA '10*]

Example (Division)

$$\mathcal{R} = \begin{cases} \mathsf{minus}(x, \mathsf{0}) & \rightarrow & x \\ \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \rightarrow & \mathsf{minus}(x, y) \\ \mathsf{quot}(\mathsf{0}, \mathsf{s}(y)) & \rightarrow & \mathsf{0} \\ \mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \rightarrow & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y))) \end{cases}$$

Term rewriting: Evaluate terms by applying rules from $\mathcal{R}$

$$\mathsf{minus}(\mathsf{s}(\mathsf{s}(\mathsf{0})), \mathsf{s}(\mathsf{0})) \rightarrow_{\mathcal{R}} \mathsf{minus}(\mathsf{s}(\mathsf{0}), \mathsf{0}) \rightarrow_{\mathcal{R}} \mathsf{s}(\mathsf{0})$$

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Termination: No infinite evaluation sequences $t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} t_3 \to_{\mathcal{R}} \ldots$

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Show termination using Dependency Pairs

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Dependency Pairs [Arts, Giesl, *TCS '00*]

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  - delete $s \to t$ with $s \succ t$ from $\mathcal{DP}$
- Find $\succ$ automatically and efficiently

Polynomial Interpretations

Get $\succ$ via polynomial interpretations $[\,\cdot\,]$ over $\mathbb{N}$ [Lankford '75]

Example

$$\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) \;\; \gtrsim \;\; \mathsf{minus}(x, y)$$

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Use $[\,\cdot\,]$ with

- $[\mathsf{minus}](x_1, x_2) = x_1$
- $[\mathsf{s}](x_1) = x_1 + 1$

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$$\forall x, y. \quad x + 1 \;=\; [\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y))] \;\geq\; [\mathsf{minus}(x, y)] \;=\; x$$

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Extend to terms:

- $[x] = x$
- $[\,f(t_1, \ldots, t_n)] = [\,f]([t_1], \ldots, [t_n])$

$\succ$ boils down to $>$ over $\mathbb{N}$

Example (Constraints for Division)

$$\mathcal{R} = \begin{cases} \mathsf{minus}(x, \mathsf{0}) & \succsim & x \\ \mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) & \succsim & \mathsf{minus}(x, y) \\ \mathsf{quot}(\mathsf{0}, \mathsf{s}(y)) & \succsim & \mathsf{0} \\ \mathsf{quot}(\mathsf{s}(x), \mathsf{s}(y)) & \succsim & \mathsf{s}(\mathsf{quot}(\mathsf{minus}(x, y), \mathsf{s}(y))) \end{cases}$$

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Use interpretation $[\,\cdot\,]$ over $\mathbb{N}$ with

$$\begin{aligned} [\mathsf{quot}^\sharp](x_1, x_2) &= x_1 & \qquad [\mathsf{quot}](x_1, x_2) &= x_1 + x_2 \\ [\mathsf{minus}^\sharp](x_1, x_2) &= x_1 & \qquad [\mathsf{minus}](x_1, x_2) &= x_1 \\ [\mathsf{0}] &= 0 & \qquad [\mathsf{s}](x_1) &= x_1 + 1 \end{aligned}$$

$\curvearrowright$ order solves all constraints

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$\curvearrowright$ $\mathcal{DP} = \emptyset$

$\curvearrowright$ termination of division algorithm proved $\square$

Remark

Polynomial interpretations play several roles for program analysis:

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- Ranking function: $[\mathsf{quot}^\sharp]$ and $[\mathsf{minus}^\sharp]$

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- Ranking function: $[\mathsf{quot}^\sharp]$ and $[\mathsf{minus}^\sharp]$
- Summary: $[\mathsf{quot}]$ and $[\mathsf{minus}]$
- Abstraction (aka norm) for data structures: $[\mathsf{0}]$ and $[\mathsf{s}]$

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1. Fix template polynomials with **parametric coefficients**, get interpretation template:

$$[\mathsf{minus}](x, y) = a_{\mathsf{m}} + b_{\mathsf{m}}\, x + c_{\mathsf{m}}\, y, \quad [\mathsf{s}](x) = a_{\mathsf{s}} + b_{\mathsf{s}}\, x$$

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$$s \gtrsim t \;\curvearrowright\; [s] \geq [t]$$

Here: $\forall x, y.\; (a_{\mathsf{s}}\, b_{\mathsf{m}} + a_{\mathsf{s}}\, c_{\mathsf{m}}) + (b_{\mathsf{s}}\, b_{\mathsf{m}} - b_{\mathsf{m}})\, x + (b_{\mathsf{s}}\, c_{\mathsf{m}} - c_{\mathsf{m}})\, y \geq 0$

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Non-linear constraints, even for linear interpretations

Automation

Task: Solve $\mathsf{minus}(\mathsf{s}(x), \mathsf{s}(y)) \succsim \mathsf{minus}(x, y)$

1. Fix template polynomials with **parametric coefficients**, get interpretation template:

$$[\mathsf{minus}](x, y) = a_{\mathsf{m}} + b_{\mathsf{m}}\, x + c_{\mathsf{m}}\, y, \quad [\mathsf{s}](x) = a_{\mathsf{s}} + b_{\mathsf{s}}\, x$$

2. From term constraint to polynomial constraint:

$$s \succsim t \;\curvearrowright\; [s] \geq [t]$$

   Here: $\forall x, y.\; (a_{\mathsf{s}}\, b_{\mathsf{m}} + a_{\mathsf{s}}\, c_{\mathsf{m}}) + (b_{\mathsf{s}}\, b_{\mathsf{m}} - b_{\mathsf{m}})\, x + (b_{\mathsf{s}}\, c_{\mathsf{m}} - c_{\mathsf{m}})\, y \geq 0$

3. Eliminate $\forall x, y$ by absolute positiveness criterion [Hong, Jakuš, *JAR '98*]:

   Here: $a_{\mathsf{s}}\, b_{\mathsf{m}} + a_{\mathsf{s}}\, c_{\mathsf{m}} \geq 0 \wedge b_{\mathsf{s}}\, b_{\mathsf{m}} - b_{\mathsf{m}} \geq 0 \wedge b_{\mathsf{s}}\, c_{\mathsf{m}} - c_{\mathsf{m}} \geq 0$

   Non-linear constraints, even for linear interpretations

Task: Show satisfiability of non-linear constraints over $\mathbb{N}$ ($\longrightarrow$ SMT solver!)
$\curvearrowright$ Prove termination of given term rewrite system

Non-Linear Constraint Solving

Satisfiability of non-linear SMT formulas over $\mathbb{N}$ undecidable (Hilbert's 10th problem)

- Restrict **unknowns** to finite domain $\{0, \ldots, n\}$
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Approach [Fuhs et al, *SAT '07*]

- Encode non-linear SMT formula to pure SAT
  $\longrightarrow$ bit-blasting for QF_NIA
- Use SAT solver to get solution
- Eager Approach to SMT, but any SMT solver will do!
- Observation: if a model over $\mathbb{N}$ exists, usually small n suffices (e.g., $n = 3$)

Extensions of Polynomial Interpretations

- Polynomials with negative coefficients and max-operator [Hirokawa, Middeldorp, *IC '07*; Fuhs et al, *SAT '07, RTA '08*]
  - can model behaviour of functions more closely: [pred]$(x_1) = \max(x_1 - 1, 0)$
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  - linear interpretation to vectors over $\mathbb{N}^k$, coefficients are matrices
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SAT and SMT Solving for Path Orders

Path orders: based on precedences on function symbols

- Knuth-Bendix Order [Knuth, Bendix, *CPAA '70*]
  $\longrightarrow$ polynomial time algorithm [Korovin, Voronkov, *IC '03*]
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- Weighted Path Order [Yamada, Kusakari, Sakabe, *SCP '15*]
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Automation of the Order Search (1/2)

Dependency Pair Framework (simplified):

while $\mathcal{DP} \neq \emptyset$ **:**

- find well-founded order $\succ$ with $\mathcal{DP} \cup \mathcal{R} \subseteq \succsim$
- delete $s \longrightarrow t$ with $s \succ t$ from $\mathcal{DP}$

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- If internal timeout elapses (or everyone says **UNSATISFIABLE**):
  $\longrightarrow$ kill all search instances; retry with larger search space.
- In addition: try non-SAT/SMT-based techniques
  $\longrightarrow$ decompose problem into Strongly Connected Components, prove non-termination, . . .

Automation of the Order Search (2/2)

Requirements on SAT/SMT solver:

- return model **quickly** (at most 5–10 seconds)
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Current SAT solver of choice in AProVE: **MiniSat 2.2** [Eén, Sörensson, *SAT '03*]
(version from around 2008; finds models quickly)

Survey among tool authors (Aug/Sep 2022):
https://lists.rwth-aachen.de/hyperkitty/list/termtools@lists.rwth-aachen.de/thread/FNDNU5Y7TGXYXX34YWKFO2ICSRT6M3ME/

Further Techniques and Settings for TRSs

- Proving **non**-termination (an infinite run is possible)
  [Giesl, Thiemann, Schneider-Kamp, *FroCoS '05*; Payet, *TCS '08*; Zankl et al, *SOFSEM '10*; Emmes, Enger, Giesl, *IJCAR '12*; . . .]

