8 Basics of Modal Logic

Modal logic subsumes propositional logic. It allows us to speak about not just one world (a specific valuation of atomic formulas) but about "possible worlds" (different valuations of atomic formulas) and their relationships (whether a possible world is an "alternative" of another one).

8.1 Syntax

Formulas are built up from atomic formulas using the connectives $\wedge$ and $\neg$ as in the case of propositional logic, but we have an additional unary connective $\Box$:

- $\Box\varphi$ is a formula for any formula φ.

The intuition is that $\Box\varphi$ holds at the current world if φ holds in all alternative worlds — see the semantics below. We will use the abbreviation $\Diamond\varphi$ for $\neg\Box\neg\varphi$.

Recall that $\mathrm{Sf}(\rho)$ denotes the set of *subformulas* of a formula ρ. $\mathrm{Sf}(\rho)$ can be defined inductively:

- $\mathrm{Sf}(p) = \{p\}$ for atomic p,
- $\mathrm{Sf}(\neg\varphi) = \mathrm{Sf}(\varphi) \cup \{\neg\varphi\}$,
- $\mathrm{Sf}(\varphi \wedge \psi) = \mathrm{Sf}(\varphi) \cup \mathrm{Sf}(\psi) \cup \{\varphi \wedge \psi\}$,
- $\mathrm{Sf}(\Box\varphi) = \mathrm{Sf}(\varphi) \cup \{\Box\varphi\}$.

Exercise 8.1 Show that a formula φ has at most $|\varphi|$ many subformulas, where $|\varphi|$ stands for the number of characters occurring in φ (as a string). **Hint: use formula induction.**

8.2 Semantics

A *(Kripke) frame* is a pair $\mathcal{F} = (S, R)$, where the set of *possible worlds* S is a non-empty set and the *accessibility relation* R is a binary relation on S (i.e., $R \subseteq S \times S$).

A *model* on a frame $\mathcal{F} = (S, R)$ is a triple $\mathcal{M} = (S, R, V)$ with $V : P \to \mathcal{P}(S)$. Hence V assigns to each atomic formula $p \in P$ a subset $V(p)$ of S — those worlds at which p is "true".

Let $\mathcal{M} = (S, R, V)$ be a model and $s \in S$ be a world; we define ρ *is true in* $\mathcal{M} = (S, R, V)$ *at* s, in symbols

$$(\mathcal{M}, s) \models \rho$$

as follows.

$$\begin{array}{lll}
(\mathcal{M}, s) \models p & \text{iff} & s \in V(p) \\
(\mathcal{M}, s) \models \neg\varphi & \text{iff} & \text{not } (\mathcal{M}, s) \models \varphi \quad (\text{i.e., } (\mathcal{M}, s) \not\models \varphi) \\
(\mathcal{M}, s) \models (\varphi \wedge \psi) & \text{iff} & (\mathcal{M}, s) \models \varphi \text{ and } (\mathcal{M}, s) \models \psi \\
(\mathcal{M}, s) \models \Box\varphi & \text{iff} & \text{for all } t \in S \text{ such that } sRt, \text{ we have } (\mathcal{M}, t) \models \varphi
\end{array}$$

Note that

$$(\mathcal{M}, s) \models \Diamond\varphi \text{ iff there exists } t \in S \text{ with } sRt \text{ and } (\mathcal{M}, t) \models \varphi.$$

Prove it!

A formula φ is *true in model* $\mathcal{M} = (S, R, V)$, denoted $\mathcal{M} \models \varphi$, if it is true at all worlds in $\mathcal{M}$ (i.e., $(\mathcal{M}, s) \models \varphi$ for all $s \in S$). Note that the meaning of a formula in a model (i.e., those possible worlds that satisfy the formula) is determined by the meaning of its subformulas (and the accessibility relation).

