Tableau Calculus for Local Cubic Modal Logic and it's Implementation

MAARTEN MARX, *Department of Artificial Intelligence, Faculty of Sciences, Vrije Universiteit Amsterdam, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands, marx@cs.vu.nl*

SZABOLCS MIKULÁS, STEFAN SCHLOBACH, *Department of Computer Science, King's College London, Strand, London WC2R 2LS, UK, {szabolcs,schlobac}@dcs.kcl.ac.uk*

Abstract

In this paper we investigate the two-variable fragment of modal logics of relations interpreted on local squares, LC_2. A labelled tableau calculus is presented and its soundness and completeness are proven. Further, a termination proof enables us to use the calculus as a theorem prover. The prover has been implemented in Prolog, and we give a short system description. The paper also contains examples for how the system works including translations from other modal logics into LC_2.[1]

Keywords: modal logic, tableau system, theorem proving, guarded fragment

Introduction

Modal versions of first-order logics with n variables have been investigated, e.g., in [8, 10]. These systems are in fact multi-modal logics with a relational semantics, i.e., the meanings of formulas are n-ary relations. Modal logic of n-ary relations, called MLR_n, can be given different semantics, corresponding to different fragments of the language of first-order logic. The correspondence with first-order logic is worked out in detail in [8]; in [9] the correspondence between the so called local cubic semantics (see the definition below) and the guarded fragment of first order logic (cf. [2, 3, 11]) is provided. There are several versions of relativized semantics according to what is required from the universe of the model. Here we will concentrate on the local cubic version (see the definition below), since this version is computationally well behaved [7] and inherits some essential properties of first-order logic [10]. In other words, the limited interaction between the various dimensions yields a compromise between complexity and expressive power.

The aim of this paper is to show that local cubic modal logic of relations LC is expressive enough to interpret multi-modal logics and that it has a proof theory that can be effectively implemented by a theorem prover.

Here we will focus on the case with only two variables LC_2. First — as is well known — two variables are sufficient to translate (multi-)modal logics into first-order

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logic. Secondly, the language MLR_n for $n = 2$ has a PSPACE complete satisfaction problem, while for $n > 2$ this problem becomes EXPTIME-complete [7]. But of greater importance for our purpose here is that for $n = 2$, the logic has a special finite model property which gives rise to a straightforward tableau calculus based on labelled deduction ideas. For $n > 2$, the finite model property also holds [1, 6], but a simple model creating technique (like a naive tableau system) does not work (see [9]).

We designed and implemented a sound and complete labelled tableau calculus for LC_2, which can be used via the WWW from URL

`http://www.dcs.kcl.ac.uk/cgi-bin/LC2/lc2.html`.

We will show how the system works by looking at an instructive example proof. We evaluate the strength of the logic by showing how it can be used as a decision procedure for a range of other modal logics.

The paper is organized as follows. The next section introduces two-dimensional modal logic. Section 2 shows how to interpret multi-modal logics into LC_2. Section 3 presents the tableau calculus together with some examples, and shows termination, soundness and completeness. The final section deals with its implementation in PROLOG.

1 Local cubic modal logic of two dimensions

LC_2, the local cubic modal logic of two dimensions is a special case of n-dimensional modal logics of relations MLR_n, which has been studied by Marx and Venema in [8]. Restricting the universe to 2 dimensions not only allows visualization of this logic in a quite intuitive way but also ensures termination of our simple tableau calculus LC_2-TAB. The local cube property[2] of the universe implies some nice property for MLR_n (and thus for LC_2), especially decidability and finite axiomatizability.

DEFINITION 1.1
Given a countable set of propositional variables $p_0, p_1, \ldots$, the set of LC_2*-formulas* is defined as the smallest set such that

1. the propositional variables $p_0, p_1, \ldots$ and the modal constant id are LC_2-formulas (called atomic formulas),
2. if φ and ψ are LC_2-formulas then also $\neg\varphi, \varphi \wedge \psi, \diamondvert\, \varphi, \diamondminus\, \varphi$ and $\otimes\varphi$ are LC_2-formulas.

We abbreviate $\boxbar\, \varphi \equiv \neg \diamondvert \neg\varphi$, $\boxminus\, \varphi \equiv \neg \diamondminus \neg\varphi$ and $\varphi \vee \psi \equiv \neg(\neg\psi \wedge \neg\psi)$ as usual. Two new operators are defined by $\ominus\, \varphi \equiv \diamondminus(id \wedge \varphi)$ and $\obar\, \varphi \equiv \diamondvert(id \wedge \varphi)$.

Semantically, boxes and diamonds of LC_2 will be interpreted as classical S5 modalities in two dimensions, with two interacting accessibility relations. The special feature of the logic LC_2 is that the universe $W \subseteq {}^2U$ has to be local cubic. This means that the universe always contains all 4 points of a square defined by a point[3] $\langle a, b\rangle$, the

[2] In the 2-dimensional case this property could also be called *local square* property, but we prefer using the more general denotation *local cubes*.

[3] Since we are working in the 2-dimensional space, pairs of elements of the universe can be seen as points in a Cartesian coordinate system. We will therefore use the expression points and pairs as equivalent terms. These points or pairs will correspond in the tableau system LC_2-TAB to the labels of the labelled formulas.

horizontal point on the diagonal $\langle b, b\rangle$, the vertical point on the diagonal $\langle a, a\rangle$ and the point mirrored on the diagonal $\langle b, a\rangle$.

DEFINITION 1.2
Let U be a set. The set $W = U \times U$ is called a *square*. A set $W \subseteq {}^2U$ is called a *local square* if $\langle a, b\rangle \in W$ implies that ${}^2\{a, b\} \subseteq W$.

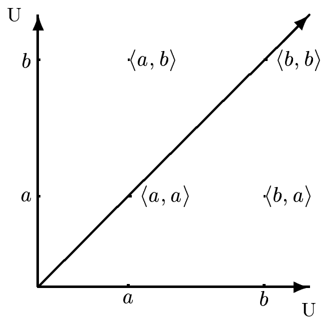

FIG. 1. The local cube property for a 2-dimensional universe.

We now define models and truth for LC_2 in the standard way.

DEFINITION 1.3
An LC_2-model is a pair $\mathcal{M}^{LC_2} = (W_{LC_2}, v_{LC_2})$, such that $W_{LC_2} \subseteq {}^2U$ is a local cubic universe for a non-empty set U and v_{LC_2} is a valuation, i.e., a function mapping every propositional variable to a subset of W_{LC_2}. *Truth* of formulas at a point $\langle a, b\rangle \in W_{LC_2}$ in such models is defined by:

$$
\begin{array}{lcl}
\mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} p & \Leftrightarrow & \langle a, b\rangle \in v_{LC_2}(p) \\
\mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} id & \Leftrightarrow & a = b \\
\mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} \neg\varphi & \Leftrightarrow & \mathcal{M}^{LC_2}, \langle a, b\rangle \not\models_{LC_2} \varphi \\
\mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} \varphi \wedge \psi & \Leftrightarrow & \mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} \varphi \text{ and } \mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} \psi \\
\mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} \otimes\varphi & \Leftrightarrow & \mathcal{M}^{LC_2}, \langle b, a\rangle \models_{LC_2} \varphi \\
\mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} \Diamond\!\!\!\mid\, \varphi & \Leftrightarrow & \text{there is a } \langle a, c\rangle \in W_{LC_2} \\
 & & \qquad \text{such that } \mathcal{M}^{LC_2}, \langle a, c\rangle \models_{LC_2} \varphi \\
\mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} \Diamond\!\!\!-\, \varphi & \Leftrightarrow & \text{there is a } \langle c, b\rangle \in W_{LC_2} \\
 & & \qquad \text{such that } \mathcal{M}^{LC_2}, \langle c, b\rangle \models_{LC_2} \varphi.
\end{array}
$$

We will in the sequel also employ the usual notions of *satisfiability* and *validity* (denoted by $\models_{LC_2} \varphi$) of a formula φ.

The semantics of the defined operators is an immediate result of definition 1.3:

$$
\begin{array}{lcl}
\mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} \obar\, \varphi & \Leftrightarrow & \mathcal{M}^{LC_2}, \langle a, a\rangle \models_{LC_2} \varphi \\
\mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2} \boxbar\, \varphi & \Leftrightarrow & \text{for all pairs } \langle a, x\rangle \in W_{LC_2} \\
 & & \qquad \text{holds that } \mathcal{M}^{LC_2}, \langle a, x\rangle \models_{LC_2} \varphi
\end{array}
$$

The semantics of $\vee, \ominus$ and $\boxminus$ is defined in a similar way.

2 Simulating other logics in $\mathbf{LC_2}$

One can use LC_2 to simulate other logics, among them the modal logics (multi-) K, KT, KB and KTB, and their tensed versions. Special properties of the accessibility relation $\mathcal{R}$ can be encoded into the translation of the diamond operator. The advantage is that we have a single proof-system which works for a wide range of logics. Using the proof-system for LC_2 is without increase in (theoretical) complexity since all other logics are PSPACE hard themselves.

MODAL LOGIC S5. Let $(\cdot)^*$ be the following translation from modal formulas to LC_2 formulas:

$$p^* = p,\ (\cdot)^* \text{ commutes with the booleans, } (\Diamond)^* = \Phi\, \varphi^*.$$

One easily shows that for all φ, φ is S5 satisfiable iff φ^* is satisfiable on a local square.

MODAL LOGICS K, KT, KB AND KTB AND THEIR TENSED VERSIONS. Now we give the translations for other modal logics. Note how we just mimic the standard translation for modal logic K, but without the cumbersome details of variables.

DEFINITION 2.1
Let $(\cdot)^{t_K}$ be the following translation from K to LC_2:

$$\begin{array}{lcll}
(p)^{t_K} & = & p & \text{for propositional variables } p \\
(\neg\varphi)^{t_K} & = & \neg(\varphi)^{t_K} & \\
(\varphi \vee \psi)^{t_K} & = & (\varphi)^{t_K} \vee (\psi)^{t_K} & \\
(\Diamond\varphi)^{t_K} & = & \Phi(r \ \wedge \ \ominus\, \varphi^{t_K}) &
\end{array}$$

where r is a new propositional variable, not occurring in the set of the propositional variables of K.

In order to simulate normal modal logics KT, KB or KTB where the accessibility relations have properties like reflexivity, symmetry or both reflexivity and symmetry, one only slightly has to change the translation of the diamond operator from definition 2.1. We use $\Diamond^{\smile}$ to denote the backwards looking (tense) modality having the same accessibility relation as $\Diamond$.

DEFINITION 2.2
Let $(\cdot)^{t_{K_t}}, (\cdot)^{t_{KB}}, (\cdot)^{t_{KT}}$ and $(\cdot)^{t_{KTB}}$ be modifications of the translation from K to LC_2, corresponding exactly to $(\cdot)^K$ for all but the diamond operator and with the following translations for K_t-, KT-, KB- and KTB-diamonds $\Diamond$:

$$\begin{array}{lcl}
(\Diamond^{\smile}\varphi)^{t_{K_t}} & = & \Phi(\otimes r \wedge \ominus\, \varphi^{t_{K_t}}) \\
(\Diamond\varphi)^{t_{KT}} & = & \Phi((r \vee id) \wedge \ominus\, \varphi^{t_{KT}}) \\
(\Diamond\varphi)^{t_{KB}} & = & \Phi(r \wedge \otimes r \wedge \ominus\, \varphi^{t_{KB}}) \\
(\Diamond\varphi)^{t_{KTB}} & = & \Phi(\ominus\, \varphi^{t_{KTB}}).
\end{array}$$

The brackets $(\cdot)$ will be omitted whenever no confusion is likely to ensue. The translations preserve satisfiability as will be shown in the following theorem.

