DOMAIN AND RANGE FOR ANGELIC AND DEMONIC COMPOSITIONS

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Abstract. We give finite axiomatizations for the varieties generated by representable domain–range algebras when the semigroup operation is interpreted as angelic or demonic composition, respectively.

1. Introduction

Any formal approach to modelling programs must encompass both logic and action. On the one hand, the role of programs is to create and change input: an action on the state space. On the other hand, the technical action of programs requires conditional tests that are logical in nature. A common algebraic formalism for this is to model programs as relations on the state space and use restrictions of the identity relation to model logical propositions. This is elegantly argued in the articles [13] and [29], where it is observed that Kozen's axiom system KAT (Kleene algebra with tests) and the program logic PDL (Propositional Dynamic Logic) can be unified by enriching the language of KAT with the introduction of unary operations modelling the domain and range of relations. This enables the modal operations of dynamic logic to be precisely captured in a one-sorted algebraic setting.

These and other articles provide simple axiomatic systems that are sound for the relational program semantics and which are sufficient to reason about many important aspects of programs. Completeness of these axioms seems more elusive. Work involving the first and second authors showed that no finite system of axioms is sufficient to capture the full first order theory of the algebra of relations under composition with domain and/or range (amongst other operations) [18, 23]. This fact is just one of a swathe of negative results relating to the theory of binary relations. For signatures involving composition, intersection and either of union or converse, not only is there no finite system of laws, but no complete system of laws can be recursively decided on finite algebras; [30, Theorem 2.5] and [16, Theorem 8.1]. Despite these negative results, in many situations it is sufficient to find systems that are complete for equations (rather than the full first order theory). Kozen's system KA [25] is precisely one such system that is complete for the equational theory of relations under composition, union and reflexive transitive closure; others include Andréka [1] and Bredkhin [2]. One of the main results of the present article is to provide a relatively simple (and finite) system of equational

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axioms that is sound and complete for the equational theory of the operations of domain, range, composition and relational union: Theorem 3.1 below.

All of the above references consider program composition modelled as conventional composition of binary relations. The validity of this approach depends on how one is to model nondeterminism: if programs are to be modelled as binary relations on the state space, then how should non-termination of programs be modelled? In particular, what if some computations terminate and some do not? The most common approach is the so-called angelic model. In the angelic model of a program p as a relation on the state space, we consider p to relate state x to state y if amongst the possible computations of p when started at x is one that terminates at y. This does not preclude the possibility that some computations of p at x do not terminate. A stricter model—the demonic model [31]—requires in addition (to some computations of p at x leading to y), that *all computations of* p *at* x *eventually terminate*.

When programs are to be modelled angelically, then the relation associated to the composite of the programs p and q is just the usual relational composite of the relations corresponding to p and q. When programs are modelled demonically, then the relation associated to the composite of the programs is the demonic composition of the relations associated to p and q. Demonic composition as a binary operation on binary relations remains associative (see [3] or [5, §5] for example), but its general algebraic properties have seen far less algebraic consideration than its angelic counterpart. A variant of the program logic PDL is developed and shown to be complete and decidable in [11], while in [12] it is observed that the algebra of binary relations under demonic composition and domain is indistinguishable from the algebra of partial maps under conventional (angelic) composition and domain. This latter system has been well-studied and has a well-known complete equational axiomatisation [37, 20, 27]. The algebraic properties of demonic relations and partial maps diverge once range information is incorporated. The second main result of the present article is to find a simple axiomatic system that is sound and complete for the equational theory of demonic composition of relations with domain and range; Theorem 4.1.

Program modelling is just one of several motivations for this work. Domain and range are already transparently natural features of both relations and functions, and the modelling of these via unary operations can be traced back at least to the work of Menger [28], through the work of Schweizer and Sklar [33, 34, 35, 36], Trokhimenko [37], Bredikhin [6] and Schein [32] and into the work of the authors and their collaborators [17, 20, 21, 22] as as in the category-theoretic work of Cockett, Lack, Guo, Hofstra, Manes amongst others [7, 8, 9, 10]. Yet another motivation comes from the structural theory of semigroups, where many authors have enriched the usual associative binary multiplication by the addition of unary operations that map onto idempotent elements; see for example Fountain [14, 15], the work of Batbedat [4], Lawson [26] and many others; a survey on aspects of this theme of research can be found in Hollings [19]. The axiom systems we investigate here appear as natural cases in this purely theoretical context.

2. Basics

Definition 2.1. Let U be a set. We define operations on elements of $\wp(U \times U)$.

Domain:

$$\mathsf{D}(X) = \{(u,u) \mid (u,v) \in X \text{ for some } v \in U\}$$

Range:

$$\mathsf{R}(X) = \{(v,v) \mid (u,v) \in X \text{ for some } u \in U\}$$

Angelic composition:

$$X \mathbin{;} Y = \{(u,v) \mid (u,w) \in X \text{ and } (w,v) \in Y \text{ for some } w \in U\}$$

Demonic composition:

$$\begin{aligned} X * Y = \{(u,v) \mid &\text{ for some } w \in U, (u,w) \in X \text{ and } (w,v) \in Y, \\ &\text{and for all } w \in U \text{ such that } (u,w) \in X, (w,w) \in \mathsf{D}(Y)\} \end{aligned}$$

for every $X, Y \subseteq U \times U$.

The class $\mathbb{R}(;, \mathsf{D}, \mathsf{R})$ of *angelic domain–range semigroups* is

$$\mathbb{IS}\{(\wp(U \times U), ;, \mathsf{D}, \mathsf{R}) \mid U \text{ a set}\}$$

while the class $\mathbb{R}(*, \mathsf{D}, \mathsf{R})$ of *demonic domain–range semigroups* is

$$\mathbb{IS}\{(\wp(U \times U), *, \mathsf{D}, \mathsf{R}) \mid U \text{ a set}\}$$

where $\mathbb{I}$ and $\mathbb{S}$ denote isomorphic copies and subalgebras, respectively.

We may call elements of $\mathbb{R}(;, \mathsf{D}, \mathsf{R})$ and $\mathbb{R}(*, \mathsf{D}, \mathsf{R})$ *representable algebras*. In this paper we give finite equational axiomatizations to the equational theories of representable domain–range algebras.

3. Angelic composition

In this section we look at angelic composition. We expand the signature of angelic domain–range semigroups with a join operation $+$ that is interpreted as union, and define $\mathbb{R}(;, \mathsf{D}, \mathsf{R}, +)$ as

$$\mathbb{IS}\{(\wp(U \times U), ;, \mathsf{D}, \mathsf{R}, +) \mid U \text{ a set}\}.$$

Let Ax^a be the following set of equations:

$$x \mathbin{;} (y \mathbin{;} z) = (x \mathbin{;} y) \mathbin{;} z \tag{1}$$
$$\mathsf{D}(x) \mathbin{;} x = x \tag{2}$$
$$x \mathbin{;} \mathsf{R}(x) = x \tag{3}$$
$$\mathsf{D}(x) \mathbin{;} \mathsf{D}(x) = \mathsf{D}(x) \tag{4}$$
$$\mathsf{R}(x) \mathbin{;} \mathsf{R}(x) = \mathsf{R}(x) \tag{5}$$
$$\mathsf{D}(x \mathbin{;} y) = \mathsf{D}(x \mathbin{;} \mathsf{D}(y)) \tag{6}$$
$$\mathsf{R}(x \mathbin{;} y) = \mathsf{R}(\mathsf{R}(x) \mathbin{;} y) \tag{7}$$
$$\mathsf{D}(\mathsf{D}(x) \mathbin{;} y) = \mathsf{D}(x) \mathbin{;} \mathsf{D}(y) \tag{8}$$
$$\mathsf{R}(x \mathbin{;} \mathsf{R}(y)) = \mathsf{R}(x) \mathbin{;} \mathsf{R}(y) \tag{9}$$
$$\mathsf{D}(\mathsf{R}(x)) = \mathsf{R}(x) \tag{10}$$
$$\mathsf{R}(\mathsf{D}(x)) = \mathsf{D}(x) \tag{11}$$
$$\mathsf{D}(x) \mathbin{;} \mathsf{D}(y) = \mathsf{D}(y) \mathbin{;} \mathsf{D}(x) \tag{12}$$
$$\mathsf{R}(x) \mathbin{;} \mathsf{R}(y) = \mathsf{R}(y) \mathbin{;} \mathsf{R}(x) \tag{13}$$
$$\mathsf{D}(x) \mathbin{;} y \leq y \tag{14}$$
$$x \mathbin{;} \mathsf{R}(y) \leq x \tag{15}$$

together with the axioms stating that join + is a semilattice (idempotent, commutative and associative) operation and that the operations are additive:

$$x \mathbin{;} (y + z) = x \mathbin{;} y + x \mathbin{;} z \tag{16}$$
$$(x + y) \mathbin{;} z = x \mathbin{;} z + y \mathbin{;} z \tag{17}$$
$$\mathsf{D}(x + y) = \mathsf{D}(x) + \mathsf{D}(y) \tag{18}$$
$$\mathsf{R}(x + y) = \mathsf{R}(x) + \mathsf{R}(y) \tag{19}$$

There was no attempt made to make the above axiom system independent. Instead we aimed for symmetry and stated both the "domain" and "range" versions of the axioms. The laws (1)–(13) can very easily be shown equivalent to the *closure semigroups with the left and right congruence conditions* in the sense of the first author and Stokes [20]. The *adequate semigroups* of Fountain [14] form a particularly well-studied special case; see Kambites [24] for example.

Our main result about angelic composition is the following finite axiomatizability theorem.

Theorem 3.1. *The variety* $\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +)$ *generated by the representation class* $\mathbb{R}(;, \mathsf{D}, \mathsf{R}, +)$ *is axiomatized by* Ax^a.

Proof. Our task is to show that, for all $(;, \mathsf{D}, \mathsf{R}, +)$-terms s, t,

$$\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s = t \text{ iff } Ax \vdash s = t$$

where $\vdash$ denotes derivability in equational logic.

