An Elementary Construction for a Non-elementary Procedure

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Abstract

We consider the problem of the product finite model property for binary products of modal logics. First we give a new proof for the product finite model property of the logic of products of Kripke frames, a result due to Shehtman. Then we modify the proof to obtain the same result for logics of products of Kripke frames satisfying any combination of seriality, reflexivity and symmetry. We do not consider the transitivity condition in isolation because it leads to infinity axioms when taking products.

Keywords: Modal logic, product of modal logics, finite model property, decidability.

1 Introduction

Taking products of modal logics is one of the most straightforward ways to combine two or more modal logics. Besides the aesthetic appeal of products of modal logics, they haved proved to be useful in reasoning about knowledge [2] and parallelism [10] and in spatio-temporal reasoning [17]. However, the nice computational behavior (low complexity, finite axiomatizability) of the component logics is not inherited by the product logic, in general.

A natural question which arises is whether a product logic has the *product finite model property* (pfmp), i.e., whether every satisfiable formula can be satisfied in a model based on the product of *finite* Kripke frames. For, if we can show the strong pfmp, i.e., that there exists a bound (in terms of the length of an input formula) on the size of the witnessing finite model, then decidability follows. Indeed, by enumerating all finite product models up to a certain size we can check if at least one of them satisfies the formula in question. An advantage of this decidability method is that it does not assume that the logic is finitely axiomatizable (although the finite product frames of the logic have to form a recursive set.)

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We note that the product finite model property and the finite model property (fmp) as normally used in modal logic are not necessarily equivalent properties. This is best explained by considering a well-known example. Take the logic **K4**. It is complete with respect to the class K_1 of all transitive Kripke frames, and also with respect to the class K_2 of all irreflexive, transitive Kripke frames. **K4** has the finite model property with respect to the class K_1, but not with respect to the class K_2. For K_2, $\Box\Diamond\top$ is an infinity axiom. With product logics the situation is similar. Many logics (like $\mathbf{K}^2$, $\mathbf{S4}^2$ and $\mathbf{S5}^2$) are complete with respect to a class of Kripke frames satisfying a number of conditions and also with respect to a subclass consisting of products of Kripke frames. Obviously, product fmp yields fmp, but not the other way round.

In this paper, we will first concentrate on the product of two **K** logics, denoted by $\mathbf{K}^2$ or $\mathbf{K}\times\mathbf{K}$. Product is defined in a semantical way — $\mathbf{K}^2$ is the bi-modal logic of the class of products of two Kripke frames. We will give a proof that every satisfiable formula of $\mathbf{K}^2$ is satisfiable in a finite product model. There are other proofs available in the literature. The result is originally due to Shehtman and appeared in [4]. All existing proofs that we know of [3, 4, 9, 16, 14] for the decidability of $\mathbf{K}^2$ yield a non-elementary upper bound for the complexity of the satisfiability problem. At the time of writing it was unknown whether this bound is tight. All we know is that for formulas of modal depth two the $\mathbf{K}^2$-satisfiability problem is already hard for non-deterministic exponential time [9].

Our proof is elementary in the sense of using only basic well-known machinery: very much like in a filtration, an arbitrary model satisfying some formula φ is turned into a finite one. By construction, the finite model is a product. That the finite model still satisfies φ is shown using a labeling of the worlds with Hintikka sets. As a corollary of the proof, we get that bi-modal logics of products of Kripke frames satisfying any combination of seriality, reflexivity and symmetry have the strong finite product model property. Hence these product logics are decidable.

The remainder of the paper is organized as follows. Next we define $\mathbf{K}^2$ and briefly recall other binary products of modal logics. In the next section, we state and prove the product finite model property for the above binary products. We finish the paper with some concluding remarks.

Binary product of K. The binary product $\mathbf{K}^2$ of **K** is defined as follows. The language consists of a countable set P of propositional variables, the propositional connectives $\wedge$ and $\neg$ and the modalities $\diamondminus$ and $\diamondvert$. We will also use the standard abbreviations, e.g., $\boxminus$ for $\neg\diamondminus\neg$.

