Axiomatizability of Algebras of Binary Relations

SZABOLCS MIKULÁS*
School of Computer Science and Information Systems
Birkbeck College, University of London
Malet Street, London WC1E 7HX, UK
szabolcs@dcs.bbk.ac.uk

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Abstract

We present an overview of the axiomatizability problem of algebras of binary relations. The focus will be on the finite and non-finite axiomatizability of several fragments of Tarski's class of representable relation algebras. We examine the step-by-step method for establishing finite axiomatizability and ultraproduct constructions for establishing non-finite axiomatizability. We conclude with some open problems that could be tackled using either of the above methods.

1 Introduction

In algebraic logic, the most extensively investigated class of algebras of binary relations is the class RRA of representable relation algebras. These are Boolean algebras equipped with some extra-Boolean operations arising from the nature of relations, like relation composition, relation converse and the identity relation. The class RRA is a variety, i.e., it is an equational class [Tar55]. When Tarski defined RRA, he raised the problem of whether RRA was *finitely* axiomatizable using equations; Monk answered the question negatively [Mon64].

An interesting question is whether algebras of binary relations with smaller clones of operations can be finitely axiomatized. See [Sch91] for a survey of axiomatizability results of *subreducts* of RRA, i.e., classes of algebras where only a subset of the operations of RRA are present. In the last decade, researchers have continued to investigate subreducts of RRA, and significant results have been obtained. In this paper, we will mainly concentrate on subreducts of RRA in which the similarity type includes relation composition and at least one of the lattice operations of meet and join. We will recall the most important finite axiomatizability results and describe the so-called *step-by-step method*, that has been successfully used to achieve finite axiomatizations. Furthermore, we will show how to use *ultraproduct constructions* to establish non-finite axiomatizability results. In both cases, we will use games to make the constructions more intuitive. At the end of the paper we will mention some open problems.

2 Basics

First we recall the definition of representable relation algebras.

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Definition 2.1 *A* representable relation algebra *is an algebra*

$$\mathfrak{A} = (A, 0, 1, \cdot, +, -, ;, \smile, 1')$$

such that $A \subseteq \mathcal{P}(W)$ (the powerset of W) for some equivalence relation W, $0 = \emptyset$, $1 = W$, $\cdot$ is intersection, $+$ is union, $-$ is complement w.r.t. W, $;$ is relation composition, $\smile$ is relation converse, and $1'$ is the identity relation. More formally, for all elements $x, y \in A$,

$$\begin{array}{rcl} x\,;y &=& \{(u,v) \in W : (u,w) \in x \text{ and } (w,v) \in y \text{ for some } w\} \\ x^{\smile} &=& \{(u,v) \in W : (v,u) \in x\} \\ 1' &=& \{(u,v) \in W : u = v\}. \end{array}$$

The set W is the unit *of $\mathfrak{A}$, and the* base *of the algebra is the field of W, i.e., the smallest set U such that $W \subseteq U \times U$.*

We denote the class of isomorphs of representable relation algebras by RRA*. We also use '*RRA*' as an abbreviation for the phrase 'representable relation algebra'.*

We will also consider classes of algebras with different similarity types than that of RRA. In general, we will say that an algebra is *representable* if it is isomorphic to a set algebra of binary relations.

As we mentioned earlier, RRA is a non-finitely axiomatizable variety. It is an interesting problem to find out precisely which combinations of operations cause non-finite axiomatizability. This is one of the motivations for investigating algebras with smaller similarity types.

Definition 2.2 *Let $\mathfrak{A} \in$ RRA and τ be a set of operations whose elements are definable by fixed terms. The* τ-reduct *of $\mathfrak{A}$ is the algebra $(A, o)_{o\in\tau}$.*

The τ-subreduct *of* RRA*,* R(τ)*, is the class of subalgebras*[1] *of the τ-reducts of elements of* RRA*.*

When forming subreducts we will look at the following operations besides the primitive operations of the similarity type of RRA. The *residuals* $\backslash$ and $/$ of $;$ are defined as follows:

$$x\backslash y = -(x^{\smile}\,;-y) \qquad x/y = -(-x\,;y^{\smile}).$$

Their semantics is given by

$$\begin{array}{rcl} x\backslash y &=& \{(u,v) \in W : \text{for all } w,\ (w,u) \in x \text{ implies } (w,v) \in y\} \\ x/y &=& \{(u,v) \in W : \text{for all } w,\ (v,w) \in y \text{ implies } (u,w) \in x\}. \end{array}$$

The *conjugates* of $;$ are $\rhd$ and $\lhd$. They are defined in RRA as

$$x \rhd y = (x^{\smile}\,;y) \qquad x \lhd y = (x\,;y^{\smile}).$$

Their meaning is

$$\begin{array}{rcl} x \rhd y &=& \{(u,v) \in W : \text{for some } w,\ (w,u) \in x \text{ and } (w,v) \in y\} \\ x \lhd y &=& \{(u,v) \in W : \text{for some } w,\ (v,w) \in y \text{ and } (u,w) \in x\}. \end{array}$$

While RRA is a variety, subreducts of RRA are not always closed under homomorphic images. However, it is not difficult to show that they are closed under forming subalgebras, products and ultraproducts. In other words, any subreduct of RRA is a quasivariety. Hence subreducts of RRA can be defined by sets of quasiequations, i.e., universal formulas of the form

$$(\sigma_1 = \rho_1 \wedge \cdots \wedge \sigma_n = \rho_n) \to \sigma_0 = \rho_0.$$

The question is for which choice of τ does there exist a *finite* set of quasiequations axiomatizing R(τ).