The right-to-left direction follows by the validity of the axioms, which can be easily checked. For the other direction we have to show that $\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s = t$ implies $Ax^a \vdash s = t$. In fact we will show its contrapositive: we will assume that $Ax^a \not\vdash s = t$ and construct a representable algebra $\mathcal{A} \in \mathbb{R}(;, \mathsf{D}, \mathsf{R}, +)$ such that $\mathcal{A} \not\models s = t$. The rest of this section is devoted to this task. □

3.1. **Elementary properties.** We start with the following easy consequences of the axioms. First, $\leq$ is indeed an ordering:

$$x \leq x \tag{20}$$
$$x \leq y \ \&\ y \leq x \Rightarrow x = y \tag{21}$$
$$x \leq y \ \&\ y \leq z \Rightarrow x \leq z \tag{22}$$

Using the additivity of the operations we get that the operations are monotonic w.r.t. $\leq$:

$$x \leq x' \Rightarrow \mathsf{D}(x) \leq \mathsf{D}(x') \tag{23}$$
$$x \leq x' \Rightarrow \mathsf{R}(x) \leq \mathsf{R}(x') \tag{24}$$
$$x \leq x' \ \&\ y \leq y' \Rightarrow x\,;y \leq x'\,;y' \tag{25}$$

Let $\mathcal{A} = (A, ;, \mathsf{D}, \mathsf{R}, +)$ be a model of Ax^a. We extend the operations to sets of elements in the obvious way:

$$\mathsf{D}(X) = \{\mathsf{D}(x) \mid x \in X\}$$
$$\mathsf{R}(X) = \{\mathsf{R}(x) \mid x \in X\}$$
$$X\,;Y = \{x\,;y \mid x \in X, y \in Y\}$$

for every $X, Y \subseteq A$. In particular, we define the set $\mathsf{D}(A)$ of *domain elements* of $\mathcal{A}$ as

$$\mathsf{D}(A) = \{\mathsf{D}(a) \mid a \in A\} = \{a \in A \mid \mathsf{D}(a) = a\}$$

Observe that range elements (defined analogously) coincide with domain elements, since $\mathsf{D}(\mathsf{R}(a)) = \mathsf{R}(a)$ and $\mathsf{R}(\mathsf{D}(a)) = \mathsf{D}(a)$.

Claim 3.2. *Let $\mathcal{A}$ be a model of Ax^a.*

(1) *The algebra $(\mathsf{D}(A), ;)$ of domain elements is a (lower) semilattice and the semilattice ordering coincides with $\leq$.*
(2) *For every $a \in A$, $\mathsf{D}(a)$ (resp. $\mathsf{R}(a)$) is the minimal element d in $\mathsf{D}(A)$ such that $d\,;a = a$ (resp. $a\,;d = a$).*
(3) *For every $a \in A$ and $d, e \in \mathsf{D}(A)$, we have $d\,;a\,;e \leq a$.*

Proof. 1. By (8) the set of domain elements is closed under ;, which is an associative (1), idempotent (2) and commutative (12) operation on domain elements. The semilattice ordering is defined by

$$\mathsf{D}(x) \leq' \mathsf{D}(y) \text{ iff } \mathsf{D}(x)\,;\mathsf{D}(y) = \mathsf{D}(x)$$

and we claim that this is equivalent to the definition of $\mathsf{D}(x) \leq \mathsf{D}(y)$ by $\mathsf{D}(x) + \mathsf{D}(y) = \mathsf{D}(x)$. Assuming $\mathsf{D}(x)\,;\mathsf{D}(y) = \mathsf{D}(x)$ we have $\mathsf{D}(x) + \mathsf{D}(y) = \mathsf{D}(x)\,;\mathsf{D}(y) + \mathsf{D}(y) = \mathsf{D}(y)$, since $\mathsf{D}(x)\,;\mathsf{D}(y) \leq \mathsf{D}(y)$ (by (14)). Assuming $\mathsf{D}(x) + \mathsf{D}(y) = \mathsf{D}(y)$ we get

$$\begin{aligned} \mathsf{D}(x)\,;\mathsf{D}(y) &= \mathsf{D}(x)\,;(\mathsf{D}(x) + \mathsf{D}(y)) \\ &= \mathsf{D}(x)\,;\mathsf{D}(x) + \mathsf{D}(x)\,;\mathsf{D}(y) && \text{by (16)} \\ &= \mathsf{D}(x) + \mathsf{D}(x)\,;\mathsf{D}(y) && \text{by (2)} \\ &= \mathsf{D}(x) \end{aligned}$$

since $\mathsf{D}(x) \geq \mathsf{D}(x)\,;\mathsf{R}(\mathsf{D}(y)) = \mathsf{D}(x)\,;\mathsf{D}(y)$ by (15) and (11).

2. Assume that $\mathsf{D}(d) \mathbin{;} a = a$. Then

$$\begin{aligned}\mathsf{D}(a) &= \mathsf{D}(\mathsf{D}(d) \mathbin{;} a)\\ &= \mathsf{D}(d) \mathbin{;} \mathsf{D}(a) && \text{by (8)}\\ &\le \mathsf{D}(d)\end{aligned}$$

as desired. The proof of the statement about range is analogous.

3. Straightforward by (14) and (15). □

3.2. **Term graphs.** Let T^-_{Var} be the set of $(;, \mathsf{D}, \mathsf{R})$-terms generated by the set of variables *Var*. We will refer to these as join-free terms. We will adopt the concept of term graphs from Andréka and Bredikhin [2] to T^-_{Var}. The operations D and R are not considered explicitly in [2], but the concepts and proofs are easily modified to cover these as well.

A *labelled graph* is a structure $G = (V, E)$ where V is a set of vertices and $E \subseteq V \times Var \times V$ is a set of labelled edges. Given two labelled graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, a *homomorphism* $h\colon G_1 \to G_2$ is a map from V_1 to V_2 that preserves labelled edges: if $(u, x, v) \in E_1$, then $(h(u), x, h(v)) \in E_2$. Given an equivalence relation θ on V, the *quotient graph* is $G/\theta = (V/\theta, E/\theta)$ where V/θ is the set of equivalence classes of V and

$$E/\theta = \{(u/\theta, x, v/\theta) : (u, x, v) \in E \text{ for some } u \in u/\theta \text{ and } v \in v/\theta\}$$

A *2-pointed graph* is a labelled graph $G = (V, E)$ with two (not necessarily distinct) distinguished vertices $\iota, o \in V$. We will call ι the *input* and o the *output* vertex of G, respectively, and denote 2-pointed graphs as $G = (V, E, \iota, o)$. In the case of 2-pointed graphs, we require that a homomorphism preserves input and output vertices as well.

Let $G_1 \oplus G_2$ denote the disjoint union of G_1 and G_2. For 2-pointed graphs $G_1 = (V_1, E_1, \iota_1, o_1)$ and $G_2 = (V_2, E_2, \iota_2, o_2)$, we define their *composition* as

$$G_1 \mathbin{;} G_2 = (((V_1, E_1) \oplus (V_2, E_2))/\theta, \iota_1/\theta, o_2/\theta)$$

where θ is the smallest equivalence relation on the disjoint union V_1 and V_2 that identifies o_1 with ι_2. When no confusion is likely we will identify an equivalence class u/θ with u, hence ι_i/θ with ι_i and o_i/θ with o_i for $i \in \{1, 2\}$.

We define *term graphs* as special 2-pointed graphs by induction on the complexity of terms. For variable x, we choose distinct points ι, o and let

$$G_x = (\{\iota, o\}, \{(\iota, x, o)\}, \iota, o)$$

Let s be a term and assume that $G_s = (V, E, \iota, o)$. We define

$$G_{\mathsf{D}(s)} = (V, E, \iota, \iota) \text{ and } G_{\mathsf{R}(s)} = (V, E, o, o)$$

Finally, for terms s and t, we set

$$G_{s;t} = G_s \mathbin{;} G_t$$

For any term s, we can consider $G_s = (V_s, E_s, \iota_s, o_s)$ as a representable algebra. To this end let $\sharp\colon Var \to \wp(E_s)$ be a valuation of variables such that

$$x^\sharp = \{(u, v) \in V_s \times V_s \mid (u, x, v) \in E_s\}$$

for every variable x occurring in s (notation $x \in s$). We define the representable algebra $\mathcal{G}_s$ as the subalgebra of $(\wp(V_s \times V_s), ;, \mathsf{D}, \mathsf{R}, +)$ generated by $\{x^\sharp \mid x \in s\}$.

The *universe* W_s of $\mathcal{G}_s$ is the reflexive–transitive closure of $\bigcup\{x^\sharp \mid x \in s\}$. Observe that, by the construction of the term graph G_s, W_s is an antisymmetric relation.

We extend $\sharp$ to an interpretation of complex terms in the obvious way:

$$(\mathsf{D}(t))^\sharp = \mathsf{D}(t^\sharp) \quad (\mathsf{R}(t))^\sharp = \mathsf{R}(t^\sharp) \quad (t_1 \,;\, t_2)^\sharp = t_1^\sharp \,;\, t_2^\sharp$$

By an easy induction on the complexity of terms we get the following.

Claim 3.3. *In $\mathcal{G}_s$, we have $(\iota_s, o_s) \in s^\sharp$.*

Next we recall a characterization of validities using graph homomorphisms from [2, Theorem 1].

Theorem 3.4. *The inequality $s \leq t$ is valid in representable algebras iff there is a homomorphism from G_t to G_s.*

Observe that $s \leq t$ implies that all the variables in t must occur in s. The key step in proving Theorem 3.4 is the following lemma, see [2, Lemma 3].