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- **Complexity analysis** [Hirokawa, Moser, *IJCAR '08*; Noschinski, Emmes, Giesl, *JAR '13*; ...]
  Can re-use termination machinery to infer and prove statements like "runtime complexity of this TRS is in $\mathcal{O}(n^3)$"

SMT Solvers *from* Termination Analysis

Annual SMT-COMP, division QF_NIA (Quantifier-Free Non-linear Integer Arithmetic)

Year	Winner
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2011	AProVE
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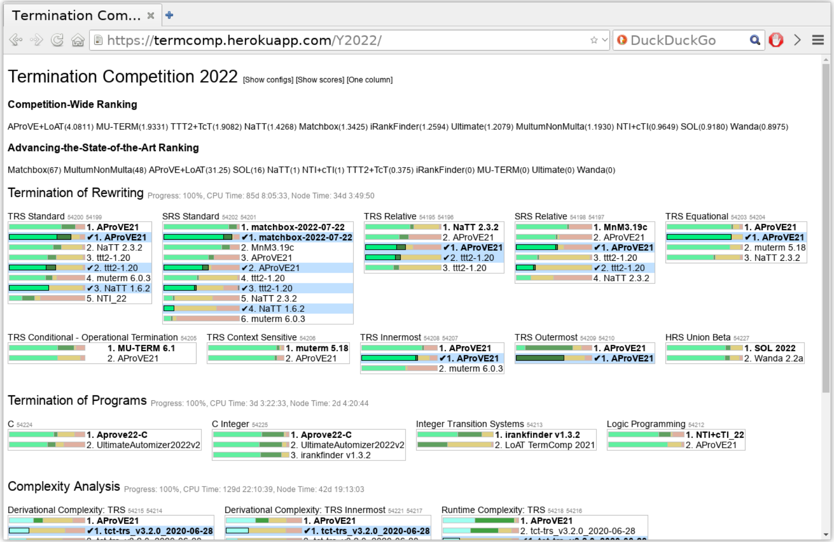
(disclaimer: Z3 participated only *hors concours*)

The Termination Competition (termCOMP) (1/3)



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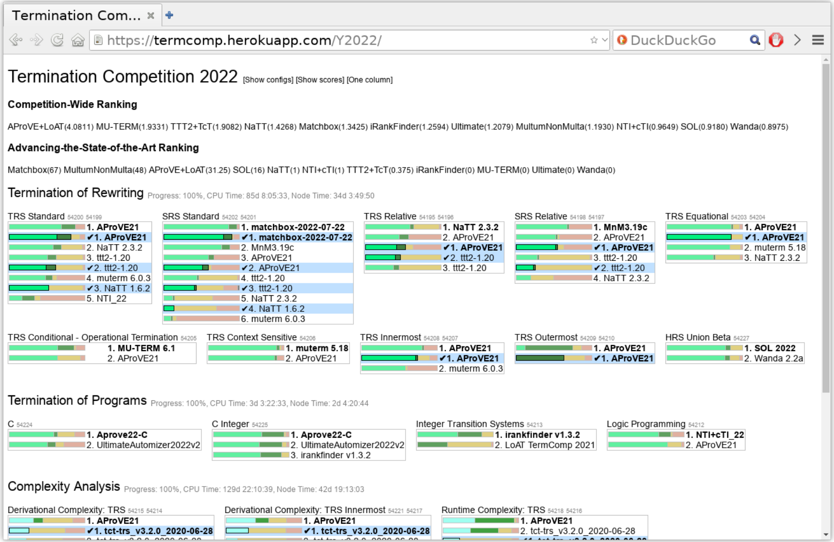

https://termination-portal.org/wiki/Termination_Competition

The Termination Competition (termCOMP) (2/3)

termCOMP 2022 participants (2024 similar):

- AProVE (RWTH Aachen, Birkbeck U London, U Innsbruck, ...)
- iRankFinder (UC Madrid)
- LoAT (RWTH Aachen)
- Matchbox (HTWK Leipzig)
- Mu-Term (UP Valencia)
- MultumNonMulta (BA Saarland)
- NaTT (AIST Tokyo)
- NTI+cTI (U Réunion)
- SOL (Gunma U)
- TcT (U Innsbruck, INRIA Sophia Antipolis)
- $\mathsf{T_TT_2}$ (U Innsbruck)
- Ultimate Automizer (U Freiburg)
- Wanda (RU Nijmegen)

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- Part of the Olympic Games at the Federated Logic Conference

Input for Automated Tools

Web interfaces available:

- AProVE: https://aprove.informatik.rwth-aachen.de/interface
- iRankFinder: http://irankfinder.loopkiller.com:8081/
- Mu-Term: http://zenon.dsic.upv.es/muterm/index.php/web-interface/
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Input format for termination of TRSs:

```
(VAR x y)
(RULES
  plus(0, y) -> y
  plus(s(x), y) -> s(plus(x, y))
)
```

I.2 Termination Analysis of Programs on Integers

Papers on termination of imperative programs often about **integers** as data

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Example (Imperative Program)

if $(\mathsf{x} \geq 0)$
  while $(\mathsf{x} \neq 0)$
    $\mathsf{x} = \mathsf{x} - 1;$

Does this program terminate?
(x ranges over $\mathbb{Z}$)

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Example (Equivalent Translation to an Integer Transition System, cf. [McCarthy, *CACM '60*])

$$\begin{array}{lcll}
\ell_0(x) & \rightarrow & \ell_1(x) & [x \geq 0] \\
\ell_0(x) & \rightarrow & \ell_3(x) & [x < 0] \\
\ell_1(x) & \rightarrow & \ell_2(x) & [x \neq 0] \\
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$\Rightarrow$ **Restrict initial states** to $\ell_0(z)$ for $z \in \mathbb{Z}$

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⇒ **Restrict initial states** to $\ell_0(z)$ for $z \in \mathbb{Z}$
⇒ Find **invariant** $x \geq 0$ at ℓ_1, ℓ_2 (exercise)

Papers on termination of imperative programs often about **integers** as data

Example (Imperative Program)

```
ℓ0:  if (x ≥ 0)
ℓ1:      while (x ≠ 0)
ℓ2:          x = x − 1;
```

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Proving Termination with Invariants

Example (Transition system with invariants)

$$
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Nowadays all SMT-based!

Extensions

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- Beyond sequential programs on integers:
  - structs/classes [Berdine et al, *CAV '06*; Otto et al, *RTA '10*; . . .]
  - arrays (pointer arithmetic) [Ströder et al, *JAR '17*, . . .]
  - multi-threaded programs [Cook et al, *PLDI '07*, . . .]
  - . . .

Why Care about Termination of Term Rewriting?

- Termination needed by theorem provers
- Translate program P with inductive data structures (trees) to TRS, represent data structures as terms

  $\Rightarrow$ Termination of TRS implies termination of P

  - Logic programming: Prolog
    [van Raamsdonk, *ICLP '97*; Schneider-Kamp et al, *TOCL '09*; Giesl et al, *PPDP '12*]
  - (Lazy) functional programming: Haskell [Giesl et al, *TOPLAS '11*]
  - Object-oriented programming: Java [Otto et al, *RTA '10*]

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Solution: use **constrained term rewriting**

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Term rewriting "with batteries included"

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- For program termination: use term rewriting with **integers** [Falke, Kapur, *CADE '09*; Fuhs et al, *RTA '09*; Giesl et al, *JAR '17*]

Logically Constrained TRSs: Adoption

Analysis techniques for Logically Constrained TRSs:

- Termination [Kop, *WST '13*; Nishida, Winkler, *VSTTE '18*]
- Complexity [Winkler, Moser, *LOPSTR '20*]
- Equivalence [Fuhs, Kop, Nishida, *TOCL '17*; Ciobâcă, Lucanu, Buruiana, *JLAMP '23*]
- Confluence [Schöpf, Middeldorp, *CADE '23*; Schöpf, Mitterwallner, Middeldorp, *IJCAR '24*]
- Reachability / Safety [Ciobâcă, Lucanu, *IJCAR '18*; Kojima, Nishida, *JLAMP '23*]

Constrained Rewriting by Example

Example (Constrained Rewrite System)

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\begin{array}{rcll}
\ell_0(n, r) & \to & \ell_1(n, r, \mathsf{Nil}) & \\
\ell_1(n, r, xs) & \to & \ell_1(n-1, r+1, \mathsf{Cons}(r, xs)) & [n > 0] \\
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Termination proof: reuse techniques for TRSs and integer programs

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... there is a powerful SAT / SMT solver!

I.3 Termination Analysis of Java programs

Front-End: from Program to Constrained Term Rewriting, high-level

- execute program **symbolically** from initial states of the program, handle language peculiarities here ($\longrightarrow$ Java: sharing, cyclicity analysis)

```
f : if ...
        ...
    else
        ...
        g : while ...
                ...
```

Front-End: from Program to Constrained Term Rewriting, high-level

- execute program **symbolically** from initial states of the program, handle language peculiarities here (⟶ Java: sharing, cyclicity analysis)

init(...)

```
f : if ...
        ...
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```

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$\downarrow$
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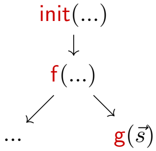

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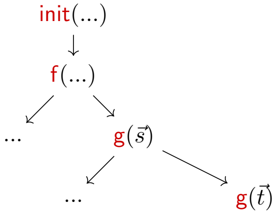

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- closely related: Abstract Interpretation

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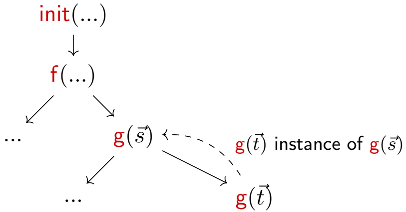

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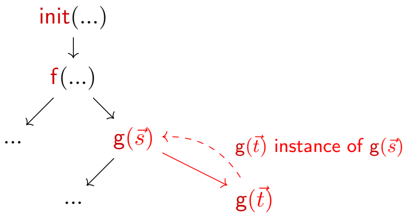

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- **extract TRS** from cycles in the representation
- if TRS terminates
  $\Rightarrow$ any **concrete program execution** can use cycles only finitely often
  $\Rightarrow$ the program **must terminate**

```
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        ...
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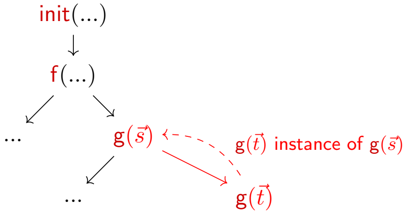

Application: Termination Analysis of Programs

Recipe for proving program termination by reusing TRS termination provers

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- Extract **rewrite rules** that “over-approximate” program executions in strongly-connected components of graph
- Prove **termination** of these rewrite rules
  $\Rightarrow$ implies termination of program from initial states

Java Challenges

Java: object-oriented imperative language

- sharing and aliasing (several references to the same object)
- side effects
- cyclic data objects (e.g., `list.next == list`)
- object-orientation with inheritance
- ...

Java Example

```
public class MyInt {

  // only wrap a primitive int
  private int val;

  // count "num" up to the value in "limit"
  public static void count(MyInt num, MyInt limit) {
    if (num == null || limit == null) {
      return;
    }
    // introduce sharing
    MyInt copy = num;
    while (num.val < limit.val) {
      copy.val++;
    }
  }
}
```

Does **`count`** terminate for all inputs? Why (not)?

(Assume that **`num`** and **`limit`** are not references to the same object.)

Approach to Termination Analysis of Java

Tailor two-stage approach to Java [Otto et al, *RTA '10*]

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Implemented in the tool AProVE ($\longrightarrow$ web interface)

http://aprove.informatik.rwth-aachen.de/

Java: Source Code vs Bytecode

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- desugared machine code for a (virtual) stack machine,
  still has all the (relevant) information from source code
- input for Java interpreter and for many program analysis tools
- somewhat inconvenient for presentation, though ...