THEOREM 2.3
For all modal formulas φ holds:

$$\models_{K^*} \varphi \Leftrightarrow \models_{LC_2} id \to \varphi^{t_{K^*}}$$

where K^* is any of the logics K, K_t, KT, KB or KTB.

The idea of the proof is to construct a locally square model $\mathcal{M} = (W, v)$ satisfying $id \wedge \varphi$ from a given K^*-model $\mathcal{M}^{K^*} = (W_{K^*}, \mathcal{R}, v_{K^*})$ for φ and vice versa. The most interesting part of the proof of theorem 2.3 is the proof for the translation of modal formulas into LC_2, because of the explicit use of the local square property. The construction of the models in the other cases is similar.

PROOF. (KTB) ($\Rightarrow$) Given a locally square model $\mathcal{M}$, we construct a new KTB-model $\mathcal{M}^{KTB} = (W_{KTB}, \mathcal{R}, v_{KTB})$ where the universe $W_{KTB} = \{u \mid \langle u, u\rangle \in W\}$ is the set of all points on the diagonal in W, and with the following valuation $v_{KTB}(p) = \{u \mid \langle u, u\rangle \in v(p)\}$. The new accessibility relation $\mathcal{R}$ is defined by $\mathcal{R} = W$, which forces $\mathcal{R}$ to be reflexive and symmetric. It remains to show, that for this model and for all modal formulas φ and pairs $\langle a, a\rangle \in W_{LC_2}$: $\mathcal{M}, \langle a, a\rangle \models_{LC_2} \varphi^{t_{KTB}} \Leftrightarrow \mathcal{M}^{KTB}, a \models_{KTB} \varphi$, which is done by induction over the structure of the formula φ.

($\Leftarrow$) Given a KTB-model $\mathcal{M}^{KTB}$, we construct $\mathcal{M}' = (W', v')$, where the universe $W' = \mathcal{R}$, which is by the reflexivity and symmetry of $\mathcal{R}$ a local square. The valuation $v'(p) = \{\langle a, a\rangle \mid a \in v_{KTB}(p)\}$ maps all propositional variables p to the set of diagonal points of W' corresponding to the elements mapped by $v_{KTB}(p)$. For this model, we prove for all $a \in W_{KTB}$ and all modal formulas φ, that $\mathcal{M}^{KTB}, a \models_{KTB} \varphi \Leftrightarrow \mathcal{M}', \langle a, a\rangle \models \varphi^{t_{KTB}}$ by induction on the structure of the formula φ. ∎

DESCRIPTION LOGICS. If one uses different new propositional variables r_i it is possible to simulate multi-modal logic K^{multi} within the same translational framework. For each different diamond $\Diamond_i$ the translation function $(\cdot)^{t_{multiK}}$ is defined by:

$$(\Diamond_i \varphi)^{t_{multiK}} = \diamondminus(r_i \wedge \ominus\, \varphi^{t_{multiK}})$$

the intuition of which is shown in Figure 2 for two different diamond formulas $\Diamond_1\varphi$ and $\Diamond_2\psi$, that are multi K-satisfiable at a world $a \in W_K$ and the translated LC_2-formulas $(\Diamond_1\varphi)^{t_{multiK}}$ and $(\Diamond_2\psi)^{t_{multiK}}$ are LC_2-satisfiable in $\langle a, a\rangle \in W$. This shows that LC_2 is also well suited to perform proofs in the basic description logic $\mathcal{ALC}$. Obviously, even satisfiability proofs in the logic $\mathcal{ALCR}$ can be mimicked in LC_2. Conjunctions of roles are translated as follows:

$$(\exists(R_1 \sqcap R_2).C)^t = \diamondbar(r_1 \wedge r_2 \wedge \ominus\, C^t).$$

Note that LC_2 contains expressive power not available in $\mathcal{ALCR}$. For instance the concept *self-employed* is definable in LC_2 but not in $\mathcal{ALCR}$. The following first order definition (2.1) is translated to the LC_2 definition (2.2). Obviously both express the same concept.

$$\begin{aligned} \text{self-employed}(x, x) \quad &\equiv \quad \text{employed-by}(x, x) \wedge \\ &\qquad \forall y(\text{employed-by}(x, y) \to x = y) \qquad (2.1) \\ \text{self-employed} \quad &\equiv \quad \obar\, \text{employed-by} \wedge \boxbar(\text{employed-by} \to id). \qquad (2.2) \end{aligned}$$

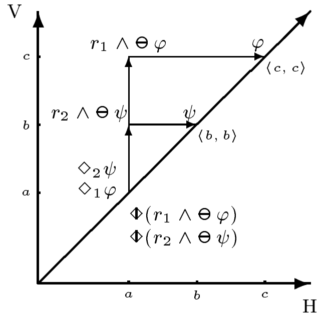

FIG. 2. Simulation of multi-K with two diamonds $\Diamond_1$ and $\Diamond_2$ by LC_2.

3 $\mathbf{LC_2}$-TAB, a tableau-like decision procedure for $\mathbf{LC_2}$

Next we define a labelled tableau calculus. Formulas will be labelled by pairs $\langle a, b\rangle$ and a local square V such that $\langle a, b\rangle \in V$. The tableau calculus LC_2-TAB follows roughly the semantics given in definition 1.3. In order to show for example that a branch of a tableau containing the formula $\otimes\varphi$ labelled with $\langle a, b\rangle$ closes, it is enough to show that a branch containing φ labelled with $\langle b, a\rangle$ closes. The tableau rule for converse formulas thus only adds $\varphi, \langle b, a\rangle, V$ to the actual branch and ticks $\otimes\varphi, \langle a, b\rangle, V$ (meaning that no more rule is to be applied on this formula). This closely corresponds to the `converse rule` in our PROLOG implementation:

```
prove(TickedFormulas,Domain,[lf(⊗φ, X, Y)|Formulas],Result) :-
prove([(⊗φ, X, Y)|TickedFormulas],Domain,[(φ, Y, X)|Formulas],Result).
```

The predicate `prove([],[(0,0)],[lf(¬⬙φ,0,0)],Result)` will succeed and yield `Result = theorem` (meaning that $\diamondminus\, \varphi$ is a theorem) if and only if contradictions in all possible branches are detected.

The notions of open and closed branches and tableau will be defined as usual, where $\perp$ is a special labelled formula, defined in definition 3.2.

DEFINITION 3.1
A branch of a tableau is *closed* if and only if it contains $\perp$. A tableau is *closed* if and only if all branches are closed; a branch is *open* if it is not closed and no rule can be applied on any of its nodes; and finally, a tableau is *open* if it contains at least one open branch.

Now we have to give some more specific definitions for the calculus LC_2-TAB about labelled formulas, universes, labels and satisfiability of labelled formulas. The set of constants which has been chosen as universe for the construction of a possible countermodel are the natural numbers $\mathbb{N}$ and x is a variable over $\mathbb{N}$.

DEFINITION 3.2
A *labelled formula* is a triple of the form $\varphi, \langle a, b\rangle, V$, where φ is the formula, the pair $\langle a, b\rangle \in {}^2(\mathbb{N} \cup \{x\})$ is the *label* and V is a local square. We call a labelled formula *ground* if its label does not contain the variable x.

A ground labelled formula $\varphi, \langle a, b\rangle, V$ is *satisfiable* if and only if there is a local cubic model[4] $\mathcal{M} = (W, v)$ and a pair $\langle a, b\rangle \in W$ such that $\mathcal{M}, \langle a, b\rangle \models \varphi$ and $V \subseteq$

[4] Since in the sequel, we will only talk about local cubic models, we omit the index LC_2 from now on and write

W. A non-ground labelled formula $\varphi, \langle a, x\rangle, V$ is *satisfiable* if there is an LC_2-model $\mathcal{M} = (W, v)$, such that $V \subseteq W$ and for all c such that $\langle a, c\rangle \in W$: $\mathcal{M}, \langle a, c\rangle \models \varphi$ (and similarly for formulas $\varphi, \langle x, b\rangle, V$). $\bot$ is a special labelled formula, which is by definition unsatisfiable. A set of labelled formulas is satisfiable if there is a model satisfying all labelled formulas in the set.

We will identify a branch B with the set of labelled formulas $\varphi, \langle a, b\rangle, V$ occurring in the branch and write $\varphi, \langle a, b\rangle, V \in B$.

3.1 The Tableau system LC_2-TAB

The idea of LC_2-TAB is to start with the negation of a formula φ theoremhood of which has to be proven, and to label it with an arbitrary point $\langle a, b\rangle$. Since $\neg\varphi$ is satisfiable at $\langle a, b\rangle$ iff $\diamondminus\neg\varphi$ is satisfiable at $\langle a, a\rangle$, we start the tableau with $\diamondminus\neg\varphi$ labelled with $\langle 0, 0\rangle$ and the smallest universe containing this point.[5] This universe $V = {}^2\{0\}$ is a local square. The proof consists of explicitly constructing a tableau with an open branch corresponding to a countermodel for the original formula. Thus the formula φ is a theorem if and only if there is a closed tableau for $\diamondminus\neg\varphi$ labelled with $\langle 0, 0\rangle$.

To be able to construct a countermodel and to keep track of all existing pairs in a branch, we also label the formulas in the calculus with a set of pairs V. At any node $\varphi, \langle a, b\rangle, V$ of the tableau, we can think of V as the *current universe*, and the current universe at the end of the open branch will be the universe of the countermodel. New pairs are created if and only if formulas of the form $\diamondminus\varphi$ or $\diamondvert\varphi$ occur and there are no witnessing pair for φ in the current universe. In that case, a "fresh" witness pair $\langle c, b\rangle \notin V$ for the labelled formula $\diamondminus\varphi, \langle a, b\rangle, V$ is created.

If a contradiction occurs, one of the closure rules put $\bot$ in the branch. ($\checkmark$) means that a formula (or a set of formulas) is marked *after* application of the rule and the formula(s) cannot be used any more. We follow Fitting [5] in his uniform notation for conjunctions and disjunctions. Fitting groups all propositional formulas of the form $X \circ Y$ and $\neg(X \circ Y)$ into two categories: those who act conjunctively, which are called α formulas, and those who act disjunctively and are called β formulas. He then defines components $\alpha_1, \alpha_2, \beta_1$ and β_2 such that $\alpha \equiv \alpha_1 \wedge \alpha_2$ and $\beta \equiv \beta_1 \vee \beta_2$. The advantage of this approach is that you only need two tableau rules for all propositional connectives: $\vee, \wedge, \rightarrow, \leftrightarrow$ and their negations.

All LC_2-TAB rules are so called extension rules, which are displayed as:

$$
\begin{array}{c}
\xi_1 \\ \vdots \\ \xi_k \\ \hline
\begin{array}{c|c|c}
\zeta_{11} & \cdots & \zeta_{n1} \\
\vdots & \vdots & \vdots \\
\zeta_{1m_1} & & \zeta_{nm_n}
\end{array}
\end{array}
\qquad \text{or} \qquad
\begin{array}{c}
B \\ \xi \\ \hline
\begin{array}{c|c}
B_1 & B_2 \\
\zeta_1 & \zeta_2
\end{array}
\end{array}
$$

$\mathcal{M}, \langle a, b\rangle \models, W$ and v instead of $\mathcal{M}^{LC_2}, \langle a, b\rangle \models_{LC_2}, W_{LC_2}$ and v_{LC_2}.