Lemma 3.5. *Let s, t be terms and consider $G_s = (V_s, E_s, \iota_s, o_s)$. Then $(\iota_s, o_s) \in t^\sharp$ iff there is a homomorphism from G_t to G_s.*

3.3. **Eliminating join.** Recall that our task is to show that for any $(;, \mathsf{D}, \mathsf{R}, +)$-terms s, t,

$$\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s = t \text{ implies } Ax^a \vdash s = t.$$

This is obviously equivalent to the statements

$$\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s \leq t \text{ implies } Ax^a \vdash s \leq t,$$
$$\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s \geq t \text{ implies } Ax^a \vdash s \geq t.$$

Thus we assume that $\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s \leq t$ and want to show $Ax^a \vdash s \leq t$.

Next we show that the above can be reduced to join-free terms. Since the operations are additive, every term s can be equivalently written in the form $s_1 + \ldots + s_n$ for some join-free terms $s_1, \ldots, s_n$, whence we have

$$\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s_1 + \ldots + s_n = s \leq t = t_1 + \ldots + t_m$$

for some join-free terms $s_1, \ldots, s_n, t_1, \ldots, t_m$. Thus we have

$$\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s_i \leq t_1 + \ldots + t_m$$

for every $1 \leq i \leq n$. Recall the term graph $G_{s_i} = (V_{s_i}, E_{s_i}, \iota_{s_i}, o_{s_i})$ and let $\sharp\colon \mathit{Var} \to \wp(E_{s_i})$ be a valuation of the variables occurring in $s_i, t_1, \ldots, t_m$ such that $x^\sharp = \{(u, v) \in V_{s_i} \times V_{s_i} \mid (u, x, v) \in E_{s_i}\}$ for the variables x occurring in s_i. Consider the generated algebra $\mathcal{G}_{s_i}$. Since $\mathcal{G}_{s_i}$ is representable, we get

$$\mathcal{G}_{s_i} \models s_i \leq t_1 + \ldots + t_m$$

By Claim 3.3, we get

$$(\iota_{s_i}, o_{s_i}) \in s_i^\sharp$$

whence

$$(\iota_{s_i}, o_{s_i}) \in t_1^\sharp + \ldots + t_m^\sharp$$

and thus

$$(\iota_{s_i}, o_{s_i}) \in t_j^\sharp$$

for some $1 \le j \le m$. By Lemma 3.5, there is a homomorphism from G_{t_j} to G_{s_i}. By Theorem 3.4 we get that $\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s_i \le t_j$. Now if we manage to show that this implies $Ax^a \vdash s_i \le t_j$, then we get

$$Ax^a \vdash s = s_1 + \ldots + s_n \le t_1 + \ldots + t_m = t$$

as desired.

Thus it suffices to show

$$\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s \le t \text{ implies } Ax^a \vdash s \le t$$

for join-free terms s, t.

3.4. **The free algebra.** We will need some properties of the free algebra of the variety defined by Ax^a.

Let *Var* be a countable set of variables and let $\mathcal{F}_{Var} = (F_{Var}, ;, \mathsf{D}, \mathsf{R}, +)$ be the free algebra of the variety defined by Ax^a freely generated by *Var*. Recall that the elements of $\mathcal{F}_{Var}$ are the equivalence classes of terms, where two terms s and t are equivalent if the equation $s = t$ is derivable from Ax^a using equational logic. When we want to emphasize the difference between a term t and its equivalence class, we will write $\bar{t}$ for the latter. The operations in $\mathcal{F}_{Var}$ are defined in the obvious way:

$$\bar{t}\,;\bar{s} = \overline{t\,;s} \quad \mathsf{D}(\bar{t}) = \overline{\mathsf{D}(t)} \quad \mathsf{R}(\bar{t}) = \overline{\mathsf{R}(t)} \quad \bar{t} + \bar{s} = \overline{t + s}$$

and recall that this definition is indeed independent of the choice of terms s, t from the equivalence classes $\bar{s}, \bar{t}$.

Claim 3.6. *Let s, t be join-free terms such that $\mathcal{F}_{Var} \models \mathsf{D}(s) \le t$. Then t is a composition of domain and range terms, i.e., it has the syntactical form $\mathsf{D}(t_1)\,;\mathsf{R}(t_2)\,;\ldots;\mathsf{D}(t_{n-1})\,;\mathsf{R}(t_n)$ for some terms $t_1, t_2, \ldots, t_{n-1}, t_n$ (allowing some, but not all, of the terms being empty).*

Proof. Let s, t be as in the claim. Since the axioms are valid in representable algebras and the term graphs are representable, we get $\mathcal{G}_{\mathsf{D}(s)} \models \mathsf{D}(s) \le t$. Towards a contradiction, assume that t does not have the form $\mathsf{D}(t_1);\mathsf{R}(t_2);\ldots;\mathsf{D}(t_{n-1});\mathsf{R}(t_n)$. Then t can be written in the form $r_1;r_2;\ldots;r_m$ such that at least one r_i is a variable, say, x. Since the universe $W_{\mathsf{D}(s)}$ of $\mathcal{G}_{\mathsf{D}(s)}$ is an antisymmetric relation and there are no loops labelled by variables (edges of the form (u, x, u)) in $G_{\mathsf{D}(s)}$, it follows that $(\iota_{\mathsf{D}(s)}, \iota_{\mathsf{D}(s)}) \notin t^\sharp$. On the other hand, by Claim 3.3 we have $(\iota_{\mathsf{D}(s)}, \iota_{\mathsf{D}(s)}) \in (\mathsf{D}(s))^\sharp$. Thus $\mathcal{G}_{\mathsf{D}(s)} \not\models \mathsf{D}(s) \le t$. $\square$

Claim 3.7. *Let r, s, t be join-free terms such that $\mathcal{F}_{Var} \models \mathsf{D}(r) \le s\,;t$. Then $\mathcal{F}_{Var} \models \mathsf{D}(r) \le s = \mathsf{D}(s)$ and $\mathcal{F}_{Var} \models \mathsf{D}(r) \le t = \mathsf{D}(t)$.*

Proof. By Claim 3.6 we have that $s\,;t$ is a composition of domain and range terms, whence so are both of them. Since domain elements are closed under the operation ; (Claim 3.2), s and t are clearly domain elements when interpreted in the free algebra. Thus $\mathcal{F}_{Var} \models \mathsf{D}(r) \le \mathsf{D}(s)\,;\mathsf{D}(t)$. Since $(\mathsf{D}(F_{Var}), ;)$ is a semilattice with the ordering $\le$ (Claim 3.2), we get $\mathcal{F}_{Var} \models \mathsf{D}(r) \le \mathsf{D}(s) = s$ and $\mathcal{F}_{Var} \models \mathsf{D}(r) \le \mathsf{D}(t) = t$. $\square$

Claim 3.8. *Let s, t be join-free terms such that $\mathcal{F}_{Var} \models s \le \mathsf{D}(t)$. Then s is a composition of domain and range terms, i.e., it has the syntactical form $\mathsf{D}(s_1)\,;\mathsf{R}(s_2)\,;\ldots;\mathsf{D}(s_{n-1})\,;\mathsf{R}(s_n)$ for some terms $s_1, s_2, \ldots, s_{n-1}, s_n$ (allowing some, but not all, of the terms being empty).*

Proof. Let s, t be as in the claim. Since the axioms are valid in representable algebras and the term graphs are representable, we get $\mathcal{G}_s \models s \leq \mathsf{D}(t)$. Towards a contradiction, assume that s does not have the form $\mathsf{D}(s_1);\mathsf{R}(s_2);\ldots;\mathsf{D}(s_{n-1});\mathsf{R}(s_n)$. Then s can be written in the form $r_1;r_2;\ldots;r_m$ such that at least one r_i is a variable, say, x. Since the universe W_s of $\mathcal{G}_s$ is an antisymmetric relation and there are no loops labelled by variables (edges of the form (u, x, u)) in $G_s = G(r_1) ; \ldots ; G(r_m)$, it follows that $\iota_s \neq o_s$. By Claim 3.3 we have $(\iota_s, o_s) \in s^\sharp$. On the other hand, since $\mathcal{G}_s$ is representable, we have $(\iota_s, o_s) \notin (\mathsf{D}(t))^\sharp$. Thus $\mathcal{G}_s \not\models s \leq \mathsf{D}(t)$. □

Claim 3.9. *Let s, t be join-free terms such that $\mathcal{F}_{Var} \models s \leq \mathsf{D}(t)$. Then $\mathcal{F}_{Var} \models s = \mathsf{D}(s)$.*

Proof. By Claim 3.8 we have that s is a composition of domain and range terms. Since domain elements are closed under the operation ; (Claim 3.2), s is clearly a domain element when interpreted in the free algebra. □

3.5. **The construction.** Using the results of the previous sections, we assume that, for some join-free terms s, t, we have $\mathbb{V}(;, \mathsf{D}, \mathsf{R}, +) \models s \leq t$ and we have to show show that this implies $Ax^a \vdash s \leq t$. Actually we will show the contrapositive, so we assume that $Ax^a \not\vdash s \leq t$ and we will construct a representable algebra $\mathcal{A} \in \mathbb{R}(;, \mathsf{D}, \mathsf{R}, +)$ witnessing $s \not\leq t$: $\mathcal{A} \not\models s \leq t$.

Our assumption that $Ax^a \not\vdash s \leq t$ is equivalent to $\mathcal{F}_{Var} \not\models s \leq t$. Instead of $\mathcal{F}_{Var} \models s \leq t$ we will sometimes write $\overline{s} \leq \overline{t}$. Let F^-_{Var} be the set of equivalence classes of join-free terms.

We will define a labelled, directed graph G_ω as the union of a chain of labelled, directed graphs $G_n = (U_n, \ell_n, W_n)$ for $n \in \omega$, where

- U_n is the set of nodes,
- $\ell_n : U_n \times U_n \to \wp(F^-_{Var})$ is a labelling of edges,
- $W_n \subseteq U_n \times U_n$ is a set of *witness edges.*

Define $E_n = \{(u, v) \in U_n \times U_n : \ell_n(u, v) \neq \emptyset\}$ as the set of edges with non-empty labels.