Java: Source Code vs Bytecode

```
00: aload_0
01: ifnull 8
04: aload_1
05: ifnonnull 9
08: return
09: aload_0
10: astore_2
11: aload_0
12: getfield val
15: aload_1
16: getfield val
19: if_icmpge 35
22: aload_2
23: aload_2
24: getfield val
27: iconst_1
28: iadd
29: putfield val
32: goto 11
35: return
```

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Here: **Java source code**

Ingredients for the Abstract Domain

1. program counter value (line number)
2. values of variables (treating `int` as $\mathbb{Z}$)
3. over-approximating info on possible variable values
   - integers: use intervals, e.g. x $\in [4,\ 7]$ or y $\in [0,\ \infty)$
   - heap memory with objects, **no sharing** unless stated otherwise
   - `MyInt`(?): maybe `null`, maybe a `MyInt` object

   Heap predicates:

   - Two references may be equal: $o_1 =^? o_2$

`03`	num : o_1, `limit` : o_2
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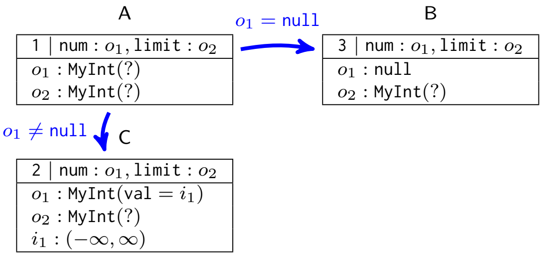

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means: refine X with $cond$, then evaluate to Y; here combined for brevity (narrowing)

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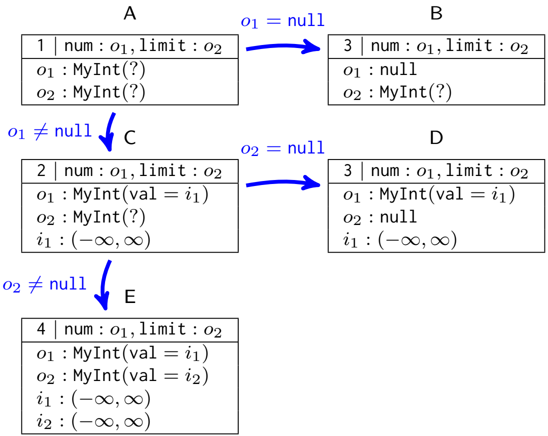

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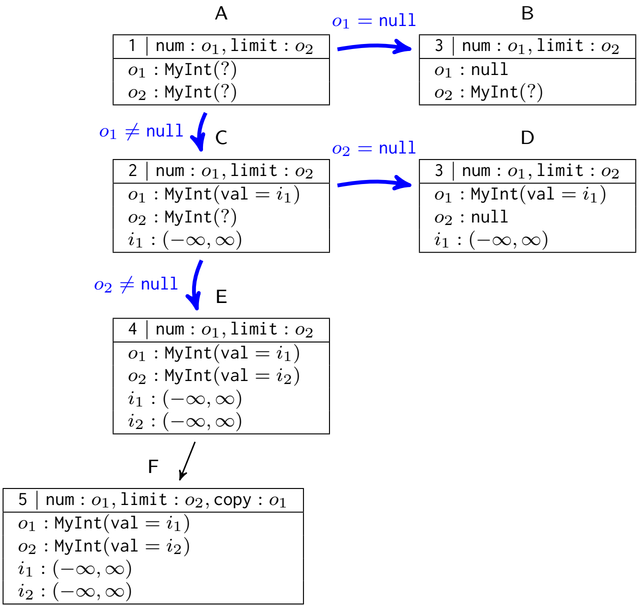

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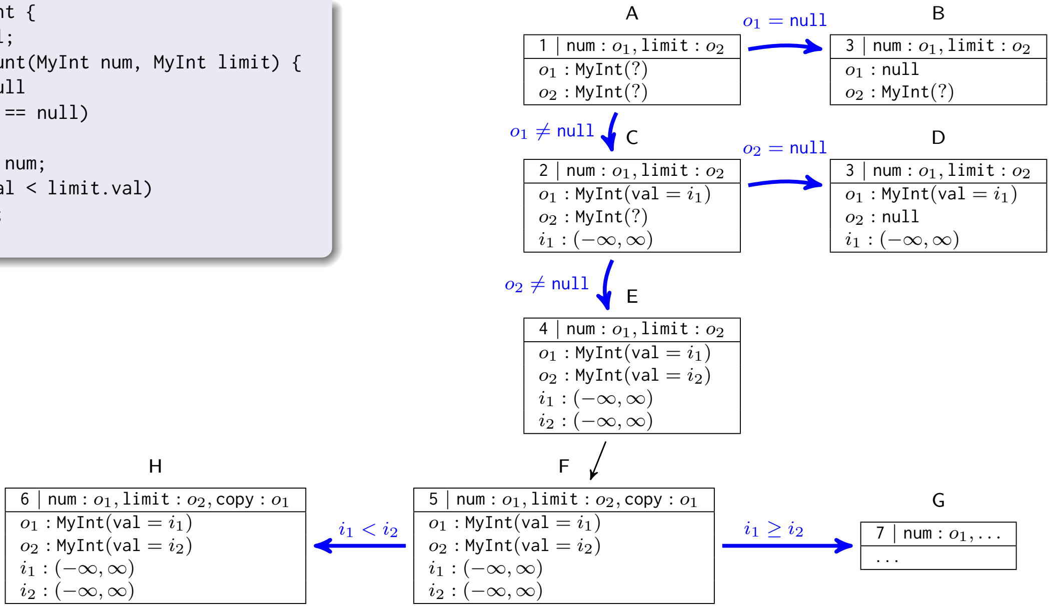

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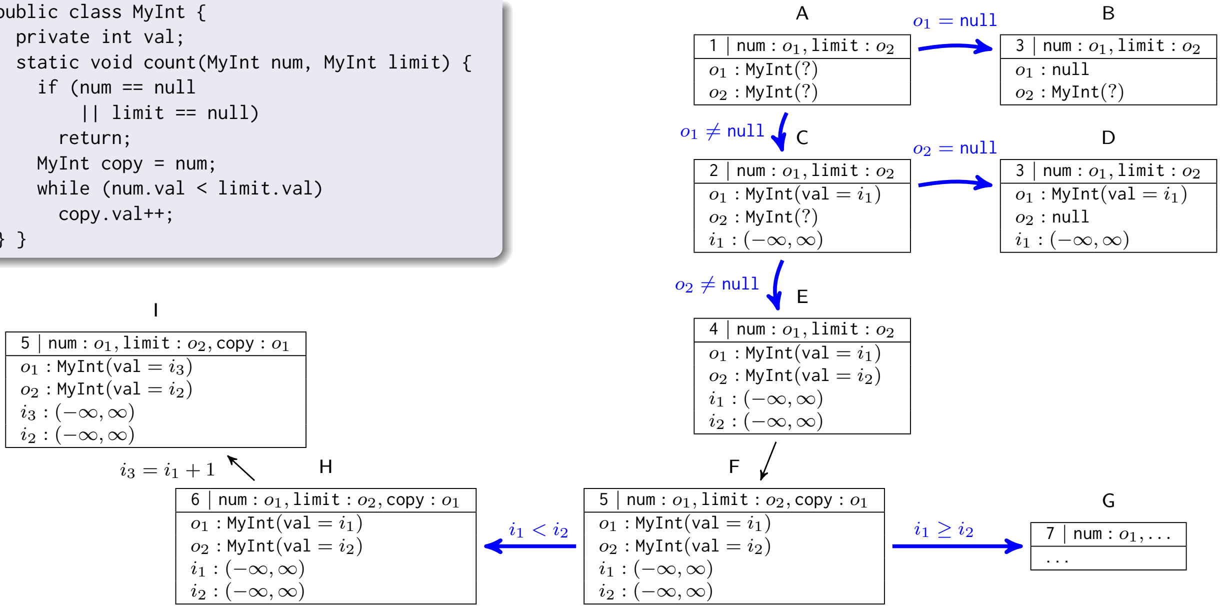

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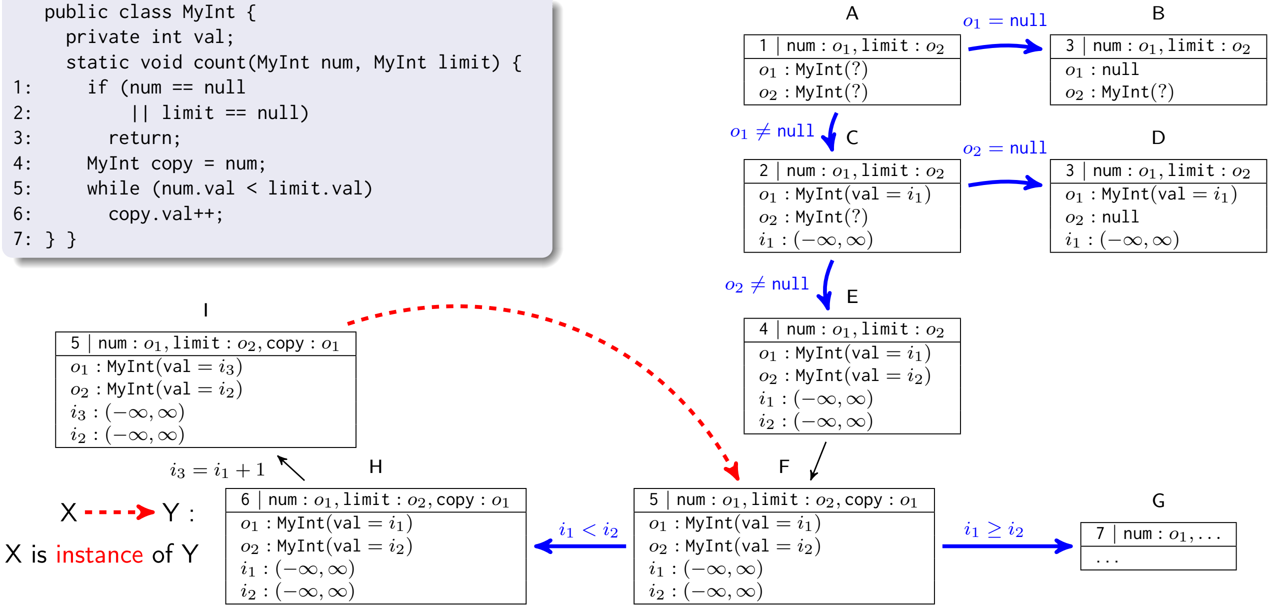

Symbolic Execution Graphs

- symbolic over-approximation of all computations (abstract interpretation)
- expand nodes until all leaves correspond to program ends
- by suitable generalisation steps (widening), one can always get a **finite** symbolic execution graph
- state s_1 is instance of state s_2 if all concrete states described by s_1 are also described by s_2

From Java to Symbolic Execution Graphs

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Using Symbolic Execution Graphs for Termination Proofs