A formula φ is *valid in frame* $\mathcal{F} = (R, S)$, denoted $\mathcal{F} \models \varphi$, if it is true in all models $\mathcal{M} = (S, R, V)$ based on $\mathcal{F}$.

If $\mathcal{C}$ is a class of frames (respectively, models), then φ is *valid* (respectively, *true*) *in* $\mathcal{C}$, denoted as $\mathcal{C} \models \varphi$, if φ is valid (respectively, true) in all elements of $\mathcal{C}$. Truth and validity are defined for sets of formulas in the obvious way.

8.3 Frame conditions

Different applications require different versions of modal logic. For instance, we can require that the set of possible worlds is a certain fixed set (e.g., the set of natural numbers) and that the accessibility relation satisfies certain conditions (e.g., transitivity). In particular, the modal logic of all frames is denoted by **K** and the modal logic of all frames with equivalence accessibility relation by **S5**.

Consider the following *frame conditions*:

reflexivity: $\forall s\ sRs$

symmetry: $\forall s, t(sRt \to tRs)$

transitivity: $\forall s, t, u[(sRt \wedge tRu) \to sRu]$.

Now look at these formulas:

refl: $\Box\varphi \to \varphi$

symm: $\varphi \to \Box\Diamond\varphi$

trans: $\Box\varphi \to \Box\Box\varphi$.

Example 8.2 Show that in transitive frames the formula $\Box\varphi \to \Box\Box\varphi$ is valid.

Solution: Let $\mathcal{F} = (S, R)$ be an arbitrary transitive frame, i.e., for every x, y and z, we have that xRy and yRz imply xRz. Let $\mathcal{M} = (S, R, V)$ be a model on $\mathcal{F}$ defined by the valuation $V : P \to \mathcal{P}(S)$. We have to prove that, for every world $s \in S$, we have

$$(\mathcal{M}, s) \models \Box\varphi \to \Box\Box\varphi.$$

Take a world $s \in S$ and assume that $(\mathcal{M}, s) \models \Box\varphi$. We have to show that $(\mathcal{M}, s) \models \Box\Box\varphi$, which means that, for every $t \in S$, we have

$$sRt \text{ implies } (\mathcal{M}, t) \models \Box\varphi$$

or, in other words, that for every t and u in S, we have

$$sRt \text{ implies } (tRu \text{ implies } (\mathcal{M}, u) \models \varphi).$$

So, suppose sRt. Then if tRu, we have sRu by transitivity. Then we have $(\mathcal{M}, u) \models \varphi$, since $(\mathcal{M}, s) \models \Box\varphi$ by assumption.

Example 8.3 Show that the formula $\Box\varphi \to \Box\Box\varphi$ is not valid in non-transitive frames.

Solution: Let $\mathcal{F} = (S, R)$ be an arbitrary frame where R is not transitive, say, we have $s, t, u \in S$ such that sRt and tRu but not sRu. Let p be a propositional atom and V be a valuation such that $u \notin V(p)$ but $v \in V(p)$ for all $v \in S$ such that sRv. If we let $\mathcal{M} = (S, R, V)$, then $(\mathcal{M}, s) \models \Box\varphi$ but $(\mathcal{M}, s) \not\models \Box\Box\varphi$.

Exercise 8.4 Show that each frame condition above is defined by the corresponding formula. That is, show that for any frame, the accessibility relation is reflexive, symmetric, and transitive if and only if, respectively, *refl*, *symm*, and *trans* is valid in the frame.

8.4 Model checking

Given a finite model $\mathcal{M} = (S, R, V)$ and a formula χ, we are interested in precisely which worlds $s \in S$ make the formula χ true.

First we define the *modal depth*, $d(\varphi)$ of the formula φ:

- $d(p) = 0$ for atomic p,
- $d(\neg\varphi) = d(\varphi)$,
- $d(\varphi \wedge \psi) = \max\{d(\varphi), d(\psi)\}$,
- $d(\Box\varphi) = d(\varphi) + 1$.