We will make sure that the following *coherence conditions* are maintained during the construction:

GenC: W_n is a reflexive and antisymmetric relation on U_n and its transitive closure is E_n.

PriC: For every $(u, v) \in E_n$, $\ell_n(u, v)$ is a principal upset: $\ell_n(u, v) = a^\uparrow = \{x \in F^-_{Var} \mid a \leq x\}$ for some $a \in F^-_{Var}$.

CompC: For all $(u, v), (u, w), (w, v) \in U_n \times U_n$ and $a, b \in F^-_{Var}$, if $a \in \ell_n(u, w)$ and $b \in \ell_n(w, v)$, then $a ; b \in \ell_n(u, v)$.

DomC: For all $(u, v) \in U_n \times U_n$ and $a \in F^-_{Var}$, if $\ell_n(u, v) = a^\uparrow$, then $\ell_n(u, u) = \mathsf{D}(a)^\uparrow$.

RanC: For all $(u, v) \in U_n \times U_n$ and $a \in F^-_{Var}$, if $\ell_n(u, v) = a^\uparrow$, then $\ell_n(v, v) = \mathsf{R}(a)^\uparrow$.

IdeC: For all $(u, v) \in U_n \times U_n$, $u = v$ iff $\ell_n(u, v) = \mathsf{D}(a)^\uparrow$ for some $a \in F^-_{Var}$.

The construction will terminate in ω steps, yielding $G_\omega = (U_\omega, \ell_\omega, W_\omega)$ where $U_\omega = \bigcup_n U_n$, $\ell_\omega = \bigcup_n \ell_n$, $W_\omega = \bigcup_n W_n$ and we also let $E_\omega = \bigcup_n E_n$. By the end of the construction we will achieve the following *saturation conditions*:

CompS: For all $(u, v) \in U_\omega \times U_\omega$ and $a, b \in F^-_{Var}$, if $a ; b \in \ell_\omega(u, v)$, then $a \in \ell_\omega(u, w)$ and $b \in \ell_\omega(w, v)$ for some $w \in U_\omega$.

DomS: For all $(u,u) \in U_\omega \times U_\omega$ and $a \in F^-_{Var}$, if $\mathsf{D}(a) \in \ell_\omega(u,u)$, then $a \in \ell_\omega(u,w)$ for some $w \in U_\omega$.

RanS: For all $(u,u) \in U_\omega \times U_\omega$ and $a \in F^-_{Var}$, if $\mathsf{R}(a) \in \ell_\omega(u,u)$, then $a \in \ell_\omega(w,u)$ for some $w \in U_\omega$.

Let Σ be a fair scheduling function $\Sigma\colon \omega \to 3 \times \omega \times \omega \times F^-_{Var} \times F^-_{Var}$, i.e., every element of $3 \times \omega \times \omega \times F^-_{Var} \times F^-_{Var}$ appears infinitely often in the range of Σ. The role of Σ is to ensure that we deal with every potential "defect".

Initial step: In the 0th step of the step-by-step construction we define $G_0 = (U_0, \ell_0, W_0)$ by creating an edge for every element of F^-_{Var}. We define U_0 by choosing elements $u_a, v_a, \ldots \in \omega$ so that $\{u_a, v_a\} \cap \{u_b, v_b\} = \emptyset$ for distinct a, b in F^-_{Var}, and $u_a = v_a$ iff $\mathsf{D}(a) = a$ (i.e., a is a domain element of $\mathcal{F}_{Var}$). We can assume that $|\omega \setminus U_0| = \omega$. We define

$$\ell_0(u_a, v_a) = a^\uparrow$$
$$\ell_0(u_a, u_a) = \mathsf{D}(a)^\uparrow$$
$$\ell_0(v_a, v_a) = \mathsf{R}(a)^\uparrow$$

and we label all other edges by $\emptyset$. All non-empty edges constructed so far will be witness edges: $W_0 = \{(u_a, u_a), (u_a, v_a), (v_a, v_a) \mid a \in F^-_{Var}\}$.

Observe that the labels are well defined: when $u_a = v_a$ then a is a domain element, i.e., $\mathsf{D}(a) = a = \mathsf{R}(a)$.

Lemma 3.10. *G_0 is coherent.*

Proof. Conditions GenC and PriC are obvious. For CompC note that $\mathsf{D}(a) ; \mathsf{D}(a) = \mathsf{D}(a)$, $\mathsf{D}(a) ; a = a = a ; \mathsf{R}(a)$ and $\mathsf{R}(a) ; \mathsf{R}(a) = \mathsf{R}(a)$. DomC and RanC are straightforward by the definition of the labels. IdeC follows by the choice of u_a and v_a. □

For the successor step $m+1$ we assume inductively that a finite, coherent graph G_m has been constructed. Let $\Sigma(m+1) = (i, u, v, a, b)$. We will have three types of successor steps depending on the value of i. In each case we will assume that $(u,v) \in E_m$ — otherwise we define $G_{m+1} = G_m$. We can assume, by the induction hypothesis PriC, that $\ell_m(u,v) = c^\uparrow$ for some $c \in F^-_{Var}$.

Successor step when $i = 0$: Our aim is to extend G_m to create an edge (u,w) witnessing a, provided $\mathsf{D}(a) \in \ell_m(u,v)$. Thus we assume that $c \leq \mathsf{D}(a)$ — otherwise we define $G_{m+1} = G_m$. Observe that c must be a domain element (by Claim 3.8), i.e., $\mathsf{D}(c) = c$. Thus, by IdeC for G_m, we have that $u = v$.

We also assume that $\mathsf{D}(c) ; a$ is not a domain element — otherwise we define $G_{m+1} = G_m$. Indeed, if $\mathsf{D}(c) ; a = \mathsf{D}(\mathsf{D}(c) ; a)$ then using (8) we get

$$\begin{aligned}\mathsf{D}(c) &= \mathsf{D}(c) ; \mathsf{D}(a)\\ &= \mathsf{D}(\mathsf{D}(c) ; a)\\ &= \mathsf{D}(c) ; a\\ &\leq a\end{aligned}$$

whence $c = \mathsf{D}(c) \leq a$, i.e., $a \in \ell_m(u,u)$.

Thus, by the above assumptions, we have a loop (u, u) labelled by the upset of a domain element $c = \mathsf{D}(c) \leq a$ such that $\mathsf{D}(c) \mathbin{;} a$ is not a domain element, but we may miss an edge (u, w) witnessing a.

We choose $w \in \omega \setminus U_m$, extend ℓ_m by

$$\ell_{m+1}(u, w) = (\mathsf{D}(c) \mathbin{;} a)^{\uparrow}$$
$$\ell_{m+1}(w, w) = (\mathsf{R}(\mathsf{D}(c) \mathbin{;} a))^{\uparrow}$$

and for every $(p, u) \in E_m$ with $\ell_m(p, u) = d^{\uparrow}$ (some $d \in F^{-}_{\mathit{Var}}$)

$$\ell_{m+1}(p, w) = (d \mathbin{;} a)^{\uparrow}$$

All other edges involving the point w have empty labels. We define $W_{m+1} = W_m \cup \{(w, w), (u, w)\}$. See Figure 1, where we show the elements whose upsets provide the labels for the edges.

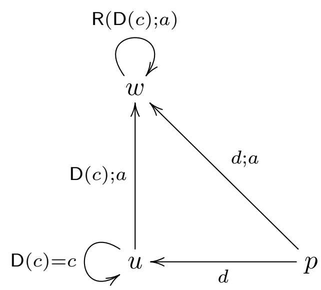

FIGURE 1. Step for domain

Successor step when $i = 1$: This is the mirror image of the previous step for range. We give an outline, since the reader should not have any difficulty in working out the details. Our assumption is that we have a loop (u, u) labelled by the upset of a range element $c = \mathsf{R}(c) \leq a$ such that $a \mathbin{;} \mathsf{R}(c)$ is not a range element, but we may miss an edge (w, u) witnessing a.

We choose $w \in \omega \setminus U_m$, extend ℓ_m by

$$\ell_{m+1}(w, u) = (a \mathbin{;} \mathsf{R}(c))^{\uparrow}$$
$$\ell_{m+1}(w, w) = (\mathsf{D}(a \mathbin{;} \mathsf{R}(c)))^{\uparrow}$$

and for every $(u, p) \in E_m$ with $\ell_m(u, p) = d^{\uparrow}$ (some $d \in F^{-}_{\mathit{Var}}$)

$$\ell_{m+1}(w, p) = (a \mathbin{;} d)^{\uparrow}$$

All other edges involving the point w have empty labels. We define $W_{m+1} = W_m \cup \{(w, w), (w, u)\}$. See Figure 2, where we show the elements whose upsets provide the labels for the edges.

Successor step when $i = 2$: In this case our aim is to extend G_m to create edges (u, w) and (w, v) witnessing a and b, provided $a \mathbin{;} b \in \ell_m(u, v)$. Thus we assume that $c \leq a \mathbin{;} b$ — otherwise we define $G_{m+1} = G_m$.

We can also assume that $u \neq v$ because of the following. If $u = v$, then, by IdeC for G_m, the element c must be a domain element $c = \mathsf{D}(c)$.

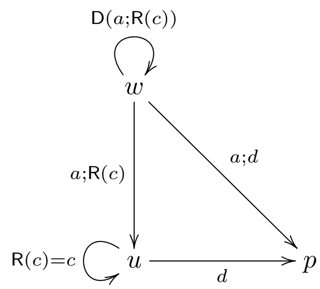

FIGURE 2. Step for range

Thus we have $c = \mathsf{D}(c) \leq a\,;b$. Using Claim 3.7 we get that both a and b are domain elements and that $c = \mathsf{D}(c) \leq a$ and $c = \mathsf{D}(c) \leq b$, whence $a, b \in \ell_m(u, u)$.

