Singular-value Decomposition

DSTA

1 Foundations

1.1 Remember eigenpairs?

Matrix A has a real λ and a vector $\mathbf{v}$ s.t.

$$A\mathbf{v} = \lambda\mathbf{v}$$

We think of A as **scaling** space with a factor λ in direction $\mathbf{v}$.

Singular values uncover categories and their strenghts.

The Eigen-decomposition of a square matrix seen in Goodfellow et al. can be extended to arbitrary matrices!

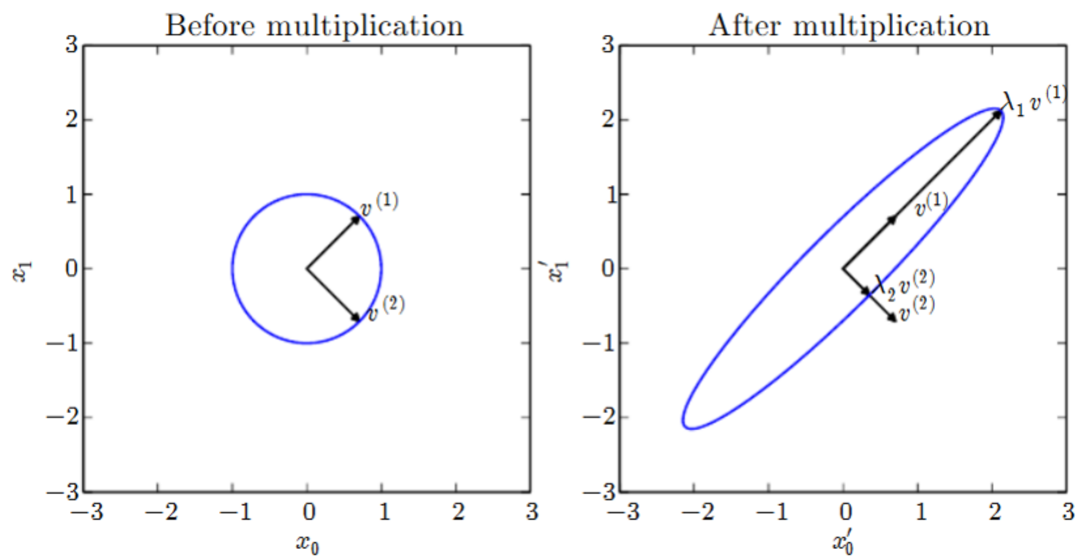

Figure 2.3: An example of the effect of eigenvectors and eigenvalues. Here, we have a matrix $\boldsymbol{A}$ with two orthonormal eigenvectors, $\boldsymbol{v}^{(1)}$ with eigenvalue λ_1 and $\boldsymbol{v}^{(2)}$ with eigenvalue λ_2. *(Left)*We plot the set of all unit vectors $\boldsymbol{u} \in \mathbb{R}^2$ as a unit circle. *(Right)*We plot the set of all points $\boldsymbol{Au}$. By observing the way that $\boldsymbol{A}$ distorts the unit circle, we can see that it scales space in direction $\boldsymbol{v}^{(i)}$ by λ_i.

We think of A as **scaling** space with a factor λ in direction $\mathbf{v}$.

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$$

For unit vectors the max (resp. min) of $f(\cdot)$ corresponds to λ_1 (resp. λ_n).

1.2 Decompose the "effect" of A

Let the square matrix A have n

- linearly-independent e-vectors $\{\mathbf{v}^{(1)} \dots \mathbf{v}^{(n)}\}$
- corresponding e-values $\{\lambda_1 \geq \lambda_1 \geq \dots \lambda_n\}$. Then

. . .

$$A = V\text{diag}(\lambda)V^T$$

where $V = [\mathbf{v}^{(1)}\mathbf{v}^{(1)} \dots \mathbf{v}^{(n)}]$

$\lambda = [\lambda_1\lambda_2 \dots \lambda_n]$.

$$\text{diag}(\lambda) = \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$A = \begin{pmatrix} \uparrow\uparrow & \vdots & \uparrow \\ \mathbf{v}^1\mathbf{v}^2 & \dots & \mathbf{v}^n \\ \downarrow\downarrow & \vdots & \downarrow \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \longleftarrow & \mathbf{v}^1 & \longrightarrow \\ \longleftarrow & \mathbf{v}^2 & \longrightarrow \\ \dots & \dots & \dots \\ \longleftarrow & \mathbf{v}^n & \longrightarrow \end{pmatrix}$$

1.3 A general form for real symmetric Ms

$$A = Q\Lambda Q^T$$

where Q is an orthogonal matrix of e-vectors and Λ is a diagonal m.

For repeated λ values the decomposition is not unique.

2 Singular-Value Decomposition

2.1 Definition

Singular-value decomp. generalises eigen-decomp.:

- any real matrix has one
- even non-square m. admit one

. . .

$$A = V\text{diag}(\lambda)V^{-1}$$

$$A_{(n\times m)} = U_{(n\times n)}D_{(n\times m)}V^T_{(m\times m)}$$

- U is a orthogonal m. of *left-singular* (col.) vectors
- D is a diagonal matrix of *singular values*
- V is a orthogonal m. of *right-singular* (col.) vectors

. . .

Where does all this come from?

2.2 Interpreting SV-decomposition

- cols. of U will be the e-vectors of AA^T
- $D_{ii} = \sqrt{\lambda_i}$ the i-th e-value of A^TA (same for AA^T)
- cols. of V will be the e-vectors of A^TA

Please see § 2.7 of [Goodfellow et al.]

3 Moore-Penrose pseudo-inverse

3.1 Motivations

solve linear systems

$$A\mathbf{x} = \mathbf{y}$$

for non-square (rectangular) matrices:

- n >m: the problem is overconstrained (no solution?)
- n < m: the problem is overparametrized (many sols.?)

3.2 Ideal procedure

If A is squared (n=m) and non-singular ($|A| \neq 0$) then

$$A\mathbf{x} = \mathbf{y}$$

. . .

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{y}$$

. . .

$$I\mathbf{x} = A^{-1}\mathbf{y}$$

Compute once, run for different values of **y.**

3.3 Define the pseudo-inverse

$$A^{+} = \lim_{\alpha \to 0} (A^T A + \alpha I)^{-1} A^T$$

It is proved that $A^{+}A \approx I$ so A^{+} will work as the left-inverse of A

Consequence: over-constrained linear systems can now be solved w. approximation.

3.4 SVD leads to approx. inversion

for the decomposition

$$A = UDV^T$$

$$A^+ = VD^+U^T$$

where D^+, such that $D^+D = I$ is easy to calculate: D is diagonal.

Does $A^+A \approx I$?

Yes, because U and V are s. t. $U^TU = VV^T = I$.

. . .

$$VD^+U^T \cdot UDV^T \;=$$

. . .

$$VD^+IDV^T \;=$$

. . .

$$VD^+DV^T \;=$$

. . .

$$VIV^T = VV^T = I$$