Let X be the set of subformulas of a fixed formula χ, and let n be the modal depth $d(\chi)$ of χ. We define labels $\ell_i(s)$ for every $s \in S$ and $0 \leq i \leq n$ such that

$$\ell_i(s) = \{\varphi \in X : d(\varphi) \leq i \ \& \ (\mathcal{M}, s) \models \varphi\}$$

Let $X_i = \{\varphi \in X : d(\varphi) = i\}$.

It is easy to compute $\ell_0(s)$: take those elements of X_0 which are (propositionally) true at s. In more detail: first take those atomic propositions p which are true at s (i.e., $s \in V(p)$); this determines which propositional combinations (using the connectives $\wedge$ and $\neg$) from X_0 of atomic propositions are true at s.

Given $\ell_i(s)$ for every $s \in S$, we can compute $\ell_{i+1}(s)$ for every $s \in S$ as follows. Consider

$$\ell_i(s) \cup \{\Box\varphi \in X_{i+1} : \varphi \in \ell_i(t) \text{ for every } t \in S \text{ such that } sRt\}$$

Assuming that these formulas are true at s, we can compute those propositional combinations from X_{i+1} of these formulas which must be true at s (see the computation of $\ell_0(s)$ above). This defines ℓ_{i+1}.

Lemma 8.5 *For every $\varphi \in X_n$ and $s \in S$,*

$$\varphi \in \ell_n(s) \Leftrightarrow (\mathcal{M}, s) \models \varphi$$

Proof: By induction on the modal depth of φ. ∎

Exercise 8.6 Perform model checking for

- $\chi = \Box(p \wedge \neg\Box q)$ and $\mathcal{M} = (S, R, V)$ where $S = \{s_0, s_1, s_2, s_3\}$, $R = \{(s_0, s_1), (s_0, s_3), (s_1, s_2), (s_3, s_3)\}$ and $V(p) = \{s_1, s_3\}$, $V(q) = \{s_2, s_3\}$.

8.5 Mosaic method for modal logic

Let ξ be a formula of modal logic and let $X = \pm\mathrm{Sf}(\xi) = \{\varphi, \neg\varphi : \varphi \in \mathrm{SF}(\xi)\}$ the set of subformulas of ξ closed under single negation.

Definition 8.7 *Let $G = (U, E, \ell)$ be a labeled, directed graph (LDG for short), i.e., U is a set of* nodes, *$E \subseteq U \times U$ is the set of* directed edges *and $\ell : E \to \mathcal{P}(X)$ is a* labeling function. *We say that G is* coherent *if it satisfies the following* coherency *conditions: for every $u, v \in U$ and formula in X,*

Coh1 $\varphi \in \ell(u) \iff \neg\varphi \notin \ell(u)$

Coh2 $\varphi \wedge \psi \in \ell(u) \iff \varphi, \psi \in \ell(u)$

Coh3 $\Box\varphi \in \ell(u)$ *and* $(u, v) \in E \Rightarrow \varphi \in \ell(v)$

By a mosaic *we mean a coherent LDG such that $|U| \leq 2$.*

Definition 8.8 *By a* saturated set of mosaics, *an SSM for short, we mean a set M of mosaics satisfying the following* saturation *condition: for every $\mu = (U_\mu, E_\mu, \ell_\mu) \in M$ and $u \in U_\mu$,*

Sat1 $\Diamond\varphi \in \ell_\mu(u) \Rightarrow$ *there are* $\nu = (U_\nu, E_\nu, \ell_\nu) \in M$, $u', v' \in U_\nu$ *such that* $\ell_\mu(u) = \ell_\nu(u')$, $(u', v') \in E_\nu$ *and* $\varphi \in \ell_\nu(v')$.

A ξ-SSM is an SSM M such that there is $\mu = (U_\mu, E_\mu, \ell_\mu) \in M$ with $\xi \in \ell_\mu(u)$ for some $u \in U_\mu$.