- every concrete Java computation corresponds to a **computation path** in the symbolic execution graph
- symbolic execution graph is called **terminating**
  iff it has no infinite computation path

Transformation of Objects to Terms (1/2)

Q

16 | num : o_1, limit : o_2, x : o_3, y : o_4, z : i_1
o_1 : MyInt$(?)$
o_2 : MyInt(val $= i_2$)
o_3 : null
o_4 : MyList$(?)$
o_4 !
i_1 : $[7, \infty)$
i_2 : $(-\infty, \infty)$

For every class C with n fields, introduce an n-ary function symbol C

- **term** for o_1: o_1
- **term** for o_2: MyInt(i_2)
- **term** for o_3: null
- **term** for o_4: x (new variable)
- **term** for i_1: i_1 with **side constraint** $i_1 \geq 7$

(add invariant $i_1 \geq 7$ to constrained rewrite rules from state Q)

Transformation of Objects to Terms (2/2)

```
public class A {
  int a;
}

public class B extends A {
  int b;
}

...
A x = new A();
x.a = 1;

B y = new B();
y.a = 2;
y.b = 3;
```

Dealing with **subclasses**:

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- first argument: part of the object corresponding to subclasses of `C`
- **term** for `x`: $\mathsf{A}(\mathsf{eoc}, 1)$

  $\longrightarrow$ eoc for end of class
- **term** for `y`: $\mathsf{A}(\mathsf{B}(\mathsf{eoc}, 3), 2)$

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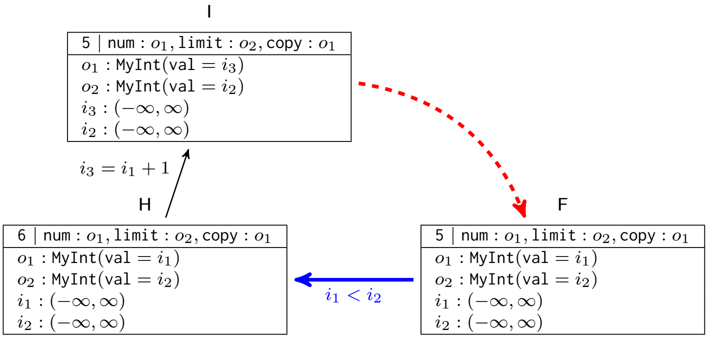
  $\longrightarrow$ eoc for end of class
- **term** for `y`: $\mathsf{jlO}(\mathsf{A}(\mathsf{B}(\mathsf{eoc}, 3), 2))$
- every class extends `Object`!
  ($\longrightarrow$ jlO $\equiv$ `java.lang.Object`)

From the Symbolic Execution Graph to Terms and Rules



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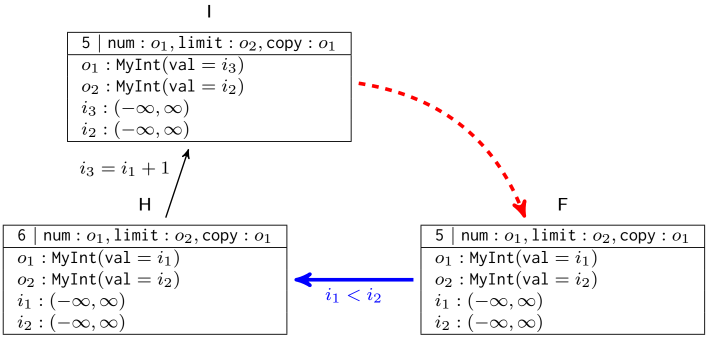

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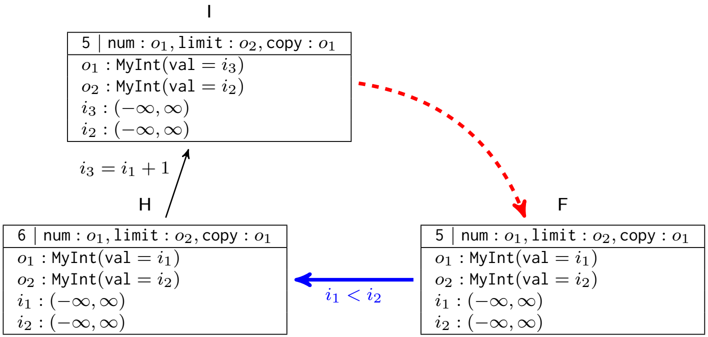

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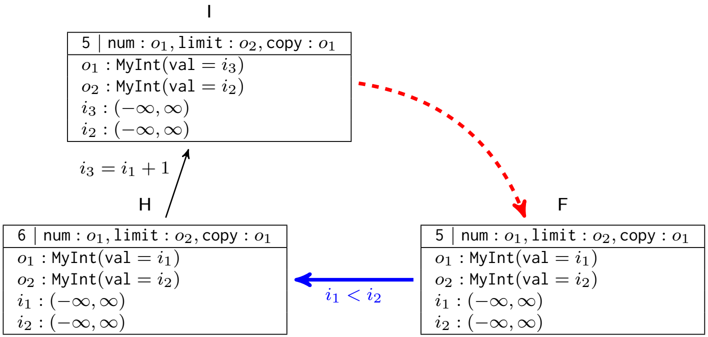

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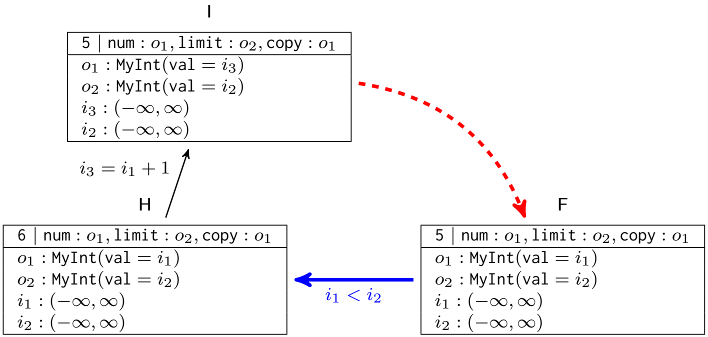

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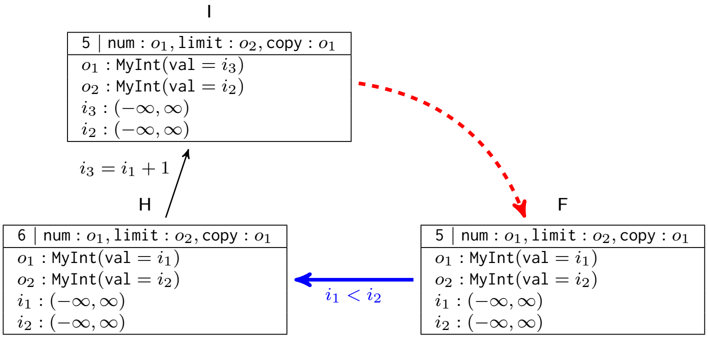
From the Symbolic Execution Graph to Terms and Rules



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- Termination easy to show (intuitively: $i_2 - i_1$ decreases against bound 0)

Extensions

- **modular** termination proofs and **recursion** [Brockschmidt et al, *RTA '11*]
- proving **reachability** and **non-termination** (uses only symbolic execution graph) [Brockschmidt et al, *FoVeOOS '11*]
- proving termination with **cyclic data objects** (preprocessing in symbolic execution graph) [Brockschmidt et al, *CAV '12*]
- proving upper bounds for **time complexity** (abstracts terms to numbers) [Frohn and Giesl, *iFM '17*]

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⇒ abstract domain based on equivalent **linear** Prolog semantics [Ströder et al, *LOPSTR '11*], tracks which variables are for ground terms vs arbitrary terms

Front-End for LLVM

LLVM [Ströder et al, *JAR '17*]

- LLVM bitcode: intermediate language of LLVM compiler framework
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Extensions:

- bitvector `int` semantics [Hensel et al, *JLAMP '18*]
- linked lists [Hensel, Giesl, *CADE '23*]

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- Works across paradigms: Java, C, Haskell, Prolog

II. Complexity Analysis

II.1 Complexity Analysis for Programs on Integers

What Do You Mean by Complexity?

Literature uses many alternative names:

- (Computational/Algorithmic) complexity analysis
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Given: Program P.
Task: Provide **upper/lower bounds** on the resource use of running P as a function of the input (size) **in the worst case**

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- More: see Section 1.1.2 of PhD thesis by Alicia Merayo Corcoba[1]

[1] A. Merayo Corcoba: *Resource analysis of integer and abstract programs*, PhD thesis, U Complutense Madrid, 2022

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  http://aprove.informatik.rwth-aachen.de/eval/IntegerComplexity-Journal

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Idea: **Countdown**.

For each loop find a **ranking function** f on the variables:

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Best runtime bound: $\mathcal{O}(x^2 + z)$

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Are There Other Techniques and Tools?

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---

[2] R. Blanc, T. Henzinger, L. Kovács: *ABC: Algebraic Bound Computation for Loops*, LPAR (Dakar) '10

[3] A. Flores-Montoya and R. Hähnle: *Resource Analysis of Complex Programs with Cost Equations*, APLAS '14

[4] E. Albert, P. Arenas, S. Genaim, G. Puebla, D.Zanardini: *Cost analysis of object-oriented bytecode programs*, TCS '12

[5] M. Sinn, F. Zuleger, H. Veith: *A Simple and Scalable Static Analysis for Bound Analysis and Amortized Complexity Analysis*, CAV '14

[6] C. Alias, A. Darte, P. Feautrier, L. Gonnord: *Multi-dimensional Rankings, Program Termination, and Complexity Bounds of Flowchart Programs*, SAS '10

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[9] E. Çiçek, M. Bouaziz, S. Cho, D. Distefano: *Static Resource Analysis at Scale (Extended Abstract)*, SAS '20

[10] J. Hoffmann, S. Jost: *Two decades of automatic amortized resource analysis*, MSCS '22

[11] J. Hoffmann, K. Aehlig, M. Hofmann: *Resource Aware ML*, CAV '12

Current Developments

- Precise handling of loops with computable complexity in the KoAT approach[12]

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[16] P. Wang, H. Fu, A. Goharshady, K. Chatterjee, X. Qin, W. Shi: *Cost analysis of nondeterministic probabilistic programs*, PLDI '19

[17] F. Meyer, M. Hark, J. Giesl: *Inferring Expected Runtimes of Probabilistic Integer Programs Using Expected Sizes*, TACAS '21

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Complexity of Integer Programs: What to Take Home?

Key insights:

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Solution:

- Alternating size/runtime analysis
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II.2 Complexity Analysis for Term Rewriting

What is *Term Rewriting*?