[5] We use the natural numbers as base for the universe which is constructed in the step-by-step approach of the calculus LC_2-TAB, because they reflect the geometrical nature of the construction in a very natural way. Due to the structure of our rules, we do only need one variable over $\mathbb{N}$, which we will call x.

The left-hand side rule means that if there are labelled formulas $\xi_1, \dots, \xi_k$ in the branch, then the branch will be split into n sub-branches and all labelled formulas $\zeta_{i1}, \dots, \zeta_{im_i}$ will be added to the i-th branch. n and the m_i may be 1. Since we identify the branch B with the set of labelled formulas occurring in B, the meaning of a rule of the right-hand type is that a new set B_1 (B_2) of labelled formulas is added to the first (second) branch. These B_1 and B_2 are usually defined from B in one of the conditions of the rules. Most of the rules come with additional restrictions for their application.

To avoid cumbersome notation we use the following convention for application of the simple rules. A rule of the form $B, \varphi / B, \varphi', \varphi(\checkmark)$ will be abbreviated by $\varphi(\checkmark)/\varphi'$. The only rules which are not of this form are the diamond rules and the label rules.

Remark 3.3
In the following, we treat the defined connectives $\ominus$, ⦶, $\boxminus$ and ◫ as primitives.

Definition 3.4
A formula φ is an *LC_2-theorem* (denoted by $\vdash \varphi$) if and only if there is tableau proof for it, where a ***tableau proof*** for a formula φ in the calculus LC_2-TAB is a closed tableau starting with the labelled formula ⟠$(\neg\varphi), \langle 0,0\rangle, {}^2\{0\}$, using the tableau rules stated below.

- Propositional rules.

$(\neg)$	(α)	(β)
$\dfrac{\neg\neg Z, \langle a,b\rangle, V(\checkmark)}{Z, \langle a,b\rangle, V}$	$\dfrac{\alpha, \langle a,b\rangle, V(\checkmark)}{\begin{matrix}\alpha_1, \langle a,b\rangle, V \\ \alpha_2, \langle a,b\rangle, V\end{matrix}}$	$\dfrac{\beta, \langle a,b\rangle, V(\checkmark)}{\beta_1, \langle a,b\rangle, V \mid \beta_2, \langle a,b\rangle, V}$
where $a, b \in \mathbb{N} \cup \{x\}$	where $a, b \in \mathbb{N} \cup \{x\}$	where $a, b \in \mathbb{N}$

- Closing rules

(id_a)	(id_b)	(con)
$\dfrac{\neg id, \langle a,a\rangle, V(\checkmark)}{\perp}$	$\dfrac{id, \langle a,b\rangle, V(\checkmark)}{\perp}$	$\dfrac{\begin{matrix}\neg\varphi, \langle a,b\rangle, V(\checkmark) \\ \varphi, \langle a,b\rangle, V(\checkmark)\end{matrix}}{\perp}$
where $a \in \mathbb{N}$	where $a, b \in \mathbb{N}$ and $a \neq b$	where $a, b \in \mathbb{N}$

- Transition rules

$(\otimes)\quad \dfrac{\otimes\varphi,\langle a,b\rangle,V(\checkmark)}{\varphi,\langle b,a\rangle,V}$ where $a,b\in\{x\}\cup\mathbb{N}$	$(\neg\otimes)\quad \dfrac{\neg\otimes\varphi,\langle a,b\rangle,V(\checkmark)}{\neg\varphi,\langle b,a\rangle,V}$ where $a,b\in\{x\}\cup\mathbb{N}$
$(\obar)\quad \dfrac{\obar\,\varphi,\langle a,b\rangle,V(\checkmark)}{\varphi,\langle a,a\rangle,V}$ where $b\in\{x\}\cup\mathbb{N}, a\in\mathbb{N}$	$(\neg\obar)\quad \dfrac{\neg\obar\,\varphi,\langle a,b\rangle,V(\checkmark)}{\neg\varphi,\langle a,a\rangle,V}$ where $b\in\{x\}\cup\mathbb{N}, a\in\mathbb{N}$
$(\text{⬖})\quad \dfrac{\begin{array}{c}B(\checkmark)\\ \text{⬖}\,\varphi,\langle a,b\rangle,V(\checkmark)\end{array}}{\begin{array}{c\|c}B & B'\\ \varphi,\langle a,a\rangle,V & \varphi,\langle a,c\rangle,V'\end{array}}$	$(\boxbar)\quad \dfrac{\boxbar\,\varphi,\langle a,b\rangle,V(\checkmark)}{\varphi,\langle a,x\rangle,V}$ where $a\in\mathbb{N},\quad b\in\mathbb{N}\cup\{x\}$

The (⬖)-rule can only be applied under the conditions that:
- B is the set of all labelled formulas occurring in the branch,
- $a\in\mathbb{N},\quad b\in\mathbb{N}\cup\{x\}$, c is a **new** constant, i.e., $\langle c,c\rangle\notin V$,
- $V'=V\cup\{\langle c,a\rangle,\langle a,c\rangle,\langle c,c\rangle\}$,
- there is no pair $\langle a,d\rangle\in V$ such that $\varphi,\langle a,d\rangle,V\in B$,
- $B'=\{\psi,\langle x,y\rangle,V'\mid\psi,\langle x,y\rangle,V\in B\}\ \cup\ \{\text{⬖}\,\varphi,\langle a,b\rangle,V'(\checkmark)\}$.

By definition of $\boxbar$ (and $\boxminus$), the $(\boxbar)$-rule (and $(\boxminus)$-rule) correspond to the negation of the diamond rules (⬖)-rule (and (⬘)-rule). There are corresponding rules for the horizontal $\text{⬘},\boxminus$ and $\ominus$ modalities.

- Label rules (lr_1) and (lr_2)

$(lr_1)\quad \dfrac{\begin{array}{c}B\\ \varphi,\langle x,b\rangle,V\end{array}}{\varphi,\langle a,b\rangle,V}$	$(lr_2)\quad \dfrac{\begin{array}{c}B\\ \varphi,\langle a,x\rangle,V\end{array}}{\varphi,\langle a,b\rangle,V}$

where $a,b\in\mathbb{N},\ \langle a,b\rangle\in V$ and $\varphi,\langle a,b\rangle,V\notin B$.

The rules (lr_1) and (lr_2) are the only rules in the tableau, where the original formulas $\varphi,\langle x,b\rangle,V$ and $\varphi,\langle a,x\rangle,V$ cannot be ticked off, i.e., they can be applied as long as there are pairs in V that have not been used yet.

Let $\varphi,\langle a,b\rangle,V$ be a node in a tableau. Recall that V is called the current universe at this node. Note that the current universes occurring in a branch form a chain, since updating a universe consists of expanding it. Also, if a labelled formula $\varphi,\langle a,b\rangle,V$ occurs in a branch, and, in a later stage, we expand the universe V to V', then the labelled formula $\varphi,\langle a,b\rangle,V'$ also occurs in the branch. Indeed, all the rules but the diamond rules leave the universe V unchanged, while at any application of a diamond rule we make a copy of all formulas occurring in the branch but we label them by the updated universe V'.

Using the observation above, we define the *kernel* of a branch B: $K(B)$. First, let the *universe* of the branch be the union of all universes occurring in the branch. (We will see later that all branches are finite, so the universe of a branch is in fact the last current universe occurring in the branch.) Then we let

$$K(B) = \{\varphi, \langle a, b\rangle, V \in B : V \text{ is the universe of } B\},$$

i.e., the kernel is the set of all labelled formulas occurring in the branch which are labelled by the universe of the branch.

Let us summarize the important property of the kernel:

> FACT: For every branch B, there is a universe V such that, for every labelled formula $\varphi, \langle a, b\rangle, V'$ occurring in B, $\varphi, \langle a, b\rangle, V$ occurs in $K(B)$ and $V' \subseteq V$.

This fact will be used to prove completeness, since we can construct a model for every formula in an open branch from the formulas in the kernel alone.

3.2 Examples

Before proving soundness and completeness of LC_2-TAB with respect to the class of LC_2 models, we will first give some examples showing the intuition behind the construction of the calculus.

3.2.1 Converse formulas

A ground labelled formula $\otimes\varphi, \langle a, b\rangle, V$ is by definition 3.2 satisfiable if and only if there is a model $\mathcal{M} = (W, v)$ such that $\mathcal{M}, \langle a, b\rangle \models_{LC_2} \otimes\varphi$ and $V \subseteq W$. By the definition of the semantics of the operator $\otimes$ (see figure 3), $\mathcal{M}, \langle a, b\rangle \models_{LC_2} \otimes\varphi$ holds if and only if $\mathcal{M}, \langle b, a\rangle \models_{LC_2} \varphi$. This condition corresponds to the satisfiability of the labelled formula $\varphi, \langle b, a\rangle, V$.

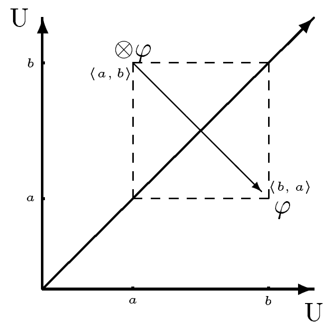

FIG. 3. The semantics of a labelled converse-formula $\otimes\varphi, \langle a, b\rangle, {}^2\{a, b\}$.

The tableau calculus LC_2-TAB follows exactly this intuitive meaning. In order to show that a branch of a tableau containing the labelled formula $\otimes\varphi, \langle a, b\rangle, V$ closes, it is enough to show that a branch containing $\varphi, \langle b, a\rangle, V$ closes. The tableau rule for converse formulas thus only adds $\varphi, \langle b, a\rangle, V$ to the actual branch and ticks $\otimes\varphi, \langle a, b\rangle, V$.

3.2.2 Diamond formulas

Tableau calculi are proof methods where a countermodel is explicitly built for a formula if there exists one. The idea of LC_2-TAB is to start with the smallest possible non-empty model, i.e., a model where the universe contains only one pair $\langle 0,0\rangle$ (the set $V = {}^2\{0\}$ is a local square), and to construct a more and more complex model where all possible contradictions are detected. New labels are only created for formulas of the form $\diamondminus\,\varphi$ or $\diamondbar\,\varphi$ if a witness label $\langle c,b\rangle$ for the labelled formula $\diamondminus\,\varphi, \langle a,b\rangle, {}^2\{a,b\}$ is needed so that φ holds in $\langle c,b\rangle$, cf. figure 4. $\langle c,b\rangle$ is either situated on the diagonal or can be represented by a completely new label not occurring in the universe of the labelled formula. The corresponding tableau rules ($\diamondbar$) and ($\diamondminus$) therefore cause branching of the tableau. The universes V of the labelled formulas in the new branch have to be updated with the newly created local square $V \cup {}^2\{c,b\}$.

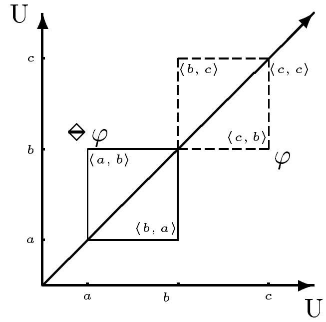

FIG. 4: The semantics of a labelled diamond formula at a label $\langle a,b\rangle$ with a newly created witness $\langle c,b\rangle$ and a new local square ${}^2\{b,c\} \cup {}^2\{a,b\}$.