[1] As usual in algebraic logic, we assume that subalgebras include isomorphic copies as well.

Remark 2.3 There is a subtle but important point worth mentioning here. Recall that, in the semantics of the extra-Boolean operations, the unit W of the algebra occurs as a parameter. Since we are talking about axiomatizing subreducts of RRA, an abstract algebra belongs to such a class if and only if it can be represented as an algebra of binary relations where the operations are computed in an *equivalence* relation.

In certain cases, it is a natural question whether an algebra can be represented as a set algebra of binary relations where the operations are computed in a transitive relation, for example. The two problems are not equivalent in general; for different choices of W we might have different semantics of the extra-Boolean operations. As an example, see the case of semilattice-ordered residuated semigroups in the next section.

3 Finite axiomatizability

In this section we look at subreducts of RRA that can be axiomatized by finite sets of quasiequations. We will use the *step-by-step method* to establish finite axiomatizability. The origin of the method goes back to the early 20th century. The same idea has been used to prove model-theoretical results (e.g., Löwenheim–Skolem theorems) and completeness theorems (e.g., Henkin's completeness proof for first-order logic). In algebraic logic, it has been used to establish axiomatizability results, see for instance [Mad82, AT88]; see also [HH97a] which contains more historical remarks about the method used in other branches of logic as well.

3.1 Semilattice-ordered semigroups

First let us look at the subreduct $\mathsf{R}(\,\cdot\,,\,;\,)$ whose similarity type consists of intersection $\cdot$ and composition $;$. Clearly $\cdot$ satisfies the semilattice axioms, viz.

$$\begin{aligned} x \cdot x &= x \\ x \cdot y &= y \cdot x \\ (x \cdot y) \cdot z &= x \cdot (y \cdot z). \end{aligned}$$

As usual in semilattices, we define

$$x \leq y \iff x = x \cdot y.$$

It is easy to check that composition is an associative operation, i.e.,

$$(x\,;y)\,;z = x\,;(y\,;z).$$

Finally, let us observe that composition is a monotone operation w.r.t. the ordering $\leq$:

$$(x \leq y \wedge u \leq v) \to x\,;u \leq y\,;v.$$

Thus $\{\,\cdot\,,\,;\}$-reducts of RRAs are (lower) semilattice-ordered semigroups. Bredikhin and Schein [BS78] showed that the converse holds as well.

Theorem 3.1 *Let Σ be the axioms for semilattice-ordered semigroups. Then we have*

$$\mathfrak{A} \in \mathsf{R}(\,\cdot\,,\,;\,) \iff \mathfrak{A} \models \Sigma.$$

Proof: To prove[2] the theorem, we have to show that every semilattice-ordered semigroup can be represented over an equivalence relation, i.e., it is isomorphic to a set algebra of binary relations where composition is computed in an equivalence relation. We will build such a representation in a step-by-step manner.

The main idea is to define a chain of partial representations such that the union of the chain defines the required isomorphism. The elements of the chain will satisfy certain *coherency conditions* that we will maintain during the construction. As we go along the chain we will approximate the structure of the abstract algebra to be represented by achieving a *saturation property*.

We will define a chain of labelled graphs

$$G_0 \subseteq \cdots \subseteq G_\alpha \subseteq \cdots$$

such that

$$G_\alpha = (U_\alpha, E_\alpha, \ell_\alpha),$$

where U_α denotes the set of nodes, $E_\alpha \subseteq U_\alpha \times U_\alpha$ is the set of edges, and $\ell_\alpha : E_\alpha \to A$ is the labelling of the edges by elements of $\mathfrak{A}$. We will maintain the following two coherency conditions:

$$E_\alpha \text{ is a transitive relation} \tag{1}$$

$$\text{for } (x,y),(y,z) \in E_\alpha,\ \ell_\alpha(x,z) \leq \ell_\alpha(x,y)\,;\ell_\alpha(y,z). \tag{2}$$

We say that G_α is *coherent* if it satisfies the two coherency conditions 1 and 2 above.

The graphs form a chain in the following sense: for $\alpha < \beta$,

$$U_\alpha \subseteq U_\beta, \quad E_\alpha \subseteq E_\beta, \quad \ell_\alpha \text{ is the restriction of } \ell_\beta \text{ to } E_\alpha.$$

Let $G = (U, E, \ell)$ be the union of the chain:

$$U = \bigcup_\alpha U_\alpha \quad E = \bigcup_\alpha E_\alpha \quad \ell = \bigcup_\alpha \ell_\alpha.$$

Clearly, G is coherent if all G_α are coherent.

The saturation property we want to achieve is: for every $a, b \in A$ and $(x,y) \in E$,

$$\ell(x,y) \leq a\,;b \text{ implies } \ell(x,z) = a, \ell(z,y) = b \text{ for some } z \in U. \tag{3}$$

Given a coherent and saturated graph $G = (U, E, \ell)$, we define $h : A \to \mathcal{P}(U \times U)$ as follows: for every $a \in A$,

$$h(a) = \{(x,y) \in U \times U : \ell(x,y) \leq a\}.$$

Let $\mathfrak{B}$ be the $\{\,\cdot\,,\,;\}$-reduct of the full RRA with unit $U \times U$. We show that h is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

By the semilattice axioms and the definition of h, h is a homomorphism w.r.t. $\cdot$. Using the coherency condition 2, one easily shows that

$$h(a)\,;h(b) \subseteq h(a\,;b),$$

while

$$h(a\,;b) \subseteq h(a)\,;h(b)$$

follows from the saturation property 3. Injectivity is guaranteed once we make sure that $\{(x,y) : \ell(x,y) \leq a\} \neq \{(x,y) : \ell(x,y) \leq b\}$ for distinct elements a and b of $\mathfrak{A}$; see the initial step of the construction below. Then h is indeed the desired embedding.