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Example (Term Rewrite System (TRS) $\mathcal{R}$)

$$\mathsf{double}(\mathsf{0}) \to \mathsf{0}$$

$$\mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))$$

Compute "double of 3 is 6":

$$\begin{aligned} & \mathsf{double}(\mathsf{s}^3(\mathsf{0})) \\ \to_{\mathcal{R}} \quad & \mathsf{s}^2(\mathsf{double}(\mathsf{s}^2(\mathsf{0}))) \\ \to_{\mathcal{R}} \quad & \mathsf{s}^4(\mathsf{double}(\mathsf{s}(\mathsf{0}))) \\ \to_{\mathcal{R}} \quad & \mathsf{s}^6(\mathsf{double}(\mathsf{0})) \\ \to_{\mathcal{R}} \quad & \mathsf{s}^6(\mathsf{0}) \end{aligned}$$

in 4 steps with $\to_{\mathcal{R}}$

What is *Complexity* of Term Rewriting?

Given: TRS $\mathcal{R}$ (e.g., $\{\ \mathsf{double}(\mathsf{0}) \to \mathsf{0},\ \ \mathsf{double}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))\ \}$)

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- **derivational complexity** $\mathrm{dc}_{\mathcal{R}}(n)$: no restrictions on start terms
- $\mathrm{dc}_{\mathcal{R}}(n)$ for **equational reasoning**: cost of solving the word problem $\mathcal{E} \models s \equiv t$ by rewriting s and t via an equivalent convergent TRS $\mathcal{R}_{\mathcal{E}}$

Complexity Analysis for TRSs: Overview

A Short Timeline (1/2)

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Some Definitions

Definition (Derivation Height dh)

For a term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ and a relation $\longrightarrow$, the **derivation height** is:

$$\mathrm{dh}(t, \longrightarrow) \;=\; \sup \{\, n \mid \exists t'.\, t \longrightarrow^{n} t' \,\}$$

If t starts an infinite $\longrightarrow$-sequence, we set $\mathrm{dh}(t, \longrightarrow) = \omega$.

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For a TRS $\mathcal{R}$, the **derivational complexity** is:

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Example: For $\mathcal{R}$ for double, we have $\mathrm{dc}_{\mathcal{R}}(n) \in \Theta(2^n)$.

Upper Bounds

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For a given TRS $\mathcal{R}$, the following questions are undecidable:

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- $\mathrm{dc}_{\mathcal{R}}(n) = \omega$ for some n? ($\longrightarrow$ termination!)
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Goal: find **approximations** for derivational complexity

Initial focus: find upper bounds

$$\mathrm{dc}_{\mathcal{R}}(n) \in \mathcal{O}(...)$$

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Derivational Complexity from Polynomial Interpretations (1/2)

Example (double)

$$\begin{aligned} \mathsf{double}(\mathsf{0}) &\to \mathsf{0} \\ \mathsf{double}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) \end{aligned}$$

Derivational Complexity from Polynomial Interpretations (1/2)

Example (double)

$$\begin{aligned} \mathsf{double}(\mathsf{0}) &\succ \mathsf{0} \\ \mathsf{double}(\mathsf{s}(x)) &\succ \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) \end{aligned}$$

Show $\mathrm{dc}_{\mathcal{R}}(n) < \omega$ by **termination proof** with reduction order $\succ$ on terms.

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Get $\succ$ via **polynomial interpretation**[25] $[\,\cdot\,]$ over $\mathbb{N}$: $\ell \succ r \iff [\ell] \succ [r]$

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Example: $[\mathsf{double}](x) = 3 \cdot x, \quad [\mathsf{s}](x) = x + 1, \quad [\mathsf{0}] = 1$

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Extend to terms:

- $[x] = x$
- $[f(t_1, \ldots, t_n)] = [f]([t_1], \ldots, [t_n])$

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Derivational Complexity from Polynomial Interpretations (1/2)

Example (double)

double(0)	$\succ$	0	3	$>$	1
double(s(x))	$\succ$	s(s(double(x)))	$3 \cdot x + 3$	$>$	$3 \cdot x + 2$

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Derivational Complexity from Polynomial Interpretations (1/2)

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$$\begin{array}{rcl|rcl} \mathsf{double}(\mathsf{0}) & \succ & \mathsf{0} & 3 & > & 1 \\ \mathsf{double}(\mathsf{s}(x)) & \succ & \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) & 3 \cdot x + 3 & > & 3 \cdot x + 2 \end{array}$$

Show $\mathrm{dc}_{\mathcal{R}}(n) < \omega$ by **termination proof** with reduction order $\succ$ on terms.
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Automated search for $[\,\cdot\,]$ via SAT[26] or SMT[27] solving

[25] D. Lankford: *Canonical algebraic simplification in computational logic*, U Texas '75
[26] C. Fuhs, J. Giesl, A. Middeldorp, P. Schneider-Kamp, R. Thiemann, H. Zankl: *SAT solving for termination analysis with polynomial interpretations*, SAT '07
[27] C. Borralleras, S. Lucas, A. Oliveras, E. Rodríguez-Carbonell, A. Rubio: *SAT modulo linear arithmetic for solving polynomial constraints*, JAR '12

Derivational Complexity from Polynomial Interpretations (2/2)

Example (double)

$$\begin{aligned} \mathsf{double}(\mathsf{0}) &\succ \mathsf{0} \\ \mathsf{double}(\mathsf{s}(x)) &\succ \mathsf{s}(\mathsf{s}(\mathsf{double}(x))) \end{aligned} \qquad \begin{aligned} 3 &> 1 \\ 3 \cdot x + 3 &> 3 \cdot x + 2 \end{aligned}$$

Example: $[\mathsf{double}](x) = 3 \cdot x$, $[\mathsf{s}](x) = x + 1$, $[\mathsf{0}] = 1$

This proves more than just termination...

Derivational Complexity from Polynomial Interpretations (2/2)

Example (double)

$\mathsf{double}(\mathsf{0}) \succ \mathsf{0}$		$3 > 1$	
$\mathsf{double}(\mathsf{s}(x)) \succ \mathsf{s}(\mathsf{s}(\mathsf{double}(x)))$		$3 \cdot x + 3 > 3 \cdot x + 2$	

Example: $[\mathsf{double}](x) = 3 \cdot x, \quad [\mathsf{s}](x) = x + 1, \quad [\mathsf{0}] = 1$

This proves more than just termination...

Theorem (Upper bounds for $\mathrm{dc}_{\mathcal{R}}(n)$ from polynomial interpretations[28])

- *Termination proof for TRS $\mathcal{R}$ with **polynomial** interpretation*

$$\Rightarrow \mathrm{dc}_{\mathcal{R}}(n) \in 2^{2^{\mathcal{O}(n)}}$$

[28] D. Hofbauer, C. Lautemann: *Termination proofs and the length of derivations*, RTA '89

Derivational Complexity from Polynomial Interpretations (2/2)

Example (double)

double(0)	$\succ$	0	3	$>$	1
double(s(x))	$\succ$	s(s(double(x)))	$3 \cdot x + 3$	$>$	$3 \cdot x + 2$

Example: [double]$(x) = 3 \cdot x$, [s]$(x) = x + 1$, [0] $= 1$

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- *Termination proof for TRS $\mathcal{R}$ with **linear polynomial** interpretation*

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[28] D. Hofbauer, C. Lautemann: *Termination proofs and the length of derivations*, RTA '89

Derivational Complexity from Termination Proofs (1/2)

Termination proof for TRS $\mathcal{R}$ with ...

- matchbounds[29] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n) \in \mathcal{O}(n)$
- arctic matrix interpretations[30] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n) \in \mathcal{O}(n)$

[29] A. Geser, D. Hofbauer, J. Waldmann: *Match-bounded string rewriting systems*, AAECC '04
[30] A. Koprowski, J. Waldmann: *Max/plus tree automata for termination of term rewriting*, Acta Cyb. '09

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- matrix interpretation of spectral radius[32] ≤ 1 $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n)$ is at most polynomial

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[32] F. Neurauter, H. Zankl, A. Middeldorp: *Revisiting matrix interpretations for polynomial derivational complexity of term rewriting*, LPAR (Yogyakarta) '10

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[33] J. Endrullis, J. Waldmann, and H. Zantema: *Matrix interpretations for proving termination of term rewriting*, JAR '08

Derivational Complexity from Termination Proofs (2/2)

Termination proof for TRS $\mathcal{R}$ with ...

- Lexicographic Path Order[34] $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n)$ is at most multiple recursive[35]

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- Dependency Pairs framework[38][39] with dependency graphs, reduction pairs, subterm criterion $\Rightarrow \mathrm{dc}_{\mathcal{R}}(n)$ is at most multiple recursive[40]

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[38] J. Giesl, R. Thiemann, P. Schneider-Kamp, S. Falke: *Mechanizing and improving dependency pairs*, JAR '06

[39] N. Hirokawa and A. Middeldorp: *Tyrolean Termination Tool: Techniques and features*, IC '07

[40] G. Moser, A. Schnabl: *Termination proofs in the dependency pair framework may induce multiple recursive derivational complexity*, RTA '11

Runtime Complexity

- So far: upper bounds for derivational complexity

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Definition (Basic Term[41])

For defined symbols $\mathcal{D}$ and constructor symbols $\mathcal{C}$, the term

$$f(t_1, \ldots, t_n)$$

is in the set $\mathcal{T}_{\mathrm{basic}}$ of **basic terms** iff $f \in \mathcal{D}$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{C}, \mathcal{V})$.

[41] N. Hirokawa, G. Moser: *Automated complexity analysis based on the dependency pair method*, IJCAR '08

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$\mathrm{rc}_{\mathcal{R}}(n)$: like derivational complexity... but for basic terms only!

[41] N. Hirokawa, G. Moser: *Automated complexity analysis based on the dependency pair method*, IJCAR '08

Runtime Complexity from Polynomial Interpretations

Polynomial interpretations can induce upper bounds to runtime complexity:[42]

Definition (Strongly linear polynomial, restricted interpretation)

- Polynomial p is **strongly linear** iff $p(x_1, \ldots, x_n) = x_1 + \cdots + x_n + a$ for some $a \in \mathbb{N}$.
- Polynomial interpretation $[\,\cdot\,]$ is **restricted** iff for all constructor symbols f, $[f](x_1, \ldots, x_n)$ is strongly linear.

Idea: $[t] \leq c \cdot |t|$ for fixed $c \in \mathbb{N}$.