The binary product of $\mathbf{K}^2$-frames are defined in the following way. Let (U_0, R_0) and (U_1, R_1) be two **K**-frames with universes U_0 and U_1 and accessibility relations R_0 and R_1, respectively. Then we form the *product* $(U_0 \times U_1, \mathsf{H}, \mathsf{V})$ of these frames by defining

$$\begin{array}{lcl}(u,v)\mathsf{H}(u',v') & \iff & u\mathsf{R}_0u' \;\&\; v = v' \\ (u,v)\mathsf{V}(u',v') & \iff & v\mathsf{R}_1v' \;\&\; u = u'.\end{array}$$

That is, the accessibility relations are defined coordinate-wise. The class of $\mathbf{K}^2$-frames consists of all binary products of **K**-frames.

A model $\mathfrak{M}$ is a frame $(U_0 \times U_1, \mathsf{H}, \mathsf{V})$ together with an evaluation $\mathfrak{m} : P \to \mathcal{P}(U_0 \times U_1)$ of the propositional variables. Truth is defined in the usual way, where the non-propositional cases are: for every $x \in U_0 \times U_1$,

$$\begin{array}{lcl}\mathfrak{M}, x \models \Diamond_h \varphi & \iff & \mathfrak{M}, y \models \varphi \text{ for some } y \text{ such that } x\mathsf{H}y \\ \mathfrak{M}, x \models \Diamond_v \varphi & \iff & \mathfrak{M}, y \models \varphi \text{ for some } y \text{ such that } x\mathsf{V}y.\end{array}$$

Recall from [3] that the logic $\mathbf{K}^2$ is finitely axiomatizable by the standard $\mathbf{K}$-axioms in both dimensions together with commutativity (Comm) and confluence (Conf):

PT: (enough) propositional tautologies,

DF: $\Box_h \varphi \leftrightarrow \neg\Diamond_h\neg\varphi$ and $\Box_v \varphi \leftrightarrow \neg\Diamond_v\neg\varphi$,

DB: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ where $\Box \in \{\Box_h, \Box_v\}$,

Comm: $\Diamond_h\Diamond_v\varphi \leftrightarrow \Diamond_v\Diamond_h\varphi$,

Conf: $\Diamond_h\Box_v\varphi \to \Box_v\Diamond_h\varphi$,

and the usual rules of Modus Ponens, Universal Generalization and Substitution.

The corresponding first-order frame conditions are:

- *commutativity*, i.e., for every w, x and y with $w\mathsf{H}x\mathsf{V}y$, there is z such that $w\mathsf{V}z\mathsf{H}y$, and for every w, x and y with $w\mathsf{V}x\mathsf{H}y$, there is z such that $w\mathsf{H}z\mathsf{V}y$;
- *confluence*, i.e., for every w, x and y with $w\mathsf{H}x$ and $w\mathsf{V}y$, there is z such that $x\mathsf{V}z$ and $y\mathsf{H}z$.

Thus this axiomatization of $\mathbf{K}^2$ is complete both with respect to the class of all frames $(W, \mathsf{H}, \mathsf{V})$ which validate the frame conditions of commutativity and confluence and with respect to its subclass of all products of $\mathbf{K}$-frames.

The product construction of frames is not limited to $\mathbf{K}$-frames, hence we can define classes of product frames for specific subclasses of all Kripke frames. Thus, given two classes K and K' of Kripke frames, we define the logic $\mathbf{L}(\mathsf{K} \times \mathsf{K}')$ as the bi-modal logic determined by the class $\{\mathfrak{F} \times \mathfrak{G} : \mathfrak{F} \in \mathsf{K}, \mathfrak{G} \in \mathsf{K}'\}$ of product frames. We will also use the notation **L**×**L'** when the logics **L** and **L'** are complete with respect to the classes K and K', respectively.

Other binary products. For a detailed picture on products of modal logics we refer the reader to [3, 4] and the forthcoming book [5].

The grid-like nature of product frames (as exemplified by the commutativity and the confluence axioms) makes decision procedures in general much more expensive in terms of time and space than procedures for the uni-modal logics separately. It is very well documented that in multi-dimensional logics, e.g., with a temporal dimension, there are non-elementary and highly undecidable systems (cf. [2] and the references therein).