Finally, we can assume that neither $\mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c))$ nor $\mathsf{R}(\mathsf{D}(c)\,;a)\,;b\,;\mathsf{R}(c)$ is a domain element because of the following. Assume that $d = \mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c))$ is a domain element, $d = \mathsf{D}(d)$. Recall that $c \leq a\,;b$, whence

$$c = \mathsf{D}(c)\,;c\,;\mathsf{R}(c) \leq \mathsf{D}(c)\,;a\,;b\,;\mathsf{R}(c)$$

Then

$$\begin{aligned}
\mathsf{D}(c) &\leq \mathsf{D}(\mathsf{D}(c)\,;a\,;b\,;\mathsf{R}(c)) \\
&= \mathsf{D}(\mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c))) && \text{by (6)} \\
&= \mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c)) && \text{by } d = \mathsf{D}(d) \\
&\leq a
\end{aligned}$$

i.e., $a \in \ell_m(u, u)$. Also,

$$\begin{aligned}
c &\leq \mathsf{D}(c)\,;a\,;b\,;\mathsf{R}(c) \\
&= \mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c))\,;b\,;\mathsf{R}(c) && \text{by (2)} \\
&= d\,;b\,;\mathsf{R}(c) \\
&\leq b && \text{by } d = \mathsf{D}(d)
\end{aligned}$$

i.e., $b \in \ell_m(u, v)$. Showing that the required witness edges already exist in G_m when $\mathsf{R}(\mathsf{D}(c)\,;a)\,;b\,;\mathsf{R}(c)$ is a domain element is completely analogous.

Summing up: we assume that

(CC1) $c \leq a\,;b$,
(CC2) $u \neq v$,
(CC3) $\mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c)) \neq \mathsf{D}(\mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c)))$,
(CC4) $\mathsf{R}(\mathsf{D}(c)\,;a)\,;b\,;\mathsf{R}(c) \neq \mathsf{R}(\mathsf{R}(\mathsf{D}(c)\,;a)\,;b\,;\mathsf{R}(c))$,

otherwise we define $G_{m+1} = G_m$. If (CC1)–(CC4) hold, then we choose $w \in \omega \setminus U_m$, extend ℓ_m by

$$\begin{aligned}
\ell_{m+1}(u, w) &= (\mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c)))^{\uparrow} \\
\ell_{m+1}(w, v) &= (\mathsf{R}(\mathsf{D}(c)\,;a)\,;b\,;\mathsf{R}(c))^{\uparrow} \\
\ell_{m+1}(w, w) &= (\mathsf{R}(\mathsf{D}(c)\,;a)\,;\mathsf{D}(b\,;\mathsf{R}(c)))^{\uparrow}
\end{aligned}$$

and for $(p, u), (v, q) \in E_m$ with $\ell_m(p, u) = d^{\uparrow}$ and $\ell_m(v, q) = e^{\uparrow}$ (some $d, e \in F^-_{Var}$)

$$\ell_{m+1}(p, w) = (d\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c)))^{\uparrow}$$
$$\ell_{m+1}(w, q) = (\mathsf{R}(\mathsf{D}(c)\,;a)\,;b\,;e)^{\uparrow}$$

All other edges involving w will have empty labels. We define $W_{m+1} = W_m \cup \{(w, w), (u, w), (w, v)\}$. See Figure 3, where the label-generating elements are indicated.

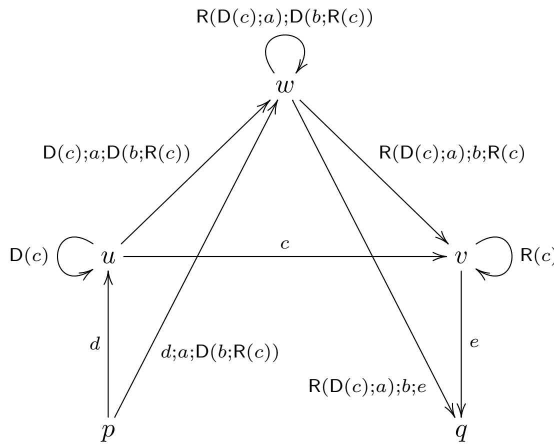

FIGURE 3. Step for composition

Lemma 3.11. *Every G_{m+1} is coherent.*

Proof. First assume that $i = 0$ and $G_m \neq G_{m+1}$. Let $p \neq u$ and d be such that $\ell_m(p, u) = d^{\uparrow}$ so that $\ell_{m+1}(u, w) = (d\,;a)^{\uparrow}$, see Figure 1.

Conditions GenC and PriC are obvious. CompC easily follows by the definition of the labels on the new edges, we just elaborate one case: the triangle consisting of the edges (p, u), (u, w) and (p, w). By RanC for G_m we have $\mathsf{D}(c) = \mathsf{R}(d)$, thus $(d\,;\mathsf{D}(c)\,;a)^{\uparrow} = (d\,;a)^{\uparrow}$, as desired.

For DomC we need first $\mathsf{D}(c) = \mathsf{D}(\mathsf{D}(c)) = \mathsf{D}(\mathsf{D}(c)\,;\mathsf{D}(a)) = \mathsf{D}(\mathsf{D}(c)\,;a)$, by $\mathsf{D}(c) \leq \mathsf{D}(a)$ and (6). Next we show $\mathsf{D}(d) = \mathsf{D}(d\,;a)$. Observe that $\mathsf{D}(c) = \mathsf{R}(d)$ by RanC for G_m, whence $\mathsf{D}(d;a) = \mathsf{D}(d;\mathsf{D}(c);a) = \mathsf{D}(d;\mathsf{D}(\mathsf{D}(c);a)) = \mathsf{D}(d;\mathsf{D}(c)) = \mathsf{D}(d)$ by (6). A similar argument using (7) shows that $\mathsf{R}(\mathsf{D}(c)\,;a) = \mathsf{R}(d\,;a)$, establishing RanC. Finally, for IdeC we need that $\mathsf{D}(c)\,;a$ is not a domain element (which is obvious by our assumption we made during the construction) and similarly for $d\,;a$. By Claim 3.7, $d\,;a \in \mathsf{D}(F_{Var})$ would imply $d, a \in \mathsf{D}(F_{Var})$. But $d \notin \mathsf{D}(F_{Var})$ by IdeC for G_m, and $a \notin \mathsf{D}(F_{Var})$, since this would imply $\mathsf{D}(c)\,;a \in \mathsf{D}(F_{Var})$.

The case for $i = 1$ is completely analogous.

Finally assume that $i = 2$ and $G_m \neq G_{m+1}$. Let $p, q \notin \{u, v\}$ and d, e be such that $\ell_m(p, u) = d^{\uparrow}$ and $\ell_m(v, q) = e^{\uparrow}$, see Figure 3.

Conditions GenC and PriC are obvious again. Next consider DomC and RanC. First we look at the edge (u, w). We need

$$\mathsf{D}(c) = \mathsf{D}(\mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c))) \tag{26}$$

and $\mathsf{R}(\mathsf{D}(c);a);\mathsf{D}(b;\mathsf{R}(c)) = \mathsf{R}(\mathsf{D}(c);a;\mathsf{D}(b;\mathsf{R}(c)))$. Now $\mathsf{D}(c) \geq \mathsf{D}(\mathsf{D}(c);a;\mathsf{D}(b;\mathsf{R}(c)))$ is obvious. For the other direction recall that we assumed that $c \leq a\,;b$, whence

$$\begin{aligned}\mathsf{D}(c) &= \mathsf{D}(\mathsf{D}(c)\,;c\,;\mathsf{R}(c))\\ &\leq \mathsf{D}(\mathsf{D}(c)\,;a\,;b\,;\mathsf{R}(c))\\ &= \mathsf{D}(\mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c)))\end{aligned}$$

by (6), and using (11) and (9) we get

$$\begin{aligned}\mathsf{R}(\mathsf{D}(c)\,;a)\,;\mathsf{D}(b\,;\mathsf{R}(c)) &= \mathsf{R}(\mathsf{D}(c)\,;a)\,;\mathsf{R}(\mathsf{D}(b\,;\mathsf{R}(c)))\\ &= \mathsf{R}(\mathsf{D}(c)\,;a\,;\mathsf{R}(\mathsf{D}(b\,;\mathsf{R}(c))))\\ &= \mathsf{R}(\mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c)))\end{aligned}$$

as desired. Next we consider the edge (p, w). Recall that $\mathsf{D}(c) = \mathsf{R}(d)$ by RanC for G_m. Then we get

$$\begin{aligned}\mathsf{D}(d) &= \mathsf{D}(d\,;\mathsf{D}(c))\\ &= \mathsf{D}(d\,;\mathsf{D}(\mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c)))) && \text{by (26)}\\ &= \mathsf{D}(d\,;\mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c))) && \text{by (6)}\\ &= \mathsf{D}(d\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c)))\end{aligned}$$

and

$$\begin{aligned}\mathsf{R}(\mathsf{D}(c)\,;a)\,;\mathsf{D}(b\,;\mathsf{R}(c)) &= \mathsf{R}(\mathsf{D}(c)\,;a)\,;\mathsf{R}(\mathsf{D}(b\,;\mathsf{R}(c))) && \text{by (11)}\\ &= \mathsf{R}(\mathsf{R}(d)\,;a)\,;\mathsf{R}(\mathsf{D}(b\,;\mathsf{R}(c)))\\ &= \mathsf{R}(d\,;a)\,;\mathsf{R}(\mathsf{D}(b\,;\mathsf{R}(c))) && \text{by (7)}\\ &= \mathsf{R}(d\,;a\,;\mathsf{R}(\mathsf{D}(b\,;\mathsf{R}(c)))) && \text{by (9)}\\ &= \mathsf{R}(d\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c))) && \text{by (11)}\end{aligned}$$

as desired. Checking DomC and RanC for other edges is completely analogous.