3.2.3 Box formulas

Box formulas ($\boxminus\,\varphi$ and $\boxbar\,\varphi$) force the formula φ to hold on complete lines or columns of the universe of the model. See figure 5 for an intuitive idea. Since we only consider local square models, the calculus LC_2-TAB can (and has to) keep track of all existing labels, and at all existing labels in a row or a column, φ has to hold. LC_2-TAB works with label rules on variables since in most of the cases it is more convenient to process a formula with variables once and for all, instead of processing the same formula (but instantiated with different points) several times. For that reason a labelled formula $\boxbar\,\varphi, \langle a,b\rangle, V$ is replaced by $\varphi, \langle a,x\rangle, V$.

3.2.4 A documented sample proof

We will construct (the open branch of) a proof in LC_2-TAB to illustrate the use of the rules. We prove, that the formula $\diamondbar\,\diamondminus\,\varphi \rightarrow \diamondminus\,\diamondbar\,\varphi$ is not a theorem in LC_2 by showing that there is an open branch for the tableau starting with the negation of the formula. For reasons of readability, we only do the proof for a point on the diagonal at $\langle 0,0\rangle$. This will prove to be sufficient to find a countermodel.

We present a complete calculation until we reach a diamond formula, which forces us to branch and update the universe of the formulas in the right-hand branch. The numbers and letters in parenthesis are not part of the calculus but make it easier to talk about the example. ($\checkmark_{(i)}$) means, that a formula is ticked after application of a rule creating the new formula (i). In addition, we give a brief explanation why we

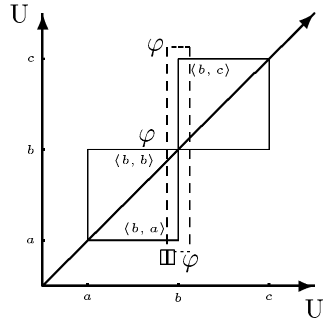

FIG. 5: The semantics of a box formula at a pair $\langle b, a\rangle$ with the universe ${}^2\{a, b\} \cup {}^2\{b, c\}$.

created a new formula on the right-hand side of the tableau.

(1)	$\neg(⬖⬘\varphi \to ⬘⬖\varphi), \langle 0,0\rangle, {}^2\{0\}(\checkmark_{(2)(3)})$			Negated input
(2)	$⬖⬘\varphi, \langle 0,0\rangle, {}^2\{0\}(\checkmark_{(5a)(5b)})$			(α) on (1)
(3)	$\neg⬘⬖\varphi, \langle 0,0\rangle, {}^2\{0\}(\checkmark_{(4)})$			(α) on (1)
(4)	$\neg⬖\varphi, \langle x,0\rangle, {}^2\{0\}(\checkmark_{(5a)(5b)})$			(⬒) on (3)
$(5a)$	$⬘\varphi, \langle 0,0\rangle, {}^2\{0\}$	$(5b)$	$⬘\varphi, \langle 0,1\rangle, {}^2\{0,1\}$	(⬖) on (2)
$(1a)$	$= (1)$	$(1b)$	$\neg(⬖⬘\varphi \to ⬘⬖\varphi), \langle 0,0\rangle, {}^2\{0,1\}(\checkmark)$	copy of (1)
$(2a)$	$= (2)$	$(2b)$	$⬖⬘\varphi, \langle 0,0\rangle, {}^2\{0,1\}(\checkmark)$	copy of (2)
$(3a)$	$= (3)$	$(3b)$	$\neg⬘⬖\varphi, \langle 0,0\rangle, {}^2\{0,1\}(\checkmark)$	copy of (3)
$(4a)$	$= (4)$	$(4b)$	$\neg⬖\varphi, \langle x,0\rangle, {}^2\{0,1\}$	copy of (4)

In the left branch (a), exact copies of the labelled formulas (1) to (4) are made. In the right-hand branch (b), the universes of (1b) to (4b) have been updated according to the definition of the (⬖)-rule. The new core of the branches are the formulas (1a) to (5a) and (1b) to 5(b). Note that the formulas (4a) and (4b) are *not* ticked. For simplicity reasons, we only proceed with branch (b). In passing we note that both subbranches of (a) close by contradiction on labels in the row $\langle x, 0\rangle$.

$(1b)$	$\neg(⬖⬘\varphi \to ⬘⬖\varphi), \langle 0,0\rangle, {}^2\{0,1\}(\checkmark)$			
$(2b)$	$⬖⬘\varphi, \langle 0,0\rangle, {}^2\{0,1\}(\checkmark)$			
$(3b)$	$\neg⬘⬖\varphi, \langle 0,0\rangle, {}^2\{0,1\}(\checkmark)$			
$(4b)$	$\neg⬖\varphi, \langle x,0\rangle, {}^2\{0,1\}(\checkmark_{(6c)(6d)})$			
$(5b)$	$⬘\varphi, \langle 0,1\rangle, {}^2\{0,1\}(\checkmark_{(6c)(6d)})$			
$(6c)$	$\varphi, \langle 1,1\rangle, {}^2\{0,1\}$	$(6d)$	$\varphi, \langle 2,1\rangle, {}^2\{0,1\} \cup {}^2\{1,2\}$	(⬘) on $(5b)$
$(1c)$	$= (1b)$	$(1d)$	$\neg(⬖⬘\varphi \to ⬘⬖\varphi),$	
			$\langle 0,0\rangle, {}^2\{0,1\} \cup {}^2\{1,2\}(\checkmark)$	copy of $(1b)$
$(2c)$	$= (2b)$	$(2d)$	$⬖⬘\varphi, \langle 0,0\rangle, {}^2\{0,1\} \cup {}^2\{1,2\}(\checkmark)$	copy of $(2b)$
$(3c)$	$= (3b)$	$(3d)$	$\neg⬘⬖\varphi, \langle 0,0\rangle, {}^2\{0,1\} \cup {}^2\{1,2\}(\checkmark)$	copy of $(3b)$
$(4c)$	$= (4b)$	$(4d)$	$\neg⬖\varphi, \langle x,0\rangle, {}^2\{0,1\} \cup {}^2\{1,2\}$	copy of $(4b)$
$(5c)$	$= (5b)$	$(5d)$	$⬘\varphi, \langle 1,0\rangle, {}^2\{0,1\} \cup {}^2\{1,2\}(\checkmark)$	copy of $(5b)$

The ⬘-formula (5b) forced another branching into (c) and (d). (1c) to (5c) are exact copies of (1b) to (5b), whereas the universe of the labelled formulas of the branch (d) is updated to ${}^2\{0,1\} \cup {}^2\{1,2\}$. Again (4c) and (4d) are not ticked. The first of these two branches (c) closes because of a contradiction at the point $\langle 1,1\rangle$:

$(1c)$	$\neg(\Diamond_{\mid}\Diamond_{-}\varphi \to \Diamond_{-}\Diamond_{\mid}\varphi), \langle 0,0\rangle,\ {}^2\{0,1\}(\checkmark)$	
$(2c)$	$\Diamond_{\mid}\Diamond_{-}\varphi, \langle 0,0\rangle,\ {}^2\{0,1\}(\checkmark)$	
$(3c)$	$\neg\Diamond_{-}\Diamond_{\mid}\varphi, \langle 0,0\rangle,\ {}^2\{0,1\}(\checkmark)$	
$(4c)$	$\neg\Diamond_{\mid}\varphi, \langle x,0\rangle,\ {}^2\{0,1\}$	
$(5c)$	$\Diamond_{-}\varphi, \langle 1,0\rangle,\ {}^2\{0,1\}(\checkmark)$	
$(6c)$	$\varphi, \langle 1,1\rangle,\ {}^2\{0,1\}$	
$(7c)$	$\neg\Diamond_{\mid}\varphi, \langle 0,0\rangle,\ {}^2\{0,1\}(\checkmark_{(8c)})$	$label-rule$ on $(4c)$
$(8c)$	$\neg\varphi, \langle 0,x\rangle,\ {}^2\{0,1\}$	$(\square_{\mid})-rule$ on $(7c)$
$(9c)$	$\neg\varphi, \langle 0,0\rangle,\ {}^2\{0,1\}$	$label-rule$ on $(8c)$
$(10c)$	$\neg\varphi, \langle 0,1\rangle,\ {}^2\{0,1\}$	$label-rule$ on $(8c)$
$(11c)$	$\neg\Diamond_{\mid}\varphi, \langle 1,0\rangle,\ {}^2\{0,1\}(\checkmark_{(12c)})$	$label-rule$ on $(4c)$
$(12c)$	$\neg\varphi, \langle 1,x\rangle,\ {}^2\{0,1\}$	$(\square_{\mid})-rule$ on $(11c)$
$(13c)$	$\neg\varphi, \langle 1,1\rangle,\ {}^2\{0,1\}$	$label-rule$ on $(12c)$
$(14c)$	$\perp$	$(13c)$ and $(6c)$

But the second branch (d) can be fully expanded so that no more rule can be applied and no contradiction occurs in the branch.

$(1d)$	$\neg(\Diamond_{\mid}\Diamond_{-}\varphi \to \Diamond_{-}\Diamond_{\mid}\varphi), \langle 0,0\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}(\checkmark)$	
$(2d)$	$\Diamond_{\mid}\Diamond_{-}\varphi, \langle 0,0\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}(\checkmark)$	
$(3d)$	$\neg\Diamond_{-}\Diamond_{\mid}\varphi, \langle 0,0\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}(\checkmark)$	
$(4d)$	$\neg\Diamond_{\mid}\varphi, \langle x,0\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}$	
$(5d)$	$\Diamond_{-}\varphi, \langle 1,0\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}(\checkmark)$	
$(6d)$	$\varphi, \langle 2,1\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}$	
$(7d)$	$\neg\Diamond_{\mid}\varphi, \langle 0,0\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}(\checkmark_{(8d)})$	$label-rule$ on $(4d)$
$(8d)$	$\neg\varphi, \langle 0,x\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}$	$(\square_{\mid})-rule$ on $(7d)$
$(9d)$	$\neg\varphi, \langle 0,0\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}$	$label-rule$ on $(8d)$
$(10d)$	$\neg\varphi, \langle 0,1\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}$	$label-rule$ on $(8d)$
$(11d)$	$\neg\Diamond_{\mid}\varphi, \langle 1,0\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}(\checkmark_{(12d)})$	$label-rule$ on $(4d)$
$(12d)$	$\neg\varphi, \langle 1,x\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}$	$(\square_{\mid})-rule$ on $(11d)$
$(13d)$	$\neg\varphi, \langle 1,0\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}$	$label-rule$ on $(12d)$
$(14d)$	$\neg\varphi, \langle 1,1\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}$	$label-rule$ on $(12d)$
$(15d)$	$\neg\varphi, \langle 1,2\rangle,\ {}^2\{0,1\}\cup{}^2\{1,2\}$	$label-rule$ on $(12d)$

Six labelled formulas are ticked off. Five of the others ($(14d), (13d), (10d), (9d)$ and $(6d)$) that are not ticked off are ground literal formulas. No more rules can applied on them. On the remaining three non-ground labelled formulas ($(12d), (8d)$ and (4d)) the label rules have been applied on all possible elements of the universe ${}^2\{0,1\}\cup{}^2\{1,2\}$. This proves that there is an open branch for $\neg(\Diamond_{\mid}\Diamond_{-}\varphi \to \Diamond_{-}\Diamond_{\mid}\varphi)$. Hence $\Diamond_{\mid}\Diamond_{-}\varphi \to \Diamond_{-}\Diamond_{\mid}\varphi$ is not a theorem in LC_2. The countermodel, which is constructed in the proof is $\mathcal{M} = (W, v)$, where $W = {}^2\{0,1\}\cup{}^2\{1,2\}$ and $v(\varphi) = \{(2,1)\}$. A visualization is given in figure 6.