In a certain sense, binary products in which one of the components is the class **S5** are well behaved. Decidability of **S5**×**S5** immediately follows from the decidability of first-order logic with two variables and without function symbols [12]; see [8] for more remarks on the history of decidability results connected to **S5**×**S5**. **S5**×**S5** also has the pfmp [13]. Grädel et al. [6] strengthen this by showing that (translated to the current modal setting) every **S5**×**S5**-satisfiable formula φ is satisfiable on a product of two finite **S5**-frames whose size is single exponential in the length of φ. For the decidability of other binary products involving **S5** see the filtrations in [3] and the mosaic/segment procedures in [10, 7].

[3, 4, 16, 14] establish decidability for several binary products of which one of the component logics is **K**. The strongest result in this direction in due to Wolter [14] who established decidability of the product of converse PDL and **K**. As a consequence of this, decidability follows for the product of **K** and any one of **K**, **T**, **K4**, **S4**, **KB**, **KTB**, **S5**. Product fmp follows from the construction in [16] for $\mathbf{K}\times\{\mathbf{K},\mathbf{S5},\mathbf{KD45}\}$.

On the other hand, the product $\mathbf{K4.3}^2$ is undecidable [11]. The decidability problem is still open for the binary products of transitive logics $\mathbf{K4}^2$ and $\mathbf{S4}^2$. These logics lack the product finite model property ([15] contains simple infinity axioms), but it might still be that they have the finite model property with respect to larger class of transitive (and reflexive) frames which validate confluence and commutativity.

2 Product Finite Model Property

In this section, we give a proof for the finite product model property for $\mathbf{K}^2$ and then generalize the result.

Theorem 2.1 *Any satisfiable formula ξ of $\mathbf{K}^2$ is satisfiable in a finite product frame whose size is computable from ξ.*

The main idea of the proof below is the following. Given a model satisfying ξ, we label each world by the set of formulas (up to a certain modal depth determined by ξ) which are true at that world. We will shrink these label sets as we go along the accessibility relations by taking into account only formulas with smaller modal depth. Thus there will be a finite set of labels and our task is to rearrange them in a finite grid that gives rise to a finite product model for ξ. Using an easy combinatorial observation, we will define a finite grid such that, for every element of the grid labeled by a set X, its successors will be labeled by precisely those sets Y such that there are two worlds in the original model that respectively define the labels X and Y and the two worlds are related via the corresponding accessibility relation. Then we will establish a truth lemma stating that, for every point in the grid, its label contains a formula if and only if the formula holds in the emerging finite model.

Proof: Let us fix a formula ξ. By $|\xi|$ we denote the number of propositional variables occurring in ξ. We define the modal depth of ξ to be the maximal number of nested diamonds. Given the language of ξ (i.e., the set of propositional variables occurring in ξ) and a natural number k, we let X_k be the set of formulas in this language up to modal depth k up to logical equivalence. That is, we assume that X_k contains precisely one element from each class of logically equivalent formulas. Note that X_k is finite (for instance, [1] Proposition 2.29).

Let $\mathfrak{F} = (U_{\mathfrak{F}}, \mathsf{H}_{\mathfrak{F}}, \mathsf{V}_{\mathfrak{F}})$ be a product frame for $\mathbf{K}^2$ and $\mathfrak{m}$ be a valuation such that $\mathfrak{M} = (\mathfrak{F}, \mathfrak{m})$ satisfies ξ, say, $\mathfrak{M}, r \models \xi$ for some $r \in U_{\mathfrak{F}}$. Let m be the modal depth of ξ. For every $0 \leq i \leq m$ and $x \in U_{\mathfrak{F}}$, we define the following label

$$\lambda^i(x) = \{\varphi \in X_i : \mathfrak{M}, x \models \varphi\}.$$

We will slightly abuse the notation and write $\varphi \in \lambda^i(x)$ if φ is logically equivalent to an element of $\lambda^i(x)$.