For CompC the main observation is the following. By our assumption $c \leq a\,;b$, whence

$$\begin{aligned}c &\leq \mathsf{D}(c)\,;a\,;b\,;\mathsf{R}(c)\\ &= \mathsf{D}(c)\,;a\,;\mathsf{R}(\mathsf{D}(c)\,;a)\,;\mathsf{D}(b\,;\mathsf{R}(c))\,;b\,;\mathsf{R}(c)\\ &= \mathsf{D}(c)\,;a\,;\mathsf{D}(b\,;\mathsf{R}(c))\,;\mathsf{R}(\mathsf{D}(c)\,;a)\,;b\,;\mathsf{R}(c)\end{aligned}$$

Thus CompC holds for the triangle consisting of the edges (u, w), (w, v) and (u, v). Checking CompC for the other triangles is easy, using the definition of the labels and the already established DomC and RanC.

Finally, checking IdeC can be done similarly to the case $i = 0$, by using the assumption that (u, w) and (w, v) are labelled by non-domain elements and that, for every $x, y \in F^-_{Var}$, we have $x\,;y \in \mathsf{D}(F_{Var})$ iff both x and y are in $\mathsf{D}(F_{Var})$. □

Lemma 3.12. *G_ω is coherent and saturated.*

Proof. Coherence of G_ω follows from the coherence of all G_m.

Let us check DomS. Let $(u, u) \in U_\omega \times U_\omega$ be such that $\mathsf{D}(a) \in \ell_\omega(u, u)$. By coherence of G_ω we have that $\ell_\omega(u, u) = \mathsf{D}(c)^\uparrow$ for some element c. If $\mathsf{D}(c)\,;a$

is a domain element, then $a \in \ell_\omega(u,u)$, as we have seen in case $i = 0$ of the successor step of the construction. If $\mathsf{D}(c) \mathbin{;} a \notin \mathsf{D}(F_{Var})$, consider $m \in \omega$ such that $\Sigma(m+1) = (0,u,u,a,a)$ and $\mathsf{D}(a) \in \ell_m(u,u)$. Then G_{m+1} contains an edge (u,w) such that $\ell_{m+1}(u,w) = (\mathsf{D}(c) \mathbin{;} a)^{\uparrow}$. By Claim 3.2 we get $\mathsf{D}(c) \mathbin{;} a \leq a$, i.e., $a \in \ell_{m+1}(u,w) = \ell_\omega(u,w)$. Checking RanS is completely analogous.

For CompS assume that $a \mathbin{;} b \in \ell_\omega(u,v) = c^{\uparrow}$, and let m be such that $\Sigma(m+1) = (2,u,v,a,b)$ and $a \mathbin{;} b \in \ell_m(u,v)$. Recall that we showed at case $i = 2$ of the successor step of the construction that the edges witnessing a and b already exists in G_m if any of the conditions (CC2)–(CC3) fails. If the conditions hold, then G_{m+1} contains w such that $(\mathsf{D}(c) \mathbin{;} a \mathbin{;} \mathsf{D}(b \mathbin{;} \mathsf{R}(c)))^{\uparrow} = \ell_{m+1}(u,w) = \ell_\omega(u,w)$ and $(\mathsf{R}(\mathsf{D}(c) \mathbin{;} a) \mathbin{;} b \mathbin{;} \mathsf{R}(c))^{\uparrow} = \ell_{m+1}(w,v) = \ell_\omega(w,v)$. Using Claim 3.2 we get $a \in \ell_\omega(u,w)$ and $b \in \ell_\omega(w,v)$ as desired. □

Next we define a valuation $\flat$ of variables. Recall that for term r, its equivalence class in F_{Var} is denoted by $\overline{r}$. We let

$$x^\flat = \{(u,v) \in U_\omega \times U_\omega : \overline{x} \in \ell_\omega(u,v)\}$$

for every variable $x \in \mathit{Var}$. Let $\mathcal{A} = (A, ;, \mathsf{D}, \mathsf{R}, +)$ be the subalgebra of the full algebra $(\wp(U_\omega \times U_\omega), ;, \mathsf{D}, \mathsf{R}, +)$ generated by $\{x^\flat : x \in \mathit{Var}\}$. Clearly $\mathcal{A}$ is representable.

Lemma 3.13. *For every join-free term r and $(u,v) \in U_\omega \times U_\omega$,*

$$(u,v) \in r^\flat \text{ iff } \overline{r} \in \ell_\omega(u,v)$$

where $r^\flat$ is the interpretation of r in $\mathcal{A}$ under the valuation $\flat$.

Proof. This is an easy induction on the complexity of terms, using that $\mathcal{A}$ is representable. For the left-to-right direction use that G_ω satisfies the coherence conditions CompC, DomC and RanC. For the right-to-left direction use that G_ω satisfies the saturation conditions CompS, DomS and RanS. □

Recall that we assumed that $\mathcal{F}_{Var} \not\models s \leq t$. In the initial step of the construction we created the edge $(u_{\overline{s}}, v_{\overline{s}})$ such that $\ell_0(u_{\overline{s}}, v_{\overline{s}}) = \overline{s}^{\uparrow}$. Thus $\overline{s} \in \ell_\omega(u_{\overline{s}}, v_{\overline{s}})$ and $\overline{t} \notin \ell_\omega(u_{\overline{s}}, v_{\overline{s}})$. Hence, by Lemma 3.13, $(u_{\overline{s}}, v_{\overline{s}}) \in s^\flat$ and $(u_{\overline{s}}, v_{\overline{s}}) \notin t^\flat$. That is, $\mathcal{A} \not\models s \leq t$, as desired. This completes the final step required in the proof of Theorem 3.1.

4. Demonic composition

In this section we look at demonic composition. We define the following set Ax^d of axioms:

$$x * (y * z) = (x * y) * z, \tag{27}$$
$$\mathsf{D}(x) * x = x, \tag{28}$$
$$\mathsf{D}(x) * \mathsf{D}(y) = \mathsf{D}(y) * \mathsf{D}(x), \tag{29}$$
$$\mathsf{D}(\mathsf{D}(x) * y) = \mathsf{D}(x) * \mathsf{D}(y), \tag{30}$$
$$x * \mathsf{D}(y) = \mathsf{D}(x * y) * x, \tag{31}$$
$$\mathsf{DR}(x) = \mathsf{R}(x), \tag{32}$$
$$\mathsf{RD}(x) = \mathsf{D}(x), \tag{33}$$
$$\mathsf{RR}(x) = \mathsf{R}(x), \tag{34}$$
$$\mathsf{R}(x) * \mathsf{R}(y) = \mathsf{R}(y) * \mathsf{R}(x), \tag{35}$$
$$\mathsf{R}(x * y) * \mathsf{R}(y) = \mathsf{R}(x * y), \tag{36}$$
$$x * \mathsf{R}(x) = x, \tag{37}$$

These axioms differ from (1)–(13) by the inclusion of the stronger domain property (31) and the weakening of the range property (7) to (36). Axioms (27)–(31) state that the R-free reduct of an algebra $\mathcal{S} = (S, *, \mathsf{D}, \mathsf{R})$ is a restriction semigroup, while the remaining laws make $\mathcal{S}$ a two-sided closure semigroup in the sense of Jackson and Stokes [20].

We claim that Ax^d is sound for demonic composition of binary relations with domain and range, i.e., for the representation class $\mathbb{R}(*, \mathsf{D}, \mathsf{R})$. Soundness of the laws (27)–(31) is observed in Desharnais, Jipsen and Struth [12]. Laws (32)–(37) simply ensure that domain and range elements coincide, and that $\mathsf{R}(x)$ is the smallest domain element (with respect to the usual order in a meet semilattice) in the abstract algebra that acts as a right identity for x; see [20, Proposition 1.2].

Observe that (31) fails for angelic composition. Indeed, we may have $(a, b) \in \mathsf{D}(x \,;\, y) \,;\, x$ (because $(a, b) \in x$ and we have some c, d such that $(a, c) \in x$ and $(c, d) \in y$) while $(a, b) \notin x \,;\, \mathsf{D}(y)$ (because we may have $(b, b) \notin \mathsf{D}(y)$).

The main result of this section is that Ax^d provides a complete equational axiomatization for demonic composition with domain and range.

Theorem 4.1. *The variety $\mathbb{V}(*, \mathsf{D}, \mathsf{R})$ generated by the representation class $\mathbb{R}(*, \mathsf{D}, \mathsf{R})$ is axiomatized by Ax^d.*

Proof. Let us say that a model of Ax^d is *cycle-free*, if it satisfies the laws

$$x * y = x \Rightarrow \mathsf{D}(y) = y \tag{38}$$
$$x * y = \mathsf{D}(z) \Rightarrow x = \mathsf{D}(x) \;\;\&\;\; y = \mathsf{D}(y) \tag{39}$$

for every x, y, z. We will show in Lemma 4.3 that the free algebras of the variety defined by Ax^d are cycle free (this is somewhat analogous to Claims 3.6–3.8 of the previous section). Furthermore, by Lemma 4.4, cycle-free models of Ax^d are in the variety $\mathbb{V}(*, \mathsf{D}, \mathsf{R})$ (in fact, they are representable). □

The rest of this section is devoted to prove these lemmas.

4.1. **The free algebra.** Let s be a term in the variables $x_1, \ldots, x_n$, and let $\mathrm{out}(s)$ denote the set of pairs (x_i, j), where $j \in \omega$ denotes the number of occurrences of the variable x_i that lie outside of any application of D or R. Let $\mathrm{out}_{x_i}(s)$ denote the value j such that $(x_i, j) \in \mathrm{out}(s)$.