3.3 Termination

We now show that, for any formula ξ, the tableau construction terminates (either with a closed or with an open tableau), and that an effective bound on the size of the tableau can be given.

First we note that the only danger for non-termination is that we create an infinite countermodel because we are forced to apply the diamond rules again and again (as a repercussion of the box and label rules).

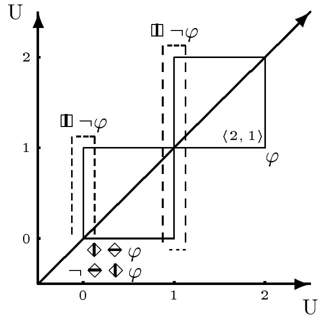

FIG. 6. A countermodel for commutativity.

Second, note that, by condition of the diamond rules, whenever we extend the current universe V we take some $\langle a, a\rangle \in V$ and some b such that $\langle b, b\rangle \notin V$ and form the new universe:

$$V' = V \cup {}^2\{a, b\}.$$

Thus, at any node of the tableau, the current universe is a local square with a tree-like form in the following sense. There is a reflexive tree $T = (I, \leq_T)$ for some finite subset I of the natural numbers such that the root is 0 and

$$V = \bigcup_{i \in I} \{{}^2\{i, j\} : i \leq_T j\}.$$

We define the distance $d(i)$ of a node i from the root 0 in T in the standard way: $d(0) = 0$ and $d(i) = d(j) + 1$ for $j <_T i$. Let $\langle x, y\rangle \in V$. We define

$$\delta(x, y) = \begin{cases} 2 \cdot \min\{d(x), d(y)\} + 1 & \text{if } x \neq y \\ 2 \cdot d(x) & \text{if } x = y. \end{cases}$$

That is, the *distance* $\delta(x, y)$ of $\langle x, y\rangle$ from the root $\langle 0, 0\rangle$ is the number of horizontal and vertical steps in the path leading to $\langle x, y\rangle$ from $\langle 0, 0\rangle$. Using this definition we will show that labelled formulas become simpler and simpler during the tableau construction.

Let ξ be a fixed formula. Below by a tableau we mean a tableau starting with $\diamondminus \neg\xi, \langle 0, 0\rangle, {}^2\{0\}$. For any formula φ, let $m(\varphi)$ denote the modal depth of φ. Let $c(m)$ be the number of diamond subformulas of $\diamondminus \xi$ of modal depth at most m. Note that any formula occurring in the tableau is a (perhaps negated) subformula of $\diamondminus \neg\xi \wedge id$.

LEMMA 3.5
Let $\varphi, \langle c, d\rangle, V'$ be a ground labelled formula such that it occurs in a tableau. Then

$$m(\varphi) + \delta(c, d) \leq m(\diamondminus \xi).$$

PROOF. The key observation is that if we have a tableau proof for $\varphi, \langle c, d\rangle, V'$ and $\langle c, d\rangle \neq \langle 0, 0\rangle$, then we can find a tableau proof for $\varphi, \langle c, d\rangle, V'$ such that there is a ground labelled formula $\psi, \langle a, b\rangle, V$ satisfying

- $\psi, \langle a, b\rangle, V$ precedes $\varphi, \langle c, d\rangle, V'$ in the branch where $\varphi, \langle c, d\rangle, V'$ occurs and

- $m(\psi) + \delta(a, b) \geq m(\varphi) + \delta(c, d)$.

We show the lemma by induction on the complexity of the proof for $\varphi, \langle c, d\rangle, V'$. The base case is when $\langle c, d\rangle = \langle 0, 0\rangle$ is obvious.

The induction hypothesis says that the lemma holds for any $\psi, \langle a, b\rangle, V$ such that $\psi, \langle a, b\rangle, V$ precedes $\varphi, \langle c, d\rangle, V'$ in a tableau.

First assume that $\varphi, \langle c, d\rangle, V'$ is the result of an application of a rule on a ground formula $\psi, \langle a, b\rangle, V$. It is easy to check that every such rule has the property that $m(\psi) + \delta(a, b) \geq m(\varphi) + \delta(c, d)$. Then, by the induction hypothesis, we have $m(\varphi) + \delta(c, d) \leq m(\diamondminus\, \xi)$.

Next assume that $\varphi, \langle c, d\rangle, V'$ is the result of applying a non-label rule ρ to a non-ground formula $\psi, \langle a, b\rangle, V$. Then consider the following modified tableau proof for $\varphi, \langle c, d\rangle, V'$: first apply the label rule to $\psi, \langle a, b\rangle, V$ resulting in a ground formula $\psi, \langle a', b'\rangle, V$ and then apply ρ to $\psi, \langle a', b'\rangle, V$ obtaining $\varphi, \langle c, d\rangle, V'$. By the induction hypothesis, $m(\psi) + \delta(a', b') \leq m(\diamondminus\, \xi)$, whence the lemma follows as in the previous paragraph.

Finally assume that $\varphi, \langle c, d\rangle, V'$ is the result of applying a label rule to a non-ground formula $\psi, \langle a, b\rangle, V$. Say, we applied the (lr_2)-rule to $\psi, \langle a, x\rangle, V$ resulting in $\psi, \langle a, b\rangle, V$. The reason why $\psi, \langle a, x\rangle, V$ occurs in the tableau is that, in a previous step, a box rule has been applied: for instance, the $(\boxbar)$-rule has been applied to the ground formula $\boxbar\, \psi, \langle a, b'\rangle, V$. By the induction hypothesis, $m(\boxbar\, \psi) + \delta(a, b') \leq m(\diamondminus\, \xi)$. Since $\delta(a, b) \leq \delta(a, b') + 1$ and $m(\psi) = m(\boxbar\, \psi) - 1$, the lemma easily follows. The other cases, when after the box rule some propositional or converse rules have been applied before the label rule, can be treated similarly. ∎

By the lemma above, every point $\langle c, d\rangle$ has at most $c(m(\diamondminus\, \xi) - \delta(c, d))$ many horizontal or vertical successors which are created to witness the diamond formulas labelled with $\langle c, d\rangle$. Indeed, if $\diamondminus\, \psi, \langle c, d\rangle, V$ occurs in the tableau and we once created a witness $\psi, \langle a, d\rangle, V'$ by an application of the $(\diamondminus)$-rule, then we do not create more witnesses of the form $\psi, \langle b, d\rangle, V''$. In fact, the fourth condition of the diamond rule forbids creating unnecessary witnesses. Hence we can find a bound on the size of the universe of any countermodel created: the size of the tree with depth $m(\diamondminus\, \xi)$ and branching factor $c(m(\diamondminus\, \xi) - k)$ on level k provides an upper bound for $\{a : \langle a, a\rangle \in V\}$ (V the universe of the created model).

3.4 Soundness and Completeness

3.4.1 Soundness

To prove soundness for LC_2-TAB, we first give a lemma that shows that the rules of the tableau system preserve satisfiability of labelled formulas.

LEMMA 3.6
For every rule

$$\frac{\begin{array}{c}(B)\\ \varphi, \langle a, b\rangle, V\end{array}}{\begin{array}{c}(B')\\ \psi, \langle a', b'\rangle, V'\end{array}}$$

with variables or constants a, b, a', b' (according to the conditions of the rule), holds that if the labelled formula $\varphi, \langle a, b\rangle, V$ is satisfiable together with the branch B, then also $\psi, \langle a', b'\rangle, V'$ is satisfiable with the possibly new branch B'.

PROOF. The proof consists of constructing a model for the result of the application of a rule from the original model for the labelled formula. Only in the case of the diamond rules a new model has to be constructed and the branch B be modified.

1. **Propositional rules:**
   - $(\neg)$-rule: $\neg\neg Z, \langle a, b\rangle, V$ is satisfiable.
     - $\langle a, b\rangle$ ground. There exists a model $\mathcal{M} = (W, v)$, such that $\mathcal{M}, \langle a, b\rangle \models \neg\neg Z$ and $V \subseteq W$. Then also $\mathcal{M}, \langle a, b\rangle \models Z$, which implies satisfiability of $Z, \langle a, b\rangle, V$.
     - $\langle a, b\rangle$ non-ground, e.g., let $b = x$. There exists a model $\mathcal{M} = (W, v)$ such that for all z whit $\langle a, z\rangle \in W$: $\mathcal{M}, \langle a, z\rangle \models \neg\neg Z$ and $V \subseteq W$. Then for all z such that $\langle a, z\rangle \in W$, also $\mathcal{M}, \langle a, z\rangle \models Z$, which implies again satisfiability of $Z, \langle a, b\rangle, V$. Similarly for $a = x$.
   - (α)-rule: $\alpha, \langle a, b\rangle, V$ is satisfiable.
     - $\langle a, b\rangle$ ground. There exists a model $\mathcal{M} = (W, v)$ such that $\mathcal{M}, \langle a, b\rangle \models \alpha$ and $V \subseteq W$. Then also $\mathcal{M}, \langle a, b\rangle \models \alpha_1$ and $\mathcal{M}, \langle a, b\rangle \models \alpha_2$ for the conjuncts α_1 and α_2, which implies satisfiability of the branch containing both $\alpha_1, \langle a, b\rangle, V$ and $\alpha_2, \langle a, b\rangle, V$.
     - $\langle a, b\rangle$ non-ground, let, e.g., $b = x$. There exists a model $\mathcal{M} = (W, v)$ such that for all z where $\langle a, z\rangle \in W$: $\mathcal{M}, \langle a, z\rangle \models \alpha$ and $V \subseteq W$. Then for all z such that $\langle a, z\rangle \in W$, also $\mathcal{M}, \langle a, z\rangle \models \alpha_1$ and $\mathcal{M}, \langle a, z\rangle \models \alpha_2$ for both components α_1 and α_1, which implies again satisfiability of $\alpha_1, \langle a, b\rangle, V$ and $\alpha_2, \langle a, b\rangle, V$. Similarly for $a = x$.
   - (β)-rule: $\beta, \langle a, b\rangle, V$ is satisfiable.
     - $\langle a, b\rangle$ ground. There exists a model $\mathcal{M} = (W, v)$ such that $\mathcal{M}, \langle a, b\rangle \models \beta$ and $V \subseteq W$. Then $\mathcal{M}, \langle a, b\rangle \models \beta_1$ or $\mathcal{M}, \langle a, b\rangle \models \beta_2$ holds for either β_1 or β_2, which implies satisfiability of at least one of the branches containing $\beta_1, \langle a, b\rangle, V$ or $\beta_2, \langle a, b\rangle, V$.
     - $\langle a, b\rangle$ non-ground. This case is excluded by the condition of the β-rule. We will give a simple counterexample: let $b = x$. The formula $id \vee \neg id, \langle 0, x\rangle, {}^2\{0, 1\}$ is trivially satisfiable, but neither $id, \langle 0, x\rangle, {}^2\{0, 1\}$ nor $\neg id, \langle 0, x\rangle, {}^2\{0, 1\}$ are satisfiable.
2. **Closure rules:** None of the labelled formulas in the premises of the closing rules is satisfiable together with the conditions for the rules, which trivially proves this case of the lemma.
3. **Transition rules:**
   - $(\otimes)$-rule: $\otimes\varphi, \langle a, b\rangle, V$ is satisfiable.
     - $\langle a, b\rangle$ ground. There exists a model $\mathcal{M} = (W, v)$ such that $\mathcal{M}, \langle a, b\rangle \models \otimes\varphi$ and $V \subseteq W$. Then also $\mathcal{M}, \langle b, a\rangle \models \varphi$, which implies satisfiability of $\varphi, \langle b, a\rangle, V$.
     - $\langle a, b\rangle$ non-ground, e.g., $b = x$. There exists a model $\mathcal{M} = (W, v)$ such that $\mathcal{M}, \langle a, z\rangle \models \otimes\varphi$ for all z where $\langle a, z\rangle \in W$ and $V \subseteq W$. Then also $\mathcal{M}, \langle z, a\rangle \models \varphi$ for all the $\langle z, a\rangle \in W$. Similarly for $a = x$.
   - $(\neg\otimes)$-rule: similar arguments hold as for the $(\otimes)$-rule.
   - $(\ominus)$-rule: $\ominus\, \varphi, \langle a, b\rangle, V$ is satisfiable.