Thus it remains to construct the chain of graphs G_α. There are three cases in the construction according to whether α is 0, a successor, or limit ordinal.

[2] The present proof is based on [AM94].

Initial step: In the base step of the construction, for every element a of the algebra $\mathfrak{A}$, we take arbitrary distinct points u_a and v_a and label the edge (u_a, v_a) by a. We choose these points such that they are distinct for different elements of the algebra, i.e., for $a \neq b$, $\{u_a, v_a\} \cap \{u_b, v_b\} = \emptyset$. We let $U_0 = \{u_a, v_a : a \in A\}$, $E_0 = \{(u_a, v_a) : a \in A\}$ and $\ell_0 = \{((u_a, v_a), a) : a \in A\}$. The coherency conditions are trivially satisfied by $G_0 = (U_0, E_0, \ell_0)$. Moreover, $\{(x, y) : \ell_0(x, y) = a\} \neq \{(x, y) : \ell_0(x, y) = b\}$ for distinct elements a and b of $\mathfrak{A}$ (thus h indeed is an injection). Finally, let D_0 be an enumeration of all "defects", i.e., those triples $((x, y), a, b)$ such that $(x, y) \in E_0$, $a, b \in A$ and $\ell_0(x, y) \leq a \,;b$.

Successor step: Let us assume that we have defined a coherently labelled graph $G_\alpha = (U_\alpha, E_\alpha, \ell_\alpha)$. Also assume that D_α is an enumeration of all triples $((x, y), a, b)$ such that $(x, y) \in E_\alpha$, $a, b \in A$ and $\ell_\alpha(x, y) \leq a \,; b$. We define

$$G_{\alpha+1} = (U_{\alpha+1}, E_{\alpha+1}, \ell_{\alpha+1})$$

as follows. Let $((x, y), a, b)$ be the first actual defect, i.e., the first element of D_α such that there is no $w \in U_\alpha$ such that $\ell_\alpha(x, w) = a$ and $\ell_\alpha(w, y) = b$. We repair this defect as follows. Choose a fresh point $z \notin U_\alpha$. We define

$$\begin{aligned}
U_{\alpha+1} &= U_\alpha \cup \{z\} \\
E_{\alpha+1} &= \{(x, z), (z, y)\} \cup \\
&\quad\ \{(r, z) : (r, x) \in E_\alpha\} \cup \{(z, s) : (y, s) \in E_\alpha\} \\
\ell_{\alpha+1} &= \ell_\alpha \cup \{((x, z), a), ((z, y), b)\} \cup \\
&\quad\ \{((r, z), \ell_\alpha(r, x) \,; a) : (r, x) \in E_\alpha\} \cup \\
&\quad\ \{((z, s), b \,; \ell_\alpha(y, s)) : (y, s) \in E_\alpha\}.
\end{aligned}$$

Clearly, $E_{\alpha+1}$ is transitive. Furthermore, using the axioms, it is routine to show that $G_{\alpha+1}$ satisfies the coherency condition 2.

Finally, we define $D_{\alpha+1}$ by appending the new possible defects to the list D_α. That is, we add the triples $((x, y), a, b)$ such that $(x, y) \in E_{\alpha+1}$, $a, b \in A$ and $\ell_{\alpha+1}(x, y) \leq a \,; b$ to the end of the enumeration D_α.

Limit step: Take the union of the chain of labelled graphs defined so far. Obviously, it satisfies the coherency conditions. Since we do not introduce any defect in the limit step, we can define the list of defects as the union of the previous lists.

The construction continues as long as there are defects that we have not dealt with yet. Note that by the careful enumeration of possible defects, every defect is repaired sooner or later. See [AM94] for a precise proof that the construction indeed terminates. Thus we can conclude that the final graph G is free from defects, i.e., it satisfies the saturation condition. ∎

Remark 3.2 1. Note that the relation E constructed above is irreflexive and asymmetric, thus it is not an equivalence relation. However, the embedding h shows that the composition operation is computed in the equivalence relation $U \times U$. In fact, using the similarity type $\{;, \cdot\}$ we cannot distinguish between representations using equivalence or transitive relations. The picture will drastically change when we include the residuals into the similarity type, see below.

2. The coherency conditions have a universal flavor; these properties talk about all edges, and they have to be maintained during the whole construction. On the other hand, one might argue that the

saturation condition is rather existential — it requires the existence of "witnessing" edges providing the decomposition of the edge with label $a\,;b$.

One way to make this idea more explicit is to introduce games. There are two players, an existential player $\exists$ and a universal player $\forall$. They start a game using an abstract algebra $\mathfrak{A}$ — $\exists$'s task is to show that $\mathfrak{A}$ can be represented, while $\forall$ tries to create obstacles. They build a chain of coherently labelled graphs. In the nth round of the game, $\forall$ points out a defect, i.e., an edge (x, y) labelled with $a\,;b$ such that there are no edges (x, z) and (z, y) labelled by a and b, respectively. Then $\exists$ tries to respond with an extension of the labelled graph containing the two labelled edges missing from the previous graph. Of course, $\exists$ has to make sure that the new graph is coherent — the coherency conditions are necessary requirements for any (partial) representation. The universal player $\forall$ wins the game if there is $n < \omega$ such that in the nth round $\exists$ cannot respond with a coherently labelled graph. Otherwise $\exists$ wins the game.