The following computation provides an upper bound on the number of possible labels of modal depth k, denoted by $f(k)$. Since every label $\lambda^0(x)$ of depth 0 is determined by the propositional variables satisfied by x, we have $f(0) \le 2^{|\xi|}$. A label $\lambda^{k+1}(x)$ is determined by the propositional variables occurring in it and by the labels $\lambda^k(y)$ and $\lambda^k(z)$ of the horizontal and vertical successors, y and z respectively, of x. Indeed, an induction on k establishes that two worlds x and x' satisfy the same formulas up to modal depth $k+1$ if and only if x and x' satisfy the same propositional variables, and they have the same vertical and horizontal successors up to labeling with formulas of modal depth at most k. Hence we have

$$f(k+1) \le 2^{(2\cdot f(k)+|\xi|)}.$$

We define

$$g(k) = \prod\{f(i) : 0 \le i < k\}.$$

Note that $g(k+1) = g(k)\cdot f(k)$ and that we have a non-elementary upper bound for $g(m)$.

First we will define two finite trees $\mathfrak{T}_H = (T_H, <_H)$ and $\mathfrak{T}_V = (T_V, <_V)$. The frame $\mathfrak{G} = (U_\mathfrak{G}, \mathsf{H}_\mathfrak{G}, \mathsf{V}_\mathfrak{G})$ is defined as the product $\mathfrak{T}_H \times \mathfrak{T}_V$. Then we will define a valuation $\mathfrak{n}$ such that the model $\mathfrak{N} = (\mathfrak{G}, \mathfrak{n})$ satisfies ξ.

Let $\mathfrak{T}_H = (T_H, <_H)$ be the following tree. Its root will be r: $T_H^0 = \{r\}$, and we let $\ell(r) = \lambda^m(r)$. The first level T_H^1 of the tree is defined as follows. Consider those $x \in U_\mathfrak{F}$ such that $\Diamond \bigwedge \lambda^{m-1}(x) \in \lambda^m(r)$. Then create $g(m-1)$ copies $(\lambda^{m-1}(x), j)$ (for $0 \le j < g(m-1)$) of each different $\lambda^{m-1}(x)$ with the above property. We label these children of r by their labels in $\mathfrak{M}$: $\ell(\lambda^{m-1}(x), j) = \lambda^{m-1}(x)$. The $(i+1)$th (for $i < m$) level T_H^{i+1} is defined as follows: for every $u \in T_H^i$ with label $\lambda^{m-i}(x)$, we create $g(m-i-1)$ copies of each label $\lambda^{m-i-1}(y)$ such that $\Diamond \bigwedge \lambda^{m-i-1}(y) \in \lambda^{m-i}(x)$. Again, we define the label ℓ for these children as the corresponding label in $\mathfrak{M}$. We let $T_H = \bigcup_{0\le i\le m} T_H^i$ and $u <_H v$ iff v is a child of u. Note that $\mathfrak{T}_H$ has a finite branching factor $(g(m-1))$ and its depth is at most m: $d(\mathfrak{T}_H) \le m$.

The vertical tree $\mathfrak{T}_V$ is defined analogously by substituting $\diamondsuit$ for $\Diamond$ above. Again, it is finite and its depth is at most m: $d(\mathfrak{T}_V) \le m$.

We define the frame $\mathfrak{G} = (U_\mathfrak{G}, \mathsf{H}_\mathfrak{G}, \mathsf{V}_\mathfrak{G})$ as the product $\mathfrak{T}_H \times \mathfrak{T}_V$:

$$\begin{aligned} U_\mathfrak{G} &= T_H \times T_V \\ (u,v)\mathsf{H}_\mathfrak{G}(u',v') &\iff u <_H u' \;\&\; v = v' \\ (u,v)\mathsf{V}_\mathfrak{G}(u',v') &\iff v <_V v' \;\&\; u = u'. \end{aligned}$$

Clearly, $\mathfrak{G}$ is a $\mathbf{K}^2$-frame, since it is a product of two Kripke frames.

For every $(x,y) \in U_\mathfrak{G}$, we define its horizontal (respectively, vertical) distance $\delta_H(x,y)$ $(\delta_V(x,y))$ from the root (r,r) as the number of horizontal (vertical) steps in the path between (r,r) and (x,y). Distance of (x,y) is defined as the sum of the vertical and horizontal distances: $\delta(x,y) = \delta_H(x,y) + \delta_V(x,y)$.