  - $\langle a,b\rangle$ ground. There exists a model $\mathcal{M}=(W,v)$ such that $\mathcal{M},\langle a,b\rangle \models \ominus\varphi$ and $V\subseteq W$. Then also $\mathcal{M},\langle b,b\rangle\models\varphi$, which implies satisfiability of $\varphi,\langle b,b\rangle,V$.
  - $\langle a,b\rangle$ non-ground: $a=x$. There exists a model $\mathcal{M}=(W,v)$ such that $\mathcal{M},\langle z,b\rangle\models\ominus\varphi$ for all z with $\langle z,b\rangle\in W$, and $V\subseteq W$. Then $\mathcal{M},\langle b,b\rangle\models\varphi$, which implies satisfiability of $\varphi,\langle b,b\rangle,V$.
  - $\langle a,b\rangle$ non-ground: $b=x$. This case is excluded by the condition of the $\ominus$-rule. Again, we will give a simple counterexample. The two labelled formulas $\ominus p,\langle 0,x\rangle,{}^2\{0,1\}\cup{}^2\{1,2\}$ and $\neg p,\langle 2,2\rangle,{}^2\{0,1\}\cup{}^2\{1,2\}$ are satisfiable in the same model, but not the formulas[6] $p,\langle x,x\rangle,{}^2\{0,1\}\cup{}^2\{1,2\}$ and $\neg p,\langle 2,2\rangle,{}^2\{0,1\}\cup{}^2\{1,2\}$.
- $(\neg\ominus)$-rule: similar arguments hold as for the $(\ominus)$-rule.
- $(\obar)$- and $(\neg\obar)$-rule: Similar to the $(\ominus)$- and the $(\neg\ominus)$-rule.
- $(\Diamondbar)$-rule: $\Diamondbar\varphi,\langle a,b\rangle,V$ is satisfiable.
  - $\langle a,b\rangle$ is ground: There is a model $\mathcal{M}=(W,v)$ such that $V\subseteq W$ and $\mathcal{M},\langle a,b\rangle\models\Diamondbar\varphi$. By definition 1.3 this means that there is a pair $\langle a,c\rangle\in W$ such that $\mathcal{M},\langle a,c\rangle\models\varphi$. There are two cases:
    (a) $a=c$, i.e., φ holds on the diagonal: $\mathcal{M},\langle a,a\rangle\models\varphi$. This corresponds to satisfiability of the labelled formula: $\varphi,\langle a,a\rangle,V$.
    (b) $a\neq c$. If $(a,c)\in W\setminus V$, we are done, since this corresponds to the application of the diamond rule introducing a *new* constant. So let's suppose that for all $(a,c)\in W$ such that $\mathcal{M},(a,c)\models\varphi$, we have $(c,c)\in V$. But by a straightforward modal extension argument we may assume that there exists a model $\mathcal{M}^+$ such that $\mathcal{M}\subseteq\mathcal{M}^+$ and in $\mathcal{M}^+$ there are infinitely many (a,d) satisfying φ. Whence there certainly is a pair $(a,d)\notin V$, and $\varphi,\langle a,d\rangle,V'$ with $V'=V\cup{}^2\{a,d\}$ is satisfiable.
  - $\langle a,b\rangle$ is non-ground: $b=x$. There is a model $\mathcal{M}=(W,v)$ such that for all z with $\langle a,z\rangle\in W:\ \mathcal{M},\langle a,z\rangle\models\Diamondbar\varphi$. This implies that there is a pair $\langle a,c\rangle\in W$ such that $\mathcal{M},\langle a,c\rangle\models\varphi$ and $V\subseteq W$. But in this case the argument above holds.
  - $\langle a,b\rangle$ is non-ground: $a=x$. This case is excluded by the conditions of the $\Diamondbar$-rule. A counterexample would be similar to the one given in the third case of the $(\ominus)$-rule.
- $(\Diamondminus)$-rule: similar arguments hold as for the $\Diamondbar$-rule.
- $(\boxminus)$-rule: $\boxminus\varphi,\langle a,b\rangle,V$ is satisfiable.
  - $\langle a,b\rangle$ ground. There exists a model $\mathcal{M}=(W,v)$ such that $\mathcal{M},\langle a,b\rangle\models\boxminus\varphi$ and $V\subseteq W$. Then for all z such that $\langle z,b\rangle\in W$: $\mathcal{M},\langle z,b\rangle\models\varphi$ which corresponds by definition to the satisfiability of the labelled formula: $\varphi,\langle x,b\rangle,V$.
  - $\langle a,b\rangle$ is non-ground: $a=x$. There exists a model $\mathcal{M}=(W,v)$ such that $\mathcal{M},\langle z,b\rangle\models\boxminus\varphi$ for all z where $\langle z,b\rangle\in W$, and $V\subseteq W$. Then also $\mathcal{M},\langle z,b\rangle\models\varphi$ for all these $\langle z,b\rangle\in W$, which implies satisfiability of $\varphi,\langle x,b\rangle,V$.
  - $\langle a,b\rangle$ is non-ground: $b=x$. This case is excluded by the condition of the $\boxminus$-rule. A counter-example would be similar to the one given in the third case of the $(\ominus)$-rule.
- $(\boxbar)$-rule: similar arguments hold as for the $\boxminus$-rule.

[6] Formulas $\varphi,\langle x,x\rangle,V$, where the label contains only the variable x in both argument positions (such a formula would be satisfiable if there is a model $\mathcal{M}=(W,v)$ such that for all a with $\langle a,a\rangle\in W:\mathcal{M},\langle a,a\rangle\models\varphi$) are not explicitly excluded by definition of labelled formulas. But by definition of the conditions of the tableau rules, it is impossible that they occur in an LC_2-TAB tableau.

4. **Label rules:** $\varphi, \langle a, x\rangle, V$ is satisfiable.
   There exists a model $\mathcal{M} = (W, v)$ such that $\mathcal{M}, \langle a, z\rangle \models \varphi$ for all z where $\langle a, z\rangle \in W$, and $V \subseteq W$. Then $\mathcal{M}, \langle a, b\rangle \models \varphi$ holds for an arbitrary $\langle a, b\rangle \in W$, which implies satisfiability of $\varphi, \langle a, b\rangle, V$ for all $\langle a, b\rangle \in V$, since $V \subseteq W$. Similarly for labelled formulas $\varphi, \langle x, b\rangle, V$.

This finishes the proof of lemma 3.6. ∎

Lemma 3.6 and the definition of an open branch imply the following lemma.

LEMMA 3.7
If a tableau starts with a satisfiable labelled formula, then there exists a branch containing only satisfiable formulas. Furthermore this branch is open.

PROOF. The proof is by contraposition. If a branch is closed, it contains by definition $\perp$, which is an unsatisfiable labelled formula. This means by lemma 3.6, that the starting formula must have been unsatisfiable. ∎

With this lemma, we can prove soundness for LC_2-TAB with respect to the class of LC_2-models.

THEOREM 3.8 (Soundness of the Tableau System)
Every theorem of the LC_2-TAB calculus is LC_2-valid, i.e.,

$$\vdash \varphi \Rightarrow \models \varphi.$$

PROOF. The proof is by contraposition. If φ is not valid in LC_2, $\neg\varphi$ is satisfiable in a model $\mathcal{M} = (W, v)$ at a world $\langle a, b\rangle \in W$. This implies satisfiability of $\diamondminus \neg\varphi$ at $\langle b, b\rangle$, which is equivalent to satisfiability of the labelled formula $\diamondminus \neg\varphi, \langle b, b\rangle, {}^2\{b\}$. By lemma 3.7 this implies that there is an open branch for $\diamondminus \neg\varphi, \langle b, b\rangle, {}^2\{b\}$. But, by simply renaming b with 0, this would mean that φ is not an LC_2-theorem. This proves theorem 3.8. ∎

3.4.2 Completeness

We will prove a lemma from which, given termination in section 3.3, completeness follows as an immediate consequence.

LEMMA 3.9
If there is an open branch B in a tableau containing a labelled formula $\varphi, \langle a, b\rangle, V'$, then there exists a model $\mathcal{M} = (V, v)$ satisfying φ, where $V' \subseteq V$.

PROOF. To prove lemma 3.9, we will construct a model $\mathcal{M}$ for the set of all labelled formulas of the open branch B. We can assume that, for every labelled formula $\varphi, \langle a, b\rangle, V' \in B$, there is a corresponding labelled formula $\varphi, \langle a, b\rangle, V \in K$ in the kernel. Recall that the universes V of all these labelled formulas in the kernel of this branch are the same.

Using this global V, we define a model $\mathcal{M} = (V, v)$ for the labelled formulas in the kernel K of an open branch B in the obvious way:

$$\langle a, b\rangle \in v(p) \Leftrightarrow p, \langle a, b\rangle, V \in K$$

for all propositional variables p and ground labels $\langle a,b\rangle \in V$. We prove the following claim:

CLAIM: For all labelled formulas $\varphi, \langle a,b\rangle, V$ occurring in the kernel K of an open branch B of a tableau and for the model $\mathcal{M} = (V, v)$ as defined above, the following hold:

$$\varphi, \langle a,b\rangle, V \in K \;\Rightarrow\; \mathcal{M}, \langle a,b\rangle \models \varphi \qquad \text{where } \langle a,b\rangle \in V \text{ is ground,}$$

and

$$\varphi, \langle a,x\rangle, V \in K \;\Rightarrow\; \text{ for all } z \text{ such that } \langle a,z\rangle \in V: \; \mathcal{M}, \langle a,z\rangle \models \varphi$$
$$\varphi, \langle x,b\rangle, V \in K \;\Rightarrow\; \text{ for all } z \text{ such that } \langle z,b\rangle \in V: \; \mathcal{M}, \langle z,b\rangle \models \varphi.$$

The proof is by induction over the structure of φ.