This idea has been made explicit in the context of relation algebras, see [HH97b]. Roughly speaking, a relation algebra is representable if and only if the existential player has a winning strategy in the representability game sketched above. In general, the existence of a winning strategy for $\exists$ can be described by a set of first-order sentences that provide an axiomatization for the class of algebras to be represented. Usually this axiomatization is infinite, though.

3.2 Semilattice-ordered residuated semigroups

Next we look at the possibility of extending the similarity type without losing finite axiomatizability.

We mentioned earlier the residuals \ and / of composition. Their abstract characterization is given by the following two formulas:

$$\begin{array}{lcl} x\,;y \leq z & \longleftrightarrow & y \leq x\backslash z \\ x\,;y \leq z & \longleftrightarrow & x \leq z/y. \end{array}$$

For instance, the first one says that $x\backslash z$ is the largest element a such that $x\,;a \leq z$. It is easy to check that the above two formulas are valid in representable algebras. It turns out that, together with the semilattice-ordered semigroup axioms Σ from Theorem 3.1, they are enough to ensure representability over transitive relations, cf. [AM94].

Theorem 3.3 *Every semilattice-ordered residuated semigroup can be represented as a set algebra of binary relations over a transitive relation.*

Proof: The proof of Theorem 3.3 is similar to that of Theorem 3.1. Again, one can define a chain of labelled graphs whose union defines the required representation of the abstract algebra. The coherency conditions are the same. But we require an additional saturation property, viz. for all $a \in A$ and $x \in U$, there exist $y, z \in U$ such that

$$\ell(y, x) = a \qquad \ell(x, z) = a. \tag{4}$$

This property can be achieved by adding the appropriate edges in the successor step $\alpha + 1$ of the construction. Of course, one has to label the other edges emerging when we make sure that $E_{\alpha+1}$ is transitive; this can be done in a similar way as above. Furthermore, the new graph has to satisfy the coherency conditions — a consequence of the axioms. Finally, one has to show that h is a homomorphism w.r.t. the residuals as well. By definition of h, $(u, v) \in h(a\backslash b)$ iff $\ell(u, v) \leq a\backslash b$. The latter holds if and only if $a\,;\ell(u, v) \leq b$ by the new axiom. Then we have that, for every z, $\ell(z, u) \leq a$

implies $\ell(z,u)\,;\ell(u,v) \leq b$. The coherency condition 2 implies that, for every z, $\ell(z,u) \leq a$ implies $\ell(z,v) \leq b$. Hence $(u,v) \in h(a)\backslash h(b)$. A similar argument using the saturation property 4 shows that $h(a)\backslash h(b) \subseteq h(a\backslash b)$. Hence h is the required embedding. ∎

3.3 Axiomatizing $\mathsf{R}(\,\cdot\,,\,;\,,\backslash,/)$

So far we proved a representation theorem for semilattice-ordered residuated semigroups over transitive relations. That is, we proved that the class of (isomorphs of) algebras of binary relations with set-theoretic operations of intersection, composition, and left and right residuals computed in a *transitive* relation is finitely axiomatizable. Consider now the problem of the finite axiomatizability of $\mathsf{R}(\,\cdot\,,\,;\,,\backslash,/)$. Recall that, in subreducts of RRA, the operations are computed in an *equivalence* relation. In other words, for axiomatizing $\mathsf{R}(\,\cdot\,,\,;\,,\backslash,/)$ we need a representation theorem with equivalence relations. The difference between the two axiomatizability problems is indicated by the following observation. In $\mathsf{R}(\,\cdot\,,\,;\,,\backslash,/)$, the identity relation is always contained in $a\backslash a$, while it clearly fails in the representation we gave in the proof above (E is irreflexive). Additional axioms ensure that we can achieve finite axiomatizability in this case as well.

Theorem 3.4 $\mathsf{R}(\,\cdot\,,\,;\,,\backslash,/)$ *is finitely axiomatizable.*

The axioms are those for semilattice-ordered residuated semigroups together with the following four additional axioms:

$$\begin{array}{ll} x \leq y \to z \leq z\,;(x\backslash y) & \qquad x \leq y \to z \leq (x\backslash y)\,;z \\ x \leq y \to z \leq z\,;(y/x) & \qquad x \leq y \to z \leq (y/x)\,;z. \end{array}$$

The proof goes along similar lines as for Theorem 3.3. However, it is more involved, since, in the constructed labelled graphs, E_α has to be not just transitive but symmetric and reflexive as well. For details we refer the interested reader to [AM94].

Of course, there are several other possible similarity types. We will look at adding union $+$ or converse $\smile$ in the next section. Discussion of the identity constant is postponed until the last section.

4 Non-finite axiomatizability

In this section we look at classes of algebras that cannot be axiomatized by finite sets of quasiequations. We will also sketch a method using ultraproduct constructions to establish non-finite axiomatizability (NFA, for short). This method dates back to the work of Tarski's school of logic in the 1950s.

The idea of establishing NFA of $\mathsf{R}(\tau)$ using ultraproducts is the following. We will define Boolean algebras with extra operators of the RRA similarity type, $\mathfrak{A}_n$ ($n < \omega$), and show that their τ-reducts are not representable while an ultraproduct of the reducts is representable. By Łoś' theorem [Hod93, Theorem 9.5.1], this is enough to show that $\mathsf{R}(\tau)$ is not finitely axiomatizable in first-order logic.