Next we define a labeling $\ell_\mathfrak{G}$ of elements of $U_\mathfrak{G}$ by recursion such that, for every $u \in U_\mathfrak{G}$, if $\delta(u) = i \le m$, then $\ell_\mathfrak{G}(u) = \lambda^{m-i}(x)$ for some $x \in U_\mathfrak{F}$. For points with zero vertical or horizontal distances, their label is given by the label in the corresponding tree: $\ell_\mathfrak{G}(u_0, r) = \ell(u_0)$ and $\ell_\mathfrak{G}(r, u_1) = \ell(u_1)$ for $u_0 \in T_H$ and $u_1 \in T_V$.

Let $0 < n < \min\{m, d(\mathfrak{T}_H)+d(\mathfrak{T}_V)\}$ and assume that we have already defined the labeling $\ell_\mathfrak{G}$ for points $(u,v) \in U_\mathfrak{G}$ with $\delta(u,v) \le n$. We assume the following induction hypothesis.

Induction hypothesis: Let $(u,v) \in U_\mathfrak{G}$, $\delta(u,v) = k < n$ and $\ell_\mathfrak{G}(u,v) = \lambda^{m-k}(x)$ for some $x \in U_\mathfrak{F}$.

(i) For every $(u',v) \in U_\mathfrak{G}$ such that $(u,v)\mathsf{H}_\mathfrak{G}(u',v)$, we have $\Diamond \bigwedge \ell_\mathfrak{G}(u',v) \in \ell_\mathfrak{G}(u,v)$.

(ii) Assume that $y \in U_\mathfrak{F}$ and that $\Diamond \bigwedge \lambda^{m-k-1}(y) \in \lambda^{m-k}(x)$. Then there are at least

$g(m-k-1)$ points $(u',v) \in U_{\mathfrak{G}}$ such that $(u,v)\mathsf{H}_{\mathfrak{G}}(u',v)$ and $\ell_{\mathfrak{G}}(u',v) = \lambda^{m-k-1}(y)$. We assume the corresponding conditions in the vertical dimension as well.

Now we are ready to define the labeling $\ell_{\mathfrak{G}}$ for the points $(u',v') \in U_{\mathfrak{G}}$ with $\delta(u',v') = n+1$. Note that, for every such (u',v'), there are unique (u,v), (u',v) and (u,v') in $U_{\mathfrak{G}}$ such that $(u,v)\mathsf{H}_{\mathfrak{G}}(u',v)\mathsf{V}_{\mathfrak{G}}(u',v')$ and $(u,v)\mathsf{V}_{\mathfrak{G}}(u,v')\mathsf{H}_{\mathfrak{G}}(u',v')$. Thus we can define the label for each (u',v') above by "completing" the triple $(\ell_{\mathfrak{G}}(u,v), \ell_{\mathfrak{G}}(u',v), \ell_{\mathfrak{G}}(u,v'))$ to a quadruple.

So let us enumerate all $(x,y) \in U_{\mathfrak{G}}$ with $\delta(x,y) = n-1$. Take the first element $L_{n-1}(0) = (u,v)$ of this list L_{n-1}. Let $a \in U_{\mathfrak{F}}$ be such that $\ell_{\mathfrak{G}}(u,v) = \lambda^{m-n+1}(a)$. Now let us enumerate all $\lambda^{m-n}(x)$ $(x \in U_{\mathfrak{F}})$ such that $\ominus \bigwedge \lambda^{m-n}(x) \in \lambda^{m-n+1}(a)$. Take the first element $\lambda^{m-n}(b)$ of this list $L^H_{(u,v)}$. We create a similar list $L^V_{(u,v)}$ (by using $\oplus$ instead of $\ominus$) and take its first element $\lambda^{m-n}(c)$. Now consider the set D of all the labels $\lambda^{m-n-1}(d)$ such that $\ominus \bigwedge \lambda^{m-n-1}(d) \in \lambda^{m-n}(c)$ and $\oplus \bigwedge \lambda^{m-n-1}(d) \in \lambda^{m-n}(b)$. Note that there are at most $f(m-n-1)$ such distinct labels. Next we show that the above set D is not empty. Recall that $\ominus \bigwedge \lambda^{m-n}(b) \in \lambda^{m-n+1}(a)$. Thus there is an $\mathsf{H}_{\mathfrak{F}}$-successor b' of a in $\mathfrak{F}$ such that $\lambda^{m-n}(b) = \lambda^{m-n}(b')$. Similarly we can find a $\mathsf{V}_{\mathfrak{F}}$-successor c' of a such that $\lambda^{m-n}(c') = \lambda^{m-n}(c)$. Since $\mathfrak{F}$ satisfies confluence, there is $d \in U_{\mathfrak{F}}$ such that $b'\mathsf{V}_{\mathfrak{F}}d$ and $c'\mathsf{H}_{\mathfrak{F}}d$. Thus $\lambda^{m-n-1}(d) \in D$.