1. **Propositional formulas:**
   - Positive literals: $p, \langle a,b\rangle, V \in K$
     - $\langle a,b\rangle$ ground. $p, \langle a,b\rangle, V \in K$ implies $\langle a,b\rangle \in v(p)$ by definition of the model. Therefore $\mathcal{M}, \langle a,b\rangle \models p$.
     - $\langle a,b\rangle$ non-ground, e.g., $b = x$. From the fact that $p, \langle a,b\rangle, V \in K$ it follows, because of the label rule, that for all z with $\langle a,z\rangle \in V$, also $p, \langle a,z\rangle, V \in K$. This implies $\langle a,z\rangle \in v(p)$ by definition of the model. Therefore $\mathcal{M}, \langle a,z\rangle \models p$ for all z with $\langle a,z\rangle \in V$.
   - Negative literals: $\neg p, \langle a,b\rangle, V \in K$
     - $\langle a,b\rangle$ ground. $\neg p, \langle a,b\rangle, V \in K$ implies $p, \langle a,b\rangle, V \notin K$ because the branch is open. This is equivalent to $\langle a,b\rangle \notin v(p)$, and thus $\mathcal{M}, \langle a,b\rangle \not\models p$. By definition of LC_2-models $\mathcal{M}, \langle a,b\rangle \models \neg p$ follows.
     - $\langle a,b\rangle$ non-ground, e.g., $b = x$. From the fact that $\neg p, \langle a,b\rangle, V \in K$ follows, because of the label rule, that for all z such that $\langle a,z\rangle \in V$, also $\neg p, \langle a,z\rangle, V \in K$. These $\langle a,z\rangle \in V$ are ground, so $\mathcal{M}, \langle a,z\rangle \models \neg p$ by the argument used above.
   - Identity formulas: $id, \langle a,b\rangle, V \in K$
     - $\langle a,b\rangle$ ground. $id, \langle a,b\rangle, V \in K$ implies, that $a = b$, since B is an open branch. Thus $\mathcal{M}, \langle a,b\rangle \models id$ follows immediately.
     - $\langle a,b\rangle$ non-ground, e.g., $b = x$ is a variable. $id, \langle a,b\rangle, V \in K$ implies that for all z with $\langle a,z\rangle \in V$, the label rule must have been applied. Hence $a = z$, since B is an open branch and thus $\mathcal{M}, \langle a,z\rangle \models id$. (This can only happen when $V = {}^2\{a\}$.)
   - Negated identity formulas: $\neg id, \langle a,b\rangle, V \in K$
     - $\langle a,b\rangle$ ground. $\neg id, \langle a,b\rangle, V \in K$ implies that $a \neq b$, since B is an open branch. Thus $\mathcal{M}, \langle a,b\rangle \models \neg id$ follows immediately.
     - $\langle a,b\rangle$ non-ground, e.g., $b = x$. This case is impossible to occur. Since V is local square, $\langle a,a\rangle \in V$, and thus $\neg id, \langle a,a\rangle, V \in K$. This would imply that $\perp \in K$ and B thus would have been closed.
   - Double negated formulas: $\neg\neg Z, \langle a,b\rangle, V \in K$
     - $\langle a,b\rangle$ ground. $\neg\neg Z, \langle a,b\rangle, V \in K$ implies that $Z, \langle a,b\rangle, V \in K$, because the $(\neg)$-rule must have been applied on $\neg\neg Z, \langle a,b\rangle, V$. By induction hypothesis we know that $\mathcal{M}, \langle a,b\rangle \models Z$, so that $\mathcal{M}, \langle a,b\rangle \models \neg\neg Z$ holds by definition 1.3.

- $\langle a,b\rangle$ non-ground, e.g., $b=x$. $\neg\neg Z,\langle a,b\rangle,V\in K$ implies that $Z,\langle a,b\rangle,V\in K$, because the $(\neg)$-rule must have been applied on $\neg\neg Z,\langle a,b\rangle,V$. By induction hypothesis we know that for all z where $\langle a,z\rangle\in V:\ \mathcal{M},\langle a,z\rangle\models Z$, which implies that also $\mathcal{M},\langle a,z\rangle\models\neg\neg Z$ for all z by definition 1.3.

- α-formulas: $\alpha,\langle a,b\rangle,V\in K$
  - $\langle a,b\rangle$ ground. $\alpha,\langle a,b\rangle,V\in K$ implies that $\alpha_1,\langle a,b\rangle,V\in K$ and $\alpha_2,\langle a,b\rangle,V\in K$, because the (α)-rule must have been applied on $\alpha,\langle a,b\rangle,V$. By induction hypothesis we know that both $\mathcal{M},\langle a,b\rangle\models\alpha_1$ and $\mathcal{M},\langle a,b\rangle\models\alpha_2$, hence $\mathcal{M},\langle a,b\rangle\models\alpha$ holds by definition 1.3.
  - $\langle a,b\rangle$ non-ground, e.g., $b=x$. $\alpha,\langle a,b\rangle,V\in K$ implies that also $\alpha_1,\langle a,b\rangle,V\in K$ and $\alpha_1,\langle a,b\rangle,V\in K$, because the (α)-rule must have been applied on $\alpha,\langle a,b\rangle,V$. By induction hypothesis we know that for all z such that $\langle a,z\rangle\in V$: $\mathcal{M},\langle a,z\rangle\models\alpha_1$ and $\mathcal{M},\langle a,z\rangle\models\alpha_2$, thus $\mathcal{M},\langle a,z\rangle\models\alpha$ for all z with $\langle a,z\rangle\in V$ by definition 1.3.
- β-formulas: $\beta,\langle a,b\rangle,V\in K$
  - $\langle a,b\rangle$ ground. $\beta,\langle a,b\rangle,V\in K$ implies that also one of the labelled formulas $\beta_1,\langle a,b\rangle,V\in K$ or $\beta_2,\langle a,b\rangle,V\in K$ are in the kernel, because the (β)-rule must have been applied on $\beta,\langle a,b\rangle,V$. By induction hypothesis we know for this disjunct β_i: $\mathcal{M},\langle a,b\rangle\models\beta_i$, whence $\mathcal{M},\langle a,b\rangle\models\beta$ holds by definition 1.3.
  - $\langle a,b\rangle$ non-ground. The (β)-rule cannot have been applied on $\beta,\langle a,b\rangle,V$ because of the conditions. Since no more rules can be applied on the formulas, a label rule must have been applied on $\beta,\langle a,b\rangle,V$ for all z such that $\langle a,z\rangle\in V$, which implies that for all z where $\langle a,z\rangle\in V$ a ground β-formula $\beta,\langle a,z\rangle,V$ is in the kernel of the branch. Using the argument of the first case proves $\mathcal{M},\langle a,z\rangle\models\beta$ for all $\langle a,z\rangle\in V$.

To avoid trivial repetitions of the same idea, we omit the proofs for negated formulas below.

2. **Modal formulas:**
- Converse formulas: $\otimes\psi,\langle a,b\rangle,V\in K$
  - $\langle a,b\rangle$ ground. $\otimes\psi,\langle a,b\rangle,V\in K$ implies that $\psi,\langle b,a\rangle,V\in K$, because the $(\otimes)$-rule must have been applied on $\otimes\psi,\langle a,b\rangle,V$. By induction hypothesis we know that $\mathcal{M},\langle b,a\rangle\models\psi$. Hence $\mathcal{M},\langle a,b\rangle\models\otimes\psi$ holds by definition 1.3.
  - $\langle a,b\rangle$ non-ground, e.g., $b=x$. $\otimes\psi,\langle a,b\rangle,V\in K$ implies that also $\psi,\langle b,a\rangle,V\in K$, because the $(\otimes)$-rule must have been applied on $\otimes\psi,\langle a,b\rangle,V$. By induction hypothesis we know that for all z where $\langle a,z\rangle\in V$: $\mathcal{M},\langle z,a\rangle\models\psi$ so that also $\mathcal{M},\langle a,z\rangle\models\otimes\psi$ for all z where $\langle a,z\rangle$ by definition 1.3.
- Circle formulas: $\ominus\psi,\langle a,b\rangle,V\in K$
  - $\langle a,b\rangle$ ground. $\ominus\psi,\langle a,b\rangle,V\in K$ implies that $\psi,\langle b,b\rangle,V\in K$, because the $(\ominus)$-rule must have been applied on $\ominus\psi,\langle a,b\rangle,V$. By induction hypothesis we know that $\mathcal{M},\langle b,b\rangle\models\psi$ so that $\mathcal{M},\langle a,b\rangle\models\ominus\psi$ holds by definition 1.3.
  - $\langle a,b\rangle$ non-ground: $a=x$. $\ominus\psi,\langle a,b\rangle,V\in K$ implies that also $\psi,\langle b,b\rangle,V\in K$, because the $(\ominus)$-rule must have been applied on $\ominus\psi,\langle a,b\rangle,V$. By induction hypothesis we know that $\mathcal{M},\langle b,b\rangle\models\psi$ which implies $\mathcal{M},\langle z,b\rangle\models\ominus\psi$ for all $\langle z,b\rangle\in V$ by definition 1.3.
  - $\langle a,b\rangle$ non-ground. $b=x$. The conditions of the $(\ominus)$-rule say that it cannot have been applied on $\ominus\psi,\langle a,b\rangle,V$. Since no more rules can be applied in the branch, one of the label rules must have been applied on $\ominus\psi,\langle a,b\rangle,V$ for all z

such that $\langle a,z\rangle \in V$. In that case all resulting formulas are ground $\ominus$-formulas $\ominus\,\psi, \langle a,z\rangle, V$. Then the first case applies.

- Similar argument works for the formulas $\obar\,\psi, \langle a,b\rangle, V \in K$.
- Diamond formulas: $\diamondminus\,\psi, \langle a,b\rangle, V \in K$

(a) ($\diamondminus$)-rule has been applied:

- $\langle a,b\rangle$ ground. $\diamondminus\,\psi, \langle a,b\rangle, V \in K$ implies that $\psi, \langle b,b\rangle, V \in K$ or $\psi, \langle c,b\rangle, V \in K$, because the ($\diamondminus$)-rule must have been applied on $\diamondminus\,\psi, \langle a,b\rangle, V$. By induction hypothesis we know that $\mathcal{M}, \langle b,b\rangle \models \psi$ or $\mathcal{M}, \langle c,b\rangle \models \psi$. In both cases $\mathcal{M}, \langle a,b\rangle \models \diamondminus\,\psi$ holds by definition 1.3.
- $\langle a,b\rangle$ non-ground: $a = x$. $\diamondminus\,\psi, \langle a,b\rangle, V \in K$ implies that either $\psi, \langle b,b\rangle, V \in K$ or $\psi, \langle c,b\rangle, V \in K$, because the ($\diamondminus$)-rule must have been applied on $\diamondminus\,\psi, \langle a,b\rangle, V$. By induction hypothesis we know that either $\mathcal{M}, \langle b,b\rangle \models \psi$ or $\mathcal{M}, \langle c,b\rangle \models \psi$ which implies $\mathcal{M}, \langle z,b\rangle \models \diamondminus\,\psi$ for all z such that $\langle z,b\rangle \in V$, by definition 1.3.

(b) ($\diamondminus$)-rule has not been applied:

- There is a labelled formula $\psi, \langle d,b\rangle, V \in K$, where $\langle d,b\rangle \in V$. Then by induction hypothesis, $\mathcal{M}, \langle d,b\rangle \models \psi$ and thus $\mathcal{M}, \langle a,b\rangle \models \diamondminus\,\psi$.
- $\langle a,b\rangle$ non-ground: $b = x$. Because of the conditions of the ($\diamondminus$)-rule, it cannot have been applied on $\diamondminus\,\psi, \langle a,b\rangle, V$. Since no more rules can be applied in the branch, one of the label-rules must have been applied on $\diamondminus\,\psi, \langle a,b\rangle, V$ for all z such that $\langle a,z\rangle \in V$. In that case all resulting formulas are ground $\diamondminus$-formulas of the form $\diamondminus\,\psi, \langle a,z\rangle, V$. Thus one of the other case applies. This implies that $\mathcal{M}, \langle a,c\rangle \models \diamondminus\,\varphi$ for all $\langle a,c\rangle \in V$ which is by definition $M, \langle a,x\rangle \models \diamondminus\,\varphi$.