Now assume that $\tau \subseteq \tau'$ and that $\mathsf{R}(\tau)$ is not finitely axiomatizable. What can we say about $\mathsf{R}(\tau')$? There is no general theorem indicating that $\mathsf{R}(\tau')$ is not finitely axiomatizable, but this is often the case. If we can show that in fact an ultraproduct of $\mathfrak{A}_n$ for $n < \omega$ (and not just an ultraproduct of their τ-reducts) is representable, then we can conclude that no $\mathsf{R}(\tau')$ with $\tau \subseteq \tau'$ can be finitely axiomatized. If this is the case, we will say that $\mathsf{R}(\tau)$ is *hereditarily non-finitely axiomatizable* (HNFA, for short).

The following results show that including union $+$ or converse $\smile$ into the similarity type leads to non-finite axiomatizability.

- $\mathsf{R}(+\,,\,;)$ is NFA, [And88];
- $\mathsf{R}(\,;\,,\smile)$ is NFA, [Bre77];
- $\mathsf{R}(\,\cdot\,,+\,,\,;)$ is HNFA, [And91];
- $\mathsf{R}(\,\cdot\,,\,;\,,\smile)$ is HNFA, cf. [HM00], see [Hai91] as well;
- $\mathsf{R}(\,\cdot\,,\,;\,,1',\rhd)$ and $\mathsf{R}(\,\cdot\,,\,;\,,1',\lhd)$ are HNFA, [HM00].

Thus, while the subreduct to intersection and composition is finitely axiomatizable, if intersection is replaced by either union or converse, the resulting subreduct is not finitely axiomatizable. If to intersection and composition we adjoin union or converse (or the identity relation and one of the conjugates), we get a hereditarily non-finitely axiomatizable class. The problem of HNFA in the first two cases seems to be open.

Below we will show how to establish the HNFA of $\mathsf{R}(\,\cdot\,,\,;\,,\smile,0)$ using a variant of the "rainbow" construction from [HM00]. This particular similarity type will make the presentation of the idea slightly simpler.

Theorem 4.1 $\mathsf{R}(\,\cdot\,,\,;\,,\smile,0)$ *is HNFA.*

Proof: We will define algebras $\mathfrak{A}_n$ $(n < \omega)$ of the RRA similarity type. Then we will show that (i) their $\{\,\cdot\,,\,;\,,\smile,0\}$-reducts are not representable, while (ii) their ultraproduct $\mathfrak{A}$ is in RRA.

We will show (i) by using the games mentioned in Remark 3.2. We will provide the universal player $\forall$ with tools (the green atoms g_i $(0 \le i \le 2^n)$ below) to gradually build a linearly ordered sequence. The algebra $\mathfrak{A}_n$ is designed so that the existential player $\exists$ has to respond with a similar linearly ordered sequence (using the red atoms r_i $(1 \le i < 2^n)$) in any representation of $\mathfrak{A}_n$. But $\forall$'s sequence is longer than the sequence $\exists$ can build (there are more green atoms than red). Hence there is a round in the game played on $\mathfrak{A}_n$ when $\exists$ is not able to respond to $\forall$'s move. Thus $\forall$ can win the game, whence $\mathfrak{A}_n$ cannot be represented. However, as n grows $\exists$ can survive more and more rounds in the representability game, since it takes more and more rounds to show that $\forall$'s sequence is longer than $\exists$'s. Then in the limit, i.e., in the game played on the ultraproduct $\mathfrak{A}$, $\exists$ can survive infinitely many rounds. It follows that $\mathfrak{A} \in \mathsf{RRA}$, establishing (ii).

We are ready to define the algebras. The construction is rather involved, since we have to make sure that $\exists$ can survive several rounds, i.e., the reason for non-representability does not become immediate. Let n be any natural number. $\mathfrak{A}_n$ has the following atoms (minimal non-zero elements):

- identity: $1'$,
- greens: $\mathsf{g}_i \quad (0 \le i \le 2^n)$,
- whites: $\mathsf{w}, \mathsf{w}_{ij} \quad (0 \le i \le j \le 2^n)$,
- yellow: y,
- black: b,
- reds: $\mathsf{r}_i \quad (0 < i < 2^n)$.

The elements of the algebra are the sums of the atoms, i.e., all possible subsets of the set of atoms. Then the Boolean operations are defined as set-theoretic union, intersection and complementation. All elements x are self-converse, i.e., $x^{\smile} = x$. Next we will define composition for atoms. Then composition for other elements of the algebra can be defined using additivity:

$$x\,;y = \sum\{u\,;v : u \leq x, v \leq y,\ u, v \text{ atoms}\}.$$

The composition of two atoms x_0 and x_1 is defined by specifying precisely which atoms are below $x_0\,;x_1$, or equivalently, by specifying which atoms are not below $x_0\,;x_1$. We say that a triple (x_0, x_1, x_2) of atoms is a *forbidden triangle* if x_2 is not one of the atoms below $x_0\,;x_1$, in symbols $(x_0\,;x_1)\cdot x_2 = 0$. Since the ultraproduct of the algebras $\mathfrak{A}_n$ is to be in RRA, we want $\mathfrak{A}_n$ to satisfy the so-called cycle law. Since every element is self-converse, it says that, for every permutation π of 3, we have $(x_{\pi(0)}\,;x_{\pi(1)})\cdot x_{\pi(2)} = 0$ as well. In other words, if (x_0, x_1, x_2) is a forbidden triangle, so is every permutation. The forbidden triangles (up to permutations) are precisely the following:

(green,green,green)
(yellow,yellow,yellow)
(green,green,white)
(yellow,yellow,black)
(r_i,r_j,r_k) unless $i + j = k$ or $i + k = j$ or $j + k = i$
(g_i,g_{i+1},r_j) unless $j = 1$
(g_i,y,w_{jk}) unless $i \in \{j, k\}$,

where, e.g., (green,green,white) stands for: $(g\,;g')\cdot w = 0$ for all green atoms g, g' and any white atom w. We also require that $(1', x, y)$ and its permutations are forbidden for all distinct atoms x and y, since we want $1'\,;x = x$.