[42] G. Bonfante, A. Cichon, J. Marion, H. Touzet: *Algorithms with polynomial interpretation termination proof*, JFP '01

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Theorem (Upper bounds for $\mathrm{rc}_\mathcal{R}(n)$ from restricted interpretations)

Termination proof for TRS $\mathcal{R}$ with ***restricted*** *interpretation $[\,\cdot\,]$ of degree at most d for $[f]$*

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Example: $[\mathsf{double}](x) = 3 \cdot x$, $[\mathsf{s}](x) = x + 1$, $[\mathsf{0}] = 1$ is restricted, degree 1
$\Rightarrow \mathrm{rc}_{\mathcal{R}}(n) \in \mathcal{O}(n)$ for TRS $\mathcal{R}$ for double

[42] G. Bonfante, A. Cichon, J. Marion, H. Touzet: *Algorithms with polynomial interpretation termination proof*, JFP '01

Dependency Tuples for *Innermost* Runtime Complexity irc

Here: innermost rewriting ($\approx$ call-by-value)

Example (reverse)

$\mathsf{app}(\mathsf{nil}, y) \to y$	$\mathsf{app}(\mathsf{add}(n, x), y) \to \mathsf{add}(n, \mathsf{app}(x, y))$
$\mathsf{reverse}(\mathsf{nil}) \to \mathsf{nil}$	$\mathsf{reverse}(\mathsf{add}(n, x)) \to \mathsf{app}(\mathsf{reverse}(x), \mathsf{add}(n, \mathsf{nil}))$

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$$\begin{array}{rcl|rcl} \mathsf{app}(\mathsf{nil}, y) & \to & y & \mathsf{app}(\mathsf{add}(n, x), y) & \to & \mathsf{add}(n, \mathsf{app}(x, y)) \\ \mathsf{reverse}(\mathsf{nil}) & \to & \mathsf{nil} & \mathsf{reverse}(\mathsf{add}(n, x)) & \to & \mathsf{app}(\mathsf{reverse}(x), \mathsf{add}(n, \mathsf{nil})) \end{array}$$

For rule $\ell \to r$, eval of ℓ costs 1 + eval of all function calls in r **together**:

[43] L. Noschinski, F. Emmes, J. Giesl: *Analyzing innermost runtime complexity of term rewriting by dependency pairs*, JAR '13

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For rule $\ell \to r$, eval of ℓ costs 1 + eval of all function calls in r **together**:

Example (Dependency Tuples[43] for reverse)

$$\begin{array}{rcl} \mathsf{app}^\sharp(\mathsf{nil}, y) & \to & \mathsf{Com}_0 \\ \mathsf{app}^\sharp(\mathsf{add}(n, x), y) & \to & \mathsf{Com}_1(\mathsf{app}^\sharp(x, y)) \\ \mathsf{reverse}^\sharp(\mathsf{nil}) & \to & \mathsf{Com}_0 \\ \mathsf{reverse}^\sharp(\mathsf{add}(n, x)) & \to & \mathsf{Com}_2(\mathsf{app}^\sharp(\mathsf{reverse}(x), \mathsf{add}(n, \mathsf{nil})), \mathsf{reverse}^\sharp(x)) \end{array}$$

- Function calls to count marked with $\sharp$
- Compound symbols Com_k group function calls together

[43] L. Noschinski, F. Emmes, J. Giesl: *Analyzing innermost runtime complexity of term rewriting by dependency pairs*, JAR '13

Example (reverse, Dependency Tuples for reverse)

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\mathsf{app}^\sharp(\mathsf{nil}, y) &\rightarrow \mathsf{Com}_0 \\
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Polynomial Interpretations for Dependency Tuples

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$\mathsf{app}(\mathsf{nil}, y) \to y$	$\mathsf{app}(\mathsf{add}(n, x), y) \to \mathsf{add}(n, \mathsf{app}(x, y))$
$\mathsf{reverse}(\mathsf{nil}) \to \mathsf{nil}$	$\mathsf{reverse}(\mathsf{add}(n, x)) \to \mathsf{app}(\mathsf{reverse}(x), \mathsf{add}(n, \mathsf{nil}))$

Use interpretation $[\,\cdot\,]$ with $[\mathsf{Com}_k](x_1, \ldots, x_k) = x_1 + \cdots + x_k$ and

$[\mathsf{nil}] = 0$	$[\mathsf{add}](x_1, x_2) = x_2 + 1$	($\leq$ restricted interpretation)
$[\mathsf{app}](x_1, x_2) = x_1 + x_2$	$[\mathsf{reverse}](x_1) = x_1$	(bounds helper function's result size)
$[\mathsf{app}^\sharp](x_1, x_2) = x_1 + 1$	$[\mathsf{reverse}^\sharp](x_1) = x_1^2 + x_1 + 1$	(complexity of function)

to show $[\ell] \geq [r]$ for all rules and $[\ell] \geq 1 + [r]$ for all Dependency Tuples

Maximum degree of $[f^\sharp]$ is 2 $\Rightarrow \mathrm{irc}_{\mathcal{R}}(n) \in \mathcal{O}(n^2)$

Related Techniques

- Dependency Tuples are an adaptation of Dependency Pairs (DPs) from termination analysis to complexity analysis, allow for **incremental** complexity proofs with several techniques

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- Further adaptation of DPs (incomparable): Weak (Innermost) Dependency Pairs for (innermost) runtime complexity[44]

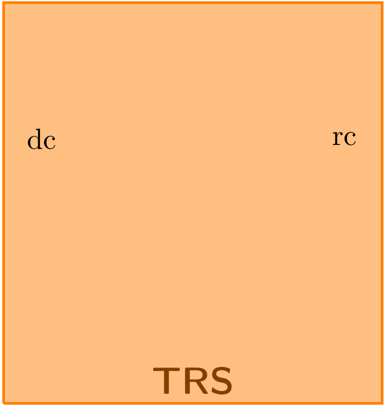
[44] N. Hirokawa, G. Moser: *Automated complexity analysis based on the dependency pair method*, IJCAR '08

Related Techniques

- Dependency Tuples are an adaptation of Dependency Pairs (DPs) from termination analysis to complexity analysis, allow for **incremental** complexity proofs with several techniques
- Further adaptation of DPs (incomparable): Weak (Innermost) Dependency Pairs for (innermost) runtime complexity[44]
- Extensions by polynomial path orders[45], usable replacement maps[46], a combination framework for complexity analysis[47], ...

[44] N. Hirokawa, G. Moser: *Automated complexity analysis based on the dependency pair method*, IJCAR '08
[45] M. Avanzini, G. Moser: *Dependency pairs and polynomial path orders*, RTA '09
[46] N. Hirokawa, G. Moser: *Automated complexity analysis based on context-sensitive rewriting*, RTA-TLCA '14
[47] M. Avanzini, G. Moser: *A combination framework for complexity*, IC '16

A Landscape of Complexity Properties and Transformations



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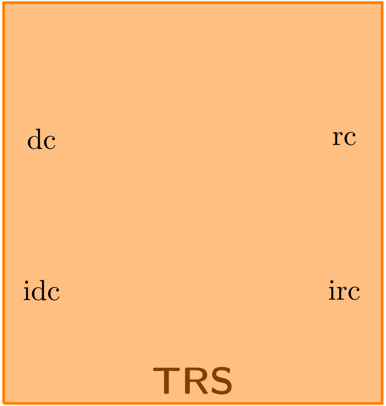

idc, irc: like dc, rc,
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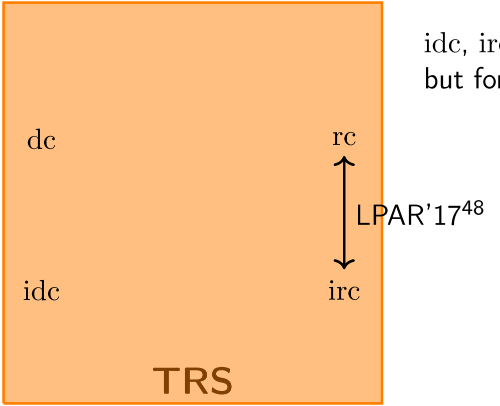

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[48] F. Frohn, J. Giesl: *Analyzing runtime complexity via innermost runtime complexity*, LPAR '17

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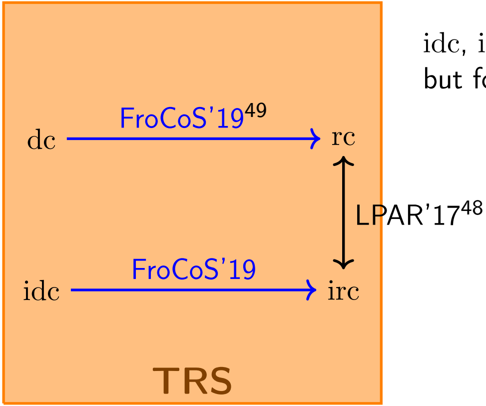

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[48] F. Frohn, J. Giesl: *Analyzing runtime complexity via innermost runtime complexity*, LPAR '17
[49] C. Fuhs: *Transforming Derivational Complexity of Term Rewriting to Runtime Complexity*, FroCoS '19

Transforming Derivational Complexity to Runtime Complexity

The big picture:

- **Have:** Tool for automated analysis of runtime complexity $\mathrm{rc}_{\mathcal{R}}$

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"$\mathrm{rc}_{\mathcal{R}}$ analysis tool + transformation on TRS $\mathcal{R}$ = $\mathrm{dc}_{\mathcal{R}}$ analysis tool"

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- **Have:** Tool for automated analysis of runtime complexity $\mathrm{rc}_{\mathcal{R}}$
- **Want:** Tool for automated analysis of derivational complexity $\mathrm{dc}_{\mathcal{R}}$
- **Idea:**

  "$\mathrm{rc}_{\mathcal{R}}$ analysis tool + transformation on TRS $\mathcal{R}$ = $\mathrm{dc}_{\mathcal{R}}$ analysis tool"

- **Benefits:**
  - Get analysis of derivational complexity "for free"
  - Progress in runtime complexity analysis automatically improves derivational complexity analysis

From dc to rc: Results

- program transformation such that runtime complexity of transformed TRS is **identical** to derivational complexity of original TRS

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- transformation correct also from idc to irc
- **implemented** in program analysis tool AProVE
- **evaluated** successfully on TPDB[50] relative to state of the art TcT

[50] Termination Problem DataBase, standard benchmark source for annual Termination Competition (termCOMP) with 1000s of problems, `http://termination-portal.org/wiki/TPDB`

From dc to rc: Transformation

Issue:

- Runtime complexity assumes **basic** terms as start terms
- We want to analyse complexity for **arbitrary** terms

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by **basic variant**

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Example (Generator rules $\mathcal{G}$)

$$\begin{aligned}
\mathsf{enc}_{\mathsf{double}}(x) &\to \mathsf{double}(\mathsf{argenc}(x))\\
\mathsf{enc}_{\mathsf{0}} &\to \mathsf{0}\\
\mathsf{enc}_{\mathsf{s}}(x) &\to \mathsf{s}(\mathsf{argenc}(x))\\
\mathsf{argenc}(\mathsf{c}_{\mathsf{double}}(x)) &\to \mathsf{double}(\mathsf{argenc}(x))\\
\mathsf{argenc}(\mathsf{0}) &\to \mathsf{0}\\
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$$\mathrm{bv}(t) = \mathsf{enc}_{\mathsf{double}}(\mathsf{c}_{\mathsf{double}}(\mathsf{c}_{\mathsf{double}}(\mathsf{s}(\mathsf{0}))))$$