Lemma 4.2. *If $Ax^d \vdash s = t$, then $\mathrm{out}(s) = \mathrm{out}(t)$, for any pair s, t of terms.*

Proof. This follows by induction on the length of a derivation: we show that the statement holds true for each individual step of deduction, and hence by a trivial induction argument, also for any finite sequence of deductions. A step of deduction starting from s involves replacing a subterm of s by the image of an axiom under some substitution. Assume we have an axiom $u = v$ (or $v = u$) from Ax^d, a substitution θ, and are replacing an instance of the subterm $\theta(u)$ of s by $\theta(v)$ to obtain a term s'. Thus it suffices to show that $\mathrm{out}(\theta(u)) = \mathrm{out}(\theta(v))$. This follows because $\mathrm{out}(u) = \mathrm{out}(v)$, as can be seen by inspection of the laws in Ax^d. □

Lemma 4.3. *The free algebras of the variety defined by Ax^d are cycle free.*

Proof. For the first law (38), assume that s and t are terms such that $s * t = s$. We show that $t = \mathsf{D}(t)$ is a consequence of Ax^d. Now, as $Ax^d \vdash s * t = s$, Lemma 4.2 shows that $\mathrm{out}(s * t) = \mathrm{out}(s)$. As for all variables x, we have $\mathrm{out}_x(s * t) = \mathrm{out}_x(s) + \mathrm{out}_x(t)$ it follows that $\mathrm{out}_x(t) = 0$ always. Thus all variables in t lie under an application of D or R, showing that $Ax^d \vdash t = \mathsf{D}(t)$, as required.

For the second law (39), assume that s, t, d are terms such that $Ax^d \vdash s*t = \mathsf{D}(d)$. An almost identical argument to before shows that all variables in s and t have all occurrences under an application of D or R, and hence $Ax^d \vdash s = \mathsf{D}(s)$ and $t = \mathsf{D}(t)$. □

4.2. **Representing cycle-free algebras.**

Lemma 4.4. *Any cycle-free model $\mathcal{S} = (S, *, \mathsf{D}, \mathsf{R})$ of Ax^d is representable: $\mathcal{S} \in \mathbb{R}(*, \mathsf{D}, \mathsf{R})$.*

Proof. We piece together a representation θ by way of an inductive gluing of pieces of the Wagner–Preston $(*, \mathsf{D})$-representations. Recall [20, Theorem 3.9] that a restriction semigroup $(S, *, \mathsf{D})$ can be represented as partial maps over itself by

$$a \mapsto \{(x * D(a), x * a) \mid x \in S\} = \{(x, x * a) \mid x \in S * \mathsf{D}(a)\}.$$

For any $s \in S$ we let F_s, the *forward closure* of s, denote the labelled directed graph obtained from this representation on the induced subgraph reachable from the point s. At each step of our inductive gluing we will have a $(*, \mathsf{D})$-representation which does not necessarily correctly represent R. A *range defect* for an element $s \in S$ will be a point p of the representation in which the domain element $\mathsf{R}(s)$ is defined, but for which p is not in the range of s^θ.

Claim 4.5. *If a restriction semigroup $(S, *, \mathsf{D})$ is cycle free then the only cycles in the Wagner–Preston representation of $(S, *, \mathsf{D})$ are loops.*

Proof. Assume for contradiction that that there is a cycle in the Wagner–Preston representation of $(S, *, \mathsf{D})$. The underlying graph of this representation is transitive, so there are $x, a, b \in S$ with $x = x * \mathsf{D}(a)$ and $x * a * \mathsf{D}(b) = x * a$ and $x * a * b = x$. Then by (38) we have $a * b = \mathsf{D}(a * b)$. By (39) we have $a = \mathsf{D}(a)$ and $b = \mathsf{D}(b)$, so that in fact $x = x * a = x * a * b$, and the cycle is a loop. □

Claim 4.6. *In a cycle-free restriction semigroup* $(S, *, \mathsf{D})$*, for any element* $s \neq \mathsf{D}(s)$*, we have* $\mathsf{D}(s) \in F_{\mathsf{D}(s)} \backslash F_s$.

Proof. Otherwise there would be b with $s * b = \mathsf{D}(s)$. Then $s = \mathsf{D}(s)$ by (39). □

Let $\mathcal{S} = (S, *, \mathsf{D}, \mathsf{R})$ be a cycle-free model of Ax^d. We will take the union over an ω-chain of partial representations $\theta_0, \theta_1, \theta_2, \ldots$ over sets $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$, where θ_{i+1} coincides with θ_i when restricted to X_i. The partial representations are constructed inductively, with the Wagner–Preston representation as the base case θ_0 (so that X_0 is the universe S of the algebra). We have the following in inductive hypothesis.

(1) Domain and demonic composition are correctly represented by θ_i.
(2) The partial representation θ_i is faithful in the sense that for $s \neq t$ there are points p, q with $(p, q) \in s^\theta \backslash t^\theta$ or $(p, q) \in t^\theta \backslash s^\theta$.

Note that by Hypothesis (1), only range might fail to be represented properly by θ_i. However, as $s * \mathsf{R}(s) = s$ and $\mathsf{R}(s)$ is a domain element (by $\mathsf{D}(\mathsf{R}(s)) = \mathsf{R}(s)$), it follows from Hypothesis (1) that we do at least have $\mathsf{R}(s^{\theta_i}) \subseteq \mathsf{R}(s)^{\theta_i}$. The construction of X_{i+1} and θ_{i+1} will be such that all range defects present in X_i under θ_i are corrected by θ_{i+1}, though new range defects may have been introduced at points in $X_{i+1} \backslash X_i$. In this way there will be no range defects in the final representation over $\bigcup_{i \in \omega} X_i$ so that we will have achieved the desired representation of $\mathcal{S}$.

We now begin the induction. The conditions hold for the base case: the inductive hypothesis simply states that we have a faithful $(*, \mathsf{D})$ representation, which holds by [20, Theorem 3.9].

Let us assume that the inductive hypothesis on the partial representation θ_i of $(S, *, \mathsf{D}, \mathsf{R})$ over X_i. If range is correctly represented by θ_i, then our proof is complete: we may let $X_{i+1} = X_i$ and $\theta_{i+1} = \theta_i$. Otherwise there are range defects in X_i under θ_i. We explain how to correct any such range defect. The set X_{i+1} and partial representation θ_{i+1} are obtained by simultaneously applying the described method to all range defects in X_i under θ_i. To avoid proliferation of indices and symbols, for the remainder of the argument we use X to abbreviate X_i and θ to abbreviate θ_i. Let $p \in X$ be a range defect for some element $s \in S$ under θ: so $p \xrightarrow{\mathsf{R}(s)} p$, but p is not in the range of s^θ; that is, no point $p' \in X$ has $p' \xrightarrow{s} p$. Note that in this instance it cannot be that s is a domain element, as then $s = \mathsf{R}(s)$ which would give $p \xrightarrow{s} p$. Thus F_s is a proper subgraph of $F_{\mathsf{D}(s)}$. Adjoin a disjoint copy of the forward closure $F_{\mathsf{D}(s)}$ to X. Retain all edges and labels already existing in X and in the newly adjoined $F_{\mathsf{D}(s)}$, but we add some new edges between $F_{\mathsf{D}(s)} \backslash F_s$ and X.

Before we describe these new *connector edges*, we observe the following lemma, which guarantees that suitable target points in X exist. The situation is also depicted in Figure 4, where boldface vertices and edges correspond to given assumptions in the lemma, and non-boldface vertices and edges are those shown to exist in the lemma.

Lemma 4.7. *If* $s \xrightarrow{a} t$ *then there is* $q \in X$ *such that* $p \xrightarrow{a} q$.

Proof. If $s \xrightarrow{a} t$ then $s * \mathsf{D}(a) = s$ so that $\mathsf{R}(s) * \mathsf{D}(a) = \mathsf{R}(s)$, as $\mathsf{R}(s)$ is the smallest domain element that acts as a right identity for s. As $\mathsf{R}(s)$ is defined at p, we also have $\mathsf{R}(s) * \mathsf{D}(a)$ defined at p, and as both $\mathsf{D}(a)$ and $\mathsf{R}(s)$ are domain elements,

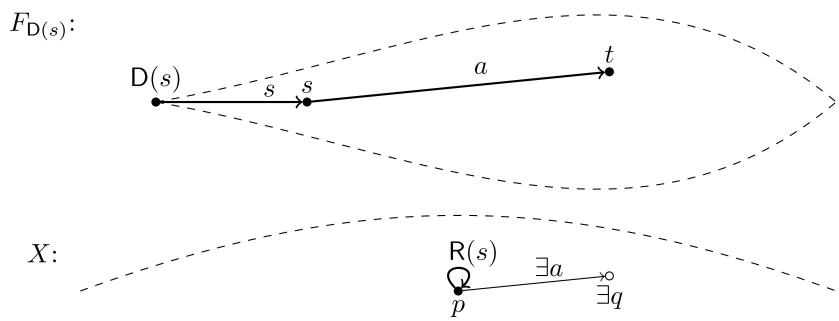

FIGURE 4. Diagram depicting Lemma 4.7. A fresh copy of $F_{\mathsf{D}(s)}$ has been placed aside X in order to eventually correct a defect in the range of s^θ at p in X. Then if $s \xrightarrow{a} t$ there exists q such that $p \xrightarrow{a} q$.

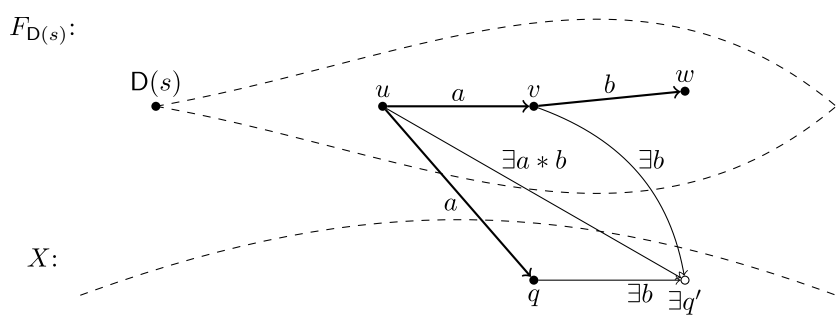

FIGURE 5. Diagram depicting Lemma 4.8. If $u \xrightarrow{a} v \xrightarrow{b} w$ in $F_{\mathsf{D}(s)}$ and $u \xrightarrow{a} q$ is a connector edge, then there exists q' in X with $q \xrightarrow{b} q'$ and such that $u \xrightarrow{a*b} q'$ is a connector edge.