As an example let us look at the composition of two green atoms with consecutive indices. We have

$$\mathsf{g}_i\,;\mathsf{g}_{i+1} = \mathsf{r}_1 + \mathsf{b} + \mathsf{y},$$

since the composition of two green atoms with consecutive indices cannot be white, green, the identity, or any red but r_1. Note that the composition of the yellow atom with itself cannot be black or yellow. Thus we have

$$(\mathsf{g}_i\,;\mathsf{g}_{i+1})\cdot(\mathsf{y}\,;\mathsf{y}) = \mathsf{r}_1.$$

Finally, let us look at

$$(\mathsf{g}_i\,;\mathsf{g}_{i+2k})\cdot(\mathsf{y}\,;\mathsf{y})\cdot(\mathsf{r}_k\,;\mathsf{r}_k)$$

for some indices i and $k > 0$ such that $i + 2k < 2^n$. We have already seen that the intersection of the composition of green atoms and of the composition of yellow with itself cannot be green, white, black or yellow, and identity is ruled out by the different indices on the green atoms; thus it must be a union of red atoms. But the only red atom below $\mathsf{r}_k\,;\mathsf{r}_k$ is r_{2k}. Thus the value of the above displayed term must be r_{2k}.

Next we show that the $\{\,\cdot\,,\,;\,,{}^{\smile},0\}$-reduct $\mathfrak{B}_n$ of $\mathfrak{A}_n$ is not representable as a set algebra of binary relations over an equivalence relation.

Lemma 4.2 *For any $n < \omega$, $\mathfrak{A}_n$ is not in* RRA. *In fact, $\mathfrak{B}_n$, the $\{\,\cdot\,,\,;\,,{}^{\smile},0\}$-reduct of $\mathfrak{A}_n$, is not representable either.*

Proof: We will show that the universal player $\forall$ can win the representability game played on $\mathfrak{A}_n$. First let us recall the coherency conditions for this game. Let us assume that we have an edge (x, y) labelled by a and let I be a finite index set such that, for each $i \in I$, the edges (x, z_i) and (z_i, y) are labelled by b_i and c_i, respectively. Then by the second coherency condition (cf. 2 in the proof of Theorem 3.1, $a \leq b_i \,;c_i$ for each $i \in I$. Hence $a \leq \prod_I (b_i \,; c_i)$ must hold as well. Since, we are considering a subreduct of RRA, the first coherency condition (cf. 1) says that the labelled graph must be an equivalence relation. Since converse is in the similarity type, if an edge (x, y) is labelled by a, then the converse edge (y, x) must be labelled by $a^{\smile}$. Since $a = a^{\smile}$, the direction of the edges will not matter;

$$\text{the label of } (x, y) \text{ must be the same as the label of } (y, x). \tag{5}$$

Furthermore, all reflexive edges must be labelled — this can be done using the identity atom $1'$. We will not bother mentioning this in the future. Consequently, in the picture illustrating the argument below, Figure 1, we do not indicate the orientation of the edges, and the reflexive edges are not shown.

The game starts with an edge (u, v) labelled by $\mathsf{w} \neq 0$. In the initial round of the game, $\forall$ points out that $\mathsf{w} \leq \mathsf{g}_0 \,; \mathsf{y}$. Thus $\exists$ has to create edges (u, u_0) and (u_0, v) labelled by g_0 and y, respectively. Since $\mathsf{w} \leq \mathsf{g}_i \,; \mathsf{y}$ for all $i \leq 2^n$, in the ith round of the game $\forall$ can force the existence of u_i $(i \leq 2^n)$ such that (u, u_i) is labelled by g_i and (u_i, v) is labelled by y. Note that by 5, (u_i, u) must be g_i and (v, u_i) must be y.

Now let us have a look at how $\exists$ can respond to the above challenges of $\forall$. In the first round of the game, $\exists$ has to label the edge (u_0, u_1). Since $(\mathsf{g}_0 \,; \mathsf{g}_1) \cdot (\mathsf{y} \,; \mathsf{y}) = \mathsf{r}_1$, the coherency requirement on the graph says that $\exists$ must label (u_0, u_1) by r_1. The same argument shows that, in the $(i+1)$th round, $\exists$ has to use r_1 to color the edge between u_i and u_{i+1} (for $i < 2^n$). Indeed, $(\mathsf{g}_i \,; \mathsf{g}_{i+1}) \cdot (\mathsf{y} \,; \mathsf{y}) = \mathsf{r}_1$ for every $i < 2^n$, whence the color of (u_i, u_{i+1}) must be r_1.