Then:

- $\mathrm{bv}(t)$ is **basic** term, size $|t|$

Example (Generator rules $\mathcal{G}$)

$$\begin{aligned}
\mathsf{enc}_{\mathsf{double}}(x) &\to \mathsf{double}(\mathsf{argenc}(x))\\
\mathsf{enc}_{\mathsf{0}} &\to \mathsf{0}\\
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\mathsf{argenc}(\mathsf{0}) &\to \mathsf{0}\\
\mathsf{argenc}(\mathsf{s}(x)) &\to \mathsf{s}(\mathsf{argenc}(x))
\end{aligned}$$

From dc to rc: Transformation

Issue:

- Runtime complexity assumes **basic** terms as start terms
- We want to analyse complexity for **arbitrary** terms

Idea:

- Introduce constructor symbol c_f for defined symbol f
- Add **generator rewrite rules** $\mathcal{G}$ to reconstruct arbitrary term with f from basic term with c_f

Represent

$$t = \mathsf{double}(\mathsf{double}(\mathsf{double}(\mathsf{s}(\mathsf{0}))))$$

by **basic variant**

$$\mathrm{bv}(t) = \mathsf{enc}_{\mathsf{double}}(\mathsf{c}_{\mathsf{double}}(\mathsf{c}_{\mathsf{double}}(\mathsf{s}(\mathsf{0}))))$$

Then:

- $\mathrm{bv}(t)$ is **basic** term, size $|t|$
- $\mathrm{bv}(t) \to_{\mathcal{G}}^{*} t$

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General Case: Relative Rewriting

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Solution:

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  $\to_{\mathcal{G}}$ steps are **not counted** for complexity analysis!
- transform $\mathcal{R}$ to $\mathcal{R}/\mathcal{G}$ ($\to_{\mathcal{R}}$ steps are counted, $\to_{\mathcal{G}}$ steps are not)

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- transform $\mathcal{R}$ to $\mathcal{R}/\mathcal{G}$ ($\to_{\mathcal{R}}$ steps are counted, $\to_{\mathcal{G}}$ steps are not)
- more generally: transform $\mathcal{R}/\mathcal{S}$ to $\mathcal{R}/(\mathcal{S} \cup \mathcal{G})$
  (input may contain relative rules $\mathcal{S}$, too)

From dc to rc: Correctness

Theorem (Derivational Complexity via Runtime Complexity)

Let $\mathcal{R}/\mathcal{S}$ be a relative TRS, let $\mathcal{G}$ be the generator rules for $\mathcal{R}/\mathcal{S}$. Then

1. $\mathrm{dc}_{\mathcal{R}/\mathcal{S}}(n) = \mathrm{rc}_{\mathcal{R}/(\mathcal{S} \cup \mathcal{G})}(n)$ *(arbitrary rewrite strategies)*
2. $\mathrm{idc}_{\mathcal{R}/\mathcal{S}}(n) = \mathrm{irc}_{\mathcal{R}/(\mathcal{S} \cup \mathcal{G})}(n)$ *(innermost rewriting)*

Note: equalities hold also non-asymptotically!

From (i)dc to (i)rc: Experiments

Experiments on TPDB, compare with state of the art in TcT:

- upper bounds idc: both AProVE and TcT with transformation are stronger than standard TcT
- upper bounds dc: TcT stronger than AProVE and TcT with transformation, but AProVE still solves some new examples
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- lower bounds idc and dc: heuristics do not seem to benefit much

$\Rightarrow$ Transformation-based approach should be part of the portfolio of analysis tools for derivational complexity

Derivational Complexity: Applications and Extensions

- **Possible applications**
  - compiler simplifications
  - SMT solver preprocessing

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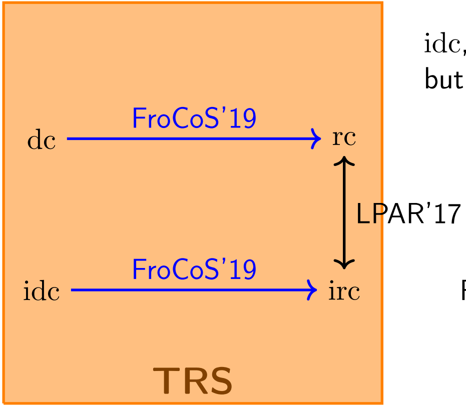
- Go **between** derivational and runtime complexity
  - So far: encode *full* term universe $\mathcal{T}$ via basic terms $\mathcal{T}_{\mathrm{basic}}$
  - Generalise: write relative rules to generate **arbitrary** set $\mathcal{U}$ of terms "between" basic and all terms ($\mathcal{T}_{\mathrm{basic}} \subseteq \mathcal{U} \subseteq \mathcal{T}$).

A Landscape of Complexity Properties and Transformations



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Rec. ITS irc ITS irc

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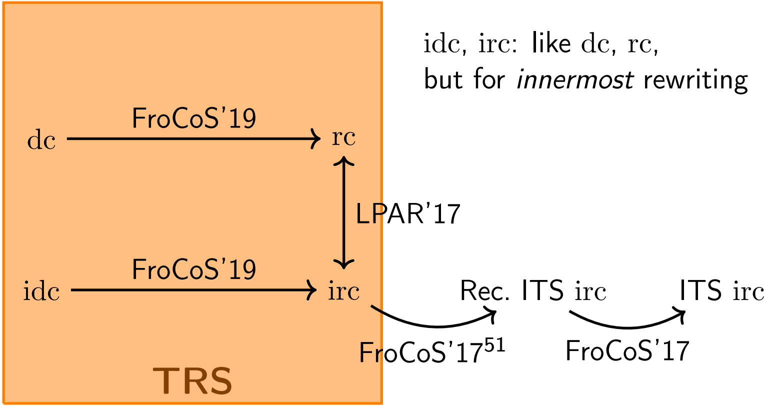

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[51] M. Naaf, F. Frohn, M. Brockschmidt, C. Fuhs, J. Giesl: *Complexity analysis for term rewriting by integer transition systems*, FroCoS '17

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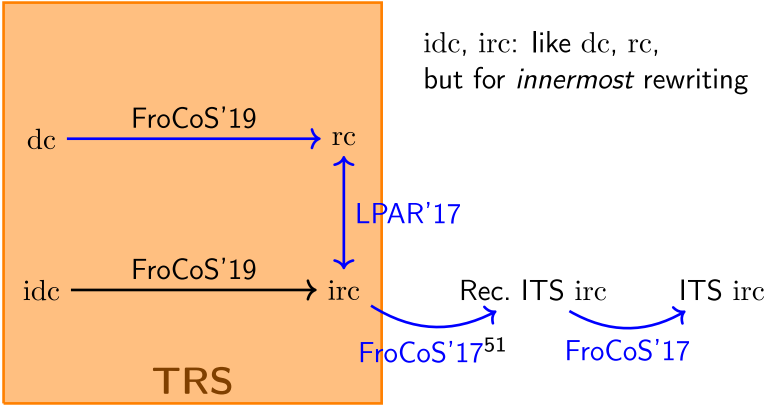

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Derivational_Complexity_Full_Rewriting/AG01/#3.12, TPDB

$$
\begin{array}{rcl|rcl}
\mathsf{app}(\mathsf{nil}, y) & \to & y & \mathsf{app}(\mathsf{add}(n, x), y) & \to & \mathsf{add}(n, \mathsf{app}(x, y)) \\
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AProVE finds lower bound $\Omega(n^3)$ for $\mathrm{dc}_{\mathcal{R}}$.[52]

[52] F. Frohn, J. Giesl, J. Hensel, C. Aschermann, and T. Ströder: *Lower bounds for runtime complexity of term rewriting*, JAR '17

Input for Automated Tools (1/4)

Automated tools for TRS Complexity at recent Termination Competitions:

- AProVE: https://aprove.informatik.rwth-aachen.de/
- TcT: https://tcs-informatik.uibk.ac.at/tools/tct/

[53] For TcT Web, use only VAR and RULES entries in the text format and configure other aspects (e.g., start terms) in the web interface.

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Web interfaces available:

- AProVE: `https://aprove.informatik.rwth-aachen.de/interface`
- TcT: `http://colo6-c703.uibk.ac.at/tct/tct-trs/`

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Input format for runtime complexity:[53]

```
(VAR x y)
(GOAL COMPLEXITY)
(STARTTERM CONSTRUCTOR-BASED)
(RULES
  plus(0, y) -> y
  plus(s(x), y) -> s(plus(x, y))
)
```

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Input for Automated Tools (2/4)

Innermost runtime complexity:

```
(VAR x y)
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(STARTTERM CONSTRUCTOR-BASED)
(STRATEGY INNERMOST)
(RULES
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)
```

Input for Automated Tools (3/4)