Lemma 8.9 *ξ is satisfiable in* $\mathbf{K} \iff$ *there is a ξ-SSM.*

Proof: First assume that ξ is satisfiable. Let $\mathcal{M} = (S, R, V)$ be a model and $s \in S$ such that $(\mathcal{M}, s) \models \xi$. For every $t \in S$, define $\ell(t) = \{\varphi \in X : (\mathcal{M}, t) \models \varphi\}$. Note that $\xi \in \ell(s)$. Let

$$M = \{(\{t, t'\}, \{t, t'\} \times \{t, t'\} \cap R, \ell) : t, t' \in S \text{ and } tRt'\}$$

It is routine to check that every element of M satisfies the coherency conditions in Definition 8.7 and that M satisfies the saturation condition in Definition 8.8. Hence M is a ξ-SSM as required.

Now assume that M is a ξ-SSM. We will define a model satisfying ξ by "glueing" together elements of M. To this end we define a LDG $G = (U, E, \ell)$. Let us call $(u, \Diamond\varphi)$ a *defect* of a LDG $G = (U, E, \ell)$ if $u \in U$, $\Diamond\varphi \in \ell(u)$ and there is no $v \in U$ such that $\varphi \in \ell(v)$ and $(u, v) \in E$.

Initial step 0**:** Since M is a ξ-SSM, there is $\mu = (U_\mu, E_\mu, \ell_\mu) \in M$ and $u \in U_\mu$ such that $\xi \in \ell_\mu(u)$. We define $G_0 = (U_0, E_0, \ell_0)$ by $U_0 = U_\mu$, $E_0 = E_\mu$ and $\ell_0 = \ell_\mu$. It is straightforward that G_0 is coherent.

Successor step $k+1$**:** Assume that we have defined a finite, coherent LDG $G_k = (U_k, E_k, \ell_k)$ such that G_k is a union of members of M, $U_k = \bigcup_{i<k} U_i$, $E_k = \bigcup_{i<k} E_i$ and $\ell_k = \bigcup_{i<k} \ell_i$. Also assume that there is no defect $(u, \Diamond\varphi)$ of G_k such that $u \in U_{k-1}$. (That is, all the defects occur in $U_k \smallsetminus U_{k-1}$.)

For each defect $(u, \Diamond\varphi)$ of G_k, we will do the following extension of G_k. Since $u \in U_k \smallsetminus U_{k-1}$, and G_k is a union of mosaics from M, there is $v \in U_{k-1}$ such that the structure $(\{v, u\}, \{v, u\} \times \{v, u\} \cap E_k, \ell_k) \in M$. Since M is saturated, there are a mosaic $\nu = (U_\nu, E_\nu, \ell_\nu) \in M$ and points $u', w \in U_\nu$ such that $\ell_k(u) = \ell_\nu(u')$, $(u', w) \in E_\nu$ and $\varphi \in \ell_\nu(w)$. We extend U_k by a node w, E_k by (u, w) and label w with the label $\ell_\nu(w)$.

We define $G_{k+1} = (U_{k+1}, E_{k+1}, \ell_{k+1})$ as the union of the above extensions of G_k when we go through all the defects of G_k. (If there is no defect in G_k, then we let $G_{k+1} = G_k$.) Since both G_k and each label $\ell_k(u)$ are finite, G_{k+1} is a finite LDG. Obviously G_{k+1} is a union of elements of M, hence it is coherent. Since we "cured" all the defects $(u, \Diamond\varphi)$ of G_k by providing the necessary witness nodes $w \in U_{k+1}$ such that $(u, w) \in E_{k+1}$ and $\varphi \in \ell_{k+1}(w)$, any defect $(v, \Diamond\psi)$ of G_k has the property $v \in E_{k+1} \smallsetminus E_k$.