So far so good, but $\exists$ has to label all the edges (u_i, u_j) for $0 \leq i, j \leq 2^n$. In particular, in the second round, $\exists$ must label (u_0, u_2). By $(\mathsf{g}_0 \,; \mathsf{g}_2) \cdot (\mathsf{y} \,; \mathsf{y}) \cdot (\mathsf{r}_1 \,; \mathsf{r}_1) = \mathsf{r}_2$, $\exists$ is forced to use r_2 to color (u_0, u_2). In general, we have that $(\mathsf{g}_i \,; \mathsf{g}_{i+2}) \cdot (\mathsf{y} \,; \mathsf{y}) \cdot (\mathsf{r}_1 \,; \mathsf{r}_1) = \mathsf{r}_2$ determines the color of (u_i, u_{i+2}) to be r_2. By induction, we get that (u_k, u_{k+2^l}) must be labelled by r_{2^l}, since $(\mathsf{g}_k \,; \mathsf{g}_{k+2^l}) \cdot (\mathsf{y} \,; \mathsf{y}) \cdot (\mathsf{r}_{2^{l-1}} \,; \mathsf{r}_{2^{l-1}}) = \mathsf{r}_{2^l}$. In particular, the label of $(u_0, u_{2^{n-1}})$ and $(u_{2^{n-1}}, u_{2^n})$ is $\mathsf{r}_{2^{n-1}}$. In the 2^nth round, $\exists$ has to find a non-zero label for (u_0, u_{2^n}). But this is impossible, since $(\mathsf{g}_0 \,; \mathsf{g}_{2^n}) \cdot (\mathsf{y} \,; \mathsf{y}) \cdot (\mathsf{r}_{2^{n-1}} \,; \mathsf{r}_{2^{n-1}}) = 0$. Thus $\exists$ cannot respond with a coherent graph in the 2^nth round.

See Figure 1 for an illustration of the argument; the solid lines represent $\forall$'s moves and the dotted lines show $\exists$'s responses. ∎

One might wonder why we made the definition of $\mathfrak{A}_n$ so complicated. Why not simply use red and green atoms and require that, say, the composition of green atoms must be red, $\mathsf{g}_i \,; \mathsf{g}_{i+1} = \mathsf{r}_1$, and $(\mathsf{r}_i \,; \mathsf{r}_j) \cdot \mathsf{r}_k = 0$ whenever the sum of two indices is different from the third one? It is not difficult to see that this would imply that $\mathsf{g}_0 \,; \mathsf{g}_{2^{n-1}} = \mathsf{g}_{2^{n-1}} \,; \mathsf{g}_{2^n} = \mathsf{r}_{2^{n-1}}$. Then $\forall$ could win the game just in two rounds independently of the value of n. Thus the ultraproduct would not be representable either.

Let i and j be different, non-consecutive indices. In our construction, $\mathsf{g}_i \,; \mathsf{g}_j$ is the sum of y, b and all the red atoms. In addition to the green atoms, $\forall$ has to use yellow to force $\exists$ to use reds: $\mathsf{y} \,; \mathsf{y}$ cannot be yellow or black. Furthermore, $\exists$ has some freedom in choosing the index of the red atom — the choice is constrained by the forbidden red triangles, i.e., by the indices of the red atoms used so far in the game.

In the proof of Lemma 4.2, it was $\exists$'s responsibility to color the (red) edges (u_i, u_j). Would not it be faster for $\forall$ to win the game by choosing some of these red colors? For instance, $\forall$ might start the game with an edge (u_0, u_2) and label it with r_3. Then $\forall$ would force $\exists$ to extend the graph by pointing

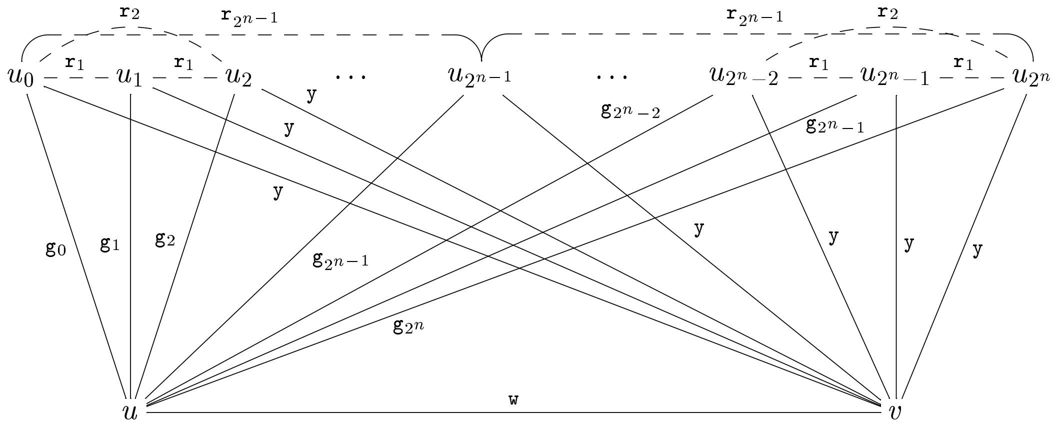

Figure 1: The reason for non-representability

out that $\mathsf{r}_3 \leq \mathsf{g}_0 \,;\mathsf{g}_2$ and that $\mathsf{r}_3 \leq \mathsf{y}\,;\mathsf{y}$. Then we would have edges $(u_0, u) \in \mathsf{g}_0$, $(u, u_2) \in \mathsf{g}_2$, $(u_0, v) \in \mathsf{y}$ and $(v, u_2) \in \mathsf{y}$. If $\forall$ could force $\exists$ to label new edges (u_0, u_1) and (u_1, u_2) by r_1 like in the above proof (by requiring that (u, u_1) be g_1 and (u_1, v) be y), it would be the end for $\exists$ (since $(\mathsf{r}_1\,;\mathsf{r}_1) \cdot \mathsf{r}_3 = 0$). This is where the indexed white atoms w_{ij} come into the picture. In the present scenario, $\exists$ is to choose the color of (u, v), and w_{02} is a possibility. Recall that $(\mathsf{g}_k\,;\mathsf{y}) \cdot \mathsf{w}_{ij} = 0$ whenever $k \notin \{i, j\}$. Hence if $\exists$ labels (u, v) by w_{02}, then $\forall$ cannot require the existence of u_1 such that (u, u_1) be labelled by g_1 and (u_1, v) by y.