Now consider the rectangle $B \times C$ determined by the set C of those vertical successors (u,y) of (u,v) such that $\ell_{\mathfrak{G}}(u,y) = \lambda^{m-n}(c)$ and the set B of those horizontal successors (x,v) of (u,v) such that $\ell_{\mathfrak{G}}(x,v) = \lambda^{m-n}(b)$. Note that both sets B and C have cardinality at least $g(m-n)$, and that each element of the rectangle has a unique horizontal predecessor in C and a unique vertical predecessor in B. Next we define the labeling for the first column $\{(u',v)\} \times C$ of the above rectangle determined by some horizontal successor $(u',v) \in B$ of (u,v) and C, by using elements of D above. Let κ be a map from this column onto the set D such that the pre-image of each element $\lambda^{m-n-1}(d)$ of D has cardinality at least $g(m-n-1)$ — since the length of the column is at least $g(m-n)$, such a map exists. Let this κ determine the labels in this column, i.e., let $\ell_{\mathfrak{G}}(u',y) = \lambda^{m-n-1}(d)$ iff $\kappa(u',y) = \lambda^{m-n-1}(d)$. For the other $\mathsf{H}_{\mathfrak{G}}$-successors (u'',v) of (u,v) such that $\ell_{\mathfrak{G}}(u'',v) = \lambda^{m-n}(b)$, we label the column $\{(u'',v)\} \times C$ by the same set D of labels, but we reshuffle them so that every label is realized in every row $B \times \{(u,v')\}$ of the rectangle at least $g(m-n-1)$ times. Since the length of the rows is at least $g(m-n)$ and we are using at most $f(m-n-1)$ different labels, this can be done (e.g., by using permutations of the first column). Thus we have defined the labeling for the rectangle determined by the first element of $L^H_{(u,v)}$ and the first element of $L^V_{(u,v)}$. Note that if there is a horizontal (vertical) successor of a point with distance n, then there are at least $g(m-n-1)$ horizontal (vertical) successors with the same label.

Repeat the same construction for every pair of labels from $L^H_{(u,v)} \times L^V_{(u,v)}$. Finally, do the same for all elements of L_{n-1}. Thus we have defined the labeling for all $(x,y) \in U_{\mathfrak{G}}$ such that $\delta(x,y) = n+1$.

It remains to show that we maintained the induction hypothesis. Item (i) should be obvious by the construction. For (ii), let $(u,v) \in U_{\mathfrak{G}}$, $\delta(u,v) = n$ and $\ell_{\mathfrak{G}}(u,v) = \lambda^{m-n}(x)$ for some $x \in U_{\mathfrak{F}}$. First assume that $y \in U_{\mathfrak{F}}$ and that $\ominus \bigwedge \lambda^{m-n-1}(y) \in \lambda^{m-n}(x)$. If the vertical distance of (u,v) is zero, then the claim follows by the definition of the labeled tree $\mathfrak{T}_H$. So assume otherwise. Then there is a unique $\mathsf{V}_{\mathfrak{G}}$-predecessor (u,w) of (u,v); let $a \in U_{\mathfrak{F}}$ be such that $\ell_{\mathfrak{G}}(u,w) = \lambda^{m-n+1}(a)$. By (i) of the induction hypothesis, we have $\oplus \bigwedge \lambda^{m-n}(x) \in \lambda^{m-n+1}(a)$. Then $\oplus\ominus \bigwedge \lambda^{m-n-1}(y) \in \lambda^{m-n+1}(a)$. Hence $\ominus\oplus \bigwedge \lambda^{m-n-1}(y) \in \lambda^{m-n+1}(a)$, since $\mathfrak{F}$ satisfies commutativity. Thus there are at least $g(m-n)$ horizontal successors (u',w)