Derivational complexity:

```
(VAR x y)
(GOAL COMPLEXITY)
(STARTTERM UNRESTRICTED)
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Input for Automated Tools (4/4)

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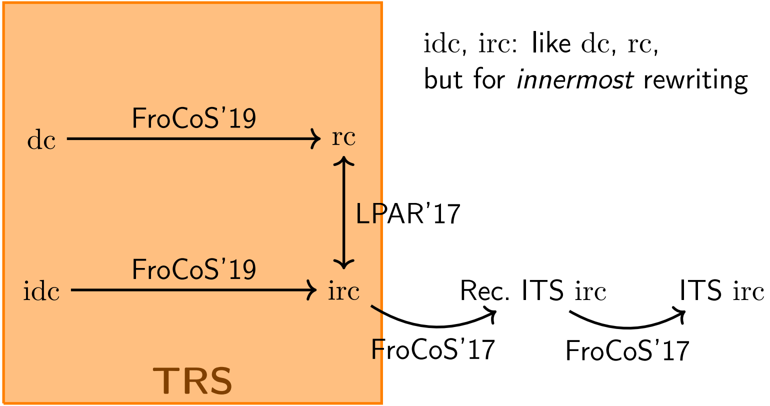

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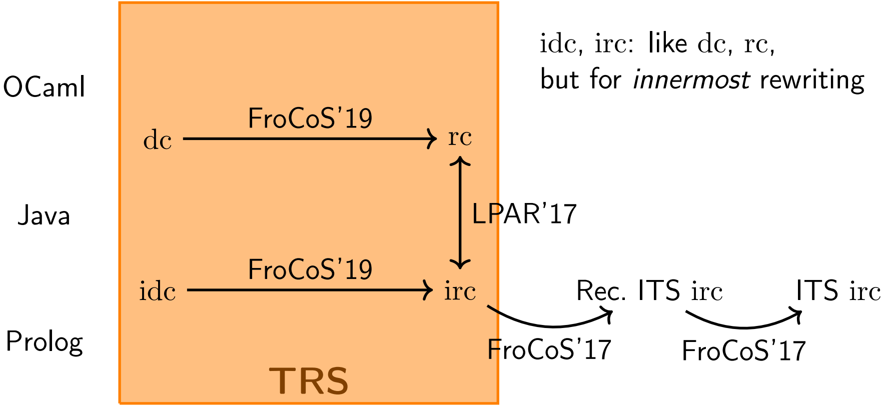

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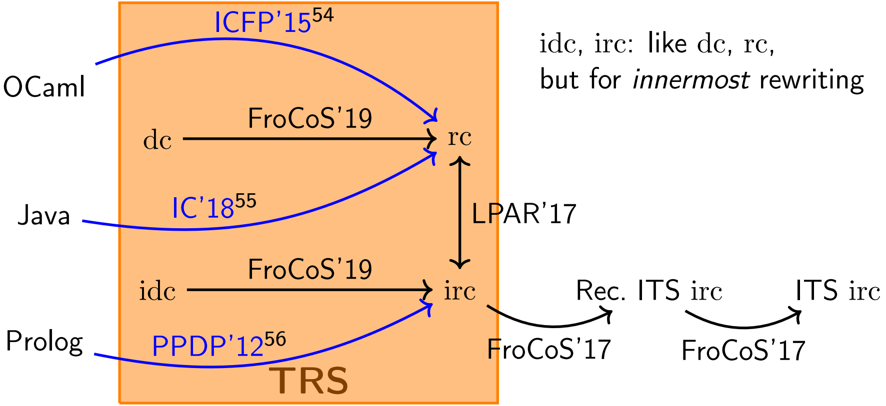

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54 M. Avanzini, U. Dal Lago, G. Moser: *Analysing the Complexity of Functional Programs: Higher-Order Meets First-Order*, ICFP '15

55 G. Moser, M. Schaper: *From Jinja bytecode to term rewriting: A complexity reflecting transformation*, IC '18

56 J. Giesl, T. Ströder, P. Schneider-Kamp, F. Emmes, C. Fuhs: *Symbolic evaluation graphs and term rewriting: A general methodology for analyzing logic programs*, PPDP '12

Program Complexity Analysis via Term Rewriting: OCaml

Complexity analysis for functional programs (OCaml) by translation to term rewriting

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Complexity analysis for functional programs (OCaml) by translation to term rewriting

Challenge for translation to TRS: OCaml is **higher-order** – functions can take functions as arguments: $\mathsf{map}(F, xs)$

Solution:

- Defunctionalisation to: $\mathsf{a}(\mathsf{a}(\mathsf{map}, F), xs)$
- Analyse start term with non-functional parameter types, then partially evaluate functions to instantiate higher-order variables
- Further program transformations

$\Rightarrow$ First-order TRS $\mathcal{R}$ with $\mathrm{rc}_{\mathcal{R}}(n)$ an upper bound for the complexity of the OCaml program

Program Complexity Analysis via Term Rewriting: Prolog and Java

Complexity analysis for Prolog programs and for Java programs by translation to term rewriting

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Complexity analysis for Prolog programs and for Java programs by translation to term rewriting

Common ideas:

- Analyse program via symbolic execution and generalisation (a form of abstract interpretation[57])
- Deal with language specifics in program analysis
- Extract TRS $\mathcal{R}$ such that $\mathrm{rc}_{\mathcal{R}}(n)$ is provably at least as high as runtime of program on input of size n
- Can represent tree structures of program as terms in TRS!

[57] P. Cousot, R. Cousot: *Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints*, POPL '77

- **amortised** complexity analysis for term rewriting[58]

[58] G. Moser, M. Schneckenreither: *Automated amortised resource analysis for term rewrite systems*, SCP '20

Current Developments

- **amortised** complexity analysis for term rewriting[58]
- **probabilistic** term rewriting $\longrightarrow$ upper bounds on **expected runtime**[59]

[58] G. Moser, M. Schneckenreither: *Automated amortised resource analysis for term rewrite systems*, SCP '20
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- analysis of **parallel**-innermost runtime complexity[62]

[58] G. Moser, M. Schneckenreither: *Automated amortised resource analysis for term rewrite systems*, SCP '20
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[60] S. Winkler, G. Moser: *Runtime complexity analysis of logically constrained rewriting*, LOPSTR '20
[61] C. Kop, D. Vale: *Tuple interpretations for higher-order rewriting*, FSCD '21
[62] T. Baudon, C. Fuhs, L. Gonnord: *Analysing parallel complexity of term rewriting*, LOPSTR '22

III. Termination and Complexity Proof Certification

Certification: Who Watches the Watchers?

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  $\longrightarrow$ CeTA tool from IsaFoR

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If certification unsuccessful:
CeTA indicates **which part** of the proof it could not follow

---

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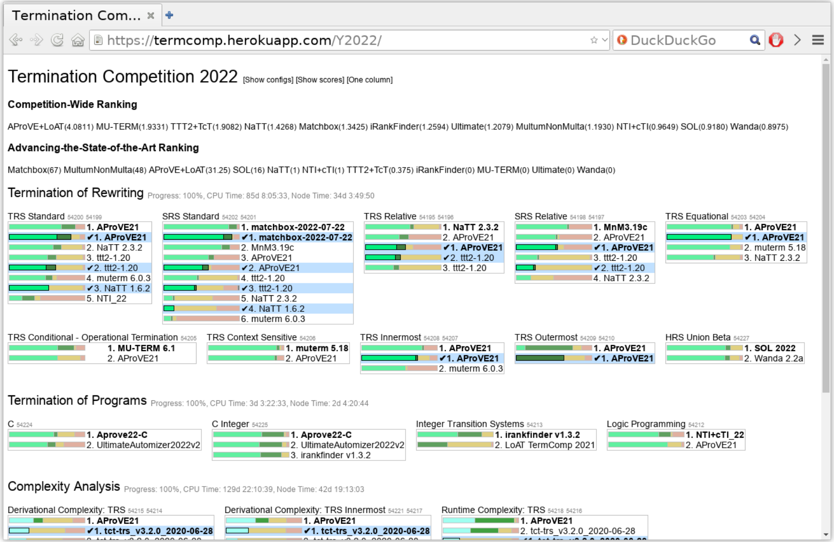
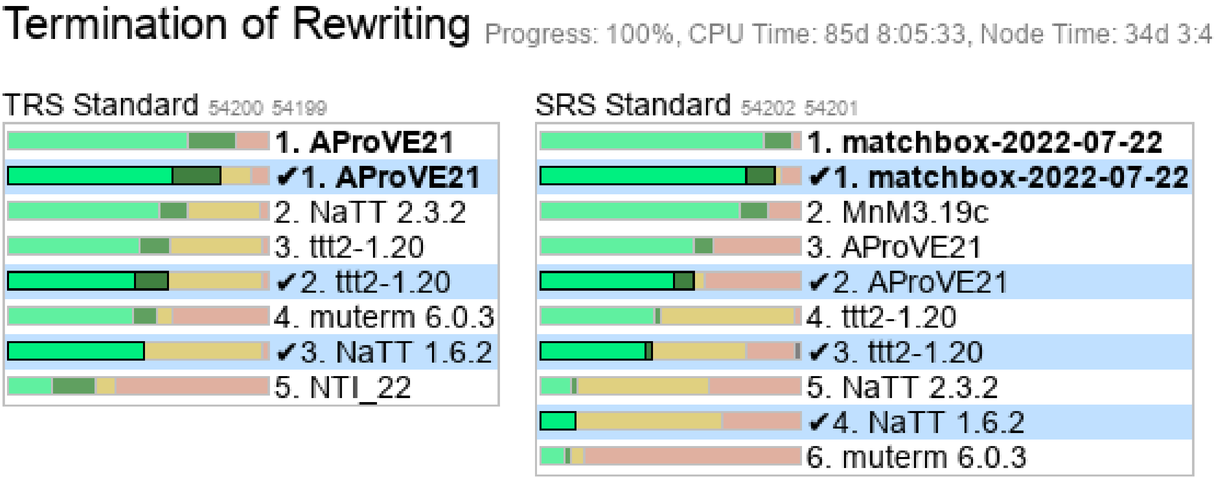
termCOMP with Certification (✓) (1/2)



termCOMP with Certification (✓) (2/2)

Let's zoom in . . .



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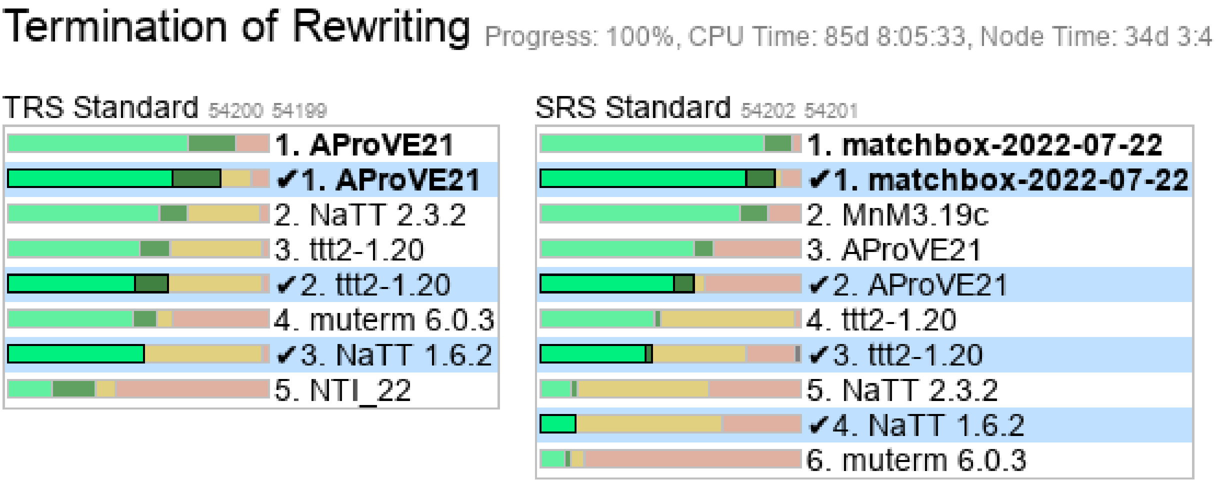

⇒ proof certification is competitive!

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Thanks a lot for your attention!

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