Looking at $\exists$'s strategy in general, it should be obvious that it is not a clever idea to use red atoms (unless it is unavoidable) and that using green and yellow atoms is not advisable either (since they can force later edges to be red). When the graph has to be extended, $\exists$'s strategy is to use an indexed white whenever this is a possible choice. The black atom gives $\exists$ another option to avoid using red, green or yellow; when $\forall$ does not use yellow and green for extending the graph and no indexed white is available, $\exists$ uses black. Finally, $\exists$ uses a red atom only if neither white nor black can be used. $\exists$ chooses the index of the red atom depending on the number of rounds played so far and on the indices of the red atoms occurring in the graph; see [HM00] for details.

Thus the black and the indexed white atoms give $\exists$ some leeway to postpone losing the game. In fact, the only way $\forall$ can win the game is to play a sequence of green and yellow atoms as described in the proof of Lemma 4.2. Furthermore, a win for $\forall$ requires more and more rounds as n grows. When the two players play the game on the ultraproduct $\mathfrak{A}$ of the algebras $\mathfrak{A}_n$ $(n < \omega)$, then $\exists$ can survive infinitely many rounds and thus has a winning strategy. Hence $\mathfrak{A}$ is representable as an RRA. We can conclude that $\mathsf{R}(\,\cdot\,,\,;\,,{}^{\smile}, 0)$ is HNFA. We refer the interested reader to [HM00] for a precise proof. ∎

Remark 4.3 1. It is not very difficult to show that the same construction can be used to establish that the $\{\,\cdot\,, \rhd, \lhd\}$-reduct of $\mathfrak{A}_n$ is not representable using an equivalence relation. Hence we have that $\mathsf{R}(\,\cdot\,, \rhd, \lhd)$ is HNFA.[3]

2. A related issue to finitely axiomatizing a class K of algebras is the problem whether the equational theory of K is finitely axiomatizable in equational logic. We will not pursue this issue here; the interested reader is referred to [HM00] where it is shown that, for all finite n, there is a valid equation

[3]A joint result with I. Hodkinson.

that fails in the nth reduct algebra, establishing that the equational theory is not finitely axiomatizable either.

5 Open problems

So far the general, if somewhat simplified, picture is that the interactions between composition $;$ and union $+$ or converse $\smile$ lead to non-finite axiomatizability. But what can we say about the interaction between $;$, $\cdot$ and identity $1'$?

Identity. The most obvious candidate as an axiom for the identity constant is

$$x\,;1' = 1'\,;x = x.$$

But it is not very hard to show that it is not enough for axiomatizing $\mathsf{R}(\cdot\,,\,;\,,1')$ over $\mathsf{R}(\cdot\,,\,;\,)$.[4] In fact, one of the most challenging open problems in this field is the following.[5]

Question 5.1 Is $\mathsf{R}(\cdot\,,\,;\,,1')$ finitely axiomatizable?

A related question is whether $\mathsf{R}(\,;\,,\cdot\,,\backslash,/,1',0)$ admits a finite axiomatization. In [AM94] a representability result was established for the specific case in which the identity constant $1'$ satisfies the axiom

$$x\,;y \leq 1' \to (x = 0 \vee y = 0 \vee x = y = 1')$$

as well. However, this axiom is usually not valid, hence the general finite axiomatizability question remains open.

Conjugates. In Remark 4.3, we mentioned the HNFA of $\mathsf{R}(\cdot\,,\rhd,\lhd)$. An alternative would be if we allow representations over transitive (not necessarily equivalence) relations. Is this class of algebras finitely axiomatizable? It can be shown that the class of algebras representable over transitive relations of similarity type $\{\cdot\,,\,;\,,\rhd,\lhd\}$ is HNFA [Mik02]. It would be interesting to find the borderline precisely where we lose finite axiomatizability.

Complexity. It is well known that the equational theory of RRA is undecidable [Tar41]. On the other hand, the equational theory of the positive fragment of RRA, $\mathsf{R}(1,0,\cdot\,,+\,,\,;\,,\smile,1')$, is decidable [AB94], but not much is known about the exact complexity of specific subreducts. As an example we mention that it follows from [Lam58] that the equational theory of $\mathsf{R}(\cdot\,,\,;\,,\backslash,/)$ is decidable in exponential time, but no matching lower bound has been found so far.

Finite base property. The step-by-step construction usually yields a representation on an infinite base. It would be interesting to find out which (classes of) algebras can be finitely represented. There are results about finite representability in certain cases, cf. [AHN99], but it is not clear if existing techniques can be applied to algebras with transitive units. A related issue is when it is enough to consider algebras with finite bases to refute non-valid formulas. Let us mention a specific problem [Ben99, p.349].

[4]There are semilattice-ordered monoids which are not representable because they fail to validate the formula $(x\,;y)\cdot 1' = 0 \to (y\,;x)\cdot 1' = 0$, for instance.

[5]This problem was mentioned by B.M. Schein at the Algebraic Logic Colloquium in Budapest in 1988.

Question 5.2 Is it the case that any non-valid equation of $\mathsf{R}(\cdot, ;, \backslash, /)$ can be refuted in an algebra represented on a *finite* equivalence relation? The same question applies when we consider algebras represented over transitive relations.

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