Limit step ω Define G_ω as the union of all G_i ($i \in \omega$):

$$U_\omega = \bigcup_{i\in\omega} U_i \quad E_i = \bigcup_{i\in\omega} E_i \quad \ell_\omega = \bigcup_{i\in\omega} \ell_i$$

We define G as G_ω. Obviously, G is a coherent LDG. It also satisfies the following saturation condition:

$$\Diamond\varphi \in \ell(u) \text{ implies that there is } v \in U \text{ such that } (u,v) \in E \text{ and } \varphi \in \ell(v). \quad (3)$$

Indeed, let $u \in U$ and $\Diamond\varphi \in \ell(u)$. Since u has been added to U during the above construction, there is $k \in \omega$ such that $u \in U_k \smallsetminus U_{k-1}$ and $\Diamond\varphi \in \ell_k(u)$. If $(u, \Diamond\varphi)$ was a defect of G_k, then it was cured in step $k+1$ of the construction, hence it is not a defect of G.

We define the model $\mathcal{M} = (U, E, V)$ where V is given by

$$V(p) = \{u \in U : p \in \ell(u)\}$$

for every propositional atom p occurring in ξ. Using that G is a coherent and saturated LDG, an easy induction on formulas yields

$$(\mathcal{M}, u) \models \varphi \iff \varphi \in \ell(u) \quad (4)$$

for every $\varphi \in X$. In particular, $(\mathcal{M}, w) \models \xi$ as desired. ∎

Corollary 8.10 **K** *has the finite model property, i.e., every satisfiable formula can be satisfied in a finite model. Furthermore, there is a bound (in terms of the length of the formula) on the size of the model.*

Proof: In the second part of the proof of Lemma 8.9, construct the model as follows. Let U be the disjoint union of the nodes of the mosaics in M. Every node u has the same label $\ell(u)$ as in the mosaic in M that contains u. Define the set of edges E as follows: for $u, v \in U$ let $(u, v) \in E$ iff $(\{u, v\}, \{(u, v)\}, \ell)$ satisfies the coherency conditions in Definition 8.7. Then it is straightforward that G is a coherent LDG. Saturation can be easily shown by using the saturation condition for M and the definition of E above. The rest of the proof is unchanged. Note that in this case the constructed model is finite. ∎

Exercise 8.11

1. Prove the equivalence (4).
2. Compute a bound on the number of mosaics for a given formula ξ and on the size of the model $\mathcal{M}$ in the proof of Corollary 8.10.

Exercise 8.12 Prove Lemma 8.9 and Corollary 8.10 for modal logic **S5**.

Hint: In Definition 8.7, make the additional requirements that $E = U \times U$ and that the following additional coherency condition holds: for every $u, v \in U$,

Coh4 for every $\Box\varphi \in X$, we have $((\Box\varphi \in \ell(u) \text{ and } (u, v) \in E) \Rightarrow \Box\varphi \in \ell(v))$

In the proof of Corollary 8.10, construct the model as follows. Let U be the disjoint union of the nodes of the mosaics in M. Every node u has the same label $\ell(u)$ as in the mosaic in M that contains u. Define the set of edges E as follows: for $u, v \in U$ let $(u, v) \in E$ iff $(\{u, v\}, \{(u, u), (u, v), (v, u), (u, u)\}, \ell)$ satisfies the coherency conditions. Then it is straightforward that G is a coherent LDG. Saturation can be easily shown by using the saturation condition for M and the definition of E above. The rest of the proof is unchanged. Note that in this case the constructed model is finite.

Theorem 8.13 *The satisfiability problem for both* **K** *and* **S5** *is decidable.*

Proof: By Lemma 8.9 it is enough to check whether there exists a ξ-SSM for a given formula ξ. Since there is a bound on the number of mosaics, there is a limited number of sets of mosaics. Then one can check whether at least one of the subsets of all mosaics satisfies the saturation condition in Definition 8.8.

Alternatively, using the upper bound from Corollary 8.10 it is enough to check whether the formula is true in finitely many finite models. ∎