of (u,w) such that $\Phi \bigwedge \lambda^{m-n-1}(y) \in \ell_{\mathfrak{G}}(u',w)$. Hence, during the construction when we filled in the row determined by the above horizontal successors of (u,w) and the single point (u,v), we created at least $g(m-n-1)$ horizontal successors of (u,v) with label $\lambda^{m-n-1}(y)$. The proof for $y \in U_{\mathfrak{F}}$ such that $\Phi \bigwedge \lambda^{m-n-1}(y) \in \lambda^{m-n}(x)$ is completely analogous. Thus we can conclude that the new labeling satisfies the induction hypothesis.

We continue the construction until we reach those points whose distance from the root (r,r) is at least $\min\{m, d(\mathfrak{T}_H)+d(\mathfrak{T}_V)\}$. We label those points which are further away from the root with arbitrary labels $\lambda^0(x)$ $(x \in U_{\mathfrak{F}})$.

We are ready to define the finite product model satisfying ξ. Let the valuation $\mathfrak{n}$ be defined as follows: for each atom p and $(u,v) \in U_{\mathfrak{G}}$,

$$(u,v) \in \mathfrak{n}(p) \Longleftrightarrow p \in \ell_{\mathfrak{G}}(u,v).$$

Define $\mathfrak{N} = (\mathfrak{G}, \mathfrak{n})$. We claim the following truth lemma: for every $0 \leq j \leq m$, $\varphi \in X_j$ and $(u,v) \in U_{\mathfrak{G}}$ with $\delta(u,v) \leq j$,

$$\varphi \in \ell_{\mathfrak{G}}(u,v) \Longleftrightarrow \mathfrak{N}, (u,v) \models \varphi.$$

The proof is a straightforward induction on the complexity of φ. The only interesting case is when φ is of the form $\ominus\psi$ or $\Phi\psi$. These cases easily follow from the fact that

$$\ominus\psi \in \ell_{\mathfrak{G}}(u,v) \Longleftrightarrow \psi \in \ell_{\mathfrak{G}}(u',v) \text{ for some } \mathsf{H}_{\mathfrak{G}}\text{-successor } (u',v) \text{ of } (u,v)$$

and

$$\Phi\psi \in \ell_{\mathfrak{G}}(u,v) \Longleftrightarrow \psi \in \ell_{\mathfrak{G}}(u,v') \text{ for some } \mathsf{V}_{\mathfrak{G}}\text{-successor } (u,v') \text{ of } (u,v).$$

Since the label of the root contains ξ, $\xi \in \ell_{\mathfrak{G}}(r,r)$, we can conclude that $\mathfrak{N}, (r,r) \models \xi$. ∎

Recall that, given two classes of Kripke frames, K_0 and K_1, we define the logic $\mathbf{L}(\mathsf{K}_0 \times \mathsf{K}_1)$ as the bi-modal logic of all product frames $\mathfrak{F} \times \mathfrak{G}$ for $\mathfrak{F} \in \mathsf{K}_0$ and $\mathfrak{G} \in \mathsf{K}_1$. Next we prove that any binary product logic from $\{\mathbf{K}, \mathbf{D}, \mathbf{T}, \mathbf{B}\}$ is decidable.

Corollary 2.2 *Let K_0 and K_1 be classes of Kripke frames defined by any combination of the frame conditions 'seriality', 'reflexivity' and 'symmetry'. Then $\mathbf{L}(\mathsf{K}_0 \times \mathsf{K}_1)$ has the finite product model property.*

Proof: We indicate the changes one has to make in the proof of Theorem 2.1 to cover the specific cases.