Hypothesis (1) again ensures that $p \xrightarrow{\mathsf{D}(a)} p$. Then by Hypothesis (1) again, there is $q \in X$ with $p \xrightarrow{a} q$. □

Now we describe the connector edges. Let $v \in F_s$ and $u \in F_{\mathsf{D}(s)}$ with $u \neq v$ and assume that $v = s * c$ for some c. For each edge $u \xrightarrow{a} v$ in $F_{\mathsf{D}(s)}$, Lemma 4.7 shows that there exists at least one $q \in X$ with $p \xrightarrow{c} q$. For every such q we add the edge $u \xrightarrow{a} q$. In particular, the edge $\mathsf{D}(s) \xrightarrow{s} s$ starts at $\mathsf{D}(s) \in F_{\mathsf{D}(s)}$ and ends at $s \in F_s$, so that the range defect for s at p is cured.

Before we verify the inductive hypotheses are maintained, we observe the following useful lemma, which is also depicted in Figure 5 (in the case when $b \neq \mathsf{D}(b)$), with the same convention on boldface edges and vertices as in Figure 4.

Lemma 4.8. *Let $u \xrightarrow{a} q$ be a connector edge (so $u \in F_{\mathsf{D}(s)}$ and $q \in X$), and $v \in F_{\mathsf{D}(s)}$ have $u \xrightarrow{a} v$. If $v \xrightarrow{b} w$ in $F_{\mathsf{D}(s)}$, then there is $q' \in X$ with $q \xrightarrow{b} q'$ in X and for every such q':*

- $u \xrightarrow{a*b} q'$
- *if b is a domain element then $q = q'$ and $v = w$, but otherwise $v \xrightarrow{b} q'$.*

Proof. Assume the hypothesis of the lemma. Because $u \xrightarrow{a} q$ is a connector edge, there must exist some $v' \in S$ and $c \in S$ with $s \xrightarrow{c} v'$, and $u \xrightarrow{a} v'$ as well as $p \xrightarrow{c} q$. Because all elements acted as functions within $F_{\mathsf{D}(s)}$ and both $u \xrightarrow{a} v$ and $u \xrightarrow{a} v'$, we must have $v = v'$. Indeed, we can write v as $s*c$ and then w as $s*c*b$, so that $s \xrightarrow{c*b} w$. Also, as $u \xrightarrow{a} v \xrightarrow{b} w$ and because demonic composition coincides with angelic composition within $F_{\mathsf{D}(s)}$ we have $u \xrightarrow{a*b} w$. Hence, by Lemma 4.7 we have that $a * b$ is defined at p in X. By Hypothesis (1), every edge labelled by a leaving p is in the domain of b. In particular, b is defined at q, so that there exists points q' such that $p \xrightarrow{b} q'$. Because $s \xrightarrow{c*b} w$ and $u \xrightarrow{a*b} w$ the definition of connector edge ensures that there is a connector edge $u \xrightarrow{a*b} q'$, for any such q'. Provided that $v \neq w$ an almost identical argument shows that there is a connector edge $v \xrightarrow{b} q'$. □

We need to verify the inductive hypothesis holds. It is the verification of demonic composition that requires particular attention, so we check the other details first.

First observe that domain is correctly represented, as this was already true in X and in the copy of $F_{\mathsf{D}(s)}$, and each connector edge $\xrightarrow{a}$ started from a point in $F_{\mathsf{D}(s)}$ that already had an outgoing edge $\xrightarrow{a}$. Faithfulness is preserved trivially, as the representation on X was already faithful, and no new edges were added to this.

Now we must check demonic composition.

Compositional witness: if $x \xrightarrow{a*b} y$, find z with $x \xrightarrow{a} z \xrightarrow{b} y$. Assume that $a * b$ labels some edge $x \xrightarrow{a*b} y$. We need to verify there is z with $x \xrightarrow{a} z \xrightarrow{b} y$. If $x, y \in X$ or $x, y \in F_{\mathsf{D}(s)}$ then we are done, as $*$ is correctly represented on these sets. As there are no edges from X to $F_{\mathsf{D}(s)}$, it remains to consider the case of a connector edge, where $x \in F_{\mathsf{D}(s)}$ and $y \in X$. In this case, $a * b$ also labels an edge from $x \in F_{\mathsf{D}(s)}$ to some point $s * c \in F_s$ (with $x \neq s * c$), and $p \xrightarrow{c} y$ in X. Because $*$ is correctly represented in $F_{\mathsf{D}(s)}$ it follows that there is a point $z' \in F_{\mathsf{D}(s)}$ with $x \xrightarrow{a} z' \xrightarrow{b} s * c$. (In fact, $z' = x * a$ by the definition of the Wagner–Preston representation.) If $z' = s * c$ (implying $z' \in F_s$), then $x * a * b = x * a$, which by (38) shows that $b = \mathsf{D}(b)$. Then $s \xrightarrow{c*b} s * c$ in $F_{\mathsf{D}(s)}$. So $c * b$ labels an edge starting at p by Lemma 4.7. Thus every edge labelled c leaving p in X is followed by one labelled b; in particular this is true for the edge $p \xrightarrow{c} y$. As b is a domain element, it follows that b labels a loop at y. Thus $x \xrightarrow{a} y \xrightarrow{b} y$ so that the required z can be chosen to be y.

Now assume that $z' \neq s * c$. In this instance, there is a connector edge $z' \xrightarrow{b} y$, so that we may choose z to be z'. This completes the check for compositional witnesses.

Demonic witness: if $x \xrightarrow{a*b} y$, verify every $x \xrightarrow{a} z$ has z in the domain of b^θ. Assume $x \xrightarrow{a*b} y$ and that $x \xrightarrow{a} z$. Note that if $x \in X$ then so also are all of x, y, z and we are done by the inductive hypothesis. So for the remainder of the proof it suffices to assume that $x \in F_{\mathsf{D}(s)}$. Now, if $y \in F_{\mathsf{D}(s)}$, then every edge in $F_{\mathsf{D}(s)}$ labelled a leaving x (and there is only one within $F_{\mathsf{D}(s)}$) can be followed by b. By Lemma 4.8, this is also true of every connector edge leaving x. As the edge $x \xrightarrow{a} z$ is either in $F_{\mathsf{D}(s)}$ or a connector edge, the verification is complete for when $y \in F_{\mathsf{D}(s)}$. Now assume that $y \in X$. Then $x \xrightarrow{a*b} y$ is a connector edge, so there is a point y' in F_s with $x \xrightarrow{a*b} y'$. Then we are in the previous case and deduce that every edge labelled by a leaving x (be it in $F_{\mathsf{D}(s)}$ or a connector edge) is followed by one labelled b.

Composition: if $(x, z) \in a^\theta * b^\theta$**, verify that** $x \xrightarrow{a*b} z$**.** Assume $x \xrightarrow{a} y \xrightarrow{b} z$ and every z' with $x \xrightarrow{a} y'$ has z' in the domain of b^θ. We need to show that $a \xrightarrow{a*b} z$. If $x, y, z \in F_{\mathsf{D}(s)}$ or $x, y, z \in X$ then we are done, because demonic composition is correctly represented in $F_{\mathsf{D}(s)}$ and in X, and we did not change the domains of any elements when adding new edges. (Note that this is the case even if $x, y, z \in F_{\mathsf{D}(s)}$ but we consider some $y' \in X$ that happens to lie in the domain of b^θ: it remains true that every $x \xrightarrow{a} y''$ in $F_{\mathsf{D}(s)}$ also is in the domain of b^θ, so we would still have $x \xrightarrow{a*b} z$. Alternatively, use the fact that composition is functional in $F_{\mathsf{D}(s)}$.) Thus we may assume that $x \in F_{\mathsf{D}(s)}$ but at least one of $y, z \in X$. If $y \in F_{\mathsf{D}(s)}$ then $y \xrightarrow{b} z$ is a connector edge, and the definition of such edges implies that there is $z' \in F_s$ with $x \xrightarrow{a} y \xrightarrow{b} z'$. Then $x \xrightarrow{a*b} z'$, as composition in $F_{\mathsf{D}(s)}$ is functional. But then $x \xrightarrow{a*b} z$ also, as required. Thus we may assume that both y and z lie in X (there are no edges from X to $F_{\mathsf{D}(s)}$, so if y in X then $z \in X$ also).

In this instance, there is $y' \in F_s$ and $x \xrightarrow{a} y'$. We are assuming that every such point y' is in the domain of b^θ, so it follows that there is z' with $y' \xrightarrow{b} z'$. Moreover, as $y' \in F_s$ we can select $z' \in F_s$ also. T hen $x \xrightarrow{a*b} z'$, as composition in $F_{\mathsf{D}(s)}$ is functional. We have not yet shown that $x \xrightarrow{a*b} z$. There is $c \in S$ such that $s \xrightarrow{c} s * c = y'$ in $F_{\mathsf{D}(s)}$, and therefore $s \xrightarrow{c*b} z'$. Thus $(c * b)^\theta$ is defined at p also, and hence we have $p \xrightarrow{c*b} z$ (as $p \xrightarrow{c} y \xrightarrow{b} z$). Thus the definition of connector edges ensures that $x \xrightarrow{a*b} z$ also. This completes the proof of Lemma 4.4. □

By Lemma 4.3, Theorem 4.1 now follows immediately from Lemma 4.4.

Question 4.9. *The class of algebras of binary relations under angelic composition, domain and range has no finite axiomatisation* [18, 23]. *The class of partial maps under composition, domain and range has a finite axiomatisation* [32]. *Does the class* $\mathbb{R}(*, \mathsf{D}, \mathsf{R})$ *of algebras of binary relations with demonic composition, domain and range have no finite axiomatisation?*

Question 4.10. *Does the class* $\mathbb{R}(;, \mathsf{D}, \mathsf{R}, +)$ *have a finite axiomatisation? Does the class* $\mathbb{V}(;, \mathsf{D}, \mathsf{R})$ *have a finite axiomatisation?*

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