- Identical argument works for formulas of the form $\diamondbar\,\psi, \langle a,b\rangle, V \in K$.
- Box formulas: $\boxminus\,\psi, \langle a,b\rangle, V \in K$
  - $\langle a,b\rangle$ ground. $\boxminus\,\psi, \langle a,b\rangle, V \in K$ implies that $\psi, \langle x,b\rangle, V \in K$, because the ($\boxminus$)-rule must have been applied on $\boxminus\,\psi, \langle a,b\rangle, V$. By induction hypothesis we know that for all $\langle z,b\rangle \in V : \mathcal{M}, \langle z,b\rangle \models \psi$, so that $\mathcal{M}, \langle a,b\rangle \models \boxminus\,\psi$ holds by definition 1.3.
  - $\langle a,b\rangle$ non-ground: $a = x$. Because the ($\boxminus$)-rule must have been applied, $\boxminus\,\psi, \langle a,b\rangle, V \in K$ implies that $\psi, \langle x,b\rangle, V$ is also in the kernel K of the branch B. By induction hypothesis we know that for all z where $\langle z,b\rangle \in V$: $\mathcal{M}, \langle z,b\rangle \models \psi$. Thus $\mathcal{M}, \langle y,b\rangle \models \boxminus\,\psi$ for all y such that $\langle y,b\rangle \in V$ by definition 1.3.
  - $\langle a,b\rangle$ non-ground: $b = x$. Because of the conditions of the ($\boxminus$)-rule, it cannot have been applied on $\boxminus\,\psi, \langle a,b\rangle, V$. Since no other rules can be applied in the branch, one of the label rules must have been applied on the $\boxminus\,\psi, \langle a,b\rangle, V$ for all z with $\langle a,z\rangle \in V$. In that case all resulting formulas are ground $\boxminus$-formulas of the form $\boxminus\,\psi, \langle a,z\rangle, V$. Thus the first case applies. This implies that $\mathcal{M}, \langle a,c\rangle \models \boxminus\,\varphi$ for all $\langle a,c\rangle \in V$, which is by definition $M, \langle a,x\rangle \models \boxminus\,\varphi$.
- Finally, the same argument works for formulas $\boxbar\,\psi, \langle a,b\rangle, V \in K$.

This proves lemma 3.9, ∎

Using the termination proof, it is now easy to show full completeness of our tableau calculus with respect to the class of local square models LC_2.

THEOREM 3.10 (Completeness of the Tableau System)
Every LC_2-valid formula φ is a theorem of the tableau calculus LC_2-TAB, i.e.,

$$\models \psi \Rightarrow \vdash \psi.$$

PROOF. Since $\models \psi \Rightarrow \vdash \psi$ is equivalent to $\nvdash \psi \Rightarrow \not\models \psi$, and since termination has been proved, we know that $\nvdash \psi$ if and only if there is an open tableau for $\neg\psi, \langle 0, 0\rangle, {}^2\{0\}$. This implies that $\neg\psi, \langle 0, 0\rangle, V$ is in the kernel (where V is the universe of the branch), which implies satisfiability of $\neg\psi$ in the model $\mathcal{M} = (V, v)$ as constructed in lemma 3.9. Therefore ψ is not valid. ∎

4 System Description

IMPLEMENTATION. The system `lc2` is implemented in Prolog and Perl and has an easy-to-use web-based user interface. The proof method applied in `lc2` is a simple subgoaling method. This means that we prove satisfiability of a (labelled) formula φ by proving satisfiability for one or more formulas that have been created according to the rules of LC_2-TAB. The core of `lc2` is the Prolog program `lc2.pl` using a depth-first search method to look for an open branch in the tableau, which is constructed by the application of all possible rules on the set of actual formulas. If a contradiction is found in the branch, the search stops and is continued on a new branch. If a branch is fully evaluated, i.e., no rule can be applied and no contradiction has been detected, the search stops and a countermodel has been found. With this method no backtracking is necessary.

SYSTEM ARCHITECTURE. The system has three major parts, a Perl-based user interface, as well as a parser and the actual theorem prover written in Prolog. The *parser* translates modal formulas into LC_2 and provides an input facility for the prover. If the tableau does not close the prover provides an LC_2 countermodel which is translated into a countermodel of the original logic. The *prover* itself reflects the structure of the tableau system. For each tableau rule, a clause of the main predicate exists; for reasons of simplicity the entire branch is always represented in one of the lists of the predicate.

STATISTICS. The statistics in Figure 7 are intended to give a brief idea of the structure and the relative power of the prover, but also of the high complexity of the logic LC_2 (the satisfaction problem being PSPACE-complete [7]). The table shows some formulas and the corresponding logic, the number of closed branches in the Tableau, the maximal depth of the branches (if the formula was a theorem) or validation and domain of a countermodel (if the formula was not a theorem) and the proof time.[7] The examples for logic LC_2 are constructed from the LC_2 axiom $\otimes \Diamond\!\!\!\mid\; a \leftrightarrow \Diamond\!\!\!\!-\; \otimes a$. The K-theorem

$$\Box(p \wedge \Box p \rightarrow q) \vee \Box(q \wedge \Box q \rightarrow p) \rightarrow \Box(p \vee q) \wedge \Box(\Box p \vee q) \wedge \Box(p \vee \Box q) \rightarrow \Box p \vee \Box q$$

is taken from a test file for modal logic theorem provers given by Beckert and Goré in [4].

[7] On a SPARCstation 5 running SWI-Prolog including compile-time.

The actual version of `lc2` has not been designed with speed but with conceptual simplicity in mind. Therefore it is not supposed to compete in its pure form with more specialized theorem provers for specific modal logics like ModLeanTAB [4]. It is designed to show the expressive power of LC_2 and to offer a decision procedure for a wide range of logics including (multi-)modal logics and an interesting part of the guarded fragment of first-order logic. With its easy-to-use interface it is ideal for use in proofs of smaller formulas and, thanks to its translation facilities, it provides the user with a countermodel if there is one.

There is a version of `lc2` where various control-structures (e.g. closure-test on non-literals, search for formulas implying closure of branches, supplementary rules to avoid unnecessary branching, etc.) have been implemented. This can improve performance significantly. Other interesting ideas include LC_2 specific improvements, e.g., the implementation of a counter of the number of applications of a label rule on a non-ground labelled formula. With an upper bound for this number (which can be easily calculated) labelled formulas also could be eliminated from the list of actual formulas.

Formula (proofs for theoremhood)	Logic	No. of branches	Max. depth	Proof time
$\Diamond\!\!\!\!\mid\ \Diamond\!\!\!\!\mid\ a \to \Diamond\!\!\!\!\mid\ a$	LC_2	4	3	≤ 1 s
$\otimes \Diamond\!\!\!\!\mid\ a \to \text{⟠} \otimes a$	LC_2	2	2	≤ 1 s
$\Diamond\!\!\!\!\mid\ \text{⟠}(\text{⟠} \otimes (a \to b) \to \boxminus \text{⟠} \otimes \text{⟠}(a \to b))$	LC_2	689	8	4.41 s
$\text{⟠} \otimes (a \leftrightarrow b) \to \boxminus \text{⟠} \otimes \text{⟠}(a \leftrightarrow b)$	LC_2	484	4	5.82 s
$\Diamond\!\!\!\!\mid\ (\otimes(a \leftrightarrow b) \to \boxminus \text{⟠} \otimes (a \leftrightarrow b))$	LC_2	472	5	6.16 s
$\text{⟠}\, \Diamond\!\!\!\!\mid\ (\otimes(a \leftrightarrow b) \to \boxminus \text{⟠} \otimes (a \leftrightarrow b))$	LC_2	608	3	19.4 s
$\Box(p \wedge \Box p \to q) \vee \Box(q \wedge \Box q \to p) \to$ $\Box(p \vee q) \wedge \Box(\Box p \vee q) \wedge \Box(p \vee \Box q)$ $\to \Box p \vee \Box q$	KTB	148	5	1.67 s
see above	KB	458	5	3.3 s
see above	K	768	5	6.9 s
see above	KT	1605	5	14.8 s
Formula (non-theoremhood)	Logic	Universe	Valuation $v(a) = \ldots$	Proof time
$\Diamond\!\!\!\!\mid\ \text{⟠}\, a \to \text{⟠}\, \Diamond\!\!\!\!\mid\ a$	LC_2	${}^2\{0,1\} \cup {}^2\{1,2\}$	$\{(1,2)\}$	≤ 1 s
$\Diamond a \to \Box\Diamond a$	K	$\{0,1\}$	$\{0\}$	≤ 1 s

FIG. 7. Some statistics and examples for LC_2-TAB.

USER INTERFACE. Using `lc2` is extremely simple. The easiest way is to fill in the form on the home-page

URL: http://www.dcs.kcl.ac.uk/cgi-bin/LC2/lc2.html

of LC_2-TAB. The content is sent to a central server at King's College London, which activates `lc2`. The answer is sent back immediately to the user. There is a time limit of 10 seconds. The user can choose from a list of K, KT, KTB, S5 or LC_2 which logics he or she wants to work with. The syntax is given on another URL and is quite self-explanatory. The user can choose between 2 output formats, i.e., either only the result or the complete step-by-step calculation of `lc2`. Some links to URLs with examples of theorems and non-theorems are given on the `lc2` home-page.

5 Further research

We see two immediate directions for further research. The first is to create a tableau-like system for modal logic of relations of dimension higher than 2. In [9] it is explained why a straightforward extension of the present system would not guarantee termination. This makes this project interesting from both a theoretical and a practical point of view. The second point is more practically oriented and only of importance if the present system will be used in practice. Clearly the system can be optimized tremendously, especially if information about the input formulas is available (for instance, if we use it to decide satisfaction of a modal K formula we do not need branching for the diamond rule). We expect that many of the optimization techniques from the description logic and the modal tableau theorem proving world are applicable to the system presented here.

References

[1] H. Andréka, I. Hodkinson, and I. Németi. Finite algebras of relations are representable on finite sets. *Journal of Symbolic Logic*, 64:243–267, 1999.

[2] H. Andréka, J. van Benthem, and I. Németi. Back and forth between modal logic and classical logic. *Bulletin of the Interest Group in Pure and Applied Logics*, 3(3):685–720, 1995.

[3] H. Andréka, J. van Benthem, and I. Németi. Modal languages and bounded fragments of predicate logic. *Journal of Philosophical Logic*, 27(3):217–274, 1998.

[4] B. Beckert, R. Goré. URL: http://i12www.ira.uka.de/modlean/

[5] M. Fitting. *First Order Logic and Automated Theorem Proving*. Springer Verlag, 1990.

[6] R. Hirsch, I. Hodkinson, M. Marx, S. Mikulás, and M. Reynolds. *Mosaic and step-by-step. Remarks on "A modal logics of relations"*. In E. Orłowska, editor, *Logic at Work*, pages 158–167. Springer Verlag, 1999.

[7] M. Marx. Complexity of modal logics of relations. Technical Report ML–97–02, Institute for Logic, Language and Computation, University of Amsterdam, May 1997.

[8] M. Marx and Y. Venema. Multi-dimensional Modal Logic. Applied Logic Series. Kluwer Academic Publishers, 1997.

[9] M. Marx, S. Mikulás, S. Schlobach. Labelled deduction for the guarded fragment. In D. Basin, M. D'Agostino, D. M. Gabbay, S. Matthews and L. Viganò, editors, *Labelled Deduction*. Kluwer Academic Publishers, to appear.

[10] S. Mikulás. Taming first-order logic. *Logic Journal of the IGPL*, 6(2):305–316, 1998.

[11] J. van Benthem. *Exploring Logical Dynamics*. Studies in Logic, Language and Information. CSLI Publications, Stanford, 1996.

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