Seriality: Assume that, say, K_0 is defined by requiring seriality. Then we require that the leaves of the tree $\mathfrak{T}_H$ are reflexive, i.e., $x <_H x$ for every leaf x of $\mathfrak{T}_H$. Obviously, $\mathfrak{T}_H$ satisfies seriality, every world has a successor. Let us denote the modified finite product frame by $\mathfrak{G}'$. The truth lemma goes through for $\mathfrak{G}'$ because of the following. Let (u,v) be the endpoint of a path in the product frame $\mathfrak{G}$. By the modification in the construction, we have that (u,v) is a horizontal successor of itself in $\mathfrak{G}'$. Since (u,v) is an endpoint in $\mathfrak{G}$, its label $\ell_{\mathfrak{G}}(u,v)$ cannot contain any diamond-formula (otherwise the construction would yield a successor for this point in the frame $\mathfrak{G}$). Then there is no box-formula in $\ell_{\mathfrak{G}}(u,v)$ either, for it would imply that $\ominus\top \in \ell_{\mathfrak{G}}(u,v)$. Indeed, we label a point with a set $\lambda^i(x)$ of all (logically non-equivalent) formulas up to modal depth i which are satisfied at the world x; and $\Diamond\top \in \lambda^i(y)$ for every world y and every $i > 0$, since the product frame $\mathfrak{F}$ satisfies seriality in the horizontal direction. Hence $\ell_{\mathfrak{G}}(u,v) = \lambda^0(z)$ for some $z \in U_{\mathfrak{F}}$, i.e., $\ell_{\mathfrak{G}}(u,v)$ is a boolean combination of atoms. Then the truth lemma holds for (u,v) regardless of its successors.

Symmetry: Take the symmetric closure of the corresponding tree(s) $\mathfrak{T}_H$ and/or $\mathfrak{T}_V$. We claim that this will not effect truth in the final model. Indeed, assume that, e.g., the horizontal frame is symmetric, $(u,v)\mathsf{H}_{\mathfrak{G}}(u',v)$ in the frame $\mathfrak{G}$ and that $\boxminus\varphi \in \ell_{\mathfrak{G}}(u',v)$. To establish the truth lemma for the symmetric frame, we have to show that $\varphi \in \ell_{\mathfrak{G}}(u,v)$. We know that $\Diamond\boxminus\varphi \in \ell_{\mathfrak{G}}(u,v)$. Since, in the original frame $\mathfrak{F}$, $\mathsf{H}_{\mathfrak{F}}$ is symmetric, we have that $\varphi \in \ell_{\mathfrak{G}}(u,v)$.

Reflexivity: It is enough to note that, in a frame $\mathfrak{F}$ where, say, $\mathsf{V}_{\mathfrak{F}}$ is reflexive, we have, for any label $\lambda^i(x)$, if $\boxed{\vert}\varphi \in \lambda^i(x)$, then $\varphi \in \lambda^i(x)$.

∎

3 Conclusion

The above construction yields a model whose size is non-elementary in terms of the length of ξ (roughly, its size is a stack of 2s determined by the modal depth of ξ finished by the number of propositional variables in ξ). In fact, all known decision procedures for $\mathbf{K}^2$ are non-elementary, but it is not obvious that $\mathbf{K}^2$ has such high complexity. The greatest lower bound that we are aware of is established in [9] where we showed that the complexity of the satisfiability problem for $\mathbf{K}^2$ is non-deterministic exponential time hard even for formulas of depth two. Certainly, it would be interesting to narrow the gap between upper and lower bounds and to find out the exact complexity of the general satisfiability problem for $\mathbf{K}^2$.

Another direction of further research is to solve the challenging open problems whether $\mathbf{K4}\times\mathbf{K4}$ and $\mathbf{S4}\times\mathbf{S4}$ are decidable. These logics do not have the *product* finite model property [15], but might have the finite model property. If one can establish that every satisfiable formula is satisfied in a finite Kripke model for these logics, then decidability follows, since these logics are finitely axiomatizable [3].

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