The Mosaic Method for Temporal Logics

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Abstract. The aim of this paper is to apply the mosaic method for proving complexity, Hilbert-style and tableau completeness results for Prior's temporal logic over linear flows of time. We also show how to implement the mosaic idea for automated theorem-proving. Finally we indicate the modifications required to achieve similar results for special linear flows of time and for the more expressive logic of *until* and *since*.

1 Introduction

The aim of this paper is to apply the mosaic method for proving decidability, Hilbert-style (strong) completeness and tableau (weak) completeness of temporal logics over linear flows of time. We also consider implications for automated theorem-proving with temporal logics. Most of these results are known, but we feel that the mosaic approach serves as a general method and slight modifications should enable us to prove important new results for many different logics of linear flows of time. So the idea here is to illustrate the general method on the basic logics avoiding the detail needed for any specific new results.

The origin of the mosaic method is in algebraic logic to prove decidability of the equational theories of certain classes of algebras of relations, cf. [13], [14]. The main idea is to show that the existence of a model is equivalent to the existence of a finite set of partial models, called mosaics, satisfying certain coherency conditions. This gives us a decision procedure, and, intuitively, a systematic procedure to check the theoremhood of a certain formula. Recently the mosaic method has been applied to prove decidability, Hilbert-style axiomatizability and complexity results for various modal logics, cf., e.g., [9], [12], and [21].

The paper is organized as follows. First we give the definition of mosaics and saturated sets of mosaics, SSMs, for temporal logic with future and past operators for linear flows of time. We will prove a key lemma stating that satisfiability is equivalent to the existence of an SSM. Then we show how to achieve

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Hilbert-style and tableau completeness, decidability and complexity results using the key lemma. Section 6 is devoted to an automated theorem-prover: we will see that the mosaic approach gives a reasonably efficient standard method for automated theorem-proving in which only minor modifications need be made to move from one temporal logic to another. Next we will sketch how to modify the basic definitions to cope with special linear flows of time. Finally, we indicate the necessary modification for the more expressive temporal language of *until* and *since*.

2 Mosaics for temporal logic

In this section we show how to define mosaics and saturated sets of mosaics, SSMs, for temporal logic with future F and past P over linear flows of time. We will show that the existence of an SSM for a formula φ is equivalent to the existence of a model for φ.

Let us start with the basic definitions for temporal logic. The language consists of (countably many) propositional variables (also called atoms), propositional connectives (the primitives $\neg$ and $\wedge$ and the usual defined connectives $\vee$, $\rightarrow$ and $\leftrightarrow$) and the temporal connectives F (future) and P (past). We will also use the propositional constants $\top$ (true) and $\bot$ (false); they can be defined as $p \vee \neg p$ and $p \wedge \neg p$, respectively, for an arbitrary atom p. A frame $\mathfrak{T}$ is a non-empty set T equipped with a strict linear ordering[1] $<$: $\mathfrak{T} = (T, <)$. A model is a frame together with an evaluation v of the propositional variables: $v(p) \subseteq T$ for every propositional variable p. Truth of formulas in a model $(T, <, v)$ is defined as follows: for $t \in T$, propositional variable p and formulas φ and ψ,

$$\begin{aligned}
t \Vdash p &\iff t \in v(p) \\
t \Vdash \neg\varphi &\iff \text{not } t \Vdash \varphi \\
t \Vdash \varphi \wedge \psi &\iff t \Vdash \varphi \text{ and } t \Vdash \psi \\
t \Vdash \mathsf{F}\varphi &\iff t' \Vdash \varphi \text{ for some } t' > t \\
t \Vdash \mathsf{P}\varphi &\iff t' \Vdash \varphi \text{ for some } t' < t.
\end{aligned}$$

We will use the defined temporal connectives G (always in the future) and H (always in the past) as well. Here are their definitions:

$$\mathsf{G}\varphi \equiv \neg\mathsf{F}\neg\varphi \qquad \mathsf{H}\varphi \equiv \neg\mathsf{P}\neg\varphi.$$

Note that we assumed that the linear order is irreflexive ($t \not< t$ for every $t \in T$); the "weak" version of future (using $\leq$ instead of $<$) can be expressed as $\varphi \vee \mathsf{F}\varphi$.

We assume familiarity with the basic notions of (temporal) logic such as subformulas, consistency, satisfiability, validity, maximal consistent sets, etc. Let X be a fixed set of formulas closed under subformulas and single negation, in the language of F and P containing at most countably many atoms.

[1] an irreflexive, transitive and connected ordering

Definition 1. *By a* mosaic μ *we mean a structure* $(\langle m_0, m_1\rangle, \ell)$ *satisfying the following. The* labelling function ℓ *associates a subset of* X *to each element of the* base $\{m_0, m_1\}$ *of* μ *such that, for every formula in* X *and index* $i < 2$,

1. $\varphi \in \ell(m_i) \Longleftrightarrow \neg\varphi \notin \ell(m_i)$;
2. $\varphi \wedge \psi \in \ell(m_i) \Longleftrightarrow \{\varphi, \psi\} \subseteq \ell(m_i)$;
3. $\mathsf{G}\varphi \in \ell(m_0) \Rightarrow \mathsf{G}\varphi \in \ell(m_1)$;
4. $\mathsf{G}\varphi \in \ell(m_0) \Rightarrow \varphi \in \ell(m_1)$;
5. $\mathsf{H}\varphi \in \ell(m_1) \Rightarrow \mathsf{H}\varphi \in \ell(m_0)$;
6. $\mathsf{H}\varphi \in \ell(m_1) \Rightarrow \varphi \in \ell(m_0)$.

We also allow mosaics with a singleton base ($m_0 = m_1$), but then we require conditions 1 and 2 only.

An isomorphism *between mosaics is a label-preserving bijection between the two bases.*

We say that conditions 5 and 6 are the *mirror images* of conditions 3 and 4, respectively. In general, mirror images are defined by swapping future operators with the corresponding past operators and reversing the order.

In the Hilbert-style completeness proof, the labelling set X will be the set of all formulas and we will label with maximally consistent sets of formulas, while in the decidability and tableau completeness proofs, X can be chosen as the set of subformulas and their single negations of the formula ξ that we are interested in.

Definition 2. *A set* M *of mosaics is a* saturated set of mosaics, *an SSM for short, if it satisfies the following saturation conditions.*
For every $\mu = (\langle m_0, m_1\rangle, \ell) \in M$,

1. *if* $\mathsf{F}\varphi \in \ell(m_1)$, *then there is a mosaic* $\mu' = (\langle m_0', m_1'\rangle, \ell') \in M$ *such that* $\ell(m_1) = \ell'(m_0')$ *and* $\varphi \in \ell'(m_1')$;
2. *if* $\mathsf{F}\varphi \in \ell(m_0)$, *then either* $\mathsf{F}\varphi \in \ell(m_1)$ *or there are mosaics* $\mu' = (\langle m_0', m_1'\rangle, \ell')$ *and* $\mu'' = (\langle m_0'', m_1''\rangle, \ell'')$ *in* M *such that* $\ell(m_0) = \ell'(m_0')$, $\ell(m_1) = \ell''(m_1'')$ *and* $\varphi \in \ell'(m_1') = \ell''(m_0'')$;
3. *if* $\mathsf{P}\varphi \in \ell(m_0)$, *then there is a mosaic* $\mu' = (\langle m_0', m_1'\rangle, \ell') \in M$ *such that* $\ell(m_0) = \ell'(m_1')$ *and* $\varphi \in \ell'(m_0')$;
4. *if* $\mathsf{P}\varphi \in \ell(m_1)$, *then either* $\mathsf{P}\varphi \in \ell(m_0)$ *or there are mosaics* $\mu' = (\langle m_0', m_1'\rangle, \ell')$ *and* $\mu'' = (\langle m_0'', m_1''\rangle, \ell'')$ *in* M *such that* $\ell(m_0) = \ell'(m_0')$, $\ell(m_1) = \ell''(m_1'')$ *and* $\varphi \in \ell'(m_1') = \ell''(m_0'')$.

Given a set Γ *of formulas we say that there exists an SSM for* Γ *if there is an SSM such that* Γ *is contained in the label of one of the points of a mosaic. If* Γ *is a singleton set* $\{\gamma\}$, *then we will talk about* γ*-SSM.*

Again, 3 an 4 are the mirror images of 1 and 2, respectively. Later we will not bother with formulating the mirror images of conditions if no confusion is likely to occur.

We are ready to formulate our main lemma.

Lemma 1. *For any set Γ of formulas, Γ is satisfiable iff there exists an SSM for Γ.*

Proof. First we show the easy direction, viz., that for any satisfiable set Γ of formulas, there is an SSM for Γ. Let $(T, <, v)$ be a model and $t \in T$ such that Γ holds at t: $t \Vdash \Gamma$. Let X be a set containing Γ which is closed under subformulas and single negations. For any $s \in T$, we let $\ell(s) = \{\varphi : s \Vdash \varphi\} \cap X$. For any $u < v \in T$, we define the mosaic $\mu(u, v) = (\langle u, v\rangle, \ell(u), \ell(v))$. It is easy to check that $\mu(u, v)$ is indeed a mosaic. Furthermore, the set $\{\mu(u, v) : u < v \in T\}$ is an SSM for Γ, since it contains all the mosaics from the same model for Γ.

To prove the other direction, assume that M is an SSM for Γ. We have to show that there is a model satisfying Γ. We will construct such a model step by step using the elements of M as building blocks. The idea is to "cure the defects" of the participating mosaics by "gluing" mosaics together. The model we are going to build will be a substructure of the rational numbers. Let X be the labelling set of M (i.e., we label the points with subsets of X).

First we enumerate all the possible *defects* of labelled structures over subsets of the rational numbers $\mathbb{Q}$:

$$D = \{\langle q, \mathsf{F}\varphi\rangle, \langle q, \mathsf{P}\varphi\rangle : q \in \mathbb{Q} \text{ and } \mathsf{F}\varphi, \mathsf{P}\varphi \in X\}.$$

Note that we assume that the language contains at most countably many atoms, hence the set of formulas in X is at most countable, also the cardinality of $\mathbb{Q}$ is ω; thus $|D| \leq \omega$.

Let $\mathfrak{I} = (I, <, \ell)$ be a labelled structure such that $I \subseteq \mathbb{Q}$, $<$ is the ordering of rational numbers according to magnitude restricted to I, and $\ell : I \to \mathcal{P}(X)$. An element $\langle q, \mathsf{F}\varphi\rangle$ of D is a *future defect* of $\mathfrak{I}$ if

1. $q \in I$
2. $\mathsf{F}\varphi \in \ell(q)$ and
3. for every $p \in I$ such that $q < p$, we have $\varphi \notin \ell(p)$.

That is, a future defect is an "unfulfilled eventuality". Past defects are defined analogously.

We say that a labelled structure $\mathfrak{I} = (I, <, \ell)$ is *coherent* if it satisfies the coherency conditions as in Definition 1: the labels are propositionally consistent, $\mathsf{G}\varphi \in \ell(i)$ implies $\mathsf{G}\varphi, \varphi \in \ell(j)$ for every $j > i$, and $\mathsf{H}\varphi \in \ell(i)$ implies $\mathsf{H}\varphi, \varphi \in \ell(j)$ for every $j < i$. Note that a coherent structure may contain defects.

Below, we will define coherent labelled substructures $\mathfrak{I}_n = (I_n, <, \ell_n)$ of the rational numbers. Let $\sigma : \omega \to \omega$ be a map such that, for every $j \in \omega$, there are infinitely many $k \in \omega$ such that $\sigma(k) = j$. We will use σ as a scheduling function that tells us which defect $D(\sigma(n))$ to cure in the nth step of the construction.

0TH STEP. Take a mosaic $\mu \in M$ such that $\Gamma \subseteq \ell(m_i)$ for some $i < 2$. If $m_0 \neq m_1$, we define $\mathfrak{I}_0 = (\{0, 1\}, \langle 0, 1\rangle, \ell_0)$, where $\ell_0(j) = \ell(m_j)$ for $j < 2$. Otherwise, if the base of μ is a singleton, we define $\mathfrak{I}_0 = (\{0\}, \emptyset, \ell(m_0))$. Clearly, $\mathfrak{I}_0$ is coherent.

$n+1$ST STEP. Assume that we have already constructed a finite linear order of rational numbers $\langle i_0 < i_1 < \ldots < i_k \rangle$ with a coherent labelling ℓ_n such that, for every j, $(\langle i_j, i_{j+1}\rangle, \ell(i_j), \ell(i_{j+1}))$ is (isomorphic to) an element of M. Let us denote this structure by $\mathfrak{I}_n = (I_n, <, \ell_n)$. Let $\sigma(n+1) = l$. Consider the list D of potential defects and take its lth element $D(l)$. We assume that $D(l)$ has the form $\langle i, \mathsf{F}\varphi\rangle$; for past defects a straightforward modification of the construction below works. Next we check if $D(l)$ is an actual defect of $\mathfrak{I}_n$. If it is not (e.g., $i \notin I_n$ or there is $j > i$ in I_n such that $\varphi \in \ell_n(j)$), we define $\mathfrak{I}_{n+1} = \mathfrak{I}_n$.

So assume that $D(l)$ is in fact a defect of $\mathfrak{I}_n$. Let i_j be the greatest element of the order $i_0 < \ldots < i_k$ such that $\mathsf{F}\varphi \in \ell_n(i_j)$; since $D(l)$ is a defect of $\mathfrak{I}_n$, such i_j exists.

If i_j is the last point of I_n in the order ($j = k$), then there is a mosaic $(\langle m'_0, m'_1\rangle, \ell'(m'_0), \ell'(m'_1))$ in M that we can glue to $(\langle i_{k-1}, i_k\rangle, \ell_n(i_{k-1}), \ell_n(i_k))$ witnessing $\mathsf{F}\varphi \in \ell_n(i_k)$; that is, the two mosaics satisfy Definition 2.1, and $\ell'(m'_0) = \ell_n(i_k)$ and $\varphi \in \ell'(m'_1)$. We add a new point i_{k+1} at the end of the linear order with label $\ell'(m'_1)$ to our structure: we define $\mathfrak{I}_{n+1}$ as the order $\langle i_0 < \ldots < i_k < i_{k+1}\rangle$ with labelling $\ell_{n+1}(i_{k+1}) = \ell'(m'_1)$ and $\ell_{n+1}(i_p) = \ell_n(i_p)$ for $0 \le p \le k$.

Now assume that $i_j < i_k$. Let us consider the mosaic $(\langle i_j, i_{j+1}\rangle, \ell_n(i_j), \ell_n(i_{j+1}))$. By the maximality condition on i_j, we have that $\neg\mathsf{F}\varphi \in \ell_n(i_{j+1})$. Then we can find two mosaics $(\langle m'_0, m'_1\rangle, \ell'(m'_0), \ell'(m'_1))$ and $(\langle m''_0, m''_1\rangle, \ell''(m''_0), \ell''(m''_1))$ such that $\ell'(m'_0) = \ell_n(i_j)$, $\varphi \in \ell'(m'_1) = \ell''(m''_0)$ and $\ell''(m''_1) = \ell_n(i_{j+1})$, see Definition 2.2. Then we insert a new point i' between i_j and i_{j+1} with a label $\ell'(m'_1)$: we define $\mathfrak{I}_{n+1}$ as the order $\langle i_0 < \ldots < i_j < i' < i_{j+1} < \ldots < i_k\rangle$ with labelling $\ell_{n+1}(i') = \ell'(m'_1)$ and $\ell_{n+1}(i_p) = \ell_n(i_p)$ for $0 \le p \le k$.

Thus, we can define a structure $\mathfrak{I}_{n+1} = (I_{n+1}, <, \ell_{n+1})$ where $D(l)$ is not a defect anymore.[2] Clearly $\mathfrak{I}_{n+1}$ is a union of copies of elements of M and it is coherent. Furthermore, for every $j \in I_n$, we have that $\ell_n(j) = \ell_{n+1}(j)$; that is the new labelling ℓ_{n+1} is compatible with the old one ℓ_n. Also note that once we cured a defect by providing a witness, the same defect cannot occur in any expansion of the structure.

ωTH STEP. Take the union $\mathfrak{I} = (I, <, \ell)$ of the labelled structures defined so far (the compatibility of the labellings makes this well defined). It is easy to see that we get a coherent linear structure that does not contain any defect — every potential defect has been dealt with infinitely many times, hence we can be sure that if it is an actual defect in some stage, then we repair it in a later step.

Let the valuation v be defined as $i \in v(p) \iff p \in \ell(i)$, for every point $i \in I$ and atom p (if p is not in our labelling set X, then let $v(p)$ be arbitrary).

Consider the model $(I, <, v)$. We claim that $i \Vdash \varphi \iff \varphi \in \ell(i)$, for every $i \in I$ and $\varphi \in X$. We proceed by induction on the complexity of φ. The case of atoms and the propositional connectives are easy. Now assume that $\mathsf{F}\varphi \in \ell(i)$. Since $(I, <, \ell)$ does not contain any defect, there is $j > i$ such that $\varphi \in \ell(j)$. By the induction hypothesis, $j \Vdash \varphi$, whence $i \Vdash \mathsf{F}\varphi$. If $i \Vdash \mathsf{F}\varphi$, then there is

[2] If $D(l)$ is a past defect $\langle i, \mathsf{P}\varphi\rangle$, then use conditions 3 and 4 in Definition 2.

$j > i$ such that $j \Vdash \varphi$. Then $\varphi \in \ell(j)$ by the induction hypothesis. Hence, by coherency, $\mathsf{F}\varphi \in \ell(i)$. The proof for $\mathsf{P}\varphi$ is completely analogous.

Since the mosaic in the 0th step was labelled by Γ, we get that $(I, <, v)$ satisfies Γ. ∎

3 Hilbert-style completeness

In this section, we show how to prove strong Hilbert-style completeness (that is, we will prove that derivability in a Hilbert-style axiom system coincides with semantical consequence) using mosaics and Lemma 1. This has the advantage that we do not have to create a model for a consistent formula set, an SSM will do.

First let us recall from [5] that the minimal temporal logic K_t is the normal logic with conjugate modalities F and P interpreted over transitive frames. We define the calculus $\vdash$ as follows: its axioms include enough propositional axioms, the axioms for K_t

$$\begin{array}{ll}
(K_F) & \mathsf{G}(\varphi \to \psi) \to (\mathsf{G}\varphi \to \mathsf{G}\psi) \\
(K_P) & \mathsf{H}(\varphi \to \psi) \to (\mathsf{H}\varphi \to \mathsf{H}\psi) \\
(C_F) & \varphi \to \mathsf{GP}\varphi \\
(C_P) & \varphi \to \mathsf{HF}\varphi \\
(4_F) & \mathsf{G}\varphi \to \mathsf{GG}\varphi \\
(4_P) & \mathsf{H}\varphi \to \mathsf{HH}\varphi
\end{array}$$

plus the following linearity axioms

$$(\mathit{Lin.1}) \qquad \Box\varphi \to \mathsf{HG}\varphi$$

and

$$(\mathit{Lin.2}) \qquad \Box\varphi \to \mathsf{GH}\varphi$$

where $\Box\varphi$ stands for $\mathsf{H}\varphi \wedge \varphi \wedge \mathsf{G}\varphi$; the derivation rules are Modus Ponens and Universal Generalization (or Necessitation):

$$\text{from } \varphi \text{ infer } \mathsf{G}\varphi \text{ and } \mathsf{H}\varphi.$$

Our next aim is to show that $\vdash$ is a strongly complete inference system for linear temporal logic, i.e., for every set Γ of formulas,

$$\Gamma \text{ is consistent iff } \Gamma \text{ is satisfiable}^3.$$

In the light of Lemma 1, the strong completeness theorem is equivalent to the following.

Lemma 2. *For any set Γ of formulas, Γ is consistent iff there exists an SSM for Γ.*

[3] Recall that consistency of Γ means that it is impossible to derive contradictions from Γ by using the inference system, while Γ is satisfiable if it has a model.

Proof. By Lemma 1, if there exists an SSM for Γ, then Γ is satisfiable, and hence consistent by the soundness of $\vdash$ (we leave the easy task to check this to the reader).

For the other direction, let us assume that Γ is consistent. We have to show that there is an SSM for Γ.

Let our labelling set be the set of all formulas[4] (in the language of Γ). We define a set of mosaics as the collection of all $(\langle 0\rangle, \ell(0))$ and $(\langle 0,1\rangle, \ell(0), \ell(1))$ where $\ell(0)$ and $\ell(1)$ are maximal consistent sets, MCSs for short, satisfying

$$\{\varphi, \mathsf{G}\varphi : \mathsf{G}\varphi \in \ell(0)\} \subseteq \ell(1). \tag{1}$$

We will say that $\langle \ell(0), \ell(1)\rangle$ satisfies (1). Using the axioms for transitivity (4_F and 4_P) and for the connection between F and P (C_F and C_P), one can easily show that the conditions of Definition 1 are met. Also, (1) is equivalent to

$$\{\varphi, \mathsf{H}\varphi : \mathsf{H}\varphi \in \ell(1)\} \subseteq \ell(0). \tag{2}$$

Since we can extend Γ to a MCS, we have a mosaic witnessing Γ.

Let X, Y, Z denote MCSs. It suffices to show 1 and 2 below (and their mirror images) to prove that we have a saturated set of mosaics.

1. For every X and $\mathsf{F}\varphi \in X$, there is a Y such that $\varphi \in Y$ and $\langle X, Y\rangle$ satisfies (1).
2. Let X, Y and ψ be such that $\langle X, Y\rangle$ satisfies (1), $\mathsf{F}\psi \in X$ and both $\neg\mathsf{F}\psi$ and $\neg\psi$ are in Y. Then there is a Z such that $\psi \in Z$ and both $\langle X, Z\rangle$ and $\langle Z, Y\rangle$ satisfy (1).

1 holds by general $K4$-consideration (as in [5]).

2 holds because of the following. By 1, the set $\{\varphi, \mathsf{G}\varphi : \mathsf{G}\varphi \in X\} \cup \{\psi\}$ is consistent. Let Z be a MCS containing it. We show that Z satisfies the claim. It suffices to show that $\langle Z, Y\rangle$ satisfies (1), i.e., $\langle Y, Z\rangle$ satisfies (2). Let us assume that $\mathsf{H}\varphi \in Y$. Then we have $\mathsf{HH}\varphi \in Y$ by (4_P). Hence $\mathsf{H}(\varphi \wedge \mathsf{H}\varphi) \in Y$ and thus $\mathsf{H}(\psi \to (\varphi \wedge \mathsf{H}\varphi)) \in Y$. Recall that $\neg\psi \in Y$, whence $\psi \to (\varphi \wedge \mathsf{H}\varphi) \in Y$. As Y contains $\neg\mathsf{F}\psi$, we have that $\mathsf{G}(\psi \to (\varphi \wedge \mathsf{H}\varphi))$ as well. Putting these together we see that Y contains $\Box(\psi \to (\varphi \wedge \mathsf{H}\varphi))$. By ($Lin.1$) we can conclude that $\mathsf{HG}(\psi \to (\varphi \wedge \mathsf{H}\varphi))$ is in Y, whence $\mathsf{G}(\psi \to (\varphi \wedge \mathsf{H}\varphi))$ is in X (by (1)). Since $\langle X, Z\rangle$ satisfies (1) and $\psi \in Z$, we have that $\varphi \wedge \mathsf{H}\varphi$ is in Z as required.

The proof of the mirror images is similar.

Thus the set of all mosaics defined above forms an SSM. ∎

4 Decidability and complexity

By Lemma 1, the decidability proof amounts to proving that the existence of an SSM for a given formula is decidable.

[4] In this case, the labelling sets may be infinite sets of formulas. We allowed this in the definition of mosaics and SSM. For decidability, complexity and weak completeness purposes we will restrict the labelling sets to finite ones.

Lemma 3. *For any formula ξ, it is decidable whether there is an SSM for ξ.*

Proof. Use the set of subformulas of ξ and their negations as the labelling set. Then the usual argument works: given the complexity of ξ, we can compute the number of mosaics; since checking the saturation conditions is decidable, we can decide whether any subset of the set of all mosaics form a saturated set of mosaics for ξ. ∎

Below we show how to use the above idea to prove complexity results. One can prove NP-completeness of linear temporal logic (due to Ono and Nakamura [15]) by considering non-standard models for ξ of size $|\xi|$. However, using the mosaic idea, we can avoid non-standard (not linear) models. Below we will show that if there is an SSM for a formula ξ with complexity n, then there is an SSM with at most $n^2 + 2n + 1$ elements. Given a set of mosaics checking the saturation conditions can be done in polynomial time. Thus we get a proof for the NP-completeness[5] of linear temporal logic by guessing an SSM of size $O(n^2)$ and checking in polynomial time whether it is indeed an SSM.

Lemma 4. *Let ξ be a formula with complexity n. If ξ is satisfiable, then there is an SSM for ξ with size $O(n^2)$.*

Proof. Let $(W, <, v)$ be a (linear) model such that $w_0 \Vdash \xi$ for some $w_0 \in W$. Let X be the set of subformulas and their negations of ξ and put $Y = \{\mathsf{F}\varphi \in X : \text{ there exists } w \in W \text{ such that } w \Vdash \varphi\} \cup \{\mathsf{P}\varphi \in X : \text{ there exists } w \in W \text{ such that } w \Vdash \varphi\}$. For every $w, w' \in W$, let $\ell(w) = \{\varphi \in X : w \Vdash \varphi\}$ and let $w \equiv w'$ iff $\ell(w) = \ell(w')$. Note that there are finitely many equivalence classes, since X is finite. For every formula $\mathsf{F}\varphi \in Y$, let $W(\varphi) = \{w \in W : w \Vdash \varphi\}$. Let $\overline{w}(\varphi)$ be a *maximal* element of $W(\varphi)$ in the following sense:

$$(\forall w' \in W(\varphi))(\exists w'' \in W(\varphi)) w' \le w'' \ \&\ w'' \equiv \overline{w}(\varphi).$$

The existence of a maximal element can be shown by an easy induction on the number of equivalence classes.

For every $\mathsf{F}\varphi \in Y$ choose a maximal element from $W(\varphi)$. Similarly, for every $\mathsf{P}\varphi \in Y$, choose a *minimal* element $\underline{w}(\varphi)$ from $W(\varphi)$ (we leave the details to the reader).

Let

$$W' = \{w_0\} \cup \{\overline{w}(\varphi) : \mathsf{F}\varphi \in Y\} \cup \{\underline{w}(\varphi) : \mathsf{P}\varphi \in Y\}.$$

Note that $|W'|$ is $O(n)$. Let M be

$$\{(\langle x, y\rangle, \ell(x), \ell(y)) : x, y \in W', (\exists x' \in W)(\exists y' \in W) x \equiv x' \ \&\ y \equiv y' \ \&\ x' < y'\}.$$

We claim that M is the desired SSM for ξ. Clearly, $|M|$ is $O(n^2)$, and M consists of mosaics.

It remains to show the saturation conditions. Let $(\langle x, y\rangle, \ell(x), \ell(y)) \in M$.

[5] The problem is NP-hard as the logic embeds classical propositional logic.

First assume that $\mathsf{F}\varphi \in \ell(y)$. Let $y' \in W$ such that $y \equiv y'$. Since $y' \Vdash \mathsf{F}\varphi$, there is $z \in W$ such that $z \Vdash \varphi$ and $y' < z$. Let $\overline{w}(\varphi)$ be the maximal element in $W(\varphi)$ that we put in W'. By the maximality of $\overline{w}(\varphi)$, there is $z' \in W$ such that $z \leq z'$ and $z' \equiv \overline{w}(\varphi)$. By this observation, we get that $(\langle y, \overline{w}(\varphi)\rangle, \ell(y), \ell(\overline{w}(\varphi))) \in M$.

Now assume that $\mathsf{F}\varphi \in \ell(x)$ and $\neg\mathsf{F}\varphi, \neg\varphi \in \ell(y)$. Let $x', y' \in W$ such that $x \equiv x'$, $y \equiv y'$ and $x' < y'$. Then there is $z \in W$ such that $x' < z < y'$ and $z \Vdash \varphi$. Let $\overline{w}(\varphi)$ be the maximal element of $W(\varphi)$ that we put in W'. Then there is $z' \in W$ such that $z \leq z'$ and $z' \equiv \overline{w}(\varphi)$. Hence $(\langle x, \overline{w}(\varphi)\rangle, \ell(x), \ell(\overline{w}(\varphi))) \in M$. Since $y' \Vdash \neg\mathsf{F}\varphi$, $y' \Vdash \neg\varphi$, and $z \Vdash \varphi$ we can not have $y' \leq z'$. Thus $z' < y'$ and $(\langle \overline{w}(\varphi), y\rangle, \ell(\overline{w}(\varphi)\rangle, \ell(y)) \in M$ as well. Thus we can insert the appropriate point between x and y.

Checking the saturation conditions for past formulas is completely analogous.
∎

5 Tableau completeness

Now we show how to use the mosaic idea to give a complete tableau axiomatization for linear temporal logic. The tableau system below follows the mosaic idea and it shows that, in theory, the mosaic approach is suitable for achieving tableau completeness. We are aware that the tableau system below is not very attractive from the implementational point of view. In Section 6, we show how to design an automated theorem-prover directly from the mosaic idea without relying on the tableau formalism below.

Semantic tableaux are used in temporal logics, cf. [6–8, 10, 19, 22, 23]. [6] concentrates on the restricted language with future as the only temporal connective, and gives cut-free tableau systems for discrete and dense linear and branching time logics. Goré's systems are cut-free, but use formulas that are not subformulas of the original formula (though there is a bound on these formulas). Our system below uses a weak version of the cut rule, where the application of the rule is restricted to (perhaps negated) subformulas.[6] [10] introduces a cut-free sequent calculus for various temporal logics including linear (transitive and connected) temporal logic. A difference between Kashima's calculus and our system below is the way how linearity is handled. Intuitively, when one wants to build a model for a temporal formula, and in particular to create a witness for a future formula $\mathsf{F}\varphi$, one has to find out where to put the required φ-world into the linear order (e.g., it cannot come after a time point satisfying $\neg\mathsf{F}\varphi$). Kashima introduces a rule (called *TrCo*) with branching according to the possibilities of the location of the witnessing point. Our strategy below is firstly to decide if a future point satisfies $\mathsf{F}\varphi$ or $\neg\mathsf{F}\varphi$ and secondly to apply the corresponding rule to create a φ-witness with a suitable location. [19] gives another tableau system for linear temporal logic. Their main aim is to define a system which does not require loop-checking (cf. [22]). Their calculus uses formulas labelled with time

[6] We are currently working on a tableau system based on the mosaic idea in which the cut can be eliminated.

intervals. We also use labelled tableau, but our labels are time points reflecting the mosaic idea.[7] Finally we note that Zimmerman's tableau system [23, 8] for minimal temporal logic might be extended to cover the linear case.

We will use labelled tableaux. In general, a tableau rule in our system has the form

$$\frac{l : A;\;\; m : B;\;\; r : C}{l : A';\;\; m : B';\;\; r : C' \mid l : A'';\;\; m : B'';\;\; r : C''}$$

where $A, \ldots, C''$ are finite sets of formulas and l, m and r stand for *l*eft, *m*iddle and *r*ight, respectively. We will also allow rules where some of the formula sets are empty, e.g., the numerator may be $l : A;\;\; m : \emptyset;\;\; r : C$ that we will abbreviate as $l : A;\;\; r : C$. A model $(N, <, v)$ satisfies the labelled triple $l : A_l;\;\; m : A_m;$ $r : A_r$ if for every non-empty A_i $(i \in \{l, m, r\})$, there is $n_i \in N$ such that $n_i \Vdash A_i$ and $n_l < n_m$, $n_m < n_r$ and $n_l < n_r$. The intuition for a rule is that if the numerator $l : A;\;\; m : B;\;\; r : C$ is satisfiable in a model, then so is one of the denominators $l : A';\;\; m : B';\;\; r : C'$ or $l : A'';\;\; m : B'';\;\; r : C''$.

Now we recall the basic definitions for tableau systems, cf. [3]. A *tableau* for a set X of formulas is a finite tree with a root containing $m : X$. If a node carries $l : X;\;\; m : Y;\;\; r : Z$ and we apply the rule

$$\frac{l : A;\;\; m : B;\;\; r : C}{l : A';\;\; m : B';\;\; r : C' \mid l : A'';\;\; m : B'';\;\; r : C''}$$

then it has two successor nodes carrying $l : A';\;\; m : B';\;\; r : C'$ and $l : A''; m :$ $B'';\;\; r : C''$. A branch is *finished* at a node if there is no applicable rule for this node or the node already occurred in the same branch. A branch is *closed* if its end node carries $m : \bot$. A tableau is closed if each of its branches is closed. A set X of formulas is *consistent* if there is no closed tableau with root $m : X$. Similarly, we will say that $l : X;\;\; m : Y;\;\; r : Z$ is consistent, if it cannot occur at the root of a closed tableau.

Let X be a given set of formulas and $Sf(X)$ be the smallest set containing the subformulas of the elements of X and their negations. Let $Y \subseteq Sf(X)$. Y is *maximal* in $Sf(X)$ if each element of $Sf(X)$ or its negation is in Y. We will need maximal sets in our tableau proofs, but note that we will work *inside* $Sf(\{\xi\})$, where ξ is the formula that we want to prove. Thus, although one may consider the need for maximal consistent sets an undesirable feature, our tableau system has the $\pm$subformula property: every formula occurring in the tableau is a (perhaps negated) subformula of ξ.

So let us fix a formula ξ and define $\Xi = Sf(\{\xi\})$. We will assume that Ξ contains $\bot$; e.g., we might start with the equivalent formula $\xi \vee \bot$ instead of ξ. We have four types of rules. The first type of rules describes mosaics. These contain rules for propositional logic:

$$\frac{l : X;\;\; m : Y \cup \{\varphi, \neg\varphi\};\;\; r : Z}{m : \bot}$$

[7] It is not necessary to use labelled formulas in the tableau system below, but using labels makes the connection with the mosaic decidability proof more transparent.

$$\frac{l : X;\;\; m : Y \cup \{\varphi \wedge \psi\};\;\; r : Z}{l : X;\;\; m : Y \cup \{\varphi, \psi\};\;\; r : Z}$$

$$\frac{l : X;\;\; m : Y \cup \{\neg(\varphi \wedge \psi)\};\;\; r : Z}{l : X;\;\; m : Y \cup \{\neg\varphi\};\;\; r : Z \mid l : X;\;\; m : Y \cup \{\neg\psi\};\;\; r : Z}$$

$$\frac{l : X;\;\; m : Y \cup \{\neg\neg\varphi\};\;\; r : Z}{l : X;\;\; m : Y \cup \{\varphi\};\;\; r : Z}.$$

We have similar rules with $l : X$ and $r : Z$ instead of $m : Y$. By the coherency conditions 3 and 4 in Definition 1, if we have $l : X \cup \{\mathsf{G}\varphi\};\;\; r : Y$, then Y must not contain $\neg\mathsf{G}\varphi$ or $\neg\varphi$. The corresponding rules are

$$(C3) \quad \frac{l : X \cup \{\mathsf{G}\varphi\};\;\; r : Y \cup \{\neg\varphi\}}{m : \bot} \qquad (C4) \quad \frac{l : X \cup \{\mathsf{G}\varphi\};\;\; r : Y \cup \{\neg\mathsf{G}\varphi\}}{m : \bot}$$

We also have the "mirror" rules $(C5)$ and $(C6)$ corresponding to conditions 5 and 6 of Definition 1.

More interesting are the rules corresponding to the saturation conditions. We use the abbreviations $\mathsf{GG}^{-1}X = \{\mathsf{G}\varphi : \mathsf{G}\varphi \in X\}$ and $\mathsf{G}^{-1}X = \{\varphi : \mathsf{G}\varphi \in X\}$ and similarly for HH^{-1} and H^{-1}. Rules for saturation conditions 1 and 3 of Definition 2 are

$$(S1) \quad \frac{m : X \cup \{\mathsf{F}\varphi\}}{l : X \cup \{\mathsf{F}\varphi\};\;\; r : \mathsf{GG}^{-1}X \cup \mathsf{G}^{-1}X \cup \{\varphi\}}$$

and its mirror rule

$$(S3) \quad \frac{m : X \cup \{\mathsf{P}\varphi\}}{l : \mathsf{HH}^{-1}X \cup \mathsf{H}^{-1}X \cup \{\varphi\};\;\; r : X \cup \{\mathsf{P}\varphi\}}$$

and those corresponding to 2 and 4 of Definition 2 are $(S2)$

$$\frac{l : X \cup \{\mathsf{F}\varphi\};\; r : Y \cup \{\neg\mathsf{F}\varphi, \neg\varphi\}}{l : X \cup \{\mathsf{F}\varphi\};\; m : \mathsf{GG}^{-1}X \cup \mathsf{G}^{-1}X \cup \{\varphi\} \cup \mathsf{HH}^{-1}Y \cup \mathsf{H}^{-1}Y;\; r : Y \cup \{\neg\mathsf{F}\varphi, \neg\varphi\}}$$

and its mirror rule $(S4)$

$$\frac{l : Y \cup \{\neg\mathsf{P}\varphi, \neg\varphi\};\; r : X \cup \{\mathsf{P}\varphi\}}{l : Y \cup \{\neg\mathsf{P}\varphi, \neg\varphi\};\; m : \mathsf{GG}^{-1}Y \cup \mathsf{G}^{-1}Y \cup \{\varphi\} \cup \mathsf{HH}^{-1}X \cup \mathsf{H}^{-1}X;\; r : X \cup \{\mathsf{P}\varphi\}}.$$

The third type of rules are used to extend consistent sets of formulas to maximal ones in Ξ. The rules to construct maximal sets are: for every $\varphi \in \Xi$,

$$(Mm) \quad \frac{l : X;\;\; m : Y;\;\; r : Z}{l : X;\;\; m : Y \cup \{\varphi\};\;\; r : Z \mid l : X;\;\; m : Y \cup \{\neg\varphi\};\;\; r : Z}$$

and the similar rules (Ml) and (Mr) for $l : X$ and $r : Z$ instead of $m : Y$.

Finally, we have rules that allow us to discard some of the assumptions. The intuition is that if a triple of labelled formulas is satisfiable, then so is any subset of the triple.

$$\frac{l : X;\;\; m : Y;\;\; r : Z}{m : X} \qquad \frac{l : X;\;\; m : Y;\;\; r : Z}{m : Y} \qquad \frac{l : X;\;\; m : Y;\;\; r : Z}{m : Z}$$

$$\frac{l:X;\ \ m:Y;\ \ r:Z}{l:X;\ \ r:Y} \qquad \frac{l:X;\ \ m:Y;\ \ r:Z}{l:X;\ \ r:Z} \qquad \frac{l:X;\ \ m:Y;\ \ r:Z}{l:Y;\ \ r:Z}$$

For establishing the soundness one has to check that consistent sets are satisfiable. This amounts to proving that if the numerator of a rule is satisfiable, then so is one of the denominators — we leave this easy task to the reader. Completeness is guaranteed by the following lemma.

Lemma 5. *Every consistent formula is satisfiable.*

Proof. Let ξ be a consistent formula, i.e., assume that no tableau closes for $m:\{\xi\}$. As before, let $\Xi = Sf(\{\xi\})$.

Using the propositional rules and (Mm) (with empty left and right formula sets), let us extend $\{\xi\}$ to a consistent maximal set A in Ξ. If A does not contain any formula of the form $\mathsf{F}\varphi$ or $\mathsf{P}\varphi$, then the one element model with the obvious evaluation will do.

The interesting case is when A contains some defect. Instead of directly constructing a model, we will show that there is a saturated set of mosaics for A, where the labels of the mosaics are maximally consistent sets in Ξ.

We define M as the set of labelled pairs $(\langle 0,1\rangle, X, Y)$ such that X and Y are maximal in Ξ and no tableau closes for $l:X;\ \ r:Y$.

Using the rules for the mosaic conditions, we get that $(\langle 0,1\rangle, X, Y)$ is in fact a mosaic. Indeed, if one of the coherency conditions of Definition 1 were violated, then the rule corresponding to this condition would close the tableau.

By assumption, A contains some defect, say $\mathsf{F}\varphi \in A$. Then the application of $(S1)$ gives us a node carrying $l:A\cup\{\mathsf{F}\varphi\};\ \ r:\mathsf{G}\mathsf{G}^{-1}A\cup\mathsf{G}^{-1}A\cup\{\varphi\}$. Then using the rule (Mr) (with empty middle formula set) for creating maximal sets, we can extend $\mathsf{G}\mathsf{G}^{-1}A\cup\mathsf{G}^{-1}A\cup\{\varphi\}$ to a maximal B such that $l:A\cup\{\mathsf{F}\varphi\};\ \ r:B$ is consistent. Thus, ξ occurs in the label A of an element of M.

It remains to show that M is saturated. Let $(\langle 0,1\rangle, X, Y)$ be any element of M.

If $\mathsf{F}\varphi \in Y$, then using $(S1)$ we have $l:Y\cup\{\mathsf{F}\varphi\};\ \ r:\mathsf{G}\mathsf{G}^{-1}Y\cup\mathsf{G}^{-1}Y\cup\{\varphi\}$. Again, we can extend the set on the right to a maximal Z such that $l:Y\cup\{\mathsf{F}\varphi\};\ \ r:Z$ is consistent. This provides the necessary mosaic curing the defect as in 1 of Definition 2.

If $\mathsf{F}\varphi \in X$ but $\neg\varphi, \neg\mathsf{F}\varphi \in Y$, then we apply $(S3)$. Thus we get that

$$l:X\cup\{\mathsf{F}\varphi\};\ m:\mathsf{G}\mathsf{G}^{-1}X\cup\mathsf{G}^{-1}X\cup\{\varphi\}\cup\mathsf{H}\mathsf{H}^{-1}Y\cup\mathsf{H}^{-1}Y;\ r:Y\cup\{\neg\mathsf{F}\varphi,\neg\varphi\}$$

is consistent. Next we extend the set in the middle to a maximal Z. Thus we have a consistent $l:X;\ \ m:Z;\ \ r:Y$. Then, by the last type of rules, we get that $l:X;\ \ r:Z$ and $l:Z;\ \ r:Y$ are consistent, i.e., $(\langle 0,1\rangle, X, Z)$ and $(\langle 0,1\rangle, Z, Y)$ are in M. Then we have the required two mosaics to insert into $(\langle 0,1\rangle, X, Y)$, cf. 2 of Definition 2.

Checking conditions 2 and 4 of Definition 2 is the same, just use $(S2)$ and $(S4)$ instead of $(S1)$ and $(S3)$. ∎

6 Automated theorem-proving

A draft version of a mosaic-based theorem-prover for the temporal logic with F and P over linear flows of time is available [18]. This Java applet checks the formula typed in by the user for satisfiability. The theorem-prover only considers mosaics with a non-singleton base and so first checks whether the formula has a one point model. If there is no one point model, the theorem-prover finds all non-singleton mosaics using the closure as in Lemma 3. Then it tries to find an SSM within the set of all mosaics; by Lemma 1, this procedure gives the correct answer. If it succeeds, then the SSM is displayed, otherwise the formula is unsatisfiable.

By systematically exploring all ways to complete a propositionally consistent subset of the labelling set, for a formula φ, it finds all MPCs, maximally propositionally consistent subsets of the labelling set. When the length of the formula is less than about 27, there usually is enough memory to store an array detailing the contents of each MPC. In any mosaic the labels at each end are MPCs. So we can easily (by checking coherency conditions, i.e., the conditions 1 to 6 of Definition 1) find and store an array of all mosaics indicating which MPC is at each end. To decide whether there is an SSM we consider a set M of mosaics which starts off as the set of all mosaics and is whittled down by the following iterative process. In each iteration check whether the formula φ appears in one end or the other of some mosaic in M: if not then we can stop and conclude that there is no SSM for φ. Then go through each mosaic in M and check that the conditions of Definition 2 are satisfied: if not then remove the mosaic from M. If we go through an iteration without removing any mosaics then we can stop and conclude that we have an SSM.

Note that the current implementation can, despite the worst case exponential time behaviour, quite quickly decide formulas to a length of about 27. There may be quicker ways of deciding formulas in this particular logic but the general mosaic-based approach of this theorem-prover can, as we will see, be used for many other temporal logics with only minor modifications.

7 Other linear flows of time

In this section, we will describe how to modify the definition of SSM for special linear flows of time. We will concentrate on decidability and weak completeness, thus we can assume that the labelling set $Sf(\chi)$ is the set of subformulas of χ and their negations, where χ is the formula to be satisfied. It should be clear that the modifications below does not effect the decidability of the existence of an SSM.

Substructures of the whole numbers. We require that in item 2 of Definition 2 we have the additional condition that $\neg\mathsf{F}\varphi \in \ell'(m_1')$ (and the corresponding condition in its mirror image 4).

It is easy to check that the proof of Lemma 1 goes through without any non-trivial modification (of course, we assume that in this case the model satisfying χ is based on a substructure of whole numbers).

We claim that the structure defined using the χ-SSM is in fact a substructure of the whole numbers. The key observation is that only once we can insert a φ-point to provide a witness for a future formula $\mathsf{F}\varphi$. Indeed, let us assume that we inserted a point k, say in step $n+1$, such that $\varphi, \neg\mathsf{F}\varphi \in \ell_{n+1}(k)$ between two points $i < j$ because we had $\mathsf{F}\varphi \in \ell_n(i)$ and $\neg\mathsf{F}\varphi, \neg\varphi \in \ell_n(j)$ (recall that according to the construction, if we can avoid inserting a point, then we create the witness at the end of the finite linear order). Towards a contradiction let us assume that in a later stage $m+1$ we have to insert another φ-point because of a a future defect $\langle l, \mathsf{F}\varphi\rangle$ in $\mathfrak{I}_m$. Where can this point l be compared to k in the linear order? First assume that $l < k$: if $\mathsf{F}\varphi \in \ell_m(l)$, then $\varphi \in \ell_m(k)$ is a good witness. On the other hand, if $l \geq k$, then $\neg\mathsf{F}\varphi \in \ell_m(l)$, since $\neg\mathsf{F}\varphi \in \ell_m(k)$ and $\mathfrak{I}_m$ is coherent. (Note that we might have to insert another φ-witness because of a past defect $\langle i', \mathsf{P}\varphi\rangle$, but the same argument shows that it can happen only once.)

Since we are using a finite label set $Sf(\chi)$, there are only finitely many $\mathsf{F}\varphi$ and $\mathsf{P}\varphi$ formulas in $Sf(\chi)$. Hence we are inserting only finitely many points into the linear order that we are constructing, and all the other points that we have to add, we glue to the beginning (for past defects) or to the end (for future defects) of the linear order we constructed so far. That is, for any two points i and j, we insert only finitely many points between i and j during the construction. Thus every non-startpoint in the linear order has an immediate predecessor and every non-endpoint has an immediate successor. As the construction terminates in at most ω steps, the linear order must be a substructure of the whole numbers.

Without endpoint(s). Add the formula $\mathsf{GF}\top \wedge \mathsf{F}\top$ (and $\mathsf{HP}\top \wedge \mathsf{P}\top$) to the label containing χ. Then, during the construction, we have to add points at the end of the linear orders infinitely often, since in each step we have a future defect $\mathsf{F}\top$ at the end of the linear order (or the past defect $\mathsf{P}\top$ at the start).

With endpoint(s). Add the formula $\mathsf{FG}\bot \vee \mathsf{G}\bot$ (and $\mathsf{PH}\bot \vee \mathsf{H}\bot$) to the label containing χ — this has the effect that we cannot add new points at the end of the structure once we created the witness for $\mathsf{FG}\bot$.

Dense. Require in the definition of SSM that, for every mosaic, there exist mosaics that can be inserted, like in Definition 2.2. Then, in the construction of the model from the SSM, in each step insert the provided points between all neighbouring points. Then, in the limit step, there will not be immediate successors and predecessors.

Natural numbers. Use the combination of the strategies for substructures of the whole numbers, linear flow with starting point and without (future) endpoint.

8 Mosaics for *Until* and *Since*

Many of the actual decidability and axiomatic completeness results above are well known. What is new is the method of establishing them.

However, we claim that the mosaic method is a powerful general method for establishing such results for many temporal logics. To justify this claim we need to move beyond Prior's language of F and P. Some important new results have already been gained by using the mosaic methods as demonstrated above on more expressive logics with the connectives *until*, U, and *since*, S, of [11].

The language, as seen in [4], for example, has two-place connectives U and S along with the classical $\neg$ and $\wedge$. The formula $\mathsf{U}(\alpha, \beta)$ is supposed to capture the idea of α being true at some time in the future and β holding at all points strictly in between now and that time. So the semantic clause for *until* over a linear flow of time $(T, <)$ is:

$$\begin{array}{lll} x \Vdash \mathsf{U}(\alpha, \beta) & \text{iff} & \text{there is } y > x \text{ in } T \text{ such that } y \Vdash \alpha \\ & & \text{and for all } z \in T \text{ such that } x < z < y, \text{ we have } z \Vdash \beta. \end{array}$$

The definition of *since* is the mirror image. Note that there are slightly different versions of these connectives which are sometimes used for discrete flows of time.

There has long been interest in the expressive temporal logics using U and S over general linear time or over specific dense flows such as the rationals and the real numbers. For example ([2] or [1]), these logics are useful for reasoning about concurrency and the behaviour of analogue devices. Despite this interest the problems of the complexity of decision procedures have remained open. (In contrast, in the case of discrete natural-numbers flows of time, it was shown in [20] that the decision problem is PSPACE-complete.)

The mosaic methods as described above are now being used to answer these questions and provide a basis for automated theorem-proving techniques in this important area. Recent work using mosaics establishes a PSPACE-complete complexity for the decision procedures for (1) the logic of U over general linear time [17] and (2) the logic of U and S over the reals [16].

The general idea of the mosaics and the decision procedures are slight modifications of the ideas for F and P. However, the details get quite complicated in specific ways when we try to fix tight complexity bounds.

In future work we will be trying to extend these results to logics such as the logic of U and S over general linear time. Here, for example, is our proposal for mosaics for this logic.

Given a formula φ, as the labelling set we use the set $Sf(\varphi)$ of all subformulas of φ and their negation. We will define a mosaic to be a triple (A, B, C) of sets of formulas. The intuition is that this corresponds to two points from a structure: A is the set of formulas (from $Sf(\varphi)$) true at the earlier point, C is the set true at the later point and B is the set of formulas which hold at all points strictly in between.

Definition 3. *Fix a formula φ. A φ-*mosaic *is a triple (A, B, C) of subsets of $Sf(\varphi)$ such that:*

0.1. A and C are maximally propositionally consistent, and
0.2. for all $\beta \in Sf(\varphi)$ with $\neg\neg\beta \in Sf(\varphi)$ we have $\neg\neg\beta \in B$ iff $\beta \in B$
and the following four coherency *conditions hold:*

C1. *if $\neg\mathsf{U}(\alpha,\beta) \in A$ and $\beta \in B$ then we have both:*
C1.1. $\neg\alpha \in C$ and either $\neg\beta \in C$ or $\neg\mathsf{U}(\alpha,\beta) \in C$; and
C1.2. $\neg\alpha \in B$ and $\neg\mathsf{U}(\alpha,\beta) \in B$;

C2. *if $\mathsf{U}(\alpha,\beta) \in A$ and $\neg\alpha \in B$ then we have both:*
C2.1 either $\alpha \in C$ or both $\beta \in C$ and $\mathsf{U}(\alpha,\beta) \in C$; and
C2.2. $\beta \in B$ and $\mathsf{U}(\alpha,\beta) \in B$;

C3-4. mirror images of C1-C2.

In order fit two mosaics together end-to-end to decompose another mosaic we need a notion of composition:

Definition 4. *We say that φ-mosaics (A', B', C') and (A'', B'', C'')* compose *iff $C' = A''$. In that case, their* composition *is $(A', B' \cap C' \cap B'', C'')$.*

It is straightforward to prove that this is a mosaic and that composition of mosaics is associative.

Now we have our analogue to Definition 2:

Definition 5. *A set M of mosaics is an SSM if it satisfies the following condition. For every $(A, B, C) \in M$,*

1. *for all $\mathsf{U}(\alpha,\beta) \in A$ we have*
   1.1. $\beta \in B$ and either ($\beta \in C$ and $\mathsf{U}(\alpha,\beta) \in C$) or $\alpha \in C$,
   1.2. or there is some $(A', B', C') \in M$ and some $(A'', B'', C'') \in M$, composing to (A, B, C) such that $\alpha \in C'$ and $\beta \in B'$;
2. *the mirror image of 1.; and*
3. *for each $\neg\beta \in Sf(\varphi)$ such that $\beta \notin B$ there is some $(A', B', C') \in M$ and some $(A'', B'', C'') \in M$, composing to (A, B, C) such that $\neg\beta \in C'$.*

Immediate future work for the mosaic-theorists will be establishing that a formula in the language of U and S is satisfiable over general linear time iff there is an SSM which contains a mosaic (A, B, C) with the formula contained in A or C. Further work should be able to show that this gives a PSPACE procedure for satisfiability/validity.

As we have mentioned, these logics are useful and it should be practical to modify the mosaic-based automated theorem-proving approach to handle them. Indeed, the modification required will be minor: we keep the overall procedure the same but we make a straightforward substitution of the coherency (cf. Definition 1) and saturation (cf. Definition 2) conditions. There is even a possibility that a general temporal logic theorem-prover generator could be implemented to accept user specification of particular coherency and saturation